ALGEBRAIC GROUPS I. JORDAN DECOMPOSITION EXERCISES

The following exercises develop Jordan decomposition in additive and multiplicative forms for finite-dimensional vector spaces over any field, including what can "go wrong" over an imperfect field. (Briefly, the formation of the decomposition commutes with arbitrary field extension when the initial ground field is perfect, but not otherwise. This leads to some difficulties when working with linear algebraic groups over imperfect fields.) The additive case underlies Jordan decomposition in Lie algebras, and the multiplicative case underlies Jordan decomposition in linear algebraic groups.

Let V be a finite-dimensional nonzero vector space over a field F, with dimension $n > 0$. A linear self-map $T : V \to V$ is *semisimple* if every T-stable subspace of V admits a T-stable complementary subspace. (That is, if $T(W) \subseteq W$ then there exists a decomposition $V = W \oplus W'$ with $T(W') \subseteq W'$.) Keep in mind that such a complement is not unique in general (e.g., consider T to be a scalar multiplication with $\dim V > 1$). Let χ_T denote the characteristic polynomial of T, and m_T the minimal polynomial of T,

1. (i) For each monic irreducible $\pi \in F[t]$, define $V(\pi)$ to be the subspace of $v \in V$ killed by a power of $\pi(T)$. Prove that $V(\pi) \neq 0$ if and only if $\pi | m_T$, and that $V = \oplus_{\pi | m_T} V(\pi)$. (In case F is algebraically closed, these are the *generalized eigenspaces* of T on V.)

(ii) Use rational canonical form to prove that T is semisimple if and only if m_T has no repeated irreducible factor over F. (Hint: apply (i) to T-stable subspaces of V to reduce to the case when m_T has one monic irreducible factor.) Deduce that a Jordan block of rank > 1 is never semisimple, that m_T is the "squarefree part" of χ_T when T is semisimple, and that if $W \subsetneq V$ is a T-stable nonzero proper subspace then the induced endomorphisms $T_W : W \to W$ and $\overline{\overline{T}} : V/W \to V/W$ are semisimple when T is semisimple.

(iii) Let $T' : V' \to V'$ be another linear self-map with V' nonzero and finite-dimensional over F. Prove that T and T' are semisimple if and only if the self-map $T \oplus T'$ of $V \oplus V'$ is semisimple.

(iii) Choose $T \in \mathrm{Mat}_n(F)$, and let F'/F be an extension splitting m_T. Prove that T is semisimple as an F'-linear endomorphism of F'^n if and only if T is diagonalizable over F', and also if and only if $m_T \in F[t]$ is separable; we then say T is *absolutely semisimple* over F. Deduce that semisimplicity is equivalent to absolutely semisimplicity over F if F is perfect, and give a counterexample over *every* imperfect field.

2. (i) Using rational canonical form and Cayley-Hamilton, prove the following are equivalent: $T^N = 0$ for some $N \geq 1$, $T^n = 0$, with respect to some ordered basis of V the matrix for T is upper triangular with 0's on the diagonal, $\chi_T = t^n$. We call such T *nilpotent*.

(ii) We say that T is *unipotent* if $T - 1$ is nilpotent. Formulate characterizations of unipotence analogous to the conditions in (i), and prove that a unipotent T is invertible.

(iii) Assume F is algebraically closed. Using Jordan canonical form and generalized eigenspaces, prove that there is a unique expression $T = T_{\mathrm{ss}} + T_{\mathrm{n}}$ where T_{ss} and T_{n} are a pair of *commuting* endomorphisms of V with T_{ss} semisimple and T_{n} nilpotent. (This is the *additive Jordan decomposition* of T.) Show by example with $\dim V = 2$ that uniqueness fails if we drop the "commuting" requirement, and show in general that $\chi_T = \chi_{T_{\mathrm{ss}}}$ (so T is invertible if and only if T_{ss} is invertible).

(iv) Assume F is algebraically closed and T is invertible. Using the existence and uniqueness of additive Jordan decomposition, prove that there is a unique expression $T = T'_{\mathrm{ss}} T'_{\mathrm{u}}$ where T'_{ss} and T'_{u} are a pair of *commuting* endomorphisms of V with T'_{ss} semisimple and T'_{u} unipotent (so T'_{ss} is necessarily invertible too). This is the *multiplicative Jordan decomposition* of T.

(v) Use Galois theory with entries of matrices to prove (iii) and (iv) for any perfect F (using the result over an algebraic closure, or rather over a suitable finite Galois extension), and give counterexamples for any imperfect F.