Algebraic Groups I. Homework 9

1. Read Appendix B in the book *Pseudo-reductive groups* to learn Tits' structure theory for smooth connected unipotent groups over arbitrary fields k with positive characteristic, and how k-tori act on such groups. Especially noteworthy are the results labelled B.1.13, B.2.7, B.3.4, and B.4.3.

2. Let U be a smooth connected commutative affine k-group, and assume U is p-torsion if $\mathrm{char}(k) = p > 0$.

(i) If $\mathrm{char}(k) > 0$ and U is k-split, use B.1.12 in *Pseudo-reductive groups* to prove U is a vector group.

(ii) Assume $\mathrm{char}(k) = 0$. Prove that any short exact sequence $0 \to \mathbf{G}_a \to G \to \mathbf{G}_a \to 0$ is split. (Hint: $\log(u)$ is an "algebraic" function on the unipotent points of Mat_n.) Deduce that $U \simeq \mathbf{G}_a^N$, and prove that any action on U by a k-split torus T respects this linear structure.

3. Let k'/k be a degree-p purely inseparable extension of a field k of characteristic $p > 0$.

(i) Prove that $U = \mathrm{R}_{k'/k}(\mathbf{G}_m)/\mathbf{G}_m$ is smooth and connected of dimension $p-1$, and is p-torsion. Deduce it is unipotent.

(ii) In the handout on quotient formalism, it is proved that any commutative extension of $\mathbf{G}_a$ by $\mathbf{G}_m$ over any field is uniquely split over that field. Prove that $\mathrm{R}_{k'/k}(\mathbf{G}_m)(k_s)[p] = 1$, and deduce that U in (i) does not contain $\mathbf{G}_a$ as a k-subgroup! (For a salvage, see Lemma B.1.10 in "Pseudo-reductive groups": a p-torsion smooth connected commutative affine group over any field of characteristic $p > 0$ admits an étale isogeny onto a vector group.)

4. Let G be a smooth group of finite type over a field k, and N a commutative normal k-subgroup scheme.

(i) Prove that the left G-action on N via conjugation factors uniquely through an action of G/N on N, and if N is central in G then prove that the action of G on itself via conjugation uniquely factors through an action of G/N on G. Describe this explicitly for $G = \mathrm{SL}_n$ and $N = \mu_n$ over any field k, accounting for the fact that $\mathrm{SL}_n(k) \to \mathrm{PGL}_n(k)$ is generally *not* surjective.

(ii) Prove the commutator map $G \times G \to G$ uniquely factors through a k-morphism $(G/Z_G) \times (G/Z_G) \to \mathscr{D}(G)$.

5. Let B be a smooth connected solvable group over a field k.

(i) If $B = \mathbf{G}_m \ltimes \mathbf{G}_a$ with the standard semi-direct product structure, prove that $Z_B(t, 0)$ is the left factor for any $t \in k^\times - \{1\}$.

(ii) Deduce by inductive arguments resting on (i) that if $k = \overline{k}$ and $S \subset B(k)$ is a commutative subgroup of semisimple elements then $S \subset T(k)$ for some maximal torus $T \subset B$.

(iii) Assume $\mathrm{char}(k) \neq 2$ with $k = \overline{k}$, and let $G = \mathrm{SO}_n$ with $n \geq 3$. Let $\mu \simeq \mu_2^{n-1}$ be the "diagonal" k-subgroup $\{(\zeta_i) \in \mu_2^n \mid \prod \zeta_i = 1\}$. Prove that the disconnected μ is maximal as a solvable smooth k-subgroup of G and is not contained in any maximal k-torus of G (hint: it has too much 2-torsion), so in particular is not contained in any Borel k-subgroup (by (ii))!

6. Let G be a quasi-split smooth connected affine k-group, and $B \subset G$ a Borel k-subgroup. Let T be a maximal k-torus in B.

(i) Using conjugacy of maximal tori in $G_{\overline{k}}$, prove $g \mapsto gBg^{-1}$ is a bijection from $N_G(T)(\overline{k})/Z_G(T)(\overline{k})$ onto the set of Borel $\overline{k}$-subgroups containing $T_{\overline{k}}$. In particular, this set is *finite*.

(ii) Using HW8 Exercise 4, prove that $N_G(T)(k_s)/Z_G(T)(k_s) \to N_G(\overline{k})/Z_G(T)(\overline{k})$ is bijective, and deduce that every Borel subgroup of $G_{\overline{k}}$ containing $T_{\overline{k}}$ is defined over k_s!

(iii) Assume that T is k-split and $Z_G(T) = T$. Using Hilbert 90 and HW8 Exercise 4, prove that $N_G(T)(k)/T(k) \to N_G(T)(k_s)/Z_G(T)(k_s)$ is bijective. Deduce that every Borel subgroup of $G_{\overline{k}}$ containing $T_{\overline{k}}$ is defined over k! In each of the classical cases (GL_n, SL_n, PGL_n, Sp_{2n}, and SO_n), find all B containing the k-split maximal"diagonal" T. How many parabolic k-subgroups can you find containing one such B? (At least for GL_n, SL_n, and PGL_n, prove you have found all such parabolics.)

(iv) Prove that each maximal smooth unipotent subgroup of $G_{\overline{k}}$ admits a conjugate contained in $B_{\overline{k}}$, and deduce that if $B \cap B' = T$ for another Borel B' containing T then G is reductive. Use this with (iii) to prove reductivity for GL_n $(n \geq 1)$, SL_n $(n \geq 2)$, PGL_n $(n \geq 2)$, Sp_{2n} $(n \geq 1)$, and SO_n $(n \geq 2)$.