Algebraic Groups I. Homework 4

1. Let $T \subset \mathrm{Sp}_{2n}$ be the torus of points $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ for diagonal $t \in \mathrm{GL}_n$. Prove $Z_G(T) = T$ (so T is a maximal torus!), and deduce $Z_{\mathrm{Sp}_{2n}} = \mu_2$. See the handout on orthogonal groups for a computation of $Z_{\mathrm{SO}(q)}$.

2. Prove that PGL_n is smooth using the infinitesimal criterion, and prove that it is connected by a suitable "action" argument. Then read the handout on smoothness and connectedness for orthogonal groups.

3. Let X be a scheme over a field k, and $x \in X(k)$. Recall that $\mathrm{Tan}_x(X)$ is identified as a set with the fiber of $X(k[\epsilon]) \to X(k)$ over x. Let $k[\epsilon, \epsilon'] = k[t, t']/(t, t')^2$, so this is 3-dimensional with basis $\{1, \epsilon, \epsilon'\}$.

(i) For $c \in k$, consider the k-algebra endomorphism of $k[\epsilon]$ defined by $\epsilon \mapsto c\epsilon$. Show that the resulting endomorphism of $X(k[\epsilon])$ over $X(k)$ restricts to scalar multiplication by c on the fiber $\mathrm{Tan}_x(X)$.

(ii) Using the two natural quotient maps $k[\epsilon, \epsilon'] \twoheadrightarrow k[\epsilon]$, define a natural map

$$X(k[\epsilon, \epsilon']) \to X(k[\epsilon]) \times_{X(k)} X(k[\epsilon])$$

and prove it is bijective. Using the natural quotient map $k[\epsilon, \epsilon'] \twoheadrightarrow k[\epsilon]$, show that the resulting map

$$X(k[\epsilon]) \times_{X(k)} X(k[\epsilon]) \overset{\simeq}{\leftarrow} X(k[\epsilon, \epsilon']) \to X(k[\epsilon])$$

induces addition on $\mathrm{Tan}_x(X)$: the k-linear structure on $\mathrm{Tan}_x(X)$ is encoded by the functor of X!

(iii) For $(X, x) = (G, e)$ with a k-group G, relate addition on $\mathrm{Tan}_x(X)$ to the group law on G: for $m : G \times G \to G$, show that $\mathrm{Tan}_e(G) \times \mathrm{Tan}_e(G) = \mathrm{Tan}_{(e,e)}(G \times G) \to \mathrm{Tan}_e(G)$ is addition.

4. Let A be a finite-dimensional associative algebra over a field k. Define the ring functor $\underline{A}$ on k-algebras by $\underline{A}(R) = A \otimes_k R$ and the group functor $\underline{A}^\times$ by $\underline{A}^\times(R) = (A \otimes_k R)^\times$.

(i) Prove that $\underline{A}$ is represented by an affine space over k. Using the k-scheme map $\mathrm{N}_{A/k} : \underline{A} \to \mathbf{A}^1_k$ defined functorially by $u \mapsto \det(m_u)$, where $m_u : A \otimes_k R \to A \otimes_k R$ is left multiplication by $u \in \underline{A}(R)$, prove that $\underline{A}^\times$ is represented by the open *affine* subscheme $\mathrm{N}_{A/k}^{-1}(\mathbf{G}_m)$. (This is often called "$A^\times$ viewed as a k-group", a phrase that is, strictly speaking, meaningless, since $A^\times$ does not encode the k-algebra A.)

(ii) For $A = \mathrm{Mat}_n(k)$ show that $\underline{A}^\times = \mathrm{GL}_n$, and for $k = \mathbf{Q}$ and $A = \mathbf{Q}(\sqrt{d})$ identify it with an explicit $\mathbf{Q}$-subgroup of GL_2 (depending on d).

(iii) How does the kernel of $\mathrm{N}_{A/k} : \underline{A}^\times \to \mathbf{G}_m$ (the *group of norm*-1 *units*) relate to Exercise 4(iii) in HW1 as a special case? For $A = \mathrm{Mat}_n(k)$, show that this homomorphism is the nth power (!) of the determinant.

5. This exercise develops a very important special case of Exercise 4. Let A be a finite-dimensional central simple algebra over k. By general theory, this is exactly the condition that $A_{\overline{k}} \simeq \mathrm{Mat}_n(\overline{k})$ as $\overline{k}$-algebras (for some $n \geq 1$), and such an isomorphism is unique up to conjugation by a unit (Skolem-Noether theorem).

(i) By a clever application of the Skolem-Noether theorem (see Exercise 30, Chapter 3 of the book by Farb/Dennis on non-commutative algebra), it is a classical fact that the linear derivations of a matrix algebra over a field are precisely the inner derivations (i.e., $x \mapsto yx - xy$ for some y). Combining this with length-induction on artin local rings, prove the Skolem-Noether theorem for $\mathrm{Mat}_n(R)$ for any artin local ring R (i.e., all R-algebra automorphisms are conjugation by a unit).

(ii) Construct an affine k-scheme I of finite type such that naturally $I(R) = \mathrm{Isom}_R(A_R, \mathrm{Mat}_n(R))$, the set of R-algebra isomorphisms. Note that $I(\overline{k})$ is non-empty! Prove I is smooth by checking the infinitesimal criterion for $I_{\overline{k}}$ with the help of (i). Deduce that $A_K \simeq \mathrm{Mat}_n(K)$ for a finite *separable* extension K/k.

(iii) By (ii), we can choose a finite Galois extension K/k and a K-algebra isomorphism $\theta : A_K \simeq \mathrm{Mat}_n(K)$, and by Skolem-Noether this is unique up to conjugation by a unit. Prove that for any choice of θ, the determinant map transfers to a multiplicative map $\underline{A}_K \to \mathbf{A}^1_K$ which is independent of θ. Deduce that it is $\mathrm{Gal}(K/k)$-equivariant, and so descends to a multiplicative map $\mathrm{Nrd}_{A/k} : \underline{A} \to \mathbf{A}^1_k$ which "becomes" the determinant over *any* extension F/k for which $A_F \simeq \mathrm{Mat}_n(F)$. Prove that $\mathrm{Nrd}_{A/k}^n = \mathrm{N}_{A/k}$ (explaining the name *reduced norm* for $\mathrm{Nrd}_{A/k}$), and conclude that $\underline{A}^\times = \mathrm{Nrd}_{A/k}^{-1}(\mathbf{G}_m)$.

(iv) Let $\mathrm{SL}(A)$ denote the scheme-theoretic kernel of $\mathrm{Nrd}_{A/k} : \underline{A}^\times \to \mathbf{G}_m$. Prove that its formation commutes with any extension of the ground field, and that it becomes isomorphic to SL_n over $\overline{k}$. In particular, $\mathrm{SL}(A)$ is *smooth* and *connected*; it is a "twisted form" of SL_n. (This is false for $\ker \mathrm{N}_{A/k}$ whenever $\mathrm{char}(k) | n!$)