Algebraic Groups I. Existence of Jordan decomposition

1. Main result

In this handout, we address the key issue in the proof of existence of Jordan decomposition: if k is algebraically closed and G is a smooth closed subgroup of some GL_n via $j : G \hookrightarrow \mathrm{GL}_n$ then for any $g \in G(k)$ we claim that $j(g)_{\mathrm{ss}}$ and $j(g)_u$ lie in $G(k)$.

As was proved in class, we make choose an auxiliary closed k-subgroup inclusion $i : G' = \mathrm{GL}_n \hookrightarrow \mathrm{GL}(V)$ such that $G = N_{G'}(L)$ for a line L in V. Thus, $i(j(g))$ preserves L, so the Jordan components $i(j(g))_{ss}$ and $i(j(g))_u$ preserve L since they are k-polynomials in $i(j(g))$ (computed in the k-algebra $\mathrm{End}(V)$). Hence, $i(j(g))_{ss}$ and $i(j(g))_u$ lie in $G(k)$ *provided* they lie in $G'(k)$! Thus, by replacing j with i we are reduced to the following more concrete problem: if $j : G = \mathrm{GL}_m \to G' = \mathrm{GL}_N$ is a k-subgroup inclusion then for any $g \in G(k)$ we have $j(g)_{ss} = j(g_{ss})$ and $j(g)_u = j(g_u)$; this assertion at least *makes sense* since the groups $\mathrm{GL}_m(k)$ and $\mathrm{GL}_N(k)$ have an a-priori theory of Jordan decomposition (for which we have already seen that the classical notions of semisimplicity and unipotence are equivalent to the corresponding properties for the associated right-translation operators on the coordinate ring).

Let $g' = j(g)$. This has a Jordan decomposition, say $g' = g'_{ss}g'_u$, and the right translation $\rho_{g'}$ on $k[G']$ is equivariant with respect to ρ_g on the *quotient* $k[G]$, so it preserves the kernel $I = \ker(k[G'] \twoheadrightarrow k[G])$. But on each finite-dimensional G'-stable subspace W of $k[G']$, the operators $\rho_{g'_{ss}} = \rho_{g',ss}$ and $\rho_{g'_u} = \rho_{g',u}$ are k-polynomials in $\rho_{g'}|_W$ and thus preserve $I \cap W$. The W's exhaust $k[G']$ in the limit, so it follows that $\rho_{g'_{ss}}$ and $\rho_{g'_u}$ preserve I in $k[G']$, so they preserve the closed subscheme G in G' defined by I. That is, right translation on G' by g'_{ss} and g'_u preserve G, but these translations move 1 to g'_{ss} and g'_u respectively. Hence, $g'_{ss}, g'_u \in G(k)$! Moreover, the right-translation operators on $k[G]$ by these points are induced by the corresponding operators on $k[G']$ that *are* respectively semisimple and unipotent since by definition $g'_{ss}, g'_u \in G'(k) = \mathrm{GL}_N(k)$ are respectively semisimple and unipotent. In other words, as elements of $G(k) = \mathrm{GL}_m(k)$ the elements g'_{ss} and g'_u are respectively semisimple and unipotent. But they also commute, so they *are* the Jordan components of g. This proves that the Jordan components of g in $G(k) = \mathrm{GL}_m(k)$ are carried to those of $g' = j(g)$ in $G'(k) = \mathrm{GL}_N(k)$, so we are done.

2. Variants in linear algebra over arbitrary fields

Jordan canonical form, upon which the preceding discussion rests, takes place over algebraically closed fields. As a supplement, the following optional exercises develop a version of "Jordan decomposition" in additive and multiplicative forms for finite-dimensional vector spaces over any field, including what can go wrong over an imperfect field. (Briefly, the formation of the decomposition commutes with arbitrary field extension when the initial ground field is perfect, but not otherwise. This leads to some difficulties when working with linear algebraic groups over imperfect fields.) The additive case underlies Jordan decomposition in Lie algebras, and the multiplicative case underlies Jordan decomposition in linear algebraic groups.

Let V be a finite-dimensional nonzero vector space over a field F, with dimension $n > 0$. A linear self-map $T : V \to V$ is *semisimple* if every T-stable subspace of V admits a T-stable complementary subspace. (That is, if $T(W) \subseteq W$ then there exists a decomposition $V = W \oplus W'$ with $T(W') \subseteq W'$.) Keep in mind that such a complement is not unique in general (e.g., consider T to be a scalar multiplication with $\dim V > 1$). Let χ_T denote the characteristic polynomial of T, and m_T the minimal polynomial of T,

1. (i) For each monic irreducible $\pi \in F[t]$, define $V(\pi)$ to be the subspace of $v \in V$ killed by a power of $\pi(T)$. Prove that $V(\pi) \neq 0$ if and only if $\pi | m_T$, and that $V = \oplus_{\pi | m_T} V(\pi)$. (In case F is algebraically closed, these are the *generalized eigenspaces* of T on V.)

(ii) Use rational canonical form to prove that T is semisimple if and only if m_T has no repeated irreducible factor over F. (Hint: apply (i) to T-stable subspaces of V to reduce to the case when m_T has one monic irreducible factor.) Deduce that a Jordan block of rank > 1 is never semisimple, that m_T is the "squarefree part" of χ_T when T is semisimple, and that if $W \subseteqq V$ is a T-stable nonzero proper subspace then the induced endomorphisms $T_W : W \to W$ and $\overline{T} : V/W \to V/W$ are semisimple when T is semisimple.

(iii) Let $T' : V' \to V'$ be another linear self-map with V' nonzero and finite-dimensional over F. Prove that T and T' are semisimple if and only if the self-map $T \oplus T'$ of $V \oplus V'$ is semisimple.

(iii) Choose $T \in \mathrm{Mat}_n(F)$, and let F'/F be an extension splitting m_T. Prove that T is semisimple as an F'-linear endomorphism of ${F'}^n$ if and only if T is diagonalizable over F', and also if and only if $m_T \in F[t]$ is separable; we then say T is *absolutely semisimple* over F. Deduce that semisimplicity is equivalent to absolutely semisimplicity over F if F is perfect, and give a counterexample over *every* imperfect field.

2. (i) Using rational canonical form and Cayley-Hamilton, prove the following are equivalent: $T^N = 0$ for some $N \geq 1$, $T^n = 0$, with respect to some ordered basis of V the matrix for T is upper triangular with 0's on the diagonal, $\chi_T = t^n$. We call such T *nilpotent*.

(ii) We say that T is *unipotent* if $T - 1$ is nilpotent. Formulate characterizations of unipotence analogous to the conditions in (i), and prove that a unipotent T is invertible.

(iii) Assume F is algebraically closed. Using Jordan canonical form and generalized eigenspaces, prove that there is a unique expression $T = T_{\mathrm{ss}} + T_{\mathrm{n}}$ where T_{ss} and T_{n} are a pair of *commuting* endomorphisms of V with T_{ss} semisimple and T_{n} nilpotent. (This is the *additive Jordan decomposition* of T.) Show by example with $\dim V = 2$ that uniqueness fails if we drop the "commuting" requirement, and show in general that $\chi_T = \chi_{T_{\mathrm{ss}}}$ (so T is invertible if and only if T_{ss} is invertible).

(iv) Assume F is algebraically closed and T is invertible. Using the existence and uniqueness of additive Jordan decomposition, prove that there is a unique expression $T = T'_{\mathrm{ss}} T'_{\mathrm{u}}$ where T'_{ss} and T'_{u} are a pair of *commuting* endomorphisms of V with T'_{ss} semisimple and T'_{u} unipotent (so T'_{ss} is necessarily invertible too). This is the *multiplicative Jordan decomposition* of T.

(v) Use Galois theory with entries of matrices to prove (iii) and (iv) for any perfect F (using the result over an algebraic closure, or rather over a suitable finite Galois extension), and give counterexamples for any imperfect F.