Algebraic Groups I. Quotient formalism

Let G be a group scheme of finite type over a field k, and H a closed k-subgroup scheme (possibly not normal). We have defined a good notion of quotient $\pi : G \to G/H$ in general, and proved existence when G is smooth and affine, with G/H smooth and quasi-projective in such cases. Moreover, if H is normal we have seen that G/H is naturally a k-group if it exists, and that G/H is also affine when G is smooth and affine.

But one can ask for more: can we carry over basic manipulations with quotients as in elementary group theory? The first part of this handout addresses such questions in many cases (and the reader who is familiar with Grothendieck topologies can adapt the arguments to a more general setting, as is also treated in SGA3). In the second part of this handout, we discuss the classification of all smooth affine k-groups G fitting into a short exact sequence

$$1 \to G' \to G \to G'' \to 1$$

with $G', G'' \in \{\mathbf{G}_a, \mathbf{G}_m\}$.

In the final part, we apply these results to describe the structure of a k-split solvable group over *any* field k as a semidirect product of toric and unipotent parts. This description is not canonical (generally there are many choices for the torus subgroup), but it is a decisive tool in the proof of general results for solvable groups.

1. Coset spaces and isomorphism theorems

We begin by relating closed subschemes of G/H to certain closed subschemes of G. For this we do not need any smoothness or affineness assumption; we merely need to assume that the quotient $\pi : G \to G/H$ exists (under the definition given in class, so it is required to be separated and of finite type over k). Recall that existence has been proved when G is smooth and affine (and it is proved in general over fields in SGA3, as was noted in class).

Proposition 1.1. *If Z is a closed subscheme of G/H then $\pi^{-1}(Z)$ is a closed subscheme of G which is stable under the right-translation action of H on G, and $Z \mapsto \pi^{-1}(Z)$ is a bijective correspondence between the set of closed subschemes of G/H and the set of closed subschemes of G stable under the right translation action of H.*

We have $Z_1 \subseteq Z_2$ if and only if $\pi^{-1}(Z_1) \subseteq \pi^{-1}(Z_2)$.

Proof. By computing with the functor of points, it is clear that $\pi^{-1}(Z)$ has the asserted properties. To prove that $Z = Z'$ when $\pi^{-1}(Z) = \pi^{-1}(Z')$, recall that π is faithfully flat map between noetherian schemes, so it suffices to prove in general that if $f : X \to Y$ is a faithfully flat quasi-compact map between scheme then a closed subscheme Z in Y is uniquely determined by $f^{-1}(Z)$. We can assume Y is affine, and then by replacing X with the disjoint union of the constituents of a finite open affine covering we can assume X is also affine. But if $A \to B$ is a faithfully flat map of rings and J is an ideal of A then $A \cap JB = J$, so we get the assertion.

Now let W be a closed subscheme of G which is invariant under the right action of H. We seek to prove that $W = \pi^{-1}(Z)$ for some (necessarily unique) closed subscheme $Z \subset G/H$. Under the action isomorphism

$$G \times H \simeq G \times_{G/H} G$$

defined by $(g, h) \mapsto (g, gh)$, $W \times H$ goes over to $W \times_{G/H} W$ due to the right-invariance hypothesis on W. But $W \times_{G/H} G$ goes over to a closed subscheme of $G \times H$ which must be contained in $W \times H$ (by computing with first projections), so the containment $W \times_{G/H} W \subseteq W \times_{G/H} G$ of closed subschemes of $G \times_{G/H} G$ is an equality. Applying the "flip" automorphism, it follows that likewise $W \times_{G/H} W = G \times_{G/H} W$. Hence, $W \times_{G/H} G = G \times_{G/H} W$. In other words, if

$q_1, q_2 : G \times_{G/H} G \rightrightarrows G$ are the two projections then $q_1^{-1}(W) = q_2^{-1}(W)$. Since $\pi : G \to G/H$ is faithfully flat and quasi-compact, by descent theory for closed subschemes (which is descent theory for quasi-coherent sheaves, applied to ideal sheaves inside of the structure sheaf) it follows that $W = \pi^{-1}(Z)$ for a closed subscheme Z in G/H.

To prove that the bijective correspondence respects inclusions in both directions it suffices to prove that if $\pi^{-1}(Z_1) \subseteq \pi^{-1}(Z_2)$ then $Z_1 \subseteq Z_2$. Letting $Z = Z_1 \cap Z_2$, we have $\pi^{-1}(Z) = \pi^{-1}(Z_1) \cap \pi^{-1}(Z_2) = \pi^{-1}(Z_1)$, so $Z = Z_1$. Hence, $Z_1 \subseteq Z_2$, as required. ■

Continuing to assume that G/H exists (a hypothesis we have proved when G is smooth and affine), we get the following existence result for additional quotients by H:

Corollary 1.2. *For any closed subscheme $Z \subseteq G$ stable under the right H-action, the quotient $Z \to Z/H$ exists and is the projection from Z onto the unique closed k-subscheme $\overline{Z} \subseteq G/H$ such that $\pi^{-1}(\overline{Z}) = Z$.*

Of course, the definition of a quotient map $Z \to Z/H$ is identical to the definition of the quotient map $G \to G/H$ as in class (which never used the k-group structure on G apart from the right H-action on G arising from it). In particular, by the argument used in class, if such a quotient exists it automatically satisfies the expected universal property for H-invariant maps from Z.

Proof. Since $Z = \pi^{-1}(\overline{Z})$, the map $\pi : Z \to \overline{Z}$ is faithfully flat and quasi-compact (even finite type). This map is also clearly invariant under the right H-action on Z. By thinking in terms of functors, we see that $Z \times_{G/H} Z = Z \times_{\overline{Z}} Z$. But the preceding proof shows that $Z \times_{G/H} Z = Z \times H$ via the right action map, so $Z \times H \simeq Z \times_{\overline{Z}} Z$ via the action map. Thus, $Z \to \overline{Z}$ satisfies the requirements to be a quotient by the H-action. ■

As a nice application, we can now construct some more quotients in the affine case without smoothness hypotheses:

Example 1.3. Let H' be an affine k-group of finite type that is a closed k-subgroup of GL_n for some $n \geq 1$. Then for any closed k-subgroup $H \subseteq H'$, the quotient H'/H exists and is quasi-projective over k. Indeed, we apply the preceding corollary to $Z = H'$ and the smooth affine $G = \mathrm{GL}_n$ upon picking a faithful linear representation. (In SGA3 it is proved that every affine group of finite type over a field admits a closed k-subgroup inclusion into some GL_n. For our purposes, what matters is that if we begin life with a smooth affine k-group and then pass to collections of closed k-subgroups, the coset schemes always exist and are quasi-projective over k.)

Here is a group scheme version of some basic isomorphism nonsense from group theory.

Proposition 1.4. *Assume G is smooth and affine, and H is normal in G. Equip $\overline{G} := G/H$ with its natural k-group structure. Then $\overline{H}' \mapsto H' := \pi^{-1}(\overline{H}')$ is a bijective correspondence between closed k-subgroup schemes of G/H and closed k-subgroup schemes of G containing H. Moreover, $H' \lhd G$ if and only if $\overline{H}' \lhd \overline{G}$, and if $H' \subseteq H''$ is a containment between such k-subgroups of G then the natural map $H'' \to \overline{H}''/\overline{H}'$ is right H'-invariant and the induced map $H''/H' \to \overline{H}''/\overline{H}'$ is an isomorphism.*

Note that under the hypothesis on G, Example 1.3 applies to prove that H''/H' exists for any such pair (H'', H') inside of G. The same goes for $\overline{H}''/\overline{H}'$, since $\overline{G}$ is smooth and affine. This is the only reason for assuming G is smooth and affine (rather than merely a k-group of finite type). If we grant the existence results for quotients in the generality of SGA3 then the proof below works verbatim without these restrictions on G.

If one approaches these matters from the viewpoint of Grothendieck topologies, the following proof can be done much more easily: it is identical to the version of sheaves of groups on a topological space (if done without the crutch of stalks), which in turn is modeled on the version in ordinary group theory.

Proof. Since π is a k-homomorphism, the formation of (scheme-theoretic!) preimages under π carries closed subgroups to closed subgroups and preserves normality. To prove the converse direction, consider a closed subscheme $\overline{Z} \subseteq \overline{G}$ such that $Z := \pi^{-1}(\overline{Z})$ is a k-subgroup of G. We wish to prove that $\overline{Z}$ is a k-subgroup of $\overline{G}$, and that it is also normal if Z is normal in G. We have to check three properties: containment of the identity, stability under inversion, and stability under the ambient group law morphism.

We have $1 \in \overline{Z}(k)$ since $\pi(1) = 1$ and $1 \in Z(k)$. Also, the inversion involution of the k-scheme $\overline{G}$ is compatible via π with the inversion involution of the k-scheme G, so the fact that Z is carried isomorphically to itself under inversion on G forces the analogue for $\overline{Z}$ due to the condition $\pi^{-1}(\overline{Z}) = Z$ uniquely determining $\overline{Z}$ as a closed k-subscheme of G. Finally, to prove that $m : G \times G \to G$ carries $Z \times Z$ into Z, we reformulate it as the condition $Z \times Z \subseteq m^{-1}(Z)$. Since π is a homomorphism, it is easy to check that

$$m^{-1}(Z) = (\pi \times \pi)^{-1}(\overline{m}^{-1}(\overline{Z})) \supseteq (\pi \times \pi)^{-1}(\overline{Z} \times \overline{Z}) = Z \times Z.$$

This completes the proof of the bijective correspondence between closed k-subgroups.

To check the normality claim, we first observe that for any k-algebra R,

$$H'(R) = \{g \in G(R) \,|\, \pi(g) \in \overline{H}'(R)\}.$$

This is clearly normal in $G(R)$ when $\overline{H}'(R)$ is normal in $\overline{G}(R)$. Conversely, suppose H' is normal in G. We seek to prove that $\overline{H}'$ is normal in $\overline{G}$. In other words, we want the conjugation map $\overline{c} : \overline{G} \times \overline{H}' \to \overline{G}$ defined by $(\overline{g}, \overline{h}') \mapsto (\overline{g}\overline{h}'\overline{g}^{-1})$ to factor through $\overline{H}'$. But as we saw in our construction of the k-group structure on $\overline{G} = G/H$ in the normal case, the natural map

$$G \times H' \to \overline{G} \times \overline{H}'$$

is a quotient by the right translation action of $H \times H$. Hence, in view of the general universal mapping property of quotients, it suffices to prove that the map

$$G \times H' \to \overline{G}$$

defined by $(g, h') \mapsto \pi(gh'g^{-1})$ factors through $\overline{H}'$. But this map factors as

$$G \times H' \to H' \hookrightarrow G \to \overline{G}$$

where the first map is $(g, h') \mapsto gh'g^{-1}$ due to the normality of H in G. Since the second and third steps in this diagram have composite equal to the quotient map $H' \to \overline{H}'$ followed by the inclusion of $\overline{H}'$ into $\overline{G}$, we are done.

Finally, we prove that if $H' \subseteq H''$ is a containment between closed k-subgroups of G containing H, then $H'' \to \overline{H}''/\overline{H}'$ is right H'-invariant with the induced map $\theta : H''/H' \to \overline{H}''/\overline{H}'$ an isomorphism. Since $H'' \to \overline{H}''$ is a k-homomorphism which carries H' into $\overline{H}'$, the desired right H'-invariance is immediate since the quotient map $\overline{H}'' \to \overline{H}''/\overline{H}'$ is right $\overline{H}'$-invariant. To prove that the induced map θ is an isomorphism, it is equivalent to prove that the natural map $q : H'' \to \overline{H}''/\overline{H}'$ satisfies the requirements to be a quotient by H'.

We have just seen that q is right H'-invariant, and it is faithfully flat and quasi-compact since it is the composite of the maps $H'' \to H''/H = \overline{H}''$ and $\overline{H}'' \to \overline{H}''/\overline{H}'$ which both have these properties. Thus, it remains to check that the natural map

$$H'' \times H' \to H'' \times_{\overline{H}''/\overline{H}'} H'' \tag{1}$$

defined by $(h'', h') \mapsto (h'', h''h')$ is an isomorphism. Consider the isomorphism

$$H'' \times_{\overline{H}''/\overline{H}'} H'' = H'' \times_{\overline{H}''} (\overline{H}'' \times_{\overline{H}''/\overline{H}'} \overline{H}'') \times_{\overline{H}''} H'' = H'' \times_{\overline{H}''} (\overline{H}'' \times \overline{H}') \times_{\overline{H}''} H'',$$

where the final term has the second projection map $\overline{H}'' \times \overline{H}' \to \overline{H}''$ equal to the multiplication map in the group law of $\overline{G}$. It follows that for any k-algebra R and $h''_1, h''_2 \in H''(R)$, their images in $(\overline{H}''/\overline{H}')(R)$ coincide if and only if the image points $\overline{h}''_1, \overline{h}''_2 \in \overline{H}''(R)$ are related by the right $\overline{H}'(R)$-action. But that says precisely that the point $(h''_1)^{-1}h''_2 \in H''(R)$ lies in $\pi^{-1}(\overline{H}'(R)) = H'(R)$ (since $H' = \pi^{-1}(\overline{H}')$), so (1) is bijective on R-points for every R. Hence, (1) is an isomorphism. ■

Remark 1.5. In Proposition 1.4, it is natural to wonder about the relationship between smoothness properties for H' and $\overline{H}'$. Since $H' \to \overline{H}'$ is faithfully flat, if H' is smooth then so is $\overline{H}'$ (as its coordinate ring on small affine opens is geometrically reduced, due to the same for H'). The converse direction is more subtle, and the best that can be said in general is that if H is also smooth then smoothness of $\overline{H}'$ implies the same for H'. In other words, under the quotient map $\pi : G \twoheadrightarrow G/H$ with smooth G and H, we claim that the scheme-theoretic preimage H' in G of a smooth k-subgroup $\overline{H}' \subseteq G/H$ is again smooth. To prove this in an elementary manner (without needing the general theory of smooth morphisms), we may extend scalars to $\overline{k}$ so that k is algebraically closed. For a smooth subgroup $\overline{H}' \subseteq G/H$, consider the smooth subgroup $\pi^{-1}(\overline{H}')_{\rm red}$ in G whose image in G/H is clearly $\overline{H}'$. This subgroup of G contains $H_{\rm red} = H$ since H is smooth, so it is H-stable. Hence, under the bijective correspondence it must go over to its image $\overline{H}'_{\rm red} = \overline{H}'$ since $\pi^{-1}(\overline{H}')_{\rm red}/H$ is certainly smooth. The bijectivity therefore forces $\pi^{-1}(\overline{H}') = \pi^{-1}(\overline{H}')_{\rm red}$, so we get the claim.

The last general nonsense issue we wish to address is the "image" of the natural map $H \to G/H'$ for a smooth affine k-group G, a closed k-subgroup H', and an auxiliary closed k-subgroup H (not assumed to have any containment relation with H' in either direction). Clearly $H \cap H'$ is a closed k-subgroup of H, normal when H' is normal in G, and the quotient $H/(H \cap H')$ exists since G is smooth and affine (so Example 1.3 can be applied). We then get an induced map

$$j : H/(H \cap H') \to G/H',$$

and it is natural to wonder: *is this map a locally closed immersion*? In some nice cases things work out well:

Proposition 1.6. *Assume H is smooth. If H' is normal in G then j is a closed immersion. If instead H' is smooth then j is a locally closed immersion.*

Proof. First assume H' is normal, so j is a homomorphism between smooth k-groups of finite type. Thus, to prove it is a closed immersion we just have to prove triviality of the kernel. By the construction of quotients, it suffices to show that $H \to G/H'$ has kernel $H \cap H'$. Since $G \to G/H'$ has kernel H', we are done.

Now suppose H' is smooth. The same Galois descent technique (using that $\overline{k}/k_s$ is purely inseparable) as in our proof of the closed orbit lemma can be used to prove that the constructible image X of $H \times H' \to G/H'$ is locally closed and is smooth with its reduced structure. (We get around the fact that $H \times H'$ has no relevant group structure here by using left translation by H and right translation by H'.)

We thereby obtain a surjective map $H \to X$ between smooth equidimensional k-schemes of finite type, and by computation on $\overline{k}$-points the geometric fibers are all equidimensional of the same dimension (translates of $H \cap H'$ after a ground field extension). Thus, by the Miracle Flatness Theorem, $H \to X$ is faithfully flat. This map is visibly $H \cap H'$-invariant on the right, so it remains to check that the natural map

$$H \times (H \cap H') \to H \times_X H$$

is an isomorphism. Thinking in terms of functors, $H \times_X H = H \times_{G/H'} H$. Thus, the desired result is clear since $G \times_{G/H'} G = G \times H'$ via the natural map. ■

Example 1.7. Since H is closed in G, it may seem surprising that j may fail to be a closed immersion (and just be locally closed). But the relevant topology inside of G is not H but rather the image of $H \times H'$ under multiplication. This could be non-closed. Such a possibility happens very often in the theory of reductive groups, especially with the so-called Bruhat decomposition.

We illustrate this in the most basic (yet very important) case: $G = \mathrm{SL}_2$, $H' = B$ the upper triangular k-subgroup, and $H = U^-$ the lower triangular unipotent k-subgroup. In this case G/H' is identified with $\mathbf{P}^1_k$ and $H \cap H' = 1$, with j becoming the standard open immersion $\mathbf{A}^1_k \to \mathbf{P}^1_k$ complementary to 0. This example "works" the same way with real and complex Lie groups, so it has nothing to do with the peculiarities of algebraic geometry.

2. Classifying some extension structures

In this section we wish to describe all smooth connected affine k-groups G for which there is a short exact sequence of smooth affine k-groups

$$1 \to G' \to G \to G'' \to 1$$

with $G', G'' \in \{\mathbf{G}_a, \mathbf{G}_m\}$. We say that *$G$ is an extension of G'' by G'*.

The case $G' = G'' = \mathbf{G}_a$ is the most subtle of all (especially in nonzero characteristic), and is addressed in HW9 Exercise 2. Also, if $G' = G'' = \mathbf{G}_m$ then G must be a k-split torus of dimension 2, so there is nothing to do (as the character group explains everything in such cases). Thus, the focus of our attention is on the other two cases.

Proposition 2.1. *For any short exact sequence $1 \to \mathbf{G}_m \to G \to \mathbf{G}_a \to 1$, necessarily G is commutative and the exact sequence is uniquely split over k.*

Proof. By HW6, Exercise 3, $G' = \mathbf{G}_m$ is central in G. By HW9 Exercise 4, the commutator map $G \times G \to G$ factors through a k-scheme map $G'' \times G'' \to G'$. A calculation with (geometric) points shows that this map $\mathbf{G}_a \times \mathbf{G}_a \to \mathbf{G}_m$ is bi-additive. But the only such map is the trivial one, since there are no nontrivial homomorphisms from $\mathbf{G}_a$ to $\mathbf{G}_m$ over $\overline{k}$. Thus, the commutator map is trivial, so G is commutative.

It follows that if the given exact sequence splits then the splitting is unique, as the set of splittings is a torsor under $\mathrm{Hom}_k(\mathbf{G}_a, \mathbf{G}_m) = 1$. In view of the uniqueness, to construct the splitting over k it suffices (by Galois descent) to work over k_s. Hence, we can assume $k = k_s$. Since G is solvable and not a torus, in the decomposition $G_{\overline{k}} = T \times U$ for a torus T and $U := \mathscr{R}_u(G_{\overline{k}})$ we must have $U \neq 1$ and T is the copy of $\mathbf{G}_m$ from the given short exact sequence. Hence, the given exact sequence splits over $\overline{k}$, so it splits over some finite extension k'/k with $k = k_s$. This settles the case when k is perfect.

To handle possibly imperfect k, I do not know a way to copy the Galois descent argument by using faithfully flat descent because general nonsense does not ensure the uniqueness of a splitting over $k'' = k' \otimes_k k'$: there do exist nontrivial k''-homomorphisms $\mathbf{G}_a \to \mathbf{G}_m$! (For example, $x \mapsto 1 + \epsilon x$ for nonzero $\epsilon \in k''$ with $\epsilon^2 = 0$.) Instead, the only method I know is to use faithfully flat descent theory in a different way, as follows. The exact sequence identifies G with a $\mathbf{G}_m$-torsor over $\mathbf{G}_a$ for the fppf or étale topologies. The isomorphism class of this torsor is classified by an element in the Picard group of $\mathbf{G}_a$ relative to the fppf or étale topologies. By descent theory for quasi-coherent sheaves, this is the same as the Picard group relative to the Zariski topology, which is trivial since $k[x]$ is a PID. Hence, it follows that the quotient map $G \to \mathbf{G}_a$ admits a section σ over k as a map of k-schemes. Composing with a suitable $G(k)$-translation then brings us to the case $\sigma(0) = 1$.

To summarize, we have an isomorphism of pointed k-schemes $G = \mathbf{G}_m \times \mathbf{G}_a$ with group law

$$(t, x)(t', x') = (tt'h(x, x'), x + x')$$

where $h(0, 0) = 1$. The only units on $\mathbf{G}_a \times \mathbf{G}_a$ are the elements of $k^\times$, so $h = 1$. This is the standard group law, as desired. ■

Now consider an extension G of $G'' = \mathbf{G}_m$ by $G' = \mathbf{G}_a$. By the method of solution of HW9 Exercise 4, since G' is commutative the conjugation action of G on itself uniquely factors through an action of $G/G' = G'' = \mathbf{G}_m$ on $G' = \mathbf{G}_a$. I claim this action must be $t.x = t^n x$ for a unique $n \in \mathbf{Z}$. To prove this, let $S = \mathbf{G}_m$ and $H = S \times \mathbf{G}_a$ viewed as an S-group. The map $H \to H$ defined by $(t, x) \mapsto (t, t.x)$ is an S-group automorphism of the additive affine line H over S. Since S is *reduced*, the only such automorphisms are given by scalar of the line parameter by a unit on the base, as may be checked by working at the generic points of S and then Zariski-locally on S. Hence, $t.x = c(t)x$ for a k-scheme map $c : S \to \mathbf{G}_m$. Clearly $c(1) = 1$, and the only units on S are $k^\times$-multiples of powers of t. Hence, $c(t) = t^n$ for some $n \in \mathbf{Z}$, as desired. Observe that $n = 0$ if and only if $\mathbf{G}_a$ is central in G. It turns out that this is equivalent to the commutativity of G. More generally:

Proposition 2.2. *There is a k-group isomorphism between G and the semidirect product $\mathbf{G}_a \rtimes \mathbf{G}_m$ defined by the action $t.x = t^n x$.*

Proof. We cannot trivially use Galois descent, for the reason that the semidirect product structure is not unique when $n \neq 0$. That is, even if there is a k-group section to the quotient map $G \to \mathbf{G}_m$, we can compose it with $G(k)$-conjugations to get more such sections when G is not commutative. Thus, we need a different method (and in the end will use Galois descent, but in a manner which is less elementary than above).

It suffices to find a nontrivial k-torus T in G. Indeed, $T \cap \mathbf{G}_a = 1$ (by applying HW5 Exercise 1 after picking a faithful linear representation of G and using k-rational conjugation so that $\mathbf{G}_a$ lands in the standard upper triangular unipotent k-subgroup), so the induced nontrivial map $T \to G/\mathbf{G}_a = \mathbf{G}_m$ is an isomorphism. That would yield the desired semidirect product structure.

To construct a nontrivial k-torus in G, we first treat the case when k is not algebraic over a finite field, and then we use that case to handle the case when k is algebraic over a finite field. Now assume k is not algebraic over a finite field, so $k^\times$ contains an element c with infinite order. The fiber of $q : G \to \mathbf{G}_m$ over c is geometrically a translate of $\mathbf{G}_a$, so it is smooth and non-empty and hence has k_s-points. Thus, $q^{-1}(c)(k_s)$ is a torsor under translation by $\mathbf{G}_a(k_s) = k_s$, with torsor structure that is $\mathrm{Gal}(k_s/k)$-equivariant. Hence, the obstruction to $q^{-1}(c)$ having a k-point is a class in $\mathrm{H}^1(\mathrm{Gal}(k_s/k), k_s) = 0$. That is, there exists $g \in G(k)$ such that $q(g) = c$. If $\mathrm{char}(k) = p > 0$ then the geometric Jordan decomposition of g may have a nontrivial unipotent part. Replacing g (and so c) with a suitable p-power in such cases then brings us to the case when g is geometrically semisimple. Likewise, if $\mathrm{char}(k) = 0$ then the Jordan decomposition of g is defined over k (as k is perfect), and the unipotent part must have trivial image in $\mathbf{G}_m$. Hence, in such cases we can replace g with its semisimple part in $G(k)$. To summarize, under the assumption that k is not algebraic over a finite field, we have constructed $g \in G(k)$ that is geometrically semisimple and has image in $\mathbf{G}_m(k) = k^\times$ with infinite order.

By working with a faithful linear representation of $G_{\overline{k}}$, it follows that the closure of the cyclic subgroup generated by g in $G(k)$ has identity component that is a nontrivial k-torus. (Beware that g may not lie in the identity component of this k-group!) This settles the case when k is not algebraic over a finite field.

Suppose instead that k is algebraic over a finite field. Let $K = k(u)$ be a rational function field over K. We can apply the preceding arguments to G_K, so we get a K-subgroup $\mathbf{G}_m \hookrightarrow G_K$. This closed immersion over $K = k(u)$ "spreads out" to a closed subgroup scheme inclusion over $k[u][1/h]$ for some sufficiently divisible nonzero $h \in k[u]$. If k is infinite then we can specialize at a point $u_0 \in k$ for which $h(u_0) \neq 0$. If k is finite then such a u_0 can be found in a finite extension k'/k.

It remains to treat the case when k is finite, and we have a section $\sigma : \mathbf{G}_m \to G_{k'}$ for some finite Galois extension k'/k. We will use nothing special about finite fields. It is an elementary calculation that for any field F, the F-group sections to $\mathbf{G}_a \rtimes \mathbf{G}_m \to \mathbf{G}_m$ (using the nth-power action to define the semidirect product) are precisely the maps $t \mapsto (h(t), t)$ where $h(tt') = h(t) + t^n h(t')$ (which forces $h(1) = 0$). The regular function h on $\mathbf{G}_m$ is a Laurent polynomial over F (i.e., $h \in F[t, 1/t]$), and it is elementary to verify that the only Laurent polynomials over F which satisfy the required functional equation are $h(t) = ct^n - c$ for a *unique* $c \in F$. But this in turn is exactly the effect of applying conjugation by $(c, 1)$ to the canonical section! In other words, in our situation with G (whose k-structure is *not yet known*), any two sections to $G_{k'} \twoheadrightarrow \mathbf{G}_m$ are related via conjugation by a *unique* element of $\mathbf{G}_a(k') \subseteq G(k')$. Hence, the obstruction to the existence of a $\mathbf{G}_a(k')$-conjugate of σ that admits a k-descent (i.e., the measure of failure of σ to have a $\mathbf{G}_a(k')$-conjugate that is $\mathrm{Gal}(k'/k)$-equivariant) is an element in $\mathrm{H}^1(\mathrm{Gal}(k'/k), k') = 0$. It follows that after applying a suitable $\mathbf{G}_a(k')$-conjugation to σ it is defined over k, and so we get the desired nontrivial k-torus in G. ■

3. Structure of split solvable groups

Let G be a k-split solvable group, with k any field. (Recall this requires G to be a smooth connected affine k-group, among other things.) We wish to describe the structure of G in the form of a semidirect product $U \rtimes T$ for a smooth connected unipotent k-group U and a k-torus T, with U and T each k-split. Note that if there is such a decomposition then $U = \mathscr{R}_{u,k}(G)$ and the description persists over any extension field, so $U_K = \mathscr{R}_{u,K}(G_K)$ for any extension field K/k. In particular, $U_{\overline{k}} = \mathscr{R}_{u,\overline{k}}(G_{\overline{k}})$. We then call U the *unipotent radical* of G.

Remark 3.1. Beware that if k is not perfect, there always exist examples of smooth connected commutative affine k-groups G such that $\mathscr{R}_{u,\overline{k}}(G_{\overline{k}})$ is nontrivial and does *not* descend to a k-subgroup of G. Thus, the possibility of a semidirect product description $G = U \rtimes T$ over a general field k relies in an essential way on the k-split hypothesis.

Before we take up the general case, let's consider the low-dimensional cases. If $\dim G \leq 1$, then by the very definition of "k-split solvable" we are done (G is either $\mathbf{G}_a$ or $\mathbf{G}_m$). If $\dim G = 2$ then by the definition of being k-split solvable, G is of the sort considered in §2. In particular, if it is an extension of $\mathbf{G}_a$ by $\mathbf{G}_a$ then G is unipotent so we can take $U = G$ and $T = 1$. Likewise, if G is an extension of $\mathbf{G}_m$ by $\mathbf{G}_m$ then G is a k-split k-torus (as we noted early in §2), so we can take $T = G$ and $U = 1$. The other two possibilities are addressed in Proposition 2.2 and Proposition 2.1. The case $\dim G \geq 3$ will be deduced from these low-dimensional cases by using induction on $\dim G$ with the help of a composition series as in the definition of G being k-split solvable.

Here is the main result.

Theorem 3.2. *Let G be a k-split solvable group over a field k. Then $G = U \rtimes T$ for a k-split smooth connected unipotent k-group U and a k-split k-torus T equipped with an action on U.*

In the proof, we freely make use of §1, and especially Remark 1.5 when arguing "as if" we were using ordinary groups (especially not needing to worry about smoothness issues when forming certain preimages through quotient maps by smooth normal k-subgroups).

Proof. As we have seen in the preceding discussion, the cases $\dim G \leq 2$ are settled. We will first treat the commutative case, and then the general case (by using dimension induction and the commutative case applied to $G/\mathscr{D}(G)$ when G is non-commutative). Assuming G to be commutative, a k-split composition series for G provides a *k-split* smooth connected k-subgroup $G' \subset G$ of codimension 1 such that G/G' is either $\mathbf{G}_a$ or $\mathbf{G}_m$. Thus, dimension induction implies $G' = T' \times U'$ for a k-split k-torus T' and a k-split smooth connected commutative unipotent k-group U'. If $G/G' = \mathbf{G}_m$ then G/U' is an extension of $\mathbf{G}_m$ by $G'/U' = T'$, so G/U' is a k-split k-torus. If $G/G' = \mathbf{G}_a$ then G/T' is an extension of $\mathbf{G}_a$ by $G'/T' = U'$, so G/T' is k-split unipotent.

Thus, G is either an extension of T by U or of U by T, where T is a k-split k-torus and U is a unipotent smooth connected commutative k-group that is also k-split. That is, either there is a short exact sequence

$$1 \to T \to G \to U \to 1$$

or

$$1 \to U \to G \to T \to 1,$$

so it suffices (for the case of commutative G) to prove that any such exact sequence with commutative G is split over k.

The category of smooth connected commutative k-groups is not abelian (think of isogenies which are not isomorphisms), but it is an additive category and has a notion of short exact sequence which enjoys familiar properties as in the axioms for an "exact category". This permits us to endow the set $\mathrm{Ext}_k(H, H')$ of commutative k-group extensions of one object by another with a natural commutative group structure making it an additive bifunctor, and when given a short exact sequence in either H or H' (with the other variable fixed) we get a natural 6-term exact sequence in Hom_k's and Ext_k's. Our task in the commutative case is to prove that $\mathrm{Ext}_k(T, U)$ and $\mathrm{Ext}_k(U, T)$ both vanish. By using composition series for T and U with each successive quotient k-isomorphic to $\mathbf{G}_m$ and $\mathbf{G}_a$ respectively, the 6-term exact sequence formalism (just in the Ext_k aspect) reduces us to the case $T = \mathbf{G}_m$ and $U = \mathbf{G}_a$. Now Proposition 2.1 and Proposition 2.2 give the required vanishing (since the commutative case in Proposition 2.2 forces $n = 0$).

Moving a bit beyond the commutative case, we next treat a case with a slightly weaker hypothesis which turns out to still imply commutativity.

Lemma 3.3. *Any extension H of $\mathbf{G}_a$ by a k-split torus T is k-isomorphic to $\mathbf{G}_a \times T$.*

Proof. The same argument as at the start of the proof of Proposition 2.1 (replacing $\mathbf{G}_m$ there with T) implies that G is commutative. Thus, by the settled commutative case we have $G = S \times U$ for a k-split torus S and a k-split unipotent smooth connected k-group U. Clearly $T \subseteq S$ since $G \twoheadrightarrow G/S = U$ must kill T, and likewise $S \subseteq T$ since $G \twoheadrightarrow G/T = \mathbf{G}_a$ must kill S. Hence, $S = T$, so $U \simeq \mathbf{G}_a$. ∎

Turning to the general case, we may assume $\dim G \geq 3$. We may also assume that G is neither unipotent nor a torus. Choose a k-split composition series for G over k, so we get a codimension-1 k-split solvable k-subgroup $G' \subset G$ with G/G' isomorphic to either $\mathbf{G}_a$ or $\mathbf{G}_m$. By induction, $G' = U' \rtimes T'$ for a smooth connected unipotent k-subgroup U' and a k-torus T', and U' and T' are each k-split. Observe that since G' is normal in G and necessarily $U'_{\overline{k}} = \mathscr{R}_{u,\overline{k}}(G'_{\overline{k}})$, it is automatic that U' is also *normal in* G! Thus, G/U' makes sense and is an extension of either $\mathbf{G}_a$ or $\mathbf{G}_m$ by T'. In the latter case, G/U' is a k-split torus, so $U' = \mathscr{R}_{u,k}(G)$ and this is k-split. In the former case, G/U' is an extension of $\mathbf{G}_a$ by T', and in the latter case $G/U' = \mathbf{G}_a \times T'$ as in Lemma 3.3. Thus, in this latter case the preimage U of $\mathbf{G}_a$ in G is a k-split unipotent smooth connected k-group which is normal in G and has quotient G/U that is a k-split torus. In other words, in the general case G is an extension of a k-split torus T by a k-split unipotent smooth connected k-group U. In particular, we have shown that $\mathscr{R}_{u,k}(G)$ is k-split and $\mathscr{R}_{u,k}(G)_{\overline{k}} = \mathscr{R}_{u,\overline{k}}(G_{\overline{k}})$.

It suffices to find a k-torus S in G that maps isomorphically onto $G/U = T$. Since G is not unipotent and not a torus, we have $T \neq 1$ and $U \neq 1$. We claim that U admits a composition series $\{U_i\}$ consisting of k-split unipotent smooth connected k-subgroups which are *normal in* G and for which U_i/U_{i-1} is a vector group. Once this is proved, by induction on $\dim U$ we can then pass to the case when U is a vector group. To construct this composition series $\{U_i\}$ we treat characteristic 0 first. In this case we can take it to be the derived series, since a commutative unipotent smooth connected group in characteristic 0 is always a vector group (HW9, Exercise 2(ii)). If instead $\mathrm{char}(k) = p > 0$, we will use Tits' structure theory for unipotent smooth connected k-groups, as developed in Appendix B of "Pseudo-reductive groups". By Proposition B.3.2 there, since U is k-split it contains a central $\mathbf{G}_a$. In general, a p-torsion commutative smooth connected k-group is a vector group if and only if it is k-split. (This follows from Lemma B.1.10 and Corollary B.1.12, together with a dimension induction.) The maximal such k-subgroup U_1 in U is nontrivial, and it formation commutes with with scalar extension to k_s, due to Galois descent and the fact that the property of being a vector group is insensitive to scalar extension to k_s (Corollary B.2.6). Hence, $(U_1)_{k_s}$ is stable under all automorphisms of U_{k_s}, such as $G(k_s)$-conjugations, so U_1 is normal in G. Passing to G/U_1 and the k-split U/U_1 then allows us to construct $\{U_i\}$ by dimension induction.

Now we are in the case that U is a vector group. In particular, since U is commutative in G the natural G-action on U factors through an action of $G/U = T$ on U. We wish to describe this action in more concrete terms. If $\mathrm{char}(k) = 0$, it follows from HW9 Exercise 2(ii) that the T-action on $U \simeq \mathbf{G}_a^N$ respects the linear structure. If $\mathrm{char}(k) = p > 0$ then by Theorem B.4.3 there is a decomposition $U = U' \times U''$ with U'' a vector group admitting a linear structure respected by the T-action and U' having trivial T-action. But U' is also a vector group since it is k-split (being a quotient of the k-split U), so we conclude as in characteristic 0 that there is an isomorphism $U \simeq \mathbf{G}_a^N$ making the T-action linear. Any linear representation of a split torus is a direct sum of 1-dimensional representations, so to lift T through the quotient map $G \twoheadrightarrow G/U$ we can use a filtration by such lines to reduce to the case $U = \mathbf{G}_a$ (via induction on $\dim U$)!

Finally, we are in the case that G is an extension of a split torus T by $\mathbf{G}_a$, in which the T-action on $\mathbf{G}_a$ is given by some $\chi \in \mathrm{X}(T)$. Letting S be the k-subtorus $(\ker\chi)^0_{\mathrm{red}}$, so S is k-split. The preimage H of S in G is a *central* extension of S by $\mathbf{G}_a$. The commutator pairing argument as in the beginning of the proof of Proposition 2.1 can now be adapted to this setting to infer that H is *commutative*. Hence, $H \simeq S \times \mathbf{G}_a$. It follows that S as a k-torus in H is central in G. If $\chi = 1$ then $S = T$ and we are done. Otherwise we can pass to G/S to reduce to the case $T = \mathbf{G}_m$. Now G is in exactly the setup for Proposition 2.2. ■