# Algebraic Groups I. Homework 2

1. Let $k$ be a perfect field, and $G$ a 1-dimensional connected linear algebraic $k$-group (so $G$ is geometrically integral over $k$). Assume $G$ is in the additive case. This exercise proves $G$ is $k$-isomorphic to $\mathbf{G}_a$.

(i) Let $X$ denote its regular compactification over $k$. Prove that $X_{\overline{k}}$ is regular, so $X$ is smooth (hint: $\overline{k}$ is a direct limit of finite separable extensions of $k$, and unit discriminant is a sufficient test for integral closures in the Dedekind setting). Deduce that $X - G$ consists of a single physical point, say $\operatorname{Spec} k'$.

(ii) Prove that $k' \otimes_k \overline{k}$ is reduced and in fact equal to $\overline{k}$. Deduce $k' = k$, and prove that $X \simeq \mathbf{P}^1_k$. Show that $G \simeq \mathbf{G}_a$ as $k$-groups.

2. Let $T$ be a torus of dimension $r \geq 1$ over a field $k$ (e.g., a 1-dimensional connected linear algebraic group in the multiplicative case). This exercise proves that $T_{k'} \simeq \mathbf{G}_m^r$ for some finite separable extension $k'/k$.

(i) Prove that it suffices to treat the case $k = k_s$.

(ii) Assume $k = k_s$. We constructed an isomorphism $f : T_{k'} \simeq \mathbf{G}_m^r$ as $k'$-groups for some finite extension $k'/k$. Let $k'' = k' \otimes_k k'$, and let $p_1, p_2 : \operatorname{Spec} k'' \rightrightarrows \operatorname{Spec} k'$ be the projections. Prove that $k''$ is an artin local ring *with residue field* $k'$, and deduce that the $k''$-isomorphisms $p_i^*(f) : T_{k''} \simeq \mathbf{G}_m^r$ coincide by comparing them with $f$ on the special fiber!

(iii) For any $k$-vector space $V$, prove that the only elements of $k' \otimes_k V$ with equal images under both maps to $k'' \otimes_k V$ are the elements of $V$ (hint: reduce to the case $V = k$ and replace $k'$ with any $k$-vector space $W$, and $k''$ with $W \otimes_k W$). Deduce that $f$ uniquely descends to a $k$-isomorphism.

3. Let $X$ and $Y$ be schemes over a field $k$, $K/k$ an extension field, and $f, g : X \rightrightarrows Y$ two $k$-morphisms.

(i) Prove $f_K = g_K$ if and only if $f = g$. (Use surjectivity of $X_K \to X$ to aid in reducing to the affine case.) Likewise prove that if $Z, Z' \subseteq X$ are closed subschemes such that $Z_K = Z'_K$ inside of $X_K$ then $Z = Z'$,

(ii) If $f_K$ is an isomorphism and $X$ and $Y$ are affine, prove $f$ is an isomorphism. Then do the same without affineness (may be really hard without Serre's cohomological criterion for affineness).

(iii) Assume $K/k$ is Galois, $\Gamma = \operatorname{Gal}(K/k)$. Prove that if a map $F : X_K \to Y_K$ satisfies $\gamma^*(F) = F$ for all $\gamma \in \Gamma$, then $F = f_K$ for a unique $k$-map $f : X \to Y$. Likewise, if $Z' \subseteq X_K$ is a closed subscheme and $\gamma^*(Z') = Z'$ for all $\gamma \in \Gamma$ then prove $Z' = Z_K$ for a unique closed subscheme $Z \subseteq X$. Do the same for open subschemes.

4. Let $q : V \to k$ be a quadratic form on a finite-dimensional vector space $V$ of dimension $d \geq 2$, and let $B_q : V \times V \to k$ be the corresponding symmetric bilinear form. Let $V^\perp = \{v \in V \mid B_q(v, \cdot) = 0\}$; we call $\delta_q := \dim V^\perp$ the *defect* of $q$.

(i) Prove that $B_q$ uniquely factors through a non-degenerate symmetric bilinear form on $V/V^\perp$, and $B_q$ is non-degenerate precisely when the defect is 0. Prove that if $\operatorname{char}(k) = 2$ then $B_q$ is alternating, and deduce that $\delta_q \equiv \dim V \bmod 2$ for such $k$ (so $\delta_q \geq 1$ if $\dim V$ is odd).

(ii) Prove that if $\delta_q = 0$ then $q_{\overline{k}}$ admits one of the following "standard forms": $\sum_{i=1}^n x_i x_{i+n}$ if $\dim V = 2n$ ($n \geq 1$), and $x_0^2 + \sum_{i=1}^n x_i x_{i+n}$ if $\dim V = 2n + 1$ ($n \geq 1$). Do the same if $\operatorname{char}(k) = 2$ and $\delta_q = 1$. (Distinguish whether or not $q|_{V^\perp} \neq 0$.) How about the converse?

(iii) If $\operatorname{char}(k) \neq 2$, prove $\delta_q = 0$ if and only if $q \neq 0$ and $(q = 0) \subseteq \mathbf{P}^{d-1}$ is smooth. If $\operatorname{char}(k) = 2$ then prove $\delta_q \leq 1$ with $q|_{V^\perp} \neq 0$ when $\delta_q = 1$ if and only if $q \neq 0$ and the $(q = 0)$ is smooth. (Hint: use (ii) to simplify calculations.) We say $q$ is *non-degenerate* when $q \neq 0$ and $(q = 0)$ is smooth in $\mathbf{P}^{d-1}$.

5. Learn about separability and $\Omega^1$ by reading in Matsumura's CRT: §25 up to before 25.3 (this is better than AG15.1–15.8 in Borel's book), and read §26 up through and including Theorem 26.3.

(i) Do Exercises 25.3, 25.4 in Matsumura, and read AG17.1 in Borel's book (noting he requires $V$ to be geometrically reduced over $k$!).

(ii) Use 26.2 in Matsumura to prove that a finite type reduced $k$-scheme $X$ is smooth on a dense open if and only if all functions fields of $X$ (at its generic points) are *separable* over $k$.

(iii) Using separating transcendence bases, the primitive element theorem, and "denominator chasing", prove that if $X$ is smooth on a dense open then $X(k_s)$ is Zariski-dense in $X_{k_s}$. (Hint: it suffices to prove $X(k_s)$ is non-empty!)