Algebraic groups I

Instructor: Prof. Brian Conrad, `conrad@math.stanford.edu`
Office: 383CC, Sloan Hall
Office hours: MWF, 10–11am, and by appointment.
Course assistant: None.

Prerequisites: Familiarity with the theory of schemes at the level of 216A, and enrollment in 210B (or knowledge of its content)

Textbooks: *Algebraic groups* by Armand Borel

Homework/exams: There will be no exams, but there will be extensive weekly homework, posted each Friday and due on the following Friday.

Course description: The theory of linear algebraic groups is ultimately concerned with the general properties of "nice" smooth group varieties defined by matrix-theoretic conditions. However, one does not get very far by explicit manipulations with matrices. Incredibly, there is a completely uniform viewpoint through which we can study all reasonable connected linear algebraic groups over any field via a common language, without tedious case-by-case arguments. It is a very interesting mixture of group theory and algebraic geometry. Over finite fields it explains virtually all finite simple groups, over number fields it illuminates many important problems concerning quadratic forms and modular forms and beyond, over the real numbers it clarifies the theory of Lie groups, and so on.

We will begin with an overview of the nature of the structure theory for various classical groups, with an emphasis on some examples and the remarkable properties of tori. We will then turn to a systematic development of basic generalities in the theory of smooth affine groups over an arbitrary field: group-theoretic constructions (matrix realizations, closed orbit lemma, commutator subgroups, normalizers, centralizers, Lie algebras, quotients), Jordan decomposition, structure of solvable groups (unipotent groups, diagonalizable groups, Lie-Kolchin theorem), and the solvable and unipotent radicals.

The theory really takes off with the Borel fixed-point theorem concerning the action of a smooth connected (split) solvable group on a smooth quasi-projective variety. That will lead into the important conjugacy theorems (for maximal tori and Borel subgroups), which then takes us on the long road toward the structure theory of connected reductive groups (in terms of root systems), one of the real gems of pure mathematics. We will work through lots of examples, and begin to understand the special role of SL_2 in the general theory of split reductive groups (to be developed in a sequel course).