Algebraic Groups I. Homework 1

1. This exercise studies the endomorphism rings of the k-groups $\mathbf{G}_m$ and $\mathbf{G}_a$, with k any commutative ring.

(i) Prove that $\mathrm{End}_k(\mathbf{G}_a)$ consists of $f \in k[t]$ such that $f(x+y) = f(x)+f(y)$ in $k[x,y]$, and that $\mathrm{End}_k(\mathbf{G}_m)$ consists of $f \in k[t,t^{-1}]$ such that $f(xy) = f(x)f(y)$ in $k[x,y,x^{-1},y^{-1}]$ and f has no zeros on any geometric fibers over $\mathrm{Spec}\, k$.

(ii) Deduce that if k is a $\mathbf{Q}$-algebra then naturally $\mathrm{End}_k(\mathbf{G}_a) = k$, and that if k is a field with characteristic $p > 0$ then it consists of $f = \sum c_j t^{p^j}$ $(c_j \in k)$. What if $k = \mathbf{Z}/(p^2)$?

(iii) Prove that $\mathrm{End}_k(\mathbf{G}_m) = \mathbf{Z}$ when k is a field, and deduce the same when k is an artin local ring via induction on the length of k. (Hint: reduce to the case when f vanishes on the special fiber.)

(iv) Prove that $\mathrm{End}_k(\mathbf{G}_m) = \mathbf{Z}$ for k *any* local ring by using (iii) to settle the case of a complete local noetherian ring, then any local noetherian ring, and finally any local ring (by using local noetherian subrings of k). Deduce that if k is any ring whatsoever, an endomorphism of the k-group $\mathbf{G}_m$ is $t \mapsto t^n$ for a locally constant function $n : \mathrm{Spec}\, k \to \mathbf{Z}$.

2. Let V be a finite-dimensional vector space over a field k. This exercise develops coordinate-free versions of GL_n, SL_n, and Sp_{2n} attached to V.

(i) Elements of the graded symmetric algebra $\mathrm{Sym}(V^*)$ are called *polynomial functions on* V. Justify the name (even for finite k!) by identifying them with *functorial maps* of sets $V_R \to R$ given by polynomial expressions relative to some (equivalently, any) basis of V, with R a varying k-algebra. In particular, show that det is a polynomial function on $\mathrm{End}(V)$.

(ii) For any k-algebra R, define the functors $\underline{\mathrm{End}}(V)$ and $\underline{\mathrm{Aut}}(V)$ on k-algebras R by $R \rightsquigarrow \mathrm{End}(V_R)$, $R \rightsquigarrow \mathrm{Aut}_R(V_R)$. Using the identification $\mathrm{End}(V_R, V_R) = \mathrm{End}(V)_R$, prove that $\underline{\mathrm{End}}(V)$ is represented by $\mathrm{Sym}(\mathrm{End}(V)^*)$.

(iii) Define $\det \in \mathrm{Sym}(\mathrm{End}(V)^*)$ and prove its non-vanishing locus

$$\mathrm{GL}(V) := \mathrm{Spec}(\mathrm{Sym}(\mathrm{End}(V)^*)[1/\det])$$

represents $\underline{\mathrm{Aut}}(V)$ as subfunctor of $\underline{\mathrm{End}}(V)$. Also discuss $\mathrm{SL}(V)$ as a closed k-subgroup of $\mathrm{GL}(V)$.

(iv) Let $B : V \times V \to k$ be a bilinear form. Prove that the subfunctor $\underline{\mathrm{Aut}}(V,B)$ of points of $\underline{\mathrm{Aut}}(V)$ preserving B is represented by a closed k-subgroup of $\mathrm{GL}(V)$. (You can use coordinates in the proof!) This is pretty bad unless B is non-degenerate. (In the alternating non-degenerate case it is denoted $\mathrm{Sp}(B)$.)

Assuming non-degeneracy, a linear automorphism T of V_R is a *B-similitude* if $B_R(Tv, Tw) = \mu(T)B(v,w)$ for all $v, w \in V_R$ and some $\mu(T) \in R^\times$. Prove $\mu(T)$ is then unique, and show that the functor of B-similitudes is represented by a closed k-subgroup of $\mathrm{GL}(V) \times \mathbf{G}_m$. (In the alternating case it is denoted $\mathrm{GSp}(B)$.)

3. (i) Prove that if a connected scheme X of finite type over a field k has a k-rational point, then $X_{k'} = X \otimes_k k'$ is connected for every finite extension k'/k (hint: $X_{k'} \to X$ is open and closed; look at fiber over $X(k)$). Deduce that $X_{k'}$ is connected for *every* extension k'/k (i.e., X is *geometrically connected* over k).

(ii) Prove that if X and Y are geometrically connected of finite type over k, so is $X \times Y$; give a counterexample over $k = \mathbf{Q}$ if "geometrically" is removed. Deduce that if G is a k-group then the identity component G^0 is a k-subgroup whose formation commutes with any extension on k.

4. Let G be a group of finite type over a field k.

(i) Prove that $(G_{\overline{k}})_{\mathrm{red}}$ is a closed $\overline{k}$-subgroup of $G_{\overline{k}}$, and prove it is *smooth*. Deduce that G^0 is *geometrically irreducible.*

(ii) Over any imperfect field k, one can make a non-reduced k-group G such that G_{red} is *not* a k-subgroup. Where does an attempted proof to the contrary get stuck?

(iii) Assume k is imperfect, $\mathrm{char}(k) = p > 0$, and choose $a \in k - k^p$. Prove $x_0^p + a x_1^p + \cdots + a^{p-1} x_{p-1}^p = 1$ defines a reduced k-group (think of $\mathrm{N}_{k(a^{1/p})/k}$) that is non-reduced over $\overline{k}$ and hence not smooth!

(iv) Prove that the condition $t^n = 1$ defines a finite closed k-subgroup $\mu_n \subseteq \mathbf{G}_m$, and show its preimage G under $\det : \mathrm{GL}_N \to \mathbf{G}_m$ is a k-subgroup of GL_N. Accepting that SL_N is connected, prove $G^0 = \mathrm{SL}_N$ if $\mathrm{char}(k) \nmid n$. For $k = \mathbf{Q}$ and $n = 5$, prove that $G - G^0$ is connected but over $\overline{k}$ has 4 connected components.

Algebraic Groups I. Homework 2

1. Let k be a perfect field, and G a 1-dimensional connected linear algebraic k-group (so G is geometrically integral over k). Assume G is in the additive case. This exercise proves G is k-isomorphic to $\mathbf{G}_a$.

(i) Let X denote its regular compactification over k. Prove that $X_{\overline{k}}$ is regular, so X is smooth (hint: $\overline{k}$ is a direct limit of finite separable extensions of k, and unit discriminant is a sufficient test for integral closures in the Dedekind setting). Deduce that $X - G$ consists of a single physical point, say $\operatorname{Spec} k'$.

(ii) Prove that $k' \otimes_k \overline{k}$ is reduced and in fact equal to $\overline{k}$. Deduce $k' = k$, and prove that $X \simeq \mathbf{P}^1_k$. Show that $G \simeq \mathbf{G}_a$ as k-groups.

2. Let T be a torus of dimension $r \geq 1$ over a field k (e.g., a 1-dimensional connected linear algebraic group in the multiplicative case). This exercise proves that $T_{k'} \simeq \mathbf{G}_m^r$ for some finite separable extension k'/k.

(i) Prove that it suffices to treat the case $k = k_s$.

(ii) Assume $k = k_s$. We constructed an isomorphism $f : T_{k'} \simeq \mathbf{G}_m^r$ as k'-groups for some finite extension k'/k. Let $k'' = k' \otimes_k k'$, and let $p_1, p_2 : \operatorname{Spec} k'' \rightrightarrows \operatorname{Spec} k'$ be the projections. Prove that k'' is an artin local ring *with residue field* k', and deduce that the k''-isomorphisms $p_i^*(f) : T_{k''} \simeq \mathbf{G}_m^r$ coincide by comparing them with f on the special fiber!

(iii) For any k-vector space V, prove that the only elements of $k' \otimes_k V$ with equal images under both maps to $k'' \otimes_k V$ are the elements of V (hint: reduce to the case $V = k$ and replace k' with any k-vector space W, and k'' with $W \otimes_k W$). Deduce that f uniquely descends to a k-isomorphism.

3. Let X and Y be schemes over a field k, K/k an extension field, and $f, g : X \rightrightarrows Y$ two k-morphisms.

(i) Prove $f_K = g_K$ if and only if $f = g$. (Use surjectivity of $X_K \to X$ to aid in reducing to the affine case.) Likewise prove that if $Z, Z' \subseteq X$ are closed subschemes such that $Z_K = Z'_K$ inside of X_K then $Z = Z'$,

(ii) If f_K is an isomorphism and X and Y are affine, prove f is an isomorphism. Then do the same without affineness (may be really hard without Serre's cohomological criterion for affineness).

(iii) Assume K/k is Galois, $\Gamma = \operatorname{Gal}(K/k)$. Prove that if a map $F : X_K \to Y_K$ satisfies $\gamma^*(F) = F$ for all $\gamma \in \Gamma$, then $F = f_K$ for a unique k-map $f : X \to Y$. Likewise, if $Z' \subseteq X_K$ is a closed subscheme and $\gamma^*(Z') = Z'$ for all $\gamma \in \Gamma$ then prove $Z' = Z_K$ for a unique closed subscheme $Z \subseteq X$. Do the same for open subschemes.

4. Let $q : V \to k$ be a quadratic form on a finite-dimensional vector space V of dimension $d \geq 2$, and let $B_q : V \times V \to k$ be the corresponding symmetric bilinear form. Let $V^\perp = \{v \in V \mid B_q(v, \cdot) = 0\}$; we call $\delta_q := \dim V^\perp$ the *defect* of q.

(i) Prove that B_q uniquely factors through a non-degenerate symmetric bilinear form on $V/V^\perp$, and B_q is non-degenerate precisely when the defect is 0. Prove that if $\operatorname{char}(k) = 2$ then B_q is alternating, and deduce that $\delta_q \equiv \dim V \bmod 2$ for such k (so $\delta_q \geq 1$ if $\dim V$ is odd).

(ii) Prove that if $\delta_q = 0$ then $q_{\overline{k}}$ admits one of the following "standard forms": $\sum_{i=1}^n x_i x_{i+n}$ if $\dim V = 2n$ ($n \geq 1$), and $x_0^2 + \sum_{i=1}^n x_i x_{i+n}$ if $\dim V = 2n+1$ ($n \geq 1$). Do the same if $\operatorname{char}(k) = 2$ and $\delta_q = 1$. (Distinguish whether or not $q|_{V^\perp} \neq 0$.) How about the converse?

(iii) If $\operatorname{char}(k) \neq 2$, prove $\delta_q = 0$ if and only if $q \neq 0$ and $(q = 0) \subseteq \mathbf{P}^{d-1}$ is smooth. If $\operatorname{char}(k) = 2$ then prove $\delta_q \leq 1$ with $q|_{V^\perp} \neq 0$ when $\delta_q = 1$ if and only if $q \neq 0$ and the $(q = 0)$ is smooth. (Hint: use (ii) to simplify calculations.) We say q is *non-degenerate* when $q \neq 0$ and $(q = 0)$ is smooth in $\mathbf{P}^{d-1}$.

5. Learn about separability and Ω^1 by reading in Matsumura's CRT: §25 up to before 25.3 (this is better than AG15.1–15.8 in Borel's book), and read §26 up through and including Theorem 26.3.

(i) Do Exercises 25.3, 25.4 in Matsumura, and read AG17.1 in Borel's book (noting he requires V to be geometrically reduced over k!).

(ii) Use 26.2 in Matsumura to prove that a finite type reduced k-scheme X is smooth on a dense open if and only if all functions fields of X (at its generic points) are *separable* over k.

(iii) Using separating transcendence bases, the primitive element theorem, and "denominator chasing", prove that if X is smooth on a dense open then $X(k_s)$ is Zariski-dense in X_{k_s}. (Hint: it suffices to prove $X(k_s)$ is non-empty!)

Algebraic Groups I. Homework 3

1. Let $k[x_{ij}]$ be the polynomial ring in variables x_{ij} with $1 \le i, j \le n$. Observe that the localization $k[x_{ij}]_{\det}$ has a natural $\mathbf{Z}$-grading, since $\det \in k[x_{ij}]$ is homogeneous. Let $k[x_{ij}]_{(\det)}$ denote the degree-0 part (i.e., fractions $f/\det^e$ with f homogenous of degree $e\deg(\det) = en$, for $e \ge 0$).

(i) Define $\mathrm{PGL}_n = \mathrm{Spec}(k[x_{ij}]_{(\det)})$. Identify this with the open affine $\{\det \neq 0\}$ in $\mathbf{P}^{n^2-1}$, and construct an injective map of sets $\mathrm{GL}_n(R)/R^\times \to \mathrm{PGL}_n(R) := \mathrm{Hom}_k(\mathrm{Spec}\,R, \mathrm{PGL}_n)$ naturally in k-algebras R.

(ii) For any R and any $m \in \mathrm{PGL}_n(R)$, show that there is an affine open covering $\{\mathrm{Spec}\,R_i\}$ of $\mathrm{Spec}\,R$ such that $m|_{R_i} \in \mathrm{GL}_n(R_i)/R_i^\times$. Deduce that $\mathrm{PGL}_n(R)$ is the *sheafification* of the presheaf $U \mapsto \mathrm{GL}_n(U)/\mathrm{GL}_1(U)$ on $\mathrm{Spec}\,U$, and that PGL_n has a unique k-group structure such that $\mathrm{GL}_n \to \mathrm{PGL}_n$ is a k-homomorphism.

(iii) Prove that if R is *local* then $\mathrm{GL}_n(R)/R^\times = \mathrm{PGL}_n(R)$, and construct a *counterexample* with $n = 2$ for any Dedekind domain R whose class group has nontrivial 2-torsion. (Hint: $I \oplus I \simeq R^2$ when I is 2-torsion.)

(iv) Write out the effect of multiplication and inversion on PGL_n at the level of coordinate rings.

2. The *scheme-theoretic kernel* of a k-homomorphism $f : G' \to G$ between k-group schemes is the scheme-theoretic fiber $f^{-1}(e)$ (with $e : \mathrm{Spec}\,k \to G$ the identity). It is denoted $\ker f$.

(i) Prove that if R is any k-algebra then $(\ker f)(R) = \ker(G'(R) \to G(R))$ as subgroups of $G'(R)$; deduce that $\ker f$ is a normal k-subgroup of G'.

(ii) Prove that the homomorphism $\mathrm{GL}_n \to \mathrm{PGL}_n$ constructed in Exercise 1 is surjective with scheme-theoretic kernel equal to the k-subgroup $D \simeq \mathrm{GL}_1$ of scalar diagonal matrices.

(iii) Let $\mu_n = \ker(t^n : \mathbf{G}_m \to \mathbf{G}_m) = \mathrm{Spec}(k[t, 1/t]/(t^n - 1))$. Identify $\mu_n(R)$ with the group of nth roots of unity in $R^\times$ naturally in any k-algebra R, and prove that the homomorphism $\mathrm{SL}_n \to \mathrm{PGL}_n$ obtained by restriction of the map in (ii) to SL_n is surjective, with kernel μ_n.

3. Let G be a k-group of finite type equipped with an action on k-scheme V of finite type. Let $W, W' \subseteq V$ be closed subschemes. Define the *functorial centralizer* $\underline{Z}_G(W)$ and *functorial transporter* $\underline{\mathrm{Tran}}_G(W, W')$ as follows: for any k-scheme S, $\underline{Z}_G(W)(S)$ is the subgroup of points $g \in G(S)$ such that the g-action on V_S is trivial, and $\underline{\mathrm{Tran}}_G(W, W')(S)$ is the subset of points $g \in G(S)$ such that $g.(W_S) \subseteq W'_S$ (as closed subschemes of V_S). The *functorial normalizer* $\underline{N}_G(W)$ is $\underline{\mathrm{Tran}}_G(W, W)$.

These are of most interest when W is a smooth closed k-subgroup of $V = G$ equipped with the left translation action. Below, assume W is *geometrically reduced* and *separated* over k.

(i) Prove W is smooth on a dense open, so $W(k_s)$ is Zariski-dense in W_{k_s} (by Exercise 5(iii), HW2). Hint: if $k = k_s$ then $W_{\overline{k}} \to W$ is a homeomorphism, and in general use Galois descent (as in Exercise 3(iii), HW2).

(ii) For each $w \in W(k)$, let $\alpha_w : G \to W$ be the orbit map $g \mapsto g.w$. Define $Z_G(w) = \alpha_w^{-1}(w)$. Prove that $Z_G(w)(S)$ is the subgroup of points $g \in G(S)$ such that $g.w_S = w_S$ in $W(S)$.

(iii) If $k = k_s$ prove $\cap_{w \in W(k)} Z_G(w)$ represents $\underline{Z}_G(W)$. (You need to use separatedness.) For general k apply Galois descent to $Z_{G_{k_s}}(W_{k_s})$; the representing scheme is denoted $Z_G(W)$.

(iv) If $k = k_s$, prove that $\cap_{w \in W(k)} \alpha_w^{-1}(W')$ represents $\underline{\mathrm{Tran}}_G(W, W')$. Then use Galois descent to prove representability by a closed subscheme $\mathrm{Tran}_G(W, W')$ for any k. The representing scheme is denoted $\mathrm{Tran}_G(W, W)$, so $N_G(W) := \mathrm{Tran}_G(W, W)$ represents $\underline{N}_G(W)$.

(v) Prove that for any k-algebra R and $g \in N_G(W)(R)$, the g-action $V_R \simeq V_R$ carries W_R *isomorphically* onto itself, and deduce that $N_G(W)$ is a k-subgroup of G. (Hint: reduce to artin local R and $k = \overline{k}$.)

4. Let G be a k-group of finite type. This exercise builds on the previous one. Note G is separated: $\Delta_{G/k}$ is a base change of $e : \mathrm{Spec}\,k \to G$! If G is smooth then the *scheme-theoretic center* of G is $Z_G := Z_G(G)$.

(i) Let G be SL_n or GL_n or PGL_n, and let T be the diagonal k-torus in each case. Prove that $Z_G(T) = T$ (as subschemes of G, not just at the level of geometric points!). Hint: to deduce the PGL_n-case from the GL_n-case, prove that the diagonal k-torus in GL_n is the scheme-theoretic preimage of the one in PGL_n.

(ii) Using (i), prove $Z_{\mathrm{SL}_n} = \mu_n$, $Z_{\mathrm{PGL}_n} = 1$, and Z_{GL_n} is the k-subgroup of scalar diagonal matrices.

(iii) Prove that for a smooth closed subscheme V in G, the formation of $Z_G(V)$ and $N_G(V)$ commutes with any extension of the ground field. (Hint: use the functorial characterizations, not the explicit constructions.) This applies to Z_G when G is smooth.

Algebraic Groups I. Homework 4

1. Let $T \subset \mathrm{Sp}_{2n}$ be the points $\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$ for diagonal $t \in \mathrm{GL}_n$. Prove $Z_G(T) = T$ (so T is a maximal torus!); deduce $Z_{\mathrm{Sp}_{2n}} = \mu_2$. The Appendix "Properties of orthogonal groups" computes $Z_{\mathrm{SO}(q)}$ (see Theorem 1.7).

2. Prove that PGL_n is smooth using the infinitesimal criterion, and prove that it is connected by a suitable "action" argument. The Appendix "Properties of orthogonal groups" treats the harder analogue for $\mathrm{SO}(q)$.

3. Let X be a scheme over a field k, and $x \in X(k)$. Recall that $\mathrm{Tan}_x(X)$ is identified as a set with the fiber of $X(k[\epsilon]) \to X(k)$ over x. Let $k[\epsilon, \epsilon'] = k[t, t']/(t, t')^2$, so this is 3-dimensional with basis $\{1, \epsilon, \epsilon'\}$.

(i) For $c \in k$, consider the k-algebra endomorphism of $k[\epsilon]$ defined by $\epsilon \mapsto c\epsilon$. Show that the resulting endomorphism of $X(k[\epsilon])$ over $X(k)$ restricts to scalar multiplication by c on the fiber $\mathrm{Tan}_x(X)$.

(ii) Using the two natural quotient maps $k[\epsilon, \epsilon'] \twoheadrightarrow k[\epsilon]$, define a natural map

$$X(k[\epsilon, \epsilon']) \to X(k[\epsilon]) \times_{X(k)} X(k[\epsilon])$$

and prove it is bijective. Using the natural quotient map $k[\epsilon, \epsilon'] \twoheadrightarrow k[\epsilon]$, show that the resulting map

$$X(k[\epsilon]) \times_{X(k)} X(k[\epsilon]) \overset{\simeq}{\leftarrow} X(k[\epsilon, \epsilon']) \to X(k[\epsilon])$$

induces addition on $\mathrm{Tan}_x(X)$: the k-linear structure on $\mathrm{Tan}_x(X)$ is encoded by the functor of X!

(iii) For $(X, x) = (G, e)$ with a k-group G, relate addition on $\mathrm{Tan}_x(X)$ to the group law on G: for $m : G \times G \to G$, show that $\mathrm{Tan}_e(G) \times \mathrm{Tan}_e(G) = \mathrm{Tan}_{(e,e)}(G \times G) \to \mathrm{Tan}_e(G)$ is addition.

4. Let A be a finite-dimensional associative algebra over a field k. Define the ring functor $\underline{A}$ on k-algebras by $\underline{A}(R) = A \otimes_k R$ and the group functor $\underline{A}^\times$ by $\underline{A}^\times(R) = (A \otimes_k R)^\times$.

(i) Prove that $\underline{A}$ is represented by an affine space over k. Using the k-scheme map $\mathrm{N}_{A/k} : \underline{A} \to \mathbf{A}^1_k$ defined functorially by $u \mapsto \det(m_u)$, where $m_u : A \otimes_k R \to A \otimes_k R$ is left multiplication by $u \in \underline{A}(R)$, prove that $\underline{A}^\times$ is represented by the open *affine* subscheme $\mathrm{N}_{A/k}^{-1}(\mathbf{G}_m)$. (This is often called "$A^\times$ viewed as a k-group", a phrase that is, strictly speaking, meaningless, since $A^\times$ does not encode the k-algebra A.)

(ii) For $A = \mathrm{Mat}_n(k)$ show that $\underline{A}^\times = \mathrm{GL}_n$, and for $k = \mathbf{Q}$ and $A = \mathbf{Q}(\sqrt{d})$ identify it with an explicit $\mathbf{Q}$-subgroup of GL_2 (depending on d).

(iii) How does the kernel of $\mathrm{N}_{A/k} : \underline{A}^\times \to \mathbf{G}_m$ (the *group of norm-*1 *units*) relate to Exercise 4(iii) in HW1 as a special case? For $A = \mathrm{Mat}_n(k)$, show that this homomorphism is the nth power (!) of the determinant.

5. This exercise develops a very important special case of Exercise 4. Let A be a finite-dimensional central simple algebra over k. By general theory, this is exactly the condition that $A_{\overline{k}} \simeq \mathrm{Mat}_n(\overline{k})$ as $\overline{k}$-algebras (for some $n \geq 1$), and such an isomorphism is unique up to conjugation by a unit (Skolem-Noether theorem).

(i) By a clever application of the Skolem-Noether theorem (see Exercise 30, Chapter 3 of the book by Farb/Dennis on non-commutative algebra), it is a classical fact that the linear derivations of a matrix algebra over a field are precisely the inner derivations (i.e., $x \mapsto yx - xy$ for some y). Combining this with length-induction on artin local rings, prove the Skolem-Noether theorem for $\mathrm{Mat}_n(R)$ for any artin local ring R (i.e., all R-algebra automorphisms are conjugation by a unit).

(ii) Construct an affine k-scheme I of finite type such that naturally $I(R) = \mathrm{Isom}_R(A_R, \mathrm{Mat}_n(R))$, the set of R-algebra isomorphisms. Note that $I(\overline{k})$ is non-empty! Prove I is smooth by checking the infinitesimal criterion for $I_{\overline{k}}$ with the help of (i). Deduce that $A_K \simeq \mathrm{Mat}_n(K)$ for a finite *separable* extension K/k.

(iii) By (ii), we can choose a finite Galois extension K/k and a K-algebra isomorphism $\theta : A_K \simeq \mathrm{Mat}_n(K)$, and by Skolem-Noether this is unique up to conjugation by a unit. Prove that for any choice of θ, the determinant map transfers to a multiplicative map $\underline{A}_K \to \mathbf{A}^1_K$ which is independent of θ. Deduce that it is $\mathrm{Gal}(K/k)$-equivariant, and so descends to a multiplicative map $\mathrm{Nrd}_{A/k} : \underline{A} \to \mathbf{A}^1_k$ which "becomes" the determinant over *any* extension F/k for which $A_F \simeq \mathrm{Mat}_n(F)$. Prove that $\mathrm{Nrd}^n_{A/k} = \mathrm{N}_{A/k}$ (explaining the name *reduced norm* for $\mathrm{Nrd}_{A/k}$), and conclude that $\underline{A}^\times = \mathrm{Nrd}^{-1}_{A/k}(\mathbf{G}_m)$.

(iv) Let $\mathrm{SL}(A)$ denote the scheme-theoretic kernel of $\mathrm{Nrd}_{A/k} : \underline{A}^\times \to \mathbf{G}_m$. Prove that its formation commutes with any extension of the ground field, and that it becomes isomorphic to SL_n over $\overline{k}$. In particular, $\mathrm{SL}(A)$ is *smooth* and *connected*; it is a "twisted form" of SL_n. (This is false for $\ker \mathrm{N}_{A/k}$ whenever $\mathrm{char}(k) | n!$)

Algebraic Groups I. Homework 5

1. Let k be a field, U_n the standard strictly upper-triangular unipotent k-subgroup of GL_n. Prove that no nontrivial k-group scheme is isomorphic to closed k-subgroups of $\mathbf{G}_a$ and $\mathbf{G}_m$. (If $\mathrm{char}(k) = p > 0$, the key is to prove that μ_p is not a k-subgroup of $\mathbf{G}_a$.) Deduce that $T \cap U_n = 1$ for any k-torus T in GL_n.

2. Let a smooth finite type k-group G act linearly on a finite-dimensional V. Let $\underline{V}$ denote the affine space whose A-points are V_A. Define $\underline{V}^G(A)$ to be the set of $v \in V_A$ on which G_A acts trivially.

(i) Prove that $\underline{V}^G$ is represented by the closed subscheme associated to a k-subspace of V (denoted of course as V^G). Hint: use Galois descent to reduce to the case $k = k_s$, and then show $V^{G(k)}$ works.

(ii) For an extension field K/k, prove that $(V_K)^{G_K} = (V^G)_K$ inside of V_K.

3. This exercise develops the important concept of *Weil restriction of scalars* in the affine case. It is an analogue of viewing a complex manifold as a real manifold with twice the dimension (and "complex points" become "real points"). Let k be a field, k' a finite commutative k-algebra (*not* necessarily a field!), and X' an affine k'-scheme of finite type. Consider the functor $\mathrm{R}_{k'/k}(X') : A \rightsquigarrow X'(k' \otimes_k A)$ on k-algebras.

(i) By considering $X' = \mathbf{A}^n_{k'}$ and then any X' via a closed immersion into an affine space, prove that this functor is represented by an affine k-scheme of finite type, again denoted $\mathrm{R}_{k'/k}(X')$. Prove its formation naturally commutes with products in X', and compute $\mathrm{R}_{k'/k}(\mathbf{G}_m)$ inside $\mathrm{R}_{k'/k}(\mathbf{A}^1_{k'})$. What if $k' = 0$?

(ii) Prove $\mathrm{R}_{k'/k}(\mathrm{Spec}\, k') = \mathrm{Spec}\, k$, and explain why $\mathrm{R}_{k'/k}(X')$ is naturally a k-group when X' is a k'-group.

(iii) For an extension field K/k, prove that $\mathrm{R}_{k'/k}(X')_K \simeq \mathrm{R}_{K'/K}(X'_{K'})$ for $K' = k' \otimes_k K$. Taking $K = \overline{k}$, use the infinitesimal criterion to prove that if k' is a field then $\mathrm{R}_{k'/k}(X')$ is k-smooth when X' is k'-smooth. (Can you see it directly from the construction?) Warning: if k'/k is not separable then $\mathrm{R}_{k'/k}(X')$ can be empty (resp. disconnected) when X' is non-empty (resp. geometrically integral)!

(iv) If k'/k is a separable extension field, prove $\mathrm{R}_{k'/k}(X')_{k_s} \simeq \prod_\sigma \sigma^*(X')$ with σ varying through $\mathrm{Hom}_k(k', k_s)$. Transfer the natural $\mathrm{Gal}(k_s/k)$-action on the left over to the right and describe it.

4. Let $\Gamma = \mathrm{Gal}(k_s/k)$. For any k-torus T, define the *character group* $\mathrm{X}(T) = \mathrm{Hom}_{k_s}(T_{k_s}, \mathbf{G}_m)$. A Γ-*lattice* is a finite free $\mathbf{Z}$-module equipped with a Γ-action making an open subgroup act trivially.

(i) Prove $\mathrm{X}(T)$ is a finite free $\mathbf{Z}$-module of rank $\dim T$. Describe a natural Γ-lattice structure on $\mathrm{X}(T)$.

(ii) For a Γ-lattice Λ, prove $R \rightsquigarrow \mathrm{Hom}(\Lambda, R^\times_{k_s})^\Gamma$ is represented by a k-torus $\mathrm{D}_k(\Lambda)$, the *dual* of Λ. (Hint: use finite Galois descent to reduce to Λ with trivial Γ-action.) Prove $\Lambda \simeq \mathrm{X}(\mathrm{D}_k(\Lambda))$ naturally as Γ-lattices.

(iii) Prove $T \simeq \mathrm{D}_k(\mathrm{X}(T))$ naturally as k-tori, so the category of k-tori is anti-equivalent to the category of Γ-lattices. Describe scalar extension in such terms, and prove T is k-split if and only if $\mathrm{X}(T) = \mathrm{X}(T)^\Gamma$.

(iv) Prove a map of k-tori $T' \to T$ is surjective if and only if $\mathrm{X}(T) \to \mathrm{X}(T')$ is injective. Prove $\ker(T' \to T)$ is a k-torus (resp. finite, resp. 0) if and only if $\mathrm{coker}(X(T) \to X(T'))$ is torsion-free (resp. finite, resp. 0). Inducting on $\dim T$, prove smooth *connected* k-subgroups M of T are k-tori. (Hint: prove $M(\overline{k})$ is divisible.)

(v) If k'/k is a finite separable subextension of k_s, prove that $\mathrm{R}_{k'/k}(T')$ is a k-torus if T' is a k'-torus. (For $T' = \mathbf{G}_m$, this is "$k'^\times$ viewed as a k-group".) By functorial considerations, prove $\mathrm{X}(\mathrm{R}_{k'/k}(T')) = \mathrm{Ind}_{\Gamma'}^{\Gamma}(\mathrm{X}(T))$ with Γ' the open subgroup corresponding to k'. For every k-torus T, construct a surjective k-homomorphism $\prod_i \mathrm{Res}_{k'_i/k}(\mathbf{G}_m) \twoheadrightarrow T$ for finite separable extensions k'_i/k. Conclude that k-tori are *unirational* over k.

(vi) (optional) For a finite extension field k'/k, define a *norm* map $\mathrm{N}_{k'/k} : \mathrm{R}_{k'/k}(\mathbf{G}_m) \to \mathbf{G}_m$. Prove its kernel is a torus when k'/k is separable (e.g., $k = \mathbf{R}$!), and relate to HW1, Exercise 4(iii) for imperfect k.

5. Consider a k-torus $T \subset \mathrm{GL}(V)$, with k infinite. Let $A_T \subset \mathrm{End}(V)$ be the commutative k-subalgebra generated by $T(k)$ (Zariski-dense in T since k is infinite, due to unirationality from Exercise 4(iv)).

(i) Using Jordan decomposition, prove that all elements of $T(\overline{k})$ are semisimple in $\mathrm{End}(V_{\overline{k}})$.

(ii) Assume $k = k_s$. Prove A_T is a product of copies of k, and $T(k) = A_T^\times$ when T is maximal.

(iii) Using Galois descent and the end of 4(v), prove $(A_T)_{k_s} = A_{T_{k_s}}$, and deduce $T(k) = A_T^\times$ for maximal T. Show naturally $T \simeq \mathrm{Res}_{A_T/k}(\mathbf{G}_m)$, and that maximal k-subtori in $\mathrm{GL}(V)$ and maximal étale commutative k-subalgebras of $\mathrm{End}(V)$ are in bijective correspondence. Generalize to *finite* k with another definition of A_T, and to central simple algebras in place of $\mathrm{End}(V)$ (hint: use HW4 Exercise 5(ii) and Galois descent).

(iv) For any (possibly finite) k, prove a smooth connected *commutative* k-group is a torus if and only if its $\overline{k}$-points are semisimple. (Use the end of Exercise 4(iv).)

ALGEBRAIC GROUPS I. HOMEWORK 6

1. Use the method of proof of Proposition 4.10, Chapter I, to prove the following scheme-theoretic version: if k is a field and a smooth unipotent affine k-group G is equipped with a left action on a quasi-affine k-scheme V of finite type then for any $v \in V(k)$ the smooth locally closed image of the orbit map $G \to V$ defined by $g \mapsto gv$ is actually closed in V.

(Hint: to begin, let $k[V]$ denote the k-algebra of global functions on V and prove that $R \otimes_k k[V]$ is the R-algebra of global functions on V_R for any k-algebra R. Use this to construct a functorial k-linear representation of G on $k[V]$ respecting the k-algebra structure. Borel's K should be replaced with k after passing to the case $k = \overline{k}$. Note that it is not necessary to assume Borel's F is non-empty; the argument directly proves J meets $k^{\times}$, so $J = (1)$ and hence F is empty.)

2. A k-homomorphism $f : G' \to G$ between k-groups of finite type is an *isogeny* if it is surjective and flat with finite kernel.

(i) Prove that a surjective homomorphism between smooth finite type k-groups of the same dimension is an isogeny. (The Miracle Flatness Theorem will be useful here.)

(ii) Prove that a map $f : T' \to T$ between k-tori is an isogeny if and only if the corresponding map $\mathrm{X}(T) \to \mathrm{X}(T')$ between Galois lattices is injective with finite cokernel.

(iii) Prove the following are equivalent for a k-torus T: (a) it contains $\mathbf{G}_m$ as a k-subgroup, (b) there exists a surjective k-homomorphism $T \twoheadrightarrow \mathbf{G}_m$, and (c) $\mathrm{X}(T)_{\mathbf{Q}}$ has a nonzero $\mathrm{Gal}(k_s/k)$-invariant vector. Such T are called *k-isotropic*; otherwise we say T is *k-anisotropic*. In general, a smooth affine k-group is called *k-isotropic* if it contains $\mathbf{G}_m$ as a k-subgroup, and *k-anisotropic* otherwise.

(iv) Let T be a k-torus. Prove the existence of a k-split k-subtorus T_s that contains all others, as well as a k-anisotropic k-subtorus T_a that contains all others. Also prove that $T_s \times T_a \to T$ is an isogeny. Compute T_s and T_a for $T = \mathrm{R}_{k'/k}(\mathbf{G}_m)$ for a finite separable extension k'/k.

3. (i) For a k-torus T, prove the existence of an étale k-group $\mathrm{Aut}_{T/k}$ representing the automorphism functor $S \rightsquigarrow \mathrm{Aut}_S(T_S)$. (Hint: if T is k-split then show that the constant k-group associated to $\mathrm{Aut}(\mathrm{X}(T)) \simeq \mathrm{GL}_r(\mathbf{Z})$ does the job. In general let k'/k be finite Galois such that $T_{k'}$ is k'-split, and use Galois descent.)

(ii) Using the existence of the étale k-group $\mathrm{Aut}_{T/k}$, prove that if a connected k-group scheme G is equipped with an action on T then the action must be trivial. Deduce that if T is a normal k-subgroup of a connected finite type k-group G then it is a central k-subgroup. Give an example of a smooth connected k-group containing $\mathbf{G}_a$ as a *non-central* normal k-subgroup. (Hint: look inside SL_2.)

4. Let T be a k-torus in a k-group G of finite type. This exercise uses $\mathrm{Aut}_{T/k}$ from Exercise 3.

(i) Construct a k-morphism $N_G(T) \to \mathrm{Aut}_{T/k}$ with kernel $Z_G(T)$. Prove $W(G,T) := N_G(T)(\overline{k})/Z_G(T)(\overline{k})$ is naturally a *finite* subgroup of $\mathrm{Aut}_{\mathbf{Z}}(\mathrm{X}(T))$. If $f : G' \to G$ is surjective with finite kernel and T' is a k-torus in G' containing $\ker f$ with $f(T') = T$ then prove $W(G', T') \to W(G,T)$ is an isomorphism.

(ii) For $G = \mathrm{GL}_n, \mathrm{PGL}_n, \mathrm{SL}_n, \mathrm{Sp}_{2n}$ and T the k-split diagonal maximal k-torus (so $Z_G(T) = T$), respectively identify $\mathrm{X}(T)$ with $\mathbf{Z}^n$, $\mathbf{Z}^n/\mathrm{diag}$, $\{m \in \mathbf{Z}^n \mid \sum m_j = 0\}$, and $\mathbf{Z}^n$. Prove $N_G(T)(k)/Z_G(T)(k) \subset \mathrm{Aut}_{\mathbf{Q}}(\mathrm{X}(T)_{\mathbf{Q}})$ is S_n for the first three, and $S_n \ltimes \langle -1 \rangle^n$ for Sp_{2n}, all with natural action. (Hint: to control $N_G(T)$, via $G \hookrightarrow \mathrm{GL}(V)$ decompose V as a direct sum of T-stable lines with *distinct* eigencharacters.)

5. Let (V, q) be a non-degenerate quadratic space over a field k with $\dim V \geq 2$. This exercise proves $\mathrm{SO}(q)$ contains $\mathbf{G}_m$ (i.e., it is k-isotropic in the sense of Exercise 2(iii)) if and only if $q = 0$ has a solution in $V - \{0\}$.

(i) If $q = 0$ has a nonzero solution v in V, prove that v lies in a hyperbolic plane H with $H \oplus H^{\perp} = V$. (If $\mathrm{char}(k) = 2$ and $\dim V$ is odd, work over $\overline{k}$ to show $v \notin V^{\perp}$.) Use this to construct a $\mathbf{G}_m$ inside of $\mathrm{SO}(q)$.

(ii) If $\mathrm{SO}(q)$ contains $\mathbf{G}_m$ as a k-subgroup S, prove that $q = 0$ has a nonzero solution in V. (Hint: apply Exercise 5(iii) in HW5 to the 2-dimensional k-split k-torus T generated in $\mathrm{GL}(V)$ by S and the central $\mathbf{G}_m$. If $A \simeq k^r$ is the corresponding "k-split" commutative k-subalgebra of $\mathrm{End}(V)$, prove the resulting inclusion $\mathbf{G}_m = S \hookrightarrow T = \mathrm{R}_{A/k}(\mathbf{G}_m) = \mathbf{G}_m^r$ is $t \mapsto (t^{h_1}, \dots, t^{h_r})$. Use the A-module structure on V to find a k-basis $\{e_i\}$ that identifies S with $\mathrm{diag}(t^{n_1}, \dots, t^{n_d})$ for $n_1 \leq \cdots \leq n_d$ with $\sum n_i = 0$. Prove $n_1 < 0 < n_d$, and if $q = \sum_{i \leq j} a_{ij} x_i x_j$ in these coordinates then prove $n_i + n_j = 0$ when $a_{ij} \neq 0$. Deduce $q(v) = 0$ for any v in the span of the e_i for which $n_i < 0$, or for which $n_i > 0$.)

Algebraic Groups I. Homework 7

0. (optional) Read the proof (p. 101 in Mumford's "Abelian Varieties") of *Cartier's theorem*: group schemes G locally of finite type over a field of characteristic 0 are smooth! (This uses the left-invariant derivations.)

1. (i) Prove that ∂_x is an invariant vector field on $\mathbf{G}_a$, and $t^{-1}\partial_t$ is an invariant vector field on $\mathbf{G}_m$.

(ii) Let A be a finite-dimensional associative k-algebra, and $\underline{A}^\times$ the associated k-group of units. Prove $\mathrm{Tan}_e(\underline{A}^\times) = A$ naturally, and that the Lie algebra structure is then $[a, a'] = aa' - a'a$. Using $A = \mathrm{End}(V)$, compute $\mathfrak{gl}(V)$. Use this to compute the Lie algebras $\mathfrak{sl}(V), \mathfrak{pgl}(V), \mathfrak{sp}(B), \mathfrak{gsp}(B), \mathfrak{so}(q)$.

(iii) Read Corollary A.7.6 and Lemma A.7.13 (and the paragraph preceding it) in the book *Pseudo-reductive groups*. Compute the p-Lie algebra structure on $\mathrm{Lie}(\underline{A}^\times)$, $\mathrm{Lie}(\mathbf{G}_m)$, and $\mathrm{Lie}(\mathbf{G}_a)$ if $\mathrm{char}(k) = p > 0$.

2. Let G be a smooth group of dimension $d > 0$ over k.

(i) Define the concept of *left-invariant* differential i-form for $i \geq 0$, and prove the space $\Omega_G^{i,\ell}(G)$ of such form has dimension $\binom{d}{i}$. Compute the 1-dimensional $\Omega_G^{d,\ell}(G)$ for $\mathrm{GL}(V)$, $\mathrm{SL}(V)$, and $\mathrm{PGL}(V)$.

(ii) Using right-translation, construct a linear representation of G on $\Omega_G^{d,\ell}(G)$; the associated character $\chi_G : G \to \mathbf{G}_m$ is the *modulus character*. Prove $\chi_G|_{Z_G} = 1$ and deduce that $\chi_G = 1$ if $G/Z_G = \mathscr{D}(G/Z_G)$.

(iii) (optional) If k is local (allow $\mathbf{R}$, $\mathbf{C}$) and X is smooth, use the k-analytic inverse function theorem to equip $X(k)$ with a functorial k-analytic manifold structure, and use k-analytic Change of Variables to assign a measure on $X(k)$ to a nowhere-vanishing $\omega \in \Omega_X^{\dim X}(X)$. (Serre's "Lie groups and Lie algebras" does k-analytic foundations.) Relate with Haar measures, and prove $\chi_G^{\pm 1}|_{G(k)}$ *is* the classical modulus character.

3. Let K/k be a degree-2 finite étale algebra (i.e., a separable quadratic field extension or $k \times k$), and let σ be the unique non-trivial k-automorphism of K; note that $K^\sigma = k$. A *σ-hermitian space* is a pair (V, h) consisting of a finite free K-module equipped with a perfect σ-semilinear form $h : V \times V \to K$ (i.e., $h(cv, v') = ch(v, v')$, $h(v, cv') = \sigma h(v, v')$, and $h(v', v) = \sigma(h(v, v'))$). Note $v \mapsto h(v, v)$ is a quadratic form $q_h : V \to k$ over k satisfying $q_h(cv) = \mathrm{N}_{K/k}(c) q_h(v)$ for $c \in K$, $v \in V$, and $\dim_k V$ is even ($\mathrm{char}(k) = 2$ ok!).

The *unitary group* $\mathrm{U}(h)$ over k is the subgroup of $\mathrm{R}_{K/k}(\mathrm{GL}(V))$ preserving h. Using $\mathrm{R}_{K/k}(\mathrm{SL}(V))$ gives the *special unitary group* $\mathrm{SU}(h)$. Example: $V = F$ finite étale over K with an involution σ' lifting σ, and $h(v, v') := \mathrm{Tr}_{F/K}(v\sigma'(v'))$; e.g., F and K CM fields, k totally real, and complex conjugations σ' and σ.

(i) If $K = k \times k$, prove $V \simeq V_0 \times V_0^\vee$ with $h((v, \ell), (v', \ell')) = (\ell'(v), \ell(v'))$ for a k-vector space V_0. Identify $\mathrm{U}(h)$ with $\mathrm{GL}(V_0)$ carrying $\mathrm{SU}(h)$ to $\mathrm{SL}(V_0)$. Compute q_h and prove non-degeneracy.

(ii) In the non-split case prove that $\mathrm{U}(h)_K \simeq \mathrm{GL}_n$ carrying $\mathrm{SU}(h)$ to SL_n ($n = \dim_K V$). Prove $\mathrm{U}(h)$ is smooth and connected with derived group $\mathrm{SU}(h)$ and center $\mathbf{G}_m$, and q_h is non-degenerate. Compute $\mathfrak{su}(h)$.

(iii) Identify $\mathrm{U}(h)$ with a k-subgroup of $\mathrm{SO}(q_h)$. Discuss the split case, and all cases with $k = \mathbf{R}$.

4. Let a smooth k-group H act on a separated k-scheme Y. For a k-scheme S, let $Y^H(S)$ be the set of $y \in Y(S)$ invariant by the H_S-action on Y_S (i.e., $y_{S'}$ is $H(S')$-invariant for all S-schemes S').

(i) If $k = k_s$, prove Y^H is represented by the closed subscheme $\cap_{h \in H(k)} Y^h$ where $Y^h = \alpha_h^{-1}(\Delta_{Y/k})$ for $\alpha_h : Y \to Y \times Y$ the map $y \mapsto (y, h.y)$. Then prove representability by a closed subscheme of Y for general k by Galois descent. Relate this to Exercise 2 in HW5.

(ii) For $y \in Y^H(k)$ explain why H acts on $\mathrm{Tan}_y(Y)$ and prove $\mathrm{Tan}_y(Y^H) = \mathrm{Tan}_y(Y)^H$.

(iii) Assume H is a closed subgroup of a k-group G of finite type, $\mathfrak{g} := \mathrm{Lie}(G)$ and $\mathfrak{h} := \mathrm{Lie}(H)$. Prove $\mathrm{Tan}_e(Z_G(H)) = \mathfrak{g}^H$ via adjoint action. Also prove $\mathrm{Tan}_e(N_G(H)) = \cap_{h \in H(k)} (\mathrm{Ad}_G(h) - 1)^{-1}(\mathfrak{h})$ when $k = k_s$.

5. A diagram $1 \to G' \overset{j}{\to} G \overset{\pi}{\to} G'' \to 1$ of finite type k-groups is *exact* if π is faithfully flat and $G' = \ker \pi$.

(i) For any such diagram, prove $G'' = G/G'$ via π. Prove a diagram of k-tori $1 \to T' \to T \to T'' \to$ is exact if and only if $0 \to \mathrm{X}(T'') \to \mathrm{X}(T) \to \mathrm{X}(T') \to 0$ is exact (as $\mathbf{Z}$-modules).

(ii) If G' is finite then π is an *isogeny*. Prove that isogenies are *finite flat* with constant degree, and that $\pi_n : \mathrm{SL}_n \to \mathrm{PGL}_n$ is an isogeny of degree n. Compute $\mathrm{Lie}(\pi_n)$; when is it surjective?

(iii) Prove that a short exact sequence of finite type k-groups induces a left-exact sequence of Lie algebras, short exact if G and G' are smooth. (Smoothness of G can be dropped.)

(iv) Read §A.3 through Example A.3.4 in *Pseudo-reductive groups*, and prove $F_{X/k} : X \to X^{(p)}$ is finite flat of degree $p^{\dim X}$ for k-smooth X. Prove $\mathrm{Lie}(F_{G/k}) = 0$, and compute $F_{G/k}$ for $\mathrm{GL}(V)$ and $\mathrm{O}(q)$.

Algebraic Groups I. Homework 8

1. Let A be a central simple algebra over a field k, T a k-torus in $\underline{A}^{\times}$.

(i) Adapt Exercise 5 in HW5 to make an étale commutative k-subalgebra $A_T \subseteq A$ such that $(A_T)_{k_s}$ is generated by $T(k_s)$, and establish a bijection between the sets of maximal k-tori in $\underline{A}^{\times}$ and maximal étale commutative k-subalgebras of A. Deduce that $\mathrm{SL}(A)$ is k-anisotropic if and only if A is a division algebra.

(ii) For an étale commutative k-subalgebra $C \subseteq A$, prove $Z_A(C)$ is a semisimple k-algebra with center C.

(iv) If T is *maximal* as a k-split subtorus of $\underline{A}^{\times}$ prove T is the k-group of units in A_T and that the (central!) simple factors B_i of $B_T := Z_A(A_T)$ are *division algebras.*

(v) Fix $A \simeq \mathrm{End}_D(V)$ for a right module V over a central division algebra D, so V is a left A-module and $V = \prod V_i$ with *nonzero* left B_i-modules V_i. If T is maximal as a k-split torus in $\underline{A}^{\times}$, prove V_i has rank 1 over B_i and D, so $B_i \simeq D$. Using D-bases, deduce that *all maximal k-split tori in $\underline{A}^{\times}$ are $\underline{A}^{\times}(k)$-conjugate.*

2. For a torus T over a local field k (allow $\mathbf{R}$, $\mathbf{C}$), prove T is k-anisotropic if and only if $T(k)$ is compact.

3. Let Y be a smooth separated k-scheme locally of finite type, and T a k-torus with a left action on Y. This exercise proves that Y^T is *smooth.*

(i) Reduce to the case $k = \overline{k}$. Fix a finite local k-algebra R with residue field k, and an ideal J in R with $J\mathfrak{m}_R = 0$. Choose $\overline{y} \in Y^T(R/J)$, and for R-algebras A let $E(A)$ be the fiber of $Y(A) \twoheadrightarrow Y(A/JA)$ over $\overline{y}_{A/JA}$. Let $y_0 = \overline{y} \bmod \mathfrak{m}_R \in Y^T(k)$ and $A_0 = A/\mathfrak{m}_R A$. Prove $E(A) \neq \emptyset$ and make it a torsor over the A_0-module $F(A) := JA \otimes_k \mathrm{Tan}_{y_0}(Y) = JA \otimes_{A_0} (A_0 \otimes_k \mathrm{Tan}_{y_0}(Y))$ naturally in A (denoted $v + y$).

(ii) Define an A_0-linear $T(A_0)$-action on $F(A)$ (hence a T_R-action on F), and prove that $E(A)$ is $T(A)$-stable in $Y(A)$ with $t.(v + y) = t_0.v + t.y$ for $y \in E(A)$, $t \in T(A)$, $v \in F(A)$, and $t_0 = t \bmod \mathfrak{m}_R$.

(iii) Choose $\xi \in E(R)$ and define a map of functors $h : T_R \to F$ by $t.\xi = h(t) + \xi$ for points t of T_R; check it is a 1-cocycle, and is a 1-coboundary if and only if $E^{T_R}(R) \neq \emptyset$. For $V_0 = J \otimes_k \mathrm{Tan}_{y_0}(Y)$ use h to define a 1-cocycle $h_0 : T \to \underline{V}_0$, and prove $t.(v, c) := (t.v + ch_0(t), c)$ is a k-linear representation of T on $V_0 \oplus k$. Use a T-equivariant splitting (!) to prove h_0 (and then h) is a 1-coboundary; deduce Y^T is smooth!

4. Let G be a smooth k-group of finite type, and T a k-torus equipped with a left action on G (an interesting case being T a k-subgroup acting by conjugation, in which case $G^T = Z_G(T)$).

(i) Use Exercise 3 to show $Z_G(T)$ is smooth, and by computing its tangent space at the identity prove for *connected* G that $T \subset Z_G$ if and only if T acts trivially on $\mathfrak{g} = \mathrm{Lie}(G)$.

(ii) Assume T is a k-subgroup of G acting by conjugation. Using Exercise 4(iii) of HW7 and the semisimplicity of the restriction to T of $\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$, prove that $N_G(T)$ and $Z_G(T)$ have the same tangent space at the identity. Via (i), deduce that $Z_G(T)$ is an *open subscheme* of $N_G(T)$, so $N_G(T)$ is *smooth* and $N_G(T)/Z_G(T)$ is finite étale over k.

(iii) Assumptions as in (ii), the *Weyl group* $W = W(G,T)$ is $N_G(T)/Z_G(T)$. If T is k-split, use the equality $\mathrm{End}_k(T) = \mathrm{End}_{k_s}(T_{k_s})$ to prove that $W(k) = W(k_s)$ and deduce that W is a constant k-group. But show $N_G(T)(k)$ does *not* map onto $W(k)$ if k is infinite and K is a separable quadratic extension of k such that $-1 \notin \mathrm{N}_{K/k}(K^{\times})$ (e.g., k totally real and K a CM extension, or $k = \mathbf{Q}$ and $K = \mathbf{Q}(\sqrt{3})$) with $G = \mathrm{SL}(K) \simeq \mathrm{SL}_2$ and T the *non-split* maximal k-torus corresponding the norm-1 part of $K \subset \mathrm{End}_k(K)$.

(iv) Prove that $N_G(T)(k) \to W(k) = W(\overline{k})$ is surjective for the cases in HW6, Exercise 4(ii).

5. (i) For any field k, affine k-scheme X of finite type, and nonzero finite k-algebra k', define a natural map $j_{X,k'/k} : X \to \mathrm{Res}_{k'/k}(X_{k'})$ by $X(R) \to X(k' \otimes_k R) = X_{k'}(k' \otimes_k R)$ for k-algebras R. Prove $j_{X,k'/k}$ is a closed immersion and that its formation commutes with fiber products in X.

(ii) Let G be an affine k-group of finite type. Prove that $j_{G,k'/k}$ is a k-homomorphism.

(iii) A *vector group* over k is a k-group G admitting an isomorphism $G \simeq \mathbf{G}_a^n$, and a *linear structure* on G is the resulting $\mathbf{G}_m$-action. A *linear homomorphism* $G' \to G$ between vector groups equipped with linear structures is a k-homomorphism which respects the linear structures. For example, $(x, y) \mapsto (x, y + x^p)$ is a *non-linear* automorphism of $\mathbf{G}_a^2$ (with its usual linear structure) when $\mathrm{char}(k) = p > 0$.

For any k, prove $\mathbf{G}_a$ admits a unique linear structure and its linear endomorphism ring is k. Giving $\mathbf{G}_a^n$ and $\mathbf{G}_a^m$ their usual linear structures, prove the linear k-homomorphisms $\mathbf{G}_a^n \to \mathbf{G}_a^m$ correspond to $\mathrm{Mat}_{m\times n}(k)$. Are there non-linear homomorphisms if $\mathrm{char}(k) = 0$?

Algebraic Groups I. Homework 9

1. Read Appendix B in the book *Pseudo-reductive groups* to learn Tits' structure theory for smooth connected unipotent groups over arbitrary fields k with positive characteristic, and how k-tori act on such groups. Especially noteworthy are the results labelled B.1.13, B.2.7, B.3.4, and B.4.3.

2. Let U be a smooth connected commutative affine k-group, and assume U is p-torsion if $\mathrm{char}(k) = p > 0$.

(i) If $\mathrm{char}(k) > 0$ and U is k-split, use B.1.12 in *Pseudo-reductive groups* to prove U is a vector group.

(ii) Assume $\mathrm{char}(k) = 0$. Prove that any short exact sequence $0 \to \mathbf{G}_a \to G \to \mathbf{G}_a \to 0$ is split. (Hint: $\log(u)$ is an "algebraic" function on the unipotent points of Mat_n.) Deduce that $U \simeq \mathbf{G}_a^N$, and prove that any action on U by a k-split torus T respects this linear structure.

3. Let k'/k be a degree-p purely inseparable extension of a field k of characteristic $p > 0$.

(i) Prove that $U = \mathrm{R}_{k'/k}(\mathbf{G}_m)/\mathbf{G}_m$ is smooth and connected of dimension $p-1$, and is p-torsion. Deduce it is unipotent.

(ii) In the Appendix "Quotient formalism" it is proved that any commutative extension of $\mathbf{G}_a$ by $\mathbf{G}_m$ over any field is uniquely split over that field. Prove that $\mathrm{R}_{k'/k}(\mathbf{G}_m)(k_s)[p] = 1$, and deduce that U in (i) does not contain $\mathbf{G}_a$ as a k-subgroup! (For a salvage, see Lemma B.1.10 in *Pseudo-reductive groups*: a p-torsion smooth connected commutative affine group over any field of characteristic $p > 0$ admits an étale isogeny onto a vector group.)

4. Let G be a smooth group of finite type over a field k, and N a commutative normal k-subgroup scheme.

(i) Prove that the left G-action on N via conjugation factors uniquely through an action of G/N on N, and if N is central in G then prove that the action of G on itself via conjugation uniquely factors through an action of G/N on G. Describe this explicitly for $G = \mathrm{SL}_n$ and $N = \mu_n$ over any field k, accounting for the fact that $\mathrm{SL}_n(k) \to \mathrm{PGL}_n(k)$ is generally *not* surjective.

(ii) Prove the commutator map $G \times G \to G$ uniquely factors through a k-morphism $(G/Z_G) \times (G/Z_G) \to \mathscr{D}(G)$.

5. Let B be a smooth connected solvable group over a field k.

(i) If $B = \mathbf{G}_m \ltimes \mathbf{G}_a$ with the standard semi-direct product structure, prove that $Z_B(t, 0)$ is the left factor for any $t \in k^\times - \{1\}$.

(ii) Deduce by inductive arguments resting on (i) that if $k = \overline{k}$ and $S \subset B(k)$ is a commutative subgroup of semisimple elements then $S \subset T(k)$ for some maximal torus $T \subset B$.

(iii) Assume $\mathrm{char}(k) \neq 2$ with $k = \overline{k}$, and let $G = \mathrm{SO}_n$ with $n \geq 3$. Let $\mu \simeq \mu_2^{n-1}$ be the "diagonal" k-subgroup $\{(\zeta_i) \in \mu_2^n \mid \prod \zeta_i = 1\}$. Prove that the disconnected μ is maximal as a solvable smooth k-subgroup of G and is not contained in any maximal k-torus of G (hint: it has too much 2-torsion), so in particular is not contained in any Borel k-subgroup (by (ii))!

6. Let G be a quasi-split smooth connected affine k-group, and $B \subset G$ a Borel k-subgroup. Let T be a maximal k-torus in B.

(i) Using conjugacy of maximal tori in $G_{\overline{k}}$, prove $g \mapsto gBg^{-1}$ is a bijection from $N_G(T)(\overline{k})/Z_G(T)(\overline{k})$ onto the set of Borel $\overline{k}$-subgroups containing $T_{\overline{k}}$. In particular, this set is *finite*.

(ii) Using HW8 Exercise 4, prove that $N_G(T)(k_s)/Z_G(T)(k_s) \to N_G(\overline{k})/Z_G(T)(\overline{k})$ is bijective, and deduce that every Borel subgroup of $G_{\overline{k}}$ containing $T_{\overline{k}}$ is defined over k_s!

(iii) Assume that T is k-split and $Z_G(T) = T$. Using Hilbert 90 and HW8 Exercise 4, prove that $N_G(T)(k)/T(k) \to N_G(T)(k_s)/Z_G(T)(k_s)$ is bijective. Deduce that every Borel subgroup of $G_{\overline{k}}$ containing $T_{\overline{k}}$ is defined over k! In each of the classical cases (GL_n, SL_n, PGL_n, Sp_{2n}, and SO_n), find all B containing the k-split maximal"diagonal" T. How many parabolic k-subgroups can you find containing one such B? (At least for GL_n, SL_n, and PGL_n, prove you have found all such parabolics.)

(iv) Prove that each maximal smooth unipotent subgroup of $G_{\overline{k}}$ admits a conjugate contained in $B_{\overline{k}}$, and deduce that if $B \cap B' = T$ for another Borel B' containing T then G is reductive. Use this with (iii) to prove reductivity for GL_n $(n \geq 1)$, SL_n $(n \geq 2)$, PGL_n $(n \geq 2)$, Sp_{2n} $(n \geq 1)$, and SO_n $(n \geq 2)$.

Algebraic Groups I. Homework 10

1. Let G be a smooth connected affine group over a field k.

(i) For a maximal k-torus T in G and a smooth connected k-subgroup N in G that is normalized by T, prove that $T \cap N$ is a maximal k-torus in N (e.g., smooth and connected!). Show by example that $S \cap N$ can be disconnected for a non-maximal k-torus S. Hint: first analyze $Z_G(T) \cap N$ using $T \ltimes N$ to reduce to the case when T is central in G, and then pass to G/T.

(ii) Let H be a smooth connected normal k-subgroup of G, and P a parabolic k-subgroup. If $k = \overline{k}$ then prove $(P \cap H)^0_{\rm red}$ is a parabolic k-subgroup of H, and use Chevalley's theorem on parabolics being their own normalizers on geometric points (applied to H) to prove $P \cap H$ is connected (hint: work over $\overline{k}$).

(iii) Granting $Q = N_H(Q)$ scheme-theoretically for parabolic Q in H (Prop. 3.5.7 in *Pseudo-reductive groups*, rests on structure theory of reductive groups), prove $P \cap H$ in (ii) is smooth. (Hint: prove $(P \cap H)^0_{\rm red}$ is normal in P, hence in $P \cap H$!) In particular, $B \cap H$ is a Borel k-subgroup of H for all Borels B of G.

2. Let k be a field, and $G \in \{{\rm SL}_2, {\rm PGL}_2\}$.

(i) Define a unique ${\rm PGL}_2$-action on ${\rm SL}_2$ lifting conjugation. Prove a k-automorphism of G preserving the standard Borel k-subgroup and the diagonal k-torus is induced by the action of a diagonal k-point of ${\rm PGL}_2$.

(ii) Prove that the homomorphism ${\rm PGL}_2(k) \to {\rm Aut}_k(G)$ is an isomorphism. In particular, every k-automorphism of ${\rm PGL}_2$ is inner. Show that ${\rm SL}_2$ admits non-inner k-automorphisms if and only if $k^\times \neq (k^\times)^2$.

3. Let $\lambda : \mathbf{G}_m \to G$ be a 1-parameter k-subgroup of a smooth affine k-group G. Define $\mu : U_G(\lambda^{-1}) \times P_G(\lambda) \to G$ to be multiplication. We seek to prove it is an open immersion. Let $\mathfrak{g} = {\rm Lie}(G)$.

(i) For $n \in \mathbf{Z}$ define $\mathfrak{g}_n$ to be the n-weight space for λ (i.e., ${\rm ad}(\lambda(t)).X = t^n X$). Define $\mathfrak{g}_{\lambda \geq 0} = \oplus_{n \geq 0} \mathfrak{g}_n$, and similarly for $\mathfrak{g}_{\lambda > 0}$. Prove ${\rm Lie}(P_G(\lambda)) = \mathfrak{g}_{\lambda \geq 0}$, ${\rm Lie}(U_G(\lambda)) = \mathfrak{g}_{\lambda > 0}$, and ${\rm Tan}_{(e,e)}(\mu)$ is an isomorphism.

(ii) If $G = {\rm GL}(V)$ and the $\mathbf{G}_m$-action on V has weights $e_1 > \cdots > e_m$, justify the block-matrix descriptions of $U_G(\lambda^{\pm 1})$, $Z_G(\lambda)$, and $P_G(\lambda)$. Deduce $U_G(\lambda^{-1})$ and $P_G(\lambda)$ are smooth and have trivial intersection.

(iii) Working over $\overline{k}$ and using suitable left and right translations by geometric points, prove that ${\rm d}\mu(\xi)$ is an isomorphism for all $\overline{k}$-points ξ of $U_G(\lambda^{-1}) \times P_G(\lambda)$. Deduce that if $U_G(\lambda^{-1})$ and $P_G(\lambda)$ are smooth (OK for ${\rm GL}(V)$ by (ii)) then μ induces an isomorphism between complete local rings at all $\overline{k}$-points, and conclude that μ is flat and quasi-finite. Hence, μ has open image in such cases.

(iv) Using valuative criterion for properness, prove a flat quasi-finite separated map $f : X \to Y$ between noetherian schemes is proper if all fibers X_y have the same rank. (Hint: base change to Y the spectrum of a dvr.) By Zariski's Main Theorem, proper quasi-finite maps are finite. Deduce μ is an open immersion if $U_G(\lambda^{-1})$ and $P_G(\lambda)$ are smooth with trivial intersection. (Hint: finite flat of fiber-degree 1 is isomorphism.)

This settles ${\rm GL}(V)$; the Appendix "Dynamic approach to algebraic groups" then yields the general case!

4. Let $\lambda : \mathbf{G}_m \to G$ be a 1-parameter k-subgroup of a smooth affine k-group. For any integer $n \geq 1$, prove that $P_G(\lambda^n) = P_G(\lambda)$, $U_G(\lambda^n) = U_G(\lambda)$, and $Z_G(\lambda^n) = Z_G(\lambda)$.

5. Let G be a reductive group over a field k, and N a smooth closed normal k-subgroup. Prove N is reductive. In particular, $\mathscr{D}(G)$ is reductive.

6. Prove that $\mu_n[d] = \mu_d$ for $d|n$, and that $\mathbf{Z}/n\mathbf{Z} \to {\rm End}(\mu_n)$ is an isomorphism.

7. Prove that a rational homomorphism (defined in evident manner: respecting multiplication as rational map) between smooth connected groups over a field k extends uniquely to a k-homomorphism. (Hint: pass to the case $k = k_s$ by Galois descent, and then use suitable k-point translations.)

8. (optional) Let G be a smooth connected affine group over an algebraically closed field k, ${\rm char}(k) = 0$.

(i) If all finite-dimensional linear representations of G are completely reducible, then prove that G is reductive. (Hint: use Lie-Kolchin, and behavior of semisimplicity under restriction to a normal subgroup. This will not use characteristic 0.)

(ii) Conversely, assume that G is reductive. The structure theory of reductive groups implies that ${\rm Lie}(\mathscr{D}G)$ is a semisimple Lie algebra, and a subspace of a finite-dimensional linear representation space for G is G-stable if and only if it is $\mathfrak{g}$-stable under the induced action $\mathfrak{g} \to {\rm End}(V)$ since ${\rm char}(k) = 0$. Prove that all finite-dimensional linear representations of G are completely reducible.