Algebraic Groups I. Homework 1

1. This exercise studies the endomorphism rings of the k-groups $\mathbf{G}_m$ and $\mathbf{G}_a$, with k any commutative ring.

(i) Prove that $\mathrm{End}_k(\mathbf{G}_a)$ consists of $f \in k[t]$ such that $f(x+y) = f(x)+f(y)$ in $k[x,y]$, and that $\mathrm{End}_k(\mathbf{G}_m)$ consists of $f \in k[t,t^{-1}]$ such that $f(xy) = f(x)f(y)$ in $k[x,y,x^{-1},y^{-1}]$ and f has no zeros on any geometric fibers over $\mathrm{Spec}\, k$.

(ii) Deduce that if k is a $\mathbf{Q}$-algebra then naturally $\mathrm{End}_k(\mathbf{G}_a) = k$, and that if k is a field with characteristic $p > 0$ then it consists of $f = \sum c_j t^{p^j}$ $(c_j \in k)$. What if $k = \mathbf{Z}/(p^2)$?

(iii) Prove that $\mathrm{End}_k(\mathbf{G}_m) = \mathbf{Z}$ when k is a field, and deduce the same when k is an artin local ring via induction on the length of k. (Hint: reduce to the case when f vanishes on the special fiber.)

(iv) Prove that $\mathrm{End}_k(\mathbf{G}_m) = \mathbf{Z}$ for k *any* local ring by using (iii) to settle the case of a complete local noetherian ring, then any local noetherian ring, and finally any local ring (by using local noetherian subrings of k). Deduce that if k is any ring whatsoever, an endomorphism of the k-group $\mathbf{G}_m$ is $t \mapsto t^n$ for a locally constant function $n : \mathrm{Spec}\, k \to \mathbf{Z}$.

2. Let V be a finite-dimensional vector space over a field k. This exercise develops coordinate-free versions of GL_n, SL_n, and Sp_{2n} attached to V.

(i) Elements of the graded symmetric algebra $\mathrm{Sym}(V^*)$ are called *polynomial functions on* V. Justify the name (even for finite k!) by identifying them with *functorial maps* of sets $V_R \to R$ given by polynomial expressions relative to some (equivalently, any) basis of V, with R a varying k-algebra. In particular, show that det is a polynomial function on $\mathrm{End}(V)$.

(ii) For any k-algebra R, define the functors $\underline{\mathrm{End}}(V)$ and $\underline{\mathrm{Aut}}(V)$ on k-algebras R by $R \rightsquigarrow \mathrm{End}(V_R)$, $R \rightsquigarrow \mathrm{Aut}_R(V_R)$. Using the identification $\mathrm{End}(V_R, V_R) = \mathrm{End}(V)_R$, prove that $\underline{\mathrm{End}}(V)$ is represented by $\mathrm{Sym}(\mathrm{End}(V)^*)$.

(iii) Define $\det \in \mathrm{Sym}(\mathrm{End}(V)^*)$ and prove its non-vanishing locus

$$\mathrm{GL}(V) := \mathrm{Spec}(\mathrm{Sym}(\mathrm{End}(V)^*)[1/\det])$$

represents $\underline{\mathrm{Aut}}(V)$ as subfunctor of $\underline{\mathrm{End}}(V)$. Also discuss $\mathrm{SL}(V)$ as a closed k-subgroup of $\mathrm{GL}(V)$.

(iv) Let $B : V \times V \to k$ be a bilinear form. Prove that the subfunctor $\underline{\mathrm{Aut}}(V, B)$ of points of $\underline{\mathrm{Aut}}(V)$ preserving B is represented by a closed k-subgroup of $\mathrm{GL}(V)$. (You can use coordinates in the proof!) This is pretty bad unless B is non-degenerate. (In the alternating non-degenerate case it is denoted $\mathrm{Sp}(B)$.)

Assuming non-degeneracy, a linear automorphism T of V_R is a *B-similitude* if $B_R(Tv, Tw) = \mu(T)B(v,w)$ for all $v, w \in V_R$ and some $\mu(T) \in R^\times$. Prove $\mu(T)$ is then unique, and show that the functor of B-similitudes is represented by a closed k-subgroup of $\mathrm{GL}(V) \times \mathbf{G}_m$. (In the alternating case it is denoted $\mathrm{GSp}(B)$.)

3. (i) Prove that if a connected scheme X of finite type over a field k has a k-rational point, then $X_{k'} = X \otimes_k k'$ is connected for every finite extension k'/k (hint: $X_{k'} \to X$ is open and closed; look at fiber over $X(k)$). Deduce that $X_{k'}$ is connected for *every* extension k'/k (i.e., X is *geometrically connected* over k).

(ii) Prove that if X and Y are geometrically connected of finite type over k, so is $X \times Y$; give a counterexample over $k = \mathbf{Q}$ if "geometrically" is removed. Deduce that if G is a k-group then the identity component G^0 is a k-subgroup whose formation commutes with any extension on k.

4. Let G be a group of finite type over a field k.

(i) Prove that $(G_{\overline{k}})_{\mathrm{red}}$ is a closed $\overline{k}$-subgroup of $G_{\overline{k}}$, and prove it is *smooth*. Deduce that G^0 is *geometrically irreducible*.

(ii) Over any imperfect field k, one can make a non-reduced k-group G such that G_{red} is *not* a k-subgroup. Where does an attempted proof to the contrary get stuck?

(iii) Assume k is imperfect, $\mathrm{char}(k) = p > 0$, and choose $a \in k - k^p$. Prove $x_0^p + a x_1^p + \cdots + a^{p-1} x_{p-1}^p = 1$ defines a reduced k-group (think of $\mathrm{N}_{k(a^{1/p})/k}$) that is non-reduced over $\overline{k}$ and hence not smooth!

(iv) Prove that the condition $t^n = 1$ defines a finite closed k-subgroup $\mu_n \subseteq \mathbf{G}_m$, and show its preimage G under $\det : \mathrm{GL}_N \to \mathbf{G}_m$ is a k-subgroup of GL_N. Accepting that SL_N is connected, prove $G^0 = \mathrm{SL}_N$ if $\mathrm{char}(k) \nmid n$. For $k = \mathbf{Q}$ and $n = 5$, prove that $G - G^0$ is connected but over $\overline{k}$ has 4 connected components.