Algebraic Groups I. Homework 6

1. Use the method of proof of Proposition 4.10, Chapter I, to prove the following scheme-theoretic version: if k is a field and a smooth unipotent affine k-group G is equipped with a left action on a quasi-affine k-scheme V of finite type then for any $v \in V(k)$ the smooth locally closed image of the orbit map $G \to V$ defined by $g \mapsto gv$ is actually closed in V.

(Hint: to begin, let $k[V]$ denote the k-algebra of global functions on V and prove that $R \otimes_k k[V]$ is the R-algebra of global functions on V_R for any k-algebra R. Use this to construct a functorial k-linear representation of G on $k[V]$ respecting the k-algebra structure. Borel's K should be replaced with k after passing to the case $k = \overline{k}$. Note that it is not necessary to assume Borel's F is non-empty; the argument directly proves J meets $k^{\times}$, so $J = (1)$ and hence F is empty.)

2. A k-homomorphism $f : G' \to G$ between k-groups of finite type is an *isogeny* if it is surjective and flat with finite kernel.

(i) Prove that a surjective homomorphism between smooth finite type k-groups of the same dimension is an isogeny. (The Miracle Flatness Theorem will be useful here.)

(ii) Prove that a map $f : T' \to T$ between k-tori is an isogeny if and only if the corresponding map $\mathrm{X}(T) \to \mathrm{X}(T')$ between Galois lattices is injective with finite cokernel.

(iii) Prove the following are equivalent for a k-torus T: (a) it contains $\mathbf{G}_m$ as a k-subgroup, (b) there exists a surjective k-homomorphism $T \twoheadrightarrow \mathbf{G}_m$, and (c) $\mathrm{X}(T)_{\mathbf{Q}}$ has a nonzero $\mathrm{Gal}(k_s/k)$-invariant vector. Such T are called *k-isotropic*; otherwise we say T is *k-anisotropic*. In general, a smooth affine k-group is called *k-isotropic* if it contains $\mathbf{G}_m$ as a k-subgroup, and *k-anisotropic* otherwise.

(iv) Let T be a k-torus. Prove the existence of a k-split k-subtorus T_s that contains all others, as well as a k-anisotropic k-subtorus T_a that contains all others. Also prove that $T_s \times T_a \to T$ is an isogeny. Compute T_s and T_a for $T = \mathrm{R}_{k'/k}(\mathbf{G}_m)$ for a finite separable extension k'/k.

3. (i) For a k-torus T, prove the existence of an étale k-group $\mathrm{Aut}_{T/k}$ representing the automorphism functor $S \rightsquigarrow \mathrm{Aut}_S(T_S)$. (Hint: if T is k-split then show that the constant k-group associated to $\mathrm{Aut}(\mathrm{X}(T)) \simeq \mathrm{GL}_r(\mathbf{Z})$ does the job. In general let k'/k be finite Galois such that $T_{k'}$ is k'-split, and use Galois descent.)

(ii) Using the existence of the étale k-group $\mathrm{Aut}_{T/k}$, prove that if a connected k-group scheme G is equipped with an action on T then the action must be trivial. Deduce that if T is a normal k-subgroup of a connected finite type k-group G then it is a central k-subgroup. Give an example of a smooth connected k-group containing $\mathbf{G}_a$ as a *non-central* normal k-subgroup. (Hint: look inside SL_2.)

4. Let T be a k-torus in a k-group G of finite type. This exercise uses $\mathrm{Aut}_{T/k}$ from Exercise 3.

(i) Construct a k-morphism $N_G(T) \to \mathrm{Aut}_{T/k}$ with kernel $Z_G(T)$. Prove $W(G,T) := N_G(T)(\overline{k})/Z_G(T)(\overline{k})$ is naturally a *finite* subgroup of $\mathrm{Aut}_{\mathbf{Z}}(\mathrm{X}(T))$. If $f : G' \to G$ is surjective with finite kernel and T' is a k-torus in G' containing $\ker f$ with $f(T') = T$ then prove $W(G', T') \to W(G,T)$ is an isomorphism.

(ii) For $G = \mathrm{GL}_n, \mathrm{PGL}_n, \mathrm{SL}_n, \mathrm{Sp}_{2n}$ and T the k-split diagonal maximal k-torus (so $Z_G(T) = T$), respectively identify $\mathrm{X}(T)$ with $\mathbf{Z}^n$, $\mathbf{Z}^n/\mathrm{diag}$, $\{m \in \mathbf{Z}^n \mid \sum m_j = 0\}$, and $\mathbf{Z}^n$. Prove $N_G(T)(k)/Z_G(T)(k) \subset \mathrm{Aut}_{\mathbf{Q}}(\mathrm{X}(T)_{\mathbf{Q}})$ is S_n for the first three, and $S_n \ltimes \langle -1 \rangle^n$ for Sp_{2n}, all with natural action. (Hint: to control $N_G(T)$, via $G \hookrightarrow \mathrm{GL}(V)$ decompose V as a direct sum of T-stable lines with *distinct* eigencharacters.)

5. Let (V, q) be a non-degenerate quadratic space over a field k with $\dim V \geq 2$. This exercise proves $\mathrm{SO}(q)$ contains $\mathbf{G}_m$ (i.e., it is k-isotropic in the sense of Exercise 2(iii)) if and only if $q = 0$ has a solution in $V - \{0\}$.

(i) If $q = 0$ has a nonzero solution v in V, prove that v lies in a hyperbolic plane H with $H \oplus H^{\perp} = V$. (If $\mathrm{char}(k) = 2$ and $\dim V$ is odd, work over $\overline{k}$ to show $v \notin V^{\perp}$.) Use this to construct a $\mathbf{G}_m$ inside of $\mathrm{SO}(q)$.

(ii) If $\mathrm{SO}(q)$ contains $\mathbf{G}_m$ as a k-subgroup S, prove that $q = 0$ has a nonzero solution in V. (Hint: apply Exercise 5(iii) in HW5 to the 2-dimensional k-split k-torus T generated in $\mathrm{GL}(V)$ by S and the central $\mathbf{G}_m$. If $A \simeq k^r$ is the corresponding "k-split" commutative k-subalgebra of $\mathrm{End}(V)$, prove the resulting inclusion $\mathbf{G}_m = S \hookrightarrow T = \mathrm{R}_{A/k}(\mathbf{G}_m) = \mathbf{G}_m^r$ is $t \mapsto (t^{h_1}, \dots, t^{h_r})$. Use the A-module structure on V to find a k-basis $\{e_i\}$ that identifies S with $\mathrm{diag}(t^{n_1}, \dots, t^{n_d})$ for $n_1 \leq \cdots \leq n_d$ with $\sum n_i = 0$. Prove $n_1 < 0 < n_d$, and if $q = \sum_{i \leq j} a_{ij} x_i x_j$ in these coordinates then prove $n_i + n_j = 0$ when $a_{ij} \neq 0$. Deduce $q(v) = 0$ for any v in the span of the e_i for which $n_i < 0$, or for which $n_i > 0$.)