Algebraic Groups I. Homework 8

1. Let A be a central simple algebra over a field k, T a k-torus in $\underline{A}^{\times}$.

(i) Adapt Exercise 5 in HW5 to make an étale commutative k-subalgebra $A_T \subseteq A$ such that $(A_T)_{k_s}$ is generated by $T(k_s)$, and establish a bijection between the sets of maximal k-tori in $\underline{A}^{\times}$ and maximal étale commutative k-subalgebras of A. Deduce that $\mathrm{SL}(A)$ is k-anisotropic if and only if A is a division algebra.

(ii) For an étale commutative k-subalgebra $C \subseteq A$, prove $Z_A(C)$ is a semisimple k-algebra with center C.

(iv) If T is *maximal* as a k-split subtorus of $\underline{A}^{\times}$ prove T is the k-group of units in A_T and that the (central!) simple factors B_i of $B_T := Z_A(A_T)$ are *division algebras.*

(v) Fix $A \simeq \mathrm{End}_D(V)$ for a right module V over a central division algebra D, so V is a left A-module and $V = \prod V_i$ with *nonzero* left B_i-modules V_i. If T is maximal as a k-split torus in $\underline{A}^{\times}$, prove V_i has rank 1 over B_i and D, so $B_i \simeq D$. Using D-bases, deduce that *all maximal k-split tori in $\underline{A}^{\times}$ are $\underline{A}^{\times}(k)$-conjugate.*

2. For a torus T over a local field k (allow $\mathbf{R}$, $\mathbf{C}$), prove T is k-anisotropic if and only if $T(k)$ is compact.

3. Let Y be a smooth separated k-scheme locally of finite type, and T a k-torus with a left action on Y. This exercise proves that Y^T is *smooth.*

(i) Reduce to the case $k = \overline{k}$. Fix a finite local k-algebra R with residue field k, and an ideal J in R with $J\mathfrak{m}_R = 0$. Choose $\overline{y} \in Y^T(R/J)$, and for R-algebras A let $E(A)$ be the fiber of $Y(A) \twoheadrightarrow Y(A/JA)$ over $\overline{y}_{A/JA}$. Let $y_0 = \overline{y} \bmod \mathfrak{m}_R \in Y^T(k)$ and $A_0 = A/\mathfrak{m}_R A$. Prove $E(A) \neq \emptyset$ and make it a torsor over the A_0-module $F(A) := JA \otimes_k \mathrm{Tan}_{y_0}(Y) = JA \otimes_{A_0} (A_0 \otimes_k \mathrm{Tan}_{y_0}(Y))$ naturally in A (denoted $v + y$).

(ii) Define an A_0-linear $T(A_0)$-action on $F(A)$ (hence a T_R-action on F), and prove that $E(A)$ is $T(A)$-stable in $Y(A)$ with $t.(v + y) = t_0.v + t.y$ for $y \in E(A)$, $t \in T(A)$, $v \in F(A)$, and $t_0 = t \bmod \mathfrak{m}_R$.

(iii) Choose $\xi \in E(R)$ and define a map of functors $h : T_R \to F$ by $t.\xi = h(t) + \xi$ for points t of T_R; check it is a 1-cocycle, and is a 1-coboundary if and only if $E^{T_R}(R) \neq \emptyset$. For $V_0 = J \otimes_k \mathrm{Tan}_{y_0}(Y)$ use h to define a 1-cocycle $h_0 : T \to \underline{V}_0$, and prove $t.(v, c) := (t.v + ch_0(t), c)$ is a k-linear representation of T on $V_0 \oplus k$. Use a T-equivariant splitting (!) to prove h_0 (and then h) is a 1-coboundary; deduce Y^T is smooth!

4. Let G be a smooth k-group of finite type, and T a k-torus equipped with a left action on G (an interesting case being T a k-subgroup acting by conjugation, in which case $G^T = Z_G(T)$).

(i) Use Exercise 3 to show $Z_G(T)$ is smooth, and by computing its tangent space at the identity prove for *connected* G that $T \subset Z_G$ if and only if T acts trivially on $\mathfrak{g} = \mathrm{Lie}(G)$.

(ii) Assume T is a k-subgroup of G acting by conjugation. Using Exercise 4(iii) of HW7 and the semisimplicity of the restriction to T of $\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$, prove that $N_G(T)$ and $Z_G(T)$ have the same tangent space at the identity. Via (i), deduce that $Z_G(T)$ is an *open subscheme* of $N_G(T)$, so $N_G(T)$ is *smooth* and $N_G(T)/Z_G(T)$ is finite étale over k.

(iii) Assumptions as in (ii), the *Weyl group* $W = W(G, T)$ is $N_G(T)/Z_G(T)$. If T is k-split, use the equality $\mathrm{End}_k(T) = \mathrm{End}_{k_s}(T_{k_s})$ to prove that $W(k) = W(k_s)$ and deduce that W is a constant k-group. But show $N_G(T)(k)$ does *not* map onto $W(k)$ if k is infinite and K is a separable quadratic extension of k such that $-1 \notin \mathrm{N}_{K/k}(K^{\times})$ (e.g., k totally real and K a CM extension, or $k = \mathbf{Q}$ and $K = \mathbf{Q}(\sqrt{3})$) with $G = \mathrm{SL}(K) \simeq \mathrm{SL}_2$ and T the *non-split* maximal k-torus corresponding the norm-1 part of $K \subset \mathrm{End}_k(K)$.

(iv) Prove that $N_G(T)(k) \to W(k) = W(\overline{k})$ is surjective for the cases in HW6, Exercise 4(ii).

5. (i) For any field k, affine k-scheme X of finite type, and nonzero finite k-algebra k', define a natural map $j_{X,k'/k} : X \to \mathrm{Res}_{k'/k}(X_{k'})$ by $X(R) \to X(k' \otimes_k R) = X_{k'}(k' \otimes_k R)$ for k-algebras R. Prove $j_{X,k'/k}$ is a closed immersion and that its formation commutes with fiber products in X.

(ii) Let G be an affine k-group of finite type. Prove that $j_{G,k'/k}$ is a k-homomorphism.

(iii) A *vector group* over k is a k-group G admitting an isomorphism $G \simeq \mathbf{G}_a^n$, and a *linear structure* on G is the resulting $\mathbf{G}_m$-action. A *linear homomorphism* $G' \to G$ between vector groups equipped with linear structures is a k-homomorphism which respects the linear structures. For example, $(x, y) \mapsto (x, y + x^p)$ is a *non-linear* automorphism of $\mathbf{G}_a^2$ (with its usual linear structure) when $\mathrm{char}(k) = p > 0$.

For any k, prove $\mathbf{G}_a$ admits a unique linear structure and its linear endomorphism ring is k. Giving $\mathbf{G}_a^n$ and $\mathbf{G}_a^m$ their usual linear structures, prove the linear k-homomorphisms $\mathbf{G}_a^n \to \mathbf{G}_a^m$ correspond to $\mathrm{Mat}_{m\times n}(k)$. Are there non-linear homomorphisms if $\mathrm{char}(k) = 0$?