Algebraic Groups I. Homework 3

1. Let $k[x_{ij}]$ be the polynomial ring in variables x_{ij} with $1 \le i, j \le n$. Observe that the localization $k[x_{ij}]_{\det}$ has a natural $\mathbf{Z}$-grading, since $\det \in k[x_{ij}]$ is homogeneous. Let $k[x_{ij}]_{(\det)}$ denote the degree-0 part (i.e., fractions $f/\det^e$ with f homogenous of degree $e\deg(\det) = en$, for $e \ge 0$).

(i) Define $\mathrm{PGL}_n = \mathrm{Spec}(k[x_{ij}]_{(\det)})$. Identify this with the open affine $\{\det \ne 0\}$ in $\mathbf{P}^{n^2-1}$, and construct an injective map of sets $\mathrm{GL}_n(R)/R^\times \to \mathrm{PGL}_n(R) := \mathrm{Hom}_k(\mathrm{Spec}\,R, \mathrm{PGL}_n)$ naturally in k-algebras R.

(ii) For any R and any $m \in \mathrm{PGL}_n(R)$, show that there is an affine open covering $\{\mathrm{Spec}\,R_i\}$ of $\mathrm{Spec}\,R$ such that $m|_{R_i} \in \mathrm{GL}_n(R_i)/R_i^\times$. Deduce that $\mathrm{PGL}_n(R)$ is the *sheafification* of the presheaf $U \mapsto \mathrm{GL}_n(U)/\mathrm{GL}_1(U)$ on $\mathrm{Spec}\,U$, and that PGL_n has a unique k-group structure such that $\mathrm{GL}_n \to \mathrm{PGL}_n$ is a k-homomorphism.

(iii) Prove that if R is *local* then $\mathrm{GL}_n(R)/R^\times = \mathrm{PGL}_n(R)$, and construct a *counterexample* with $n = 2$ for any Dedekind domain R whose class group has nontrivial 2-torsion. (Hint: $I \oplus I \simeq R^2$ when I is 2-torsion.)

(iv) Write out the effect of multiplication and inversion on PGL_n at the level of coordinate rings.

2. The *scheme-theoretic kernel* of a k-homomorphism $f : G' \to G$ between k-group schemes is the scheme-theoretic fiber $f^{-1}(e)$ (with $e : \mathrm{Spec}\,k \to G$ the identity). It is denoted $\ker f$.

(i) Prove that if R is any k-algebra then $(\ker f)(R) = \ker(G'(R) \to G(R))$ as subgroups of $G'(R)$; deduce that $\ker f$ is a normal k-subgroup of G'.

(ii) Prove that the homomorphism $\mathrm{GL}_n \to \mathrm{PGL}_n$ constructed in Exercise 1 is surjective with scheme-theoretic kernel equal to the k-subgroup $D \simeq \mathrm{GL}_1$ of scalar diagonal matrices.

(iii) Let $\mu_n = \ker(t^n : \mathbf{G}_m \to \mathbf{G}_m) = \mathrm{Spec}(k[t, 1/t]/(t^n - 1))$. Identify $\mu_n(R)$ with the group of nth roots of unity in $R^\times$ naturally in any k-algebra R, and prove that the homomorphism $\mathrm{SL}_n \to \mathrm{PGL}_n$ obtained by restriction of the map in (ii) to SL_n is surjective, with kernel μ_n.

3. Let G be a k-group of finite type equipped with an action on k-scheme V of finite type. Let $W, W' \subseteq V$ be closed subschemes. Define the *functorial centralizer* $\underline{Z}_G(W)$ and *functorial transporter* $\underline{\mathrm{Tran}}_G(W, W')$ as follows: for any k-scheme S, $\underline{Z}_G(W)(S)$ is the subgroup of points $g \in G(S)$ such that the g-action on V_S is trivial, and $\underline{\mathrm{Tran}}_G(W, W')(S)$ is the subset of points $g \in G(S)$ such that $g.(W_S) \subseteq W'_S$ (as closed subschemes of V_S). The *functorial normalizer* $\underline{N}_G(W)$ is $\underline{\mathrm{Tran}}_G(W, W)$.

These are of most interest when W is a smooth closed k-subgroup of $V = G$ equipped with the left translation action. Below, assume W is *geometrically reduced* and *separated* over k.

(i) Prove W is smooth on a dense open, so $W(k_s)$ is Zariski-dense in W_{k_s} (by Exercise 5(iii), HW2). Hint: if $k = k_s$ then $W_{\overline{k}} \to W$ is a homeomorphism, and in general use Galois descent (as in Exercise 3(iii), HW2).

(ii) For each $w \in W(k)$, let $\alpha_w : G \to W$ be the orbit map $g \mapsto g.w$. Define $Z_G(w) = \alpha_w^{-1}(w)$. Prove that $Z_G(w)(S)$ is the subgroup of points $g \in G(S)$ such that $g.w_S = w_S$ in $W(S)$.

(iii) If $k = k_s$ prove $\cap_{w \in W(k)} Z_G(w)$ represents $\underline{Z}_G(W)$. (You need to use separatedness.) For general k apply Galois descent to $Z_{G_{k_s}}(W_{k_s})$; the representing scheme is denoted $Z_G(W)$.

(iv) If $k = k_s$, prove that $\cap_{w \in W(k)} \alpha_w^{-1}(W')$ represents $\underline{\mathrm{Tran}}_G(W, W')$. Then use Galois descent to prove representability by a closed subscheme $\mathrm{Tran}_G(W, W')$ for any k. The representing scheme is denoted $\mathrm{Tran}_G(W, W)$, so $N_G(W) := \mathrm{Tran}_G(W, W)$ represents $\underline{N}_G(W)$.

(v) Prove that for any k-algebra R and $g \in N_G(W)(R)$, the g-action $V_R \simeq V_R$ carries W_R *isomorphically* onto itself, and deduce that $N_G(W)$ is a k-subgroup of G. (Hint: reduce to artin local R and $k = \overline{k}$.)

4. Let G be a k-group of finite type. This exercise builds on the previous one. Note G is separated: $\Delta_{G/k}$ is a base change of $e : \mathrm{Spec}\,k \to G$! If G is smooth then the *scheme-theoretic center* of G is $Z_G := Z_G(G)$.

(i) Let G be SL_n or GL_n or PGL_n, and let T be the diagonal k-torus in each case. Prove that $Z_G(T) = T$ (as subschemes of G, not just at the level of geometric points!). Hint: to deduce the PGL_n-case from the GL_n-case, prove that the diagonal k-torus in GL_n is the scheme-theoretic preimage of the one in PGL_n.

(ii) Using (i), prove $Z_{\mathrm{SL}_n} = \mu_n$, $Z_{\mathrm{PGL}_n} = 1$, and Z_{GL_n} is the k-subgroup of scalar diagonal matrices.

(iii) Prove that for a smooth closed subscheme V in G, the formation of $Z_G(V)$ and $N_G(V)$ commutes with any extension of the ground field. (Hint: use the functorial characterizations, not the explicit constructions.) This applies to Z_G when G is smooth.