Algebraic Groups I. Homework 10

1. Let G be a smooth connected affine group over a field k.

(i) For a maximal k-torus T in G and a smooth connected k-subgroup N in G that is normalized by T, prove that $T \cap N$ is a maximal k-torus in N (e.g., smooth and connected!). Show by example that $S \cap N$ can be disconnected for a non-maximal k-torus S. Hint: first analyze $Z_G(T) \cap N$ using $T \ltimes N$ to reduce to the case when T is central in G, and then pass to G/T.

(ii) Let H be a smooth connected normal k-subgroup of G, and P a parabolic k-subgroup. If $k = \overline{k}$ then prove $(P \cap H)^0_{\mathrm{red}}$ is a parabolic k-subgroup of H, and use Chevalley's theorem on parabolics being their own normalizers on geometric points (applied to H) to prove $P \cap H$ is connected (hint: work over $\overline{k}$).

(iii) Granting $Q = N_H(Q)$ scheme-theoretically for parabolic Q in H (Prop. 3.5.7 in *Pseudo-reductive Groups*; rests on structure theory of reductive groups), prove $P \cap H$ in (ii) is smooth. (Hint: prove $(P \cap H)^0_{\mathrm{red}}$ is normal in P, hence in $P \cap H$!) In particular, $B \cap H$ is a Borel k-subgroup of H for all Borels B of G.

2. Let k be a field, and $G \in \{\mathrm{SL}_2, \mathrm{PGL}_2\}$.

(i) Define a unique PGL_2-action on SL_2 lifting conjugation. Prove a k-automorphism of G preserving the standard Borel k-subgroup and the diagonal k-torus is induced by the action of a diagonal k-point of PGL_2.

(ii) Prove that the homomorphism $\mathrm{PGL}_2(k) \to \mathrm{Aut}_k(G)$ is an isomorphism. In particular, every k-automorphism of PGL_2 is inner. Show that SL_2 admits non-inner k-automorphisms if and only if $k^\times \neq (k^\times)^2$.

3. Let $\lambda : \mathbf{G}_m \to G$ be a 1-parameter k-subgroup of a smooth affine k-group G. Define $\mu : U_G(\lambda^{-1}) \times P_G(\lambda) \to G$ to be multiplication. We seek to prove it is an open immersion. Let $\mathfrak{g} = \mathrm{Lie}(G)$.

(i) For $n \in \mathbf{Z}$ define $\mathfrak{g}_n$ to be the n-weight space for λ (i.e., $\mathrm{ad}(\lambda(t)).X = t^n X$). Define $\mathfrak{g}_{\lambda \geq 0} = \oplus_{n \geq 0} \mathfrak{g}_n$, and similarly for $\mathfrak{g}_{\lambda > 0}$. Prove $\mathrm{Lie}(P_G(\lambda)) = \mathfrak{g}_{\lambda \geq 0}$, $\mathrm{Lie}(U_G(\lambda)) = \mathfrak{g}_{\lambda > 0}$, and $\mathrm{Tan}_{(e,e)}(\mu)$ is an isomorphism.

(ii) If $G = \mathrm{GL}(V)$ and the $\mathbf{G}_m$-action on V has weights $e_1 > \cdots > e_m$, justify the block-matrix descriptions of $U_G(\lambda^{\pm 1})$, $Z_G(\lambda)$, and $P_G(\lambda)$. Deduce $U_G(\lambda^{-1})$ and $P_G(\lambda)$ are smooth and have trivial intersection.

(iii) Working over $\overline{k}$ and using suitable left and right translations by geometric points, prove that $\mathrm{d}\mu(\xi)$ is an isomorphism for all $\overline{k}$-points ξ of $U_G(\lambda^{-1}) \times P_G(\lambda)$. Deduce that if $U_G(\lambda^{-1})$ and $P_G(\lambda)$ are smooth (OK for $\mathrm{GL}(V)$ by (ii)) then μ induces an isomorphism between complete local rings at all $\overline{k}$-points, and conclude that μ is flat and quasi-finite. Hence, μ has open image in such cases.

(iv) Using valuative criterion for properness, prove a flat quasi-finite separated map $f : X \to Y$ between noetherian schemes is proper if all fibers X_y have the same rank. (Hint: base change to Y the spectrum of a dvr.) By Zariski's Main Theorem, proper quasi-finite maps are finite. Deduce μ is an open immersion if $U_G(\lambda^{-1})$ and $P_G(\lambda)$ are smooth with trivial intersection. (Hint: finite flat of fiber-degree 1 is isomorphism.)

This settles $\mathrm{GL}(V)$; handout on "dynamic approach to algebraic groups" yields the general case from this!

4. Let $\lambda : \mathbf{G}_m \to G$ be a 1-parameter k-subgroup of a smooth affine k-group. For any integer $n \geq 1$, prove that $P_G(\lambda^n) = P_G(\lambda)$, $U_G(\lambda^n) = U_G(\lambda)$, and $Z_G(\lambda^n) = Z_G(\lambda)$.

5. Let G be a reductive group over a field k, and N a smooth closed normal k-subgroup. Prove N is reductive. In particular, $\mathscr{D}(G)$ is reductive.

6. Prove that $\mu_n[d] = \mu_d$ for $d|n$, and that $\mathbf{Z}/n\mathbf{Z} \to \mathrm{End}(\mu_n)$ is an isomorphism.

7. Prove that a rational homomorphism (defined in evident manner: respecting multiplication as rational map) between smooth connected groups over a field k extends uniquely to a k-homomorphism. (Hint: pass to the case $k = k_s$ by Galois descent, and then use suitable k-point translations.)

8. (optional) Let G be a smooth connected affine group over an algebraically closed field k, $\mathrm{char}(k) = 0$.

(i) If all finite-dimensional linear representations of G are completely reducible, then prove that G is reductive. (Hint: use Lie-Kolchin, and behavior of semisimplicity under restriction to a normal subgroup. This will not use characteristic 0.)

(ii) Conversely, assume that G is reductive. The structure theory of reductive groups implies that $\mathfrak{g}$ is a semisimple Lie algebra, and a subspace of a finite-dimensional linear representation space for G is G-stable if and only if it is $\mathfrak{g}$-stable under the induced action $\mathfrak{g} \to \mathrm{End}(V)$ since $\mathrm{char}(k) = 0$. Prove that all finite-dimensional linear representations of G are completely reducible.