Algebraic Groups I. Homework 5

1. Let k be a field, U_n the standard strictly upper-triangular unipotent k-subgroup of GL_n. Prove that no nontrivial k-group scheme is isomorphic to closed k-subgroups of $\mathbf{G}_a$ and $\mathbf{G}_m$. (If $\mathrm{char}(k) = p > 0$, the key is to prove that μ_p is not a k-subgroup of $\mathbf{G}_a$.) Deduce that $T \cap U_n = 1$ for any k-torus T in GL_n.

2. Let a smooth finite type k-group G act linearly on a finite-dimensional V. Let $\underline{V}$ denote the affine space whose A-points are V_A. Define $\underline{V}^G(A)$ to be the set of $v \in V_A$ on which G_A acts trivially.

(i) Prove that $\underline{V}^G$ is represented by the closed subscheme associated to a k-subspace of V (denoted of course as V^G). Hint: use Galois descent to reduce to the case $k = k_s$, and then show $V^{G(k)}$ works.

(ii) For an extension field K/k, prove that $(V_K)^{G_K} = (V^G)_K$ inside of V_K.

3. This exercise develops the important concept of *Weil restriction of scalars* in the affine case. It is an analogue of viewing a complex manifold as a real manifold with twice the dimension (and "complex points" become "real points"). Let k be a field, k' a finite commutative k-algebra (*not* necessarily a field!), and X' an affine k'-scheme of finite type. Consider the functor $\mathrm{R}_{k'/k}(X') : A \rightsquigarrow X'(k' \otimes_k A)$ on k-algebras.

(i) By considering $X' = \mathbf{A}^n_{k'}$ and then any X' via a closed immersion into an affine space, prove that this functor is represented by an affine k-scheme of finite type, again denoted $\mathrm{R}_{k'/k}(X')$. Prove its formation naturally commutes with products in X', and compute $\mathrm{R}_{k'/k}(\mathbf{G}_m)$ inside $\mathrm{R}_{k'/k}(\mathbf{A}^1_{k'})$. What if $k' = 0$?

(ii) Prove $\mathrm{R}_{k'/k}(\mathrm{Spec}\, k') = \mathrm{Spec}\, k$, and explain why $\mathrm{R}_{k'/k}(X')$ is naturally a k-group when X' is a k'-group.

(iii) For an extension field K/k, prove that $\mathrm{R}_{k'/k}(X')_K \simeq \mathrm{R}_{K'/K}(X'_{K'})$ for $K' = k' \otimes_k K$. Taking $K = \overline{k}$, use the infinitesimal criterion to prove that if k' is a field then $\mathrm{R}_{k'/k}(X')$ is k-smooth when X' is k'-smooth. (Can you see it directly from the construction?) Warning: if k'/k is not separable then $\mathrm{R}_{k'/k}(X')$ can be empty (resp. disconnected) when X' is non-empty (resp. geometrically integral)!

(iv) If k'/k is a separable extension field, prove $\mathrm{R}_{k'/k}(X')_{k_s} \simeq \prod_\sigma \sigma^*(X')$ with σ varying through $\mathrm{Hom}_k(k', k_s)$. Transfer the natural $\mathrm{Gal}(k_s/k)$-action on the left over to the right and describe it.

4. Let $\Gamma = \mathrm{Gal}(k_s/k)$. For any k-torus T, define the *character group* $\mathrm{X}(T) = \mathrm{Hom}_{k_s}(T_{k_s}, \mathbf{G}_m)$. A Γ-*lattice* is a finite free $\mathbf{Z}$-module equipped with a Γ-action making an open subgroup act trivially.

(i) Prove $\mathrm{X}(T)$ is a finite free $\mathbf{Z}$-module of rank $\dim T$. Describe a natural Γ-lattice structure on $\mathrm{X}(T)$.

(ii) For a Γ-lattice Λ, prove $R \rightsquigarrow \mathrm{Hom}(\Lambda, R^\times_{k_s})^\Gamma$ is represented by a k-torus $\mathrm{D}_k(\Lambda)$, the *dual* of Λ. (Hint: use finite Galois descent to reduce to Λ with trivial Γ-action.) Prove $\Lambda \simeq \mathrm{X}(\mathrm{D}_k(\Lambda))$ naturally as Γ-lattices.

(iii) Prove $T \simeq \mathrm{D}_k(\mathrm{X}(T))$ naturally as k-tori, so the category of k-tori is anti-equivalent to the category of Γ-lattices. Describe scalar extension in such terms, and prove T is k-split if and only if $\mathrm{X}(T) = \mathrm{X}(T)^\Gamma$.

(iv) Prove a map of k-tori $T' \to T$ is surjective if and only if $\mathrm{X}(T) \to \mathrm{X}(T')$ is injective. Prove $\ker(T' \to T)$ is a k-torus (resp. finite, resp. 0) if and only if $\mathrm{coker}(X(T) \to X(T'))$ is torsion-free (resp. finite, resp. 0). Inducting on $\dim T$, prove smooth *connected* k-subgroups M of T are k-tori. (Hint: prove $M(\overline{k})$ is divisible.)

(v) If k'/k is a finite separable subextension of k_s, prove that $\mathrm{R}_{k'/k}(T')$ is a k-torus if T' is a k'-torus. (For $T' = \mathbf{G}_m$, this is "$k'^\times$ viewed as a k-group".) By functorial considerations, prove $\mathrm{X}(\mathrm{R}_{k'/k}(T')) = \mathrm{Ind}^{\Gamma}_{\Gamma'}(\mathrm{X}(T))$ with Γ' the open subgroup corresponding to k'. For every k-torus T, construct a surjective k-homomorphism $\prod_i \mathrm{Res}_{k'_i/k}(\mathbf{G}_m) \twoheadrightarrow T$ for finite separable extensions k'_i/k. Conclude that k-tori are *unirational* over k.

(vi) (optional) For a finite extension field k'/k, define a *norm* map $\mathrm{N}_{k'/k} : \mathrm{R}_{k'/k}(\mathbf{G}_m) \to \mathbf{G}_m$. Prove its kernel is a torus when k'/k is separable (e.g., $k = \mathbf{R}$!), and relate to HW1, Exercise 4(iii) for imperfect k.

5. Consider a k-torus $T \subset \mathrm{GL}(V)$, with k infinite. Let $A_T \subset \mathrm{End}(V)$ be the commutative k-subalgebra generated by $T(k)$ (Zariski-dense in T since k is infinite, due to unirationality from Exercise 4(iv)).

(i) Using Jordan decomposition, prove that all elements of $T(\overline{k})$ are semisimple in $\mathrm{End}(V_{\overline{k}})$.

(ii) Assume $k = k_s$. Prove A_T is a product of copies of k, and $T(k) = A_T^\times$ when T is maximal.

(iii) Using Galois descent and the end of 4(v), prove $(A_T)_{k_s} = A_{T_{k_s}}$, and deduce $T(k) = A_T^\times$ for maximal T. Show naturally $T \simeq \mathrm{Res}_{A_T/k}(\mathbf{G}_m)$, and that maximal k-subtori in $\mathrm{GL}(V)$ and maximal étale commutative k-subalgebras of $\mathrm{End}(V)$ are in bijective correspondence. Generalize to *finite* k with another definition of A_T, and to central simple algebras in place of $\mathrm{End}(V)$ (hint: use HW4 Exercise 5(ii) and Galois descent).

(iv) For any (possibly finite) k, prove a smooth connected *commutative* k-group is a torus if and only if its $\overline{k}$-points are semisimple. (Use the end of Exercise 4(iv).)