The Artin–Tate Conjecture and Finiteness of Brauer Groups

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December 12, 2015

1 The Artin–Tate Conjecture

We recall here the statement of the Artin–Tate Conjecture. Let k be a finite field of characteristic p, $q = \#k$, $G = \mathrm{Gal}(\overline{k}/k)$ the absolute Galois group of k, X a smooth proper geometrically-connected surface over k, and $\overline{X} := X \otimes_k \overline{k}$. We first review the definitions of the various quantities appearing in the conjecture, recalling along the way to which quantities in the Birch-Swinnerton-Dyer they are analogous.

Thanks to Grothendieck's cohomological interpretation, the zeta function of X is given, for $\ell \neq p$, by the formula

$$\zeta(X, s) = \frac{P_1(X, q^{-s})P_1(X, q^{1-s})}{(1 - q^{-s})P_2(X, q^{-s})(1 - q^{2-s})}$$

where $P_i(X, T) = \det(1 - FT|\mathrm{H}^i(\overline{X}, \mathbf{Q}_\ell))$ is the characteristic polynomial of geometric Frobenius in G acting on the étale cohomology of $\overline{X}$ (Here we have replaced replaced the variable in the usual definition of the zeta function by q^{-s}.). By results of Deligne ([D], Théorème 1.6), the P_i have integer coefficients independent of ℓ. The polynomial P_2 should be thought of as an analogue to the L-function of the abelian variety in BSD.

The (cohomological) *Brauer group* $\mathrm{Br}(X)$ is $\mathrm{H}^2(X, \mathbf{G}_m)$. (All cohomology in these notes is étale unless stated otherwise.). Thanks to a theorem of Grothendieck ([G], Corollaire 2.2), this is the same as the Azumaya Brauer group, though we will not use that. We will, however, use the fact that $\mathrm{Br}(X)$ is torsion (as is seen by the fact that the restriction map $\mathrm{Br}(X) \to \mathrm{Br}(k(X))$ is injective, by Grothendieck's results, and the Brauer group of the function field is torsion since it is higher Galois cohomology). The group $\mathrm{Br}(X)$ is an analogue of the Tate–Shafarevich group in BSD.

The *Néron–Severi group* $\mathrm{NS}(X)$ admits a couple of natural definitions for a variety over a general field (not assumed algebraically closed). Fortunately over a finite field these notions will be equivalent. To explain this, recall that we have an exact sequence of k-groups

$$0 \longrightarrow \mathrm{Pic}^0_{X/k} \longrightarrow \mathrm{Pic}_{X/k} \longrightarrow \mathrm{NS}_{X/k} \longrightarrow 0 \tag{1.1}$$

where $\mathrm{Pic}^0_{X/k}$ is the connected component of the identity inside $\mathrm{Pic}_{X/k}$ and the quotient $\mathrm{NS}_{X/k}$ is étale with group of $\overline{k}$-points that is finitely generated. Note that $\mathrm{Pic}^0_{X/k}$ is geometrically connected, since it is connected and possesses a rational point (namely the identity). The two usual definitions of $\mathrm{NS}(X)$ are as follows:

(i) $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$,

(ii) $\mathrm{NS}_{X/k}(k)$.

Here $\mathrm{Pic}^0(X)$ is the group of line bundles on X that are algebraically equivalent to the trivial bundle. (For a line bundle $\mathcal{L}$ to be *algebraically equivalent to* 0 means that there is a connected scheme T and a line bundle $\mathcal{M}$ on X_T such that the restriction of $\mathcal{M}$ to one of the fibers of the map $X_T \to T$ is $\mathcal{L}$ and to another is 0. This is the same as being geometrically algebraically equivalent to 0, because $\mathrm{Pic}^0_{X/k}$ is geometrically connected.).

In general these two definitions need not agree. To understand the obstruction, note that we always have an injective map

$$\frac{\mathrm{Pic}(X)}{\mathrm{Pic}^0(X)} \hookrightarrow \mathrm{NS}_{X/k}(k) \tag{1.2}$$

via the composition

$$\frac{\mathrm{Pic}(X)}{\mathrm{Pic}^0(X)} \hookrightarrow \frac{\mathrm{Pic}_{X/k}(k)}{\mathrm{Pic}^0_{X/k}(k)} \hookrightarrow \mathrm{NS}_{X/k}(k) \tag{1.3}$$

When are both maps in this composition surjective?

Via the Leray spectral sequence for $\mathbf{G}_m$ relative to the structural morphism $X \to \mathrm{Spec}(k)$, we have an exact sequence (even for X over an arbitrary scheme S if we replace $\mathrm{Br}(k)$ by $\mathrm{Br}(S)$ and replace the 0 on the left below with $\mathrm{Pic}(S)$)

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}_{X/k}(k) \longrightarrow \mathrm{Br}(k).$$

Hence, the first map in (1.3) is surjective if $\mathrm{Br}(k) = 0$. Note that this holds if k is a finite field.

Now assume that k is perfect (which is the case if k is finite). Then the sequence (1.1) is exact as a sequence of étale sheaves (since $k^{\mathrm{sep}} = \overline{k}$). Taking Galois cohomology of (1.1) therefore yields an exact sequence

$$0 \longrightarrow \mathrm{Pic}^0_{X/k}(k) \longrightarrow \mathrm{Pic}_{X/k}(k) \longrightarrow \mathrm{NS}_{X/k}(k) \longrightarrow \mathrm{H}^1(k, \mathrm{Pic}^0_{X/k})$$

For any k-scheme Y of finite type we have $Y(L) = Y_{\mathrm{red}}(L)$ for field extensions L/k, so we may replace $\mathrm{H}^1(k, \mathrm{Pic}^0_{X/k})$ with $\mathrm{H}^1(k, (\mathrm{Pic}^0_{X/k})_{\mathrm{red}})$. Hence, the second map in (1.3) is surjective if $\mathrm{H}^1(k, (\mathrm{Pic}^0_{X/k})_{\mathrm{red}}) = 0$. This holds if k is finite by Lang's Theorem.

It follows that the canonical map (1.2) is an isomorphism if k is a finite field, and that the two definitions (i) and (ii) above therefore agree in this case. As a useful corollary of the definition (ii), we obtain the fact that

$$\mathrm{NS}(\overline{X})^G = \mathrm{NS}(X). \tag{1.4}$$

It is a classical theorem (the "Theorem of the Base") due to Lang and Néron for smooth projective varieties over an algebraically closed field that $\mathrm{NS}(\overline{X})$ is finitely-generated, so likewise for $\mathrm{NS}(X)$. The group $\mathrm{NS}(X)$ should be thought of as an analogue of the Mordell–Weil group in BSD. Let $\rho(X)$ denote the rank of $\mathrm{NS}(X)$, and let $\{D_i\}$ denote a basis for $\mathrm{NS}(X)/\mathrm{NS}(X)_{\mathrm{tor}}$. Finally, for $D, D' \in \mathrm{NS}(X)/\mathrm{NS}(X)_{\mathrm{tor}}$, let $D \cdot D'$ denote their intersection product.

The last quantity that we have to define is the strangest, as it is not clear to what (if anything) it is analogous in BSD (It appears to be a purely characteristic p phenomenon, as far as I can tell.). This is

$$\alpha(X) := \chi(X, \mathcal{O}_X) - 1 + \dim(\mathrm{Pic}_{X/k}).$$

It turns out that $\alpha(X) \geq 0$ (This is equivalent to the inequality $q - s \leq p_G$ on page 73 in chapter 5 of [B].).

For a finite set A, let $[A]$ denote the size of A. The Artin–Tate conjecture is the following statement.

Conjecture 1.1 (Artin–Tate Conjecture)**.** *The group* $\mathrm{Br}(X)$ *is finite, and*

$$P_2(X, q^{-s}) \sim \frac{[\mathrm{Br}(X)]|\det(D_i \cdot D_j)|}{q^{\alpha(X)}[\mathrm{NS}(X)_{\mathrm{tor}}]^2}(1 - q^{1-s})^{\rho(x)} \quad \text{as } s \to 1.$$

Note that the coefficient ratio in the conjecture must be an integer (if the conjecture is true) since $P(X, 0) \in \mathbf{Z}$. Much more can be said about this conjecture than for BSD. Indeed, our goal in this talk is to discuss the proofs of the following two theorems.

Theorem 1.2 ((Tate) Rank Inequality)**.** *Let r denote the multiplicity of q^{-1} as a zero of $P_2(X, T)$. Then $\rho(X) \leq r$.*

We remark that one would philosophically expect this direction to be the easier one to prove, since you are giving obstructions to the existence of divisors rather than actually having to produce them (the reverse inequality), which one would expect to be more difficult.

For an abelian group A and a prime ℓ, let $A(\ell)$ denote the ℓ-primary part of A; i.e., $A(\ell) := \cup_{n \geq 1} A[\ell^n]$.

Theorem 1.3 (Tate–Milne)**.** *If* $\mathrm{Br}(X)(\ell)$ *is finite for some prime ℓ (even $\ell = p$) then the Artin–Tate conjecture (Conjecture 1.1) holds for X.*

The case $\ell \neq p$, together with the "prime-to-p" part of Theorem 1.3, was proved by Tate using ℓ-adic cohomology, as we will discuss later. What was missing when Tate proved this theorem was a good p-adic cohomology theory. The theorem "at p" was proved by Milne using Tate's methods together with suitable p-adic cohomology theories.

Of course, nothing remotely as strong as either of these two results is known for BSD over number fields (but when applied to fibered surfaces they give results concerning BSD for Jacobians over global function fields).

2 Proof of the Rank Inequality

Although it would be most efficient to embed the proof of Theorem 1.2 in the proof of Theorem 1.3, we isolate it in this section to illustrate how shockingly simple it is (since in the BSD case over number fields, or for *general* abelian varieties over global function fields, the analogous result is completely out of reach!).

Let $\ell \neq p$ be prime. We begin with the Kummer sequence, an exact sequence of sheaves on $\overline{X}_{\mathrm{et}}$:

$$0 \longrightarrow \mu_{\ell^n} \longrightarrow \mathbf{G}_m \xrightarrow{\ell^n} \mathbf{G}_m \longrightarrow 0.$$

From this we obtain an injection

$$0 \longrightarrow \mathrm{Pic}(\overline{X})/\ell^n\mathrm{Pic}(\overline{X}) \longrightarrow \mathrm{H}^2(\overline{X}, \mu_{\ell^n}) \tag{2.1}$$

Now we have an exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(\overline{X}) \longrightarrow \mathrm{Pic}(\overline{X}) \longrightarrow \mathrm{NS}(\overline{X}) \longrightarrow 0,$$

from which we obtain an exact sequence

$$\mathrm{Pic}^0(\overline{X})/\ell^n\mathrm{Pic}^0(\overline{X}) \longrightarrow \mathrm{Pic}(\overline{X})/\ell^n\mathrm{Pic}(\overline{X}) \longrightarrow \mathrm{NS}(\overline{X})/\ell^n\mathrm{NS}(\overline{X}) \longrightarrow 0.$$

The group $\mathrm{Pic}^0(\overline{X})$ is ℓ-divisible since it equals the group of $\overline{k}$-points of a smooth connected commutative algebraic $\overline{k}$-group. (Any such group has a filtration whose successive quotients are abelian varieties, $\mathbf{G}_m$'s, and $\mathbf{G}_a$'s. Actually, since $\overline{X}$ is smooth, it follows from the valuative criterion that $\mathrm{Pic}^0_{\overline{X}/\overline{k}}$ is proper, so its underlying reduced scheme is an abelian variety.) Thus, we obtain an isomorphism

$$\mathrm{Pic}(\overline{X})/\ell^n\mathrm{Pic}(\overline{X}) \simeq \mathrm{NS}(\overline{X})/\ell^n\mathrm{NS}(\overline{X}),$$

whence the injection (2.1) becomes

$$0 \longrightarrow \mathrm{NS}(\overline{X})/\ell^n\mathrm{NS}(\overline{X}) \longrightarrow \mathrm{H}^2(\overline{X}, \mu_{\ell^n})$$

Now take the inverse limit over all $n \geq 1$, and pass to G-invariants to obtain an injection

$$0 \longrightarrow (\mathrm{NS}(\overline{X}) \otimes \mathbf{Z}_\ell)^G \longrightarrow \mathrm{H}^2(\overline{X}, \mathbf{Z}_\ell(1))^G;$$

the fact that we have $\mathrm{NS}(\overline{X}) \otimes \mathbf{Z}_\ell$ on the left before taking invariants follows from the Lang–Néron finiteness theorem that $\mathrm{NS}(\overline{X})$ is finitely-generated. (The map $\mathbf{Z}_\ell \otimes A \to \varprojlim_n A/\ell^n A$ is an isomorphism for any finitely generated $\mathbf{Z}$-module A due to the compatibility of completion and tensor product for finitely generated modules over noetherian rings.) The group $\mathrm{H}^2(\overline{X}, \mathbf{Z}_\ell(1))$ is defined to be $\varprojlim_n \mathrm{H}^2(\overline{X}, \mu_{\ell^n})$, just as $\mathrm{H}^i(\overline{X}, \mathbf{Z}_\ell)$ is defined to be $\varprojlim_n \mathrm{H}^i(\overline{X}, \mathbf{Z}/\ell^n\mathbf{Z})$.

The natural map

$$\mathbf{Z}_\ell \otimes \mathrm{NS}(\overline{X})^G \to (\mathrm{NS}(\overline{X}) \otimes \mathbf{Z}_\ell)^G \tag{2.2}$$

is an isomorphism, as $A^G = \ker(\sigma - 1)$ for any Hausdorff topological G-module A, where σ is a (topological) generator for G (so (2.2) boils down to the fact that for any map $f : A \to B$ of abelian groups, $\ker(f \otimes \mathbf{Z}_\ell) = \ker(f) \otimes \mathbf{Z}_\ell$ since $\mathbf{Z}_\ell$ is a flat $\mathbf{Z}$-module. We have $(\mathrm{NS}(\overline{X}))^G = \mathrm{NS}(X)$ since G-invariant $\overline{k}$-points are k-points, so we obtain an injection

$$0 \longrightarrow \mathrm{NS}(X) \otimes \mathbf{Z}_\ell \longrightarrow \mathrm{H}^2(\overline{X}, \mathbf{Z}_\ell(1))^G$$

Now tensor up to $\mathbf{Q}_\ell$:

$$0 \longrightarrow \mathrm{NS}(X) \otimes \mathbf{Q}_\ell \longrightarrow \mathrm{H}^2(\overline{X}, \mathbf{Q}_\ell(1))^G.$$

(That the superscript G can be pulled out of the tensor product follows similarly to the argument above, since $\mathbf{Q}_\ell$ is $\mathbf{Z}_\ell$-flat.) Thus,

$$\rho(X) \le \dim \mathrm{H}^2(\overline{X}, V_\ell \mu)^G \le r, \tag{2.3}$$

where r is the multiplicity of 1 as a root of the characteristic polynomial of *geometric* Frobenius F on $\mathrm{H}^2(\overline{X}, \mathbf{Q}_\ell(1))$.

But G-equivariantly $\mathrm{H}^2(\overline{X}, \mathbf{Q}_\ell(1)) = \mathrm{H}^2(\overline{X}, \mathbf{Q}_\ell) \otimes \mathbf{Q}_\ell(1)$, where G acts on $\mathbf{Q}_\ell(1)$ through the cyclotomic character, on which F acts as multiplication by q^{-1} since $F^{-1}(\zeta) = \zeta^q$ for any root of unity $\zeta \in \overline{k}^\times$. Thus r coincides with the multiplicity of q as a root of the characteristic polynomial of F acting on $\mathrm{H}^2(\overline{X}, \mathbf{Q}_\ell)$, which is to say the multiplicity of q^{-1} as a root of $\det(1 - FT | \mathrm{H}^2(\overline{X}, \mathbf{Q}_\ell)) = P_2(X, T)$. Hence, (2.3) yields Theorem 1.2. $\square$

Unimportant Remark/Example: For any ring A, we have a canonical map

$$\mathbf{Z}_\ell \otimes A \xrightarrow{\phi} \varprojlim_n A/\ell^n A$$

We remarked earlier in this section that this map need not be either injective or surjective in general. Here is an example. Let $\overline{\mathbf{Q}_\ell}$ be the algebraic closure of $\mathbf{Q}_\ell$, and let $\overline{\mathbf{Z}_\ell}$ be its ring of integers. Then I claim that for $A = \overline{\mathbf{Z}_\ell}$, the map ϕ above is neither injective nor surjective. Indeed, we have a commutative diagram

$$\begin{tikzcd} & \overline{\mathbf{Z}_\ell} \arrow[dr, "\beta"] & \\ \mathbf{Z}_\ell \otimes \overline{\mathbf{Z}_\ell} \arrow[ur, "\alpha"] \arrow[rr, "\phi"] & & \varprojlim_n \overline{\mathbf{Z}_\ell}/\ell^n \overline{\mathbf{Z}_\ell} \end{tikzcd}$$

where α, β are the obvious maps. It therefore suffices to show that α is not injective and β is not surjective.

The map β is not surjective simply because $\overline{\mathbf{Z}_\ell}$ is not ℓ-adically complete. (Exercise, or see Proposition 5.1 in [W].) To see that α is not injective, let $x \in \mathbf{Z}_\ell - \mathbf{Z}$ be integral over $\mathbf{Z}$. (One can obtain such x by taking a monic $f(T) \in \mathbf{Z}_\ell[T]$ with no integer roots such that the reduction $\overline{f}(T) \in \mathbf{F}_\ell[T]$ has a root x_0 of multiplicity one. Then x_0 lifts to a root of f in $\mathbf{Z}_\ell$, by Hensel's Lemma. There are many f as above: For example, if $\ell \neq 2$, take $f(T) = X^2 - n$ with $\ell \nmid n \in \mathbf{Z}$ a nonsquare that is a quadratic residue mod ℓ, and if $\ell = 2$ then take $f(T) = T^3 - 3$.) Now we have an inclusion $\mathbf{Z}_\ell \otimes \mathbf{Z}[x] \subset \mathbf{Z}_\ell \otimes \overline{\mathbf{Z}_\ell}$ because $\mathbf{Z}_\ell$ is flat over $\mathbf{Z}$. It therefore suffices to show that the map

$$\mathbf{Z}_\ell \otimes \mathbf{Z}[x] \xrightarrow{\alpha'} \mathbf{Z}_\ell$$

is not injective. Let $f(T)$ be the minimal polynomial of x over $\mathbf{Z}$. Then the ring on the left is just $\mathbf{Z}_\ell[T]/(f(T))$, hence is a free $\mathbf{Z}_\ell$-module of rank greater than one. Thus the map α' cannot be injective.

There are many other examples of such rings A, some with slightly simpler arguments than the above, but I thought that the above example is good because the ring $\overline{\mathbf{Z}_\ell}$ is a natural object.

3 Some Elementary Lemmas

The proof of Theorem 1.3 will require a few lemmas of an elementary nature, and we gather these here. Let $f : A \to B$ be a homomorphism of $\mathbf{Z}_\ell$-modules. The map f is said to be a *quasi-isomorphism* if $\ker(f)$ and $\mathrm{coker}(f)$ are finite. In this case we define

$$z(f) := [\ker(f)]/[\mathrm{coker}(f)] = |[\mathrm{coker}(f)]|_\ell/|[\ker(f)]|_\ell$$

The equality holds because a finite $\mathbf{Z}_\ell$-module is necessarily an ℓ-group. Note that if A, B are finite then the exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow A \xrightarrow{f} B \longrightarrow \mathrm{coker}(f) \longrightarrow 0$$

shows that $z(f) = [A]/[B]$.

Here are the lemmas that we will need. (Our numbering of these lemmas corresponds to Tate's for ease of cross-reference.)

Lemma z.1. *Let A, B be finitely-generated $\mathbf{Z}_\ell$-modules of the same rank, and let $\{a_i\}$, $\{b_i\}$ be bases for A/A_{tor}, B/B_{tor}, respectively. Suppose we have $f : A \to B$, and let $\overline{f} : A/A_{\mathrm{tor}} \to B/B_{\mathrm{tor}}$ be the map induced by f. Then f is a quasi-isomorphism if and only if $\det(\overline{f}) \neq 0$, in which case*

$$z(f) = \frac{[A_{\mathrm{tor}}]|\det(\overline{f})|_\ell}{[B_{\mathrm{tor}}]}$$

Lemma z.2. *Suppose we have maps $f : A \to B$, $g : B \to C$. If any two of the maps f, g, gf is a quasi-isomorphism then so is the third, and then we have $z(gf) = z(g)z(f)$.*

Lemma z.3. *Let $A^* = \mathrm{Hom}_{\mathbf{Z}_\ell}(A, \mathbf{Q}_\ell/\mathbf{Z}_\ell)$. Then $f : A \to B$ is a quasi-isomorphism if and only if $f^* : B^* \to A^*$ is, in which case $z(f)z(f^*) = 1$.*

Lemma z.4. *Suppose θ is an endomorphism of a finitely-generated $\mathbf{Z}_\ell$-module A, and let $\theta \otimes 1$ denote the corresponding endomorphism of $A \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$. Let $f : \ker(\theta) \to \mathrm{coker}(\theta)$ be the map induced by the identity $A \to A$. Then f is a quasi-isomorphism if and only if $\det(T - \theta \otimes 1) = T^\rho R(T)$, with $\rho = \mathrm{rank}_{\mathbf{Z}_\ell} \ker(\theta)$ and $R(0) \neq 0$, in which case we have $z(f) = |R(0)|_\ell$.*

Proof of Lemma z.1. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_{\mathrm{tor}} & \longrightarrow & A & \longrightarrow & A/A_{\mathrm{tor}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_{\mathrm{tor}}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} & & \\
0 & \longrightarrow & B_{\mathrm{tor}} & \longrightarrow & B & \longrightarrow & B/B_{\mathrm{tor}} & \longrightarrow & 0
\end{array}$$

The snake lemma therefore yields an exact sequence

$$0 \to \ker(f_{\mathrm{tor}}) \to \ker(f) \to \ker(\overline{f}) \to \mathrm{coker}(f_{\mathrm{tor}}) \to \mathrm{coker}(f) \to \mathrm{coker}(\overline{f}) \to 0$$

It follows immediately from this sequence that f is a quasi-isomorphism if and only if $\overline{f}$ is (since $A_{\mathrm{tor}}, B_{\mathrm{tor}}$ are finite), in which case

$$z(f) = z(\overline{f})z(f_{\mathrm{tor}}) = z(\overline{f})[A_{\mathrm{tor}}]/[B_{\mathrm{tor}}]$$

the last equation coming from the fact that $A_{\mathrm{tor}}, B_{\mathrm{tor}}$ are finite. Thus, renaming A_{tor} as A and B_{tor} as B, we may assume that A, B are free $\mathbf{Z}_\ell$-modules of finite rank. In this case, it is a standard fact (valid over any PID, not just $\mathbf{Z}_\ell$) that by choosing suitable bases for A, B, the matrix of f is diagonal, and in this case the assertion of the lemma is a straightforward computation. $\square$

Proof of Lemma z.2. The lemma follows immediately from the following exact sequence (We leave the check that this is exact to the reader.):

$$0 \to \ker(f) \to \ker(gf) \xrightarrow{f} \ker(g) \to \operatorname{coker}(f) \xrightarrow{g} \operatorname{coker}(gf) \to \operatorname{coker}(g) \to 0$$

Here all of the unmarked maps are those induced by the identity. □

Proof of Lemma z.3. $\mathbf{Q}_\ell/\mathbf{Z}_\ell$ is an injective $\mathbf{Z}_\ell$-module. Indeed, a $\mathbf{Z}_\ell$-module is injective if and only if it's ℓ-divisible (same proof as for modules over $\mathbf{Z}$). Thus $A \mapsto A^*$ is an exact functor. Therefore, applying $*$ to the exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow A \xrightarrow{f} B \longrightarrow \operatorname{coker}(f) \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \operatorname{coker}(f)^* \longrightarrow B^* \xrightarrow{f^*} A^* \longrightarrow \ker(f)^* \longrightarrow 0$$

The lemma therefore follows immediately from the following claim: For any $\mathbf{Z}_\ell$-module X, X is finite if and only if X^* is finite, in which case $[X] = [X^*]$.

The equality claim is well-known and standard, and the same goes for the implication X finite $\Longrightarrow X^*$ finite. The reverse implication can be proved as follows. We have as usual a natural map $X \to X^{**}$. I claim that this map is injective. This will prove the claim, since then X^* finite $\Longrightarrow X^{**}$ finite $\Longrightarrow X$ finite. The desired injectivity is equivalent to the claim that for any $0 \neq x \in X$, there exists $\phi \in X^*$ such that $\phi(x) \neq 0$. We first note that this is true for a cyclic $\mathbf{Z}_\ell$-module and x a generator, since any such is isomorphic to either $\mathbf{Z}_\ell$ or $\mathbf{Z}/\ell^n\mathbf{Z}$ (with $x = 1$ in both cases), for which the assertion can be checked directly. Thus, letting $\langle x \rangle$ denote the submodule generated by $x \in X$, we have $\phi' \in \langle x \rangle^*$ with $\phi(x) \neq 0$. Since $\mathbf{Q}_\ell/\mathbf{Z}_\ell$ is an injective $\mathbf{Z}_\ell$-module, ϕ' extends to an element $\phi \in X^*$, and of course $\phi(x) = \phi'(x) \neq 0$. This proves the desired injectivity. □

Proof of Lemma z.4. Let $\theta_1 : \operatorname{Im}(\theta) \to \operatorname{Im}(\theta)$ denote the restriction of θ. We have $\ker(f) = \ker(\theta) \cap \operatorname{Im}(\theta) = \ker(\theta_1)$, and $\operatorname{coker}(f) = A/(\ker(\theta) + \operatorname{Im}(\theta)) \simeq \operatorname{coker}(\theta_1)$ the last isomorphism being induced by θ. Therefore f is a quasi-isomorphism if and only if θ_1 is, in which case we have $z(f) = z(\theta_1)$.

For any endomorphism $g : B \to B$ of a finitely generated $\mathbf{Z}_\ell$-module B, Lemma z.1 implies that g is a quasi-isomorphism if and only if the endomorphism $g \otimes 1$ of $B \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$ is, in which case $z(g) = z(\overline{g}) = \det(g \otimes 1)$, in the notation of that lemma (We make use here of the fact that $B \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell = (B/B_{\mathrm{tor}}) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$.). It therefore suffices to show that $\theta_1 \otimes 1$ is an isomorphism if and only if $R(T)$ as in the lemma doesn't vanish at $T = 0$, in which case $|\det(\theta_1 \otimes 1)|_\ell = |R(0)|_\ell$.

We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(\theta)\otimes_{\mathbf{Z}_\ell}\mathbf{Q}_\ell & \longrightarrow & A\otimes_{\mathbf{Z}_\ell}\mathbf{Q}_\ell & \longrightarrow & \mathrm{Im}(\theta)\otimes_{\mathbf{Z}_\ell}\mathbf{Q}_\ell & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle 0} & & \Big\downarrow{\scriptstyle \theta\otimes 1} & & \Big\downarrow{\scriptstyle \theta_1\otimes 1} & & \\
0 & \longrightarrow & \ker(\theta)\otimes_{\mathbf{Z}_\ell}\mathbf{Q}_\ell & \longrightarrow & A\otimes_{\mathbf{Z}_\ell}\mathbf{Q}_\ell & \longrightarrow & \mathrm{Im}(\theta)\otimes_{\mathbf{Z}_\ell}\mathbf{Q}_\ell & \longrightarrow & 0
\end{array}$$

Since characteristic polynomials are multiplicative in exact sequences, we therefore obtain

$$\det(T-\theta\otimes 1) = T^{\rho}\det(T-\theta_1\otimes 1)$$

Thus $R(T)=\det(T-\theta_1\otimes 1)$, hence θ_1 is an isomorphism if and only if $R(0)\neq 0$, in which case $|\det(\theta\otimes 1)|_\ell = |R(0)|_\ell$, as desired. □

4 The Prime-to-p Part of Theorem 1.3

In this section we prove Theorem 1.3 away from p. More precisely, we will show that if for some prime $\ell\neq p$ the ℓ-primary part $\mathrm{Br}(X)(\ell)$ is finite then $\mathrm{Br}(X)(\text{non-}p)$ is finite (where $\mathrm{Br}(X)(\text{non-}p) := \bigoplus_{\ell\neq p}\mathrm{Br}(X)(\ell)$) and there exists $n\in\mathbf{Z}$ such that

$$P_2(X,q^{-s}) \sim p^n\frac{[\mathrm{Br}(X)]|\det(D_i\cdot D_j)|}{q^{\alpha(X)}[\mathrm{NS}(X)_{\mathrm{tor}}]^2}(1-q^{1-s})^{\rho(x)}, \qquad \text{as } s\to 1.$$

(The factor p^n renders the $q^{\alpha(X)}$ superfluous, but we include it in order to emphasize the connection with the Artin–Tate conjecture.)

The proof rests upon the following commutative exact diagram of finite groups, valid for any positive integer m coprime to p. (By an *exact* diagram, we mean one in which every row and column is exact.)

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 & & 0 & & \mathrm{Pic}(X)/m\mathrm{Pic}(X) & \longrightarrow & (\mathrm{NS}(\overline{X})/m\mathrm{NS}(\overline{X}))^G \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{H}^1(\overline{X},\mu_m)_G & \longrightarrow & \mathrm{H}^2(X,\mu_m) & \longrightarrow & \mathrm{H}^2(\overline{X},\mu_m)^G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & (\mathrm{Pic}(\overline{X})[m])_G & & \mathrm{Br}(X)[m] & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}\qquad (4.1)$$

Here, for a G-module A, A^G denotes the G-invariants and A_G denotes the coinvariants. (The finiteness of the groups on the left and right of the middle row follows from standard finiteness theorems in étale cohomology, and the finiteness of all other groups then follows.)

Let's explain where this diagram comes from. The columns come from the Kummer sequence on $\overline{X}_{\mathrm{et}}$ and X_{et}:

$$0 \longrightarrow \mu_m \longrightarrow \mathbf{G}_m \xrightarrow{m} \mathbf{G}_m \longrightarrow 0$$

The first and third columns are then obtained by taking G-coinvariants and invariants, respectively. The map in the first row is the obvious one. For the middle row, we use the Hochschild–Serre spectral sequence

$$\mathrm{H}^p(G, \mathrm{H}^q(\overline{X}, \mu_m)) \Longrightarrow \mathrm{H}^{p+q}(X, \mu_m)$$

together with the calculation of the Galois cohomology of finite fields (or more generally, of cyclic profinite groups) with coefficients in a torsion module:

$$\mathrm{H}^i(G, A) = \begin{cases} A^G & i = 0 \\ A_G & i = 1 \\ 0 & i > 1 \end{cases}$$

(See page 104 of [MF], or many other references on group cohomology.)

Now taking the direct limit of diagram (4.1) with $m = \ell^n$ yields

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \\
 & & \mathrm{NS}(X) \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell \arrow[r] \arrow[d] \arrow[dr, "g"] & (\mathrm{NS}(\overline{X}) \otimes (\mathbf{Q}_\ell/\mathbf{Z}_\ell))^G \arrow[d] & \\
0 \arrow[r] & \mathrm{H}^1(\overline{X}, \mu(\ell))_G \arrow[r] \arrow[d, "\simeq"] & \mathrm{H}^2(X, \mu(\ell)) \arrow[r] \arrow[d] & \mathrm{H}^2(\overline{X}, \mu(\ell))^G \arrow[r] & 0 \\
 & (\mathrm{Pic}(\overline{X})(\ell))_G & \mathrm{Br}(X)(\ell) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd} \tag{4.2}$$

(Direct limits commute with invariants and coinvariants because $\varinjlim$ is an exact functor.) The main point that bears mentioning here is that the replacement of $\mathrm{Pic}(X)$ by $\mathrm{NS}(X)$ in the first row follows from the fact that $\mathrm{Pic}^0(X)$ is finite (as it coincides with the the set $\mathrm{Pic}^0_{X/k}(k)$ of rational points of a scheme of finite type over the finite field k).

Taking the inverse limit of diagram (4.1) with $m = \ell^n$ yields

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
& & \mathrm{Pic}(X)\otimes \mathbf{Z}_\ell \arrow[r] \arrow[d] & \mathrm{NS}(X)\otimes \mathbf{Z}_\ell \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & \mathrm{H}^1(\overline{X}, \mathbf{Z}_\ell(1))_G \arrow[r] \arrow[d, "\simeq"] & \mathrm{H}^2(X, \mathbf{Z}_\ell(1)) \arrow[r] \arrow[d] & \mathrm{H}^2(\overline{X}, \mathbf{Z}_\ell(1))^G \arrow[r] & 0 \\
& T_\ell \mathrm{Pic}(\overline{X})_G & T_\ell \mathrm{Br}(X) \arrow[d] & & \\
& & 0 & &
\end{tikzcd} \tag{4.3}$$

(Since all groups appearing in diagram (4.1) are finite, $\varprojlim_n$ preserves exactness.)

That we obtain the relevant groups tensored with $\mathbf{Z}_\ell$ in the first row of (4.3) follows from the fact that the relevant groups are finitely generated (see the parenthetical remark just before (2.2)). The only thing that bears mentioning here is why we may replace $(\mathrm{NS}(\overline{X}) \otimes \mathbf{Z}_\ell)^G$ in the top right of the diagram by $\mathrm{NS}(X) \otimes \mathbf{Z}_\ell$. This was explained in section 2, but we explain it again here for the reader's convenience.

For any Hausdorff topological G-module A we have $A^G = \ker(\sigma - 1)$, where σ is a (topological) generator for G, so

$$(\mathrm{NS}(\overline{X}) \otimes \mathbf{Z}_\ell)^G = \mathrm{NS}(\overline{X})^G \otimes \mathbf{Z}_\ell$$

because for any map $f : A \to B$ of abelian groups, $\ker(f \otimes \mathbf{Z}_\ell) = \ker(f) \otimes \mathbf{Z}_\ell$ since $\mathbf{Z}_\ell$ is a flat $\mathbf{Z}$-module. Now by (1.4) we have $\mathrm{NS}(\overline{X})^G = \mathrm{NS}(X)$, so $(\mathrm{NS}(\overline{X}) \otimes \mathbf{Z}_\ell)^G = \mathrm{NS}(X) \otimes \mathbf{Z}_\ell$.

We are now ready to prove the following, which implies Theorem 1.3 away from p.

Theorem 4.1 (Tate)**.** *Let $\rho(X) = \mathrm{rank}(\mathrm{NS}(X))$, and let $\ell \neq p$ be a prime. The following are equivalent:*

(*i*) $\mathrm{Br}(X)(\ell)$ *is finite,*

(*ii*) *the map h in diagram* (4.3) *is an isomorphism,*

(*iii*) $\rho(X) = \mathrm{rank}_{\mathbf{Z}_\ell} \mathrm{H}^2(\overline{X}, \mathbf{Z}_\ell(1))^G$,

(*iv*) $\rho(X)$ *is the multiplicity of q as a reciprocal root of $P_2(X, T)$.*

Moreover, if these statements hold for one $\ell \neq p$ then $\mathrm{Br}(X)$(*non-p*) *is finite and the Artin–Tate conjecture holds away from p:*

$$P_2(X, q^{-s}) \sim p^n \frac{[\mathrm{Br}(X)]|\det(D_i \cdot D_j)|}{q^{\alpha(X)}[\mathrm{NS}(X)_{\mathrm{tor}}]^2}(1 - q^{1-s})^{\rho(x)}, \qquad \text{as } s \to 1$$

for some $n = n_X \in \mathbf{Z}$.

The rest of this section is devoted to the proof. First, I claim that

$$T_\ell \mathrm{Pic}(\overline{X})_G \text{ is finite.} \tag{4.4}$$

Indeed, $T_\ell\mathrm{Pic}(\overline{X})$ is a finitely-generated $\mathbf{Z}_\ell$-module because it is an extension of $T_\ell\mathrm{Pic}^0(\overline{X})$ by $T_\ell\mathrm{NS}(\overline{X})$; the former is finitely-generated because $\mathrm{Pic}^0(\overline{X})$ is the group of $\overline{k}$-points of an algebraic $\overline{k}$-group (even an abelian variety), and the latter is 0 because $\mathrm{NS}(\overline{X})(\ell)$ is finite.

Now $T_\ell\mathrm{Pic}(\overline{X})_G$ is the cokernel of the $\mathbf{Z}_\ell$-module map

$$\sigma - 1 : T_\ell\mathrm{Pic}(\overline{X}) \longrightarrow T_\ell\mathrm{Pic}(\overline{X}),$$

so to show that this is finite it is enough to show that the kernel vanishes. But the kernel equals $T_\ell\mathrm{Pic}(\overline{X})^G = T_\ell\mathrm{Pic}(X) = 0$ because $\mathrm{Pic}(X)(\ell)$ is finite (Again, sandwich it between Pic^0 and NS.). For the first equality, we need the fact that $\mathrm{Pic}(\overline{X})^G = \mathrm{Pic}(X)$. This holds because $\mathrm{Pic}(X) = \mathrm{Pic}_{X/k}(k)$ (See discussion of $\mathrm{NS}(X)$ in section 1.).

Now the point of (4.4) is that it yields an exact sequence coming from (4.3):

$$0 \longrightarrow \mathrm{NS}(X) \otimes \mathbf{Z}_\ell \xrightarrow{h} \mathrm{H}^2(\overline{X}, \mathbf{Z}_\ell(1))^G \longrightarrow T_\ell\mathrm{Br}(X) \longrightarrow 0 \tag{4.5}$$

The map on the right is defined by lifting $\alpha \in \mathrm{H}^2(\overline{X}, \mathbf{Z}_\ell(1))^G$ to an element of $\mathrm{H}^2(X, \mathbf{Z}_\ell(1))$, and then sending this to its image in $T_\ell\mathrm{Br}(X)$. That this is well-defined boils down to the fact that the map $(T_\ell\mathrm{Pic}(\overline{X})_G \simeq)\ \mathrm{H}^1(\overline{X}, \mathbf{Z}_\ell(1))_G \to T_\ell\mathrm{Br}(X)$ is 0. This holds because the former group is torsion (because it is finite), and the latter is torsion-free.

The equivalence of (i), (ii), and (iii) in Theorem 4.1 follows immediately from the sequence (4.5); the fact that $\mathrm{Br}(X)(\ell)$ is finite if and only if $T_\ell\mathrm{Br}(X) = 0$; and the fact that $\mathrm{coker}(h) = T_\ell\mathrm{Br}(X)$ is torsion-free.

Now suppose that (iv) holds. We have

$$P_2(X, T) = \det(1 - FT | \mathrm{H}^2(\overline{X}, \mathbf{Q}_\ell)),$$

where F is the geometric Frobenius, and further $\mathrm{H}^2(\overline{X}, \mathbf{Z}_\ell(1)) = \mathrm{H}^2(\overline{X}, \mathbf{Q}_\ell) \otimes \mathbf{Z}_\ell(1)$. Since F acts on $\mathbf{Z}_\ell(1)$ via multiplication by q^{-1}, it follows that the multiplicity of q as a reciprocal root of P_2 is the same as the multiplicity of 1 as an eigenvalue of F acting on $\mathrm{H}^2(\overline{X}, \mathbf{Z}_\ell(1))$, and this is clearly at least the rank of $\mathrm{H}^2(\overline{X}, \mathbf{Z}_\ell(1))^G$. Thus, by (iv), $\rho(X) \geq \mathrm{rank}_{\mathbf{Z}_\ell}\mathrm{H}^2(\overline{X}, \mathbf{Z}_\ell(1))^G$. Since the reverse inequality $\rho(X) \leq \mathrm{rank}_{\mathbf{Z}_\ell}\mathrm{H}^2(\overline{X}, \mathbf{Z}_\ell(1))^G$ follows from (4.5), we deduce (iii).

Finally, suppose that (i), (ii), (iii) hold, and we'll deduce (iv) and the other conclusions of the theorem. We have a commutative diagram

$$\begin{array}{ccccc} \mathrm{NS}(X) \otimes \mathbf{Z}_\ell & \xrightarrow{e} & \mathrm{Hom}(\mathrm{NS}(X), \mathbf{Z}_\ell) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathbf{Z}_\ell}(\mathrm{NS}(X) \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell, \mathbf{Q}_\ell/\mathbf{Z}_\ell) \\ \big\downarrow h & & & & \big\uparrow g^* \\ \mathrm{H}^2(\overline{X}, \mathbf{Z}_\ell(1))^G & \xrightarrow{f} & \mathrm{H}^2(\overline{X}, \mathbf{Z}_\ell(1))_G & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathbf{Z}_\ell}(\mathrm{H}^2(\overline{X}, \mu(\ell))^G, \mathbf{Q}_\ell/\mathbf{Z}_\ell) \end{array} \tag{4.6}$$

where we now describe the maps. The map e comes from the intersection pairing on X, the isomorphism in the top row is the obvious one, h is the map in (4.3), f is the map induced by the identity on $\mathrm{H}^2(\overline{X}, \mathbf{Z}_\ell(1))$, g^* is the dual to the map g in diagram (4.2), and the isomorphism in the bottom row is induced by Poincaré duality on $\overline{X}$. That (4.6) commutes is a consequence (indeed, is the statement, if we remove the G subscript and superscript) of the compatibility of Poincaré duality with the intersection pairing.

I claim that e is a quasi-isomorphism. Indeed, this would follow from (and is equivalent to) the nondegeneracy of the intersection pairing on $\mathrm{NS}(X)/\mathrm{NS}(X)_{\mathrm{tor}}$. That in turn is a consequence of the non-degeneracy of the intersection pairing on $\mathrm{NS}(\overline{X})/\mathrm{NS}(\overline{X})_{\mathrm{tor}}$, as follows. Let $a \in \mathrm{NS}(X)$ be such that $a \cdot b = 0$ for all $b \in \mathrm{NS}(X)$. Then I claim that $a \cdot c = 0$ for all $c \in \mathrm{NS}(\overline{X})$, which will prove the claim.

Let σ be a (topological) generator for G. Then $\sigma^N c = c$ for some $N > 0$, and then $\sum_{i=0}^{N-1} \sigma^i c \in \mathrm{NS}(\overline{X})^G = \mathrm{NS}(X)$. Therefore, since the intersection pairing is Galois-invariant, $0 = a \cdot (\sum_{i=0}^{N-1} \sigma^i c) = N(a \cdot c) \Longrightarrow a \cdot c = 0$, as desired. By Lemma z.1,

$$z(e) = \frac{|\det(D_i \cdot D_j)|_\ell}{|[\mathrm{NS}(X)_{\mathrm{tor}}]|_\ell}$$

Now I claim that

$$|\mathrm{Pic}(\overline{X})(\ell)_G|_\ell = |[\mathrm{NS}(X)(\ell)]|_\ell \tag{4.7}$$

Indeed, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}^0(\overline{X}) & \longrightarrow & \mathrm{Pic}(\overline{X}) & \longrightarrow & \mathrm{NS}(\overline{X}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \ell^n} & & \downarrow{\scriptstyle \ell^n} & & \downarrow{\scriptstyle \ell^n} & & \\
0 & \longrightarrow & \mathrm{Pic}^0(\overline{X}) & \longrightarrow & \mathrm{Pic}(\overline{X}) & \longrightarrow & \mathrm{NS}(\overline{X}) & \longrightarrow & 0
\end{array}$$

Since $\mathrm{Pic}^0(\overline{X})$ is divisible, applying the snake lemma yields an exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(\overline{X})[\ell^n] \longrightarrow \mathrm{Pic}(\overline{X})[\ell^n] \longrightarrow \mathrm{NS}(\overline{X})[\ell^n] \longrightarrow 0. \tag{4.8}$$

Take the direct limit over all n and pass to coinvariants to get

$$\mathrm{Pic}^0(\overline{X})(\ell)_G \longrightarrow \mathrm{Pic}(\overline{X})(\ell)_G \longrightarrow \mathrm{NS}(\overline{X})(\ell)_G \longrightarrow 0.$$

Now I claim that

$$\mathrm{Pic}^0(\overline{X})(\ell)_G = 0. \tag{4.9}$$

Assuming this, we obtain an isomorphism $\mathrm{Pic}(\overline{X})(\ell)_G \xrightarrow{\sim} \mathrm{NS}(\overline{X})(\ell)_G$. But $\mathrm{NS}(\overline{X})(\ell)$ is finite, hence $|\mathrm{Pic}(\overline{X})(\ell)_G|_\ell = |\mathrm{NS}(\overline{X})(\ell)_G|_\ell = |\mathrm{NS}(\overline{X})(\ell)^G|_\ell = |\mathrm{NS}(X)(\ell)|_\ell$, which proves (4.7).

The vanishing in (4.9) is a consequence of (4.4). Indeed, I first claim that $T_\ell(\mathrm{Pic}^0(\overline{X}))_G \simeq T_\ell(\mathrm{Pic}(\overline{X}))_G$, and so is in particular finite. This follows by taking the inverse limit of

(4.8) and using the fact that $\mathrm{NS}(\overline{X})(\ell)$ is finite, forcing $T_\ell \mathrm{NS}(\overline{X}) = 0$ (Alternatively, this can be shown directly just as the finiteness of $T_\ell \mathrm{Pic}(\overline{X})_G$ was shown.). Now suppose that ℓ^n kills $T_\ell(\mathrm{Pic}^0(\overline{X}))_G$. Choose $a \in \mathrm{Pic}^0(\overline{X})(\ell)$, say $a \in \mathrm{Pic}^0(\overline{X})[\ell^r]$. Let $b_r = a$, and for all $m > r$ inductively choose $b_m \in \mathrm{Pic}^0(\overline{X})$ such that $lb_m = b_{m-1}$. Then $(b_m)_{m \geq r} \in T_\ell \mathrm{Pic}^0(\overline{X})$, so $(\ell^n b_m)_{m \geq r} = \sigma\alpha - \alpha$ for some $\alpha = (\alpha_n)_{n \geq 0} \in T_\ell \mathrm{Pic}^0(\overline{X})$. Therefore $a = \ell^n b_{n+r} = \sigma\alpha_{n+r} - \alpha_{n+r}$. Since $a \in \mathrm{Pic}^0(\overline{X})(\ell)$ was arbitrary, this shows that $(\sigma - 1) : \mathrm{Pic}^0(\overline{X})(\ell) \to \mathrm{Pic}^0(\overline{X})(\ell)$ is surjective; i.e. $\mathrm{Pic}^0(\overline{X})(\ell)_G = 0$.

The point of (4.7) is that by Lemma z.2, diagram (4.2) (going down and then right), and (i), we see that g is a quasi-isomorphism with

$$z(g) = \frac{|[\mathrm{Br}(X)(\ell)]|_\ell}{|[\mathrm{NS}(X)(\ell)]|_\ell};$$

hence, by Lemma z.3, g^* is a quasi-isomoprhism and

$$z(g^*) = \frac{|[\mathrm{NS}(X)(\ell)]|_\ell}{|[\mathrm{Br}(X)(\ell)]|_\ell}.$$

It follows again by Lemma z.3 that f is a quasi-isomorphism and

$$z(f) = z(e)/z(g^*) = \left| \frac{\det(D_i \cdot D_j)[\mathrm{Br}(X)(\ell)]}{[\mathrm{NS}(X)_{\mathrm{tor}}]^2} \right|_\ell \tag{4.10}$$

Now apply Lemma z.4 with $A = \mathrm{H}^2(\overline{X}, \mathbf{Z}_\ell(1))$ and $\theta = \sigma - 1$. We get by (iii) that (iv) holds (via the usual interpretation of the multiplicity of q^{-1} as a root of P_2 as the multiplicity of 1 as an eigenvalue of F acting on $\mathrm{H}^2(\overline{X}, \mathbf{Z}_\ell(1))$). Further, we have that

$$z(f) = |(R(q^{-1})|_\ell \tag{4.11}$$

where $R(T) := P_2(X,T)/(1-qT)^{\rho(X)}$. The two expressions (4.10) and (4.11) for $z(f)$ imply the ℓ-part of Theorem 4.1. In sum, we have shown for every ℓ the equivalence of (i) -(iv), and that these imply the ℓ-part of the Artin–Tate conjecture.

Now suppose that (i)-(iv) hold for some $\ell \neq p$. Then since (iv) is independent of ℓ, they hold for all $\ell \neq p$, and thus the ℓ-part of the Artin–Tate conjecture holds for all $\ell \neq p$. But the equality

$$|R(q^{-1})|_\ell = \left| \frac{\det(D_i \cdot D_j)[\mathrm{Br}(X)(\ell)]}{[\mathrm{NS}(X)_{\mathrm{tor}}]^2} \right|_\ell \tag{4.12}$$

then implies that $\mathrm{Br}(X)(\ell) = 0$ for all but finitely many $\ell \neq p$. Thus, since $\mathrm{Br}(X)$ is torsion, we conclude that $\mathrm{Br}(X)$ is finite, Since (4.12) holds for all $\ell \neq p$, it holds up to a factor of $\pm p^n$ for some $n \in \mathbf{Z}$.

To see that the "+" case holds, we need to check that $(-1)^{\rho(X)} P_2(X,T) > 0$ as $T \to q^{-1}$ from the right. Since, by the Riemann Hypothesis, all roots of P_2 have absolute value q^{-1},

P_2 has no roots to the right of q^{-1}. Hence, it's enough to show that $(-1)^{\rho(X)}P_2(X,T) > 0$ as $T \to \infty$.

Write $P_2(X,T) = \prod(1-\alpha_i T)$. By Poincaré duality, $\alpha \mapsto q^2/\alpha$ is a permutation of the α_i. Pairing the α_i and q^2/α_i terms, we get a contributing factor of q^2 to the leading coefficient of P_2 whenever $\alpha \neq \pm q$. It follows that the sign of the leading coefficient of P_2 is $(-1)^{\rho'}$, where ρ' is the multiplicity of q as a reciprocal root of P_2. But we've already shown that $\rho' = \rho(X)$ (this is (iv)), so the "+" sign holds. This completes the proof of Theorem 4.1. □

5 Some brief remarks on the p-part of the conjecture

We will not prove the p-part of Theorem 1.3 here, but we would like to at least say something (rather vague) about what goes into the proof. For full details, see Milne's paper [M].

The broad outline of the proof is basically the same as the proof of Theorem 4.1 above. There are essentially three main differences/additional ingredients that we'll mention here.

(i) The étale cohomology groups are replaced by flat cohomology groups. This is so that one still has the Kummer sequence, which plays an essential role in the proof above. Milne also makes use of a flat duality theorem of his for surfaces, which plays the role of Poincaré duality.

(ii) At one point Milne has to compute the size of the group $\mathrm{H}^3(\overline{X}, T_p(\mathbf{G}_m))^G$ (Lemma 5.2 in [M]). This is accomplished by invoking results of his that relate this flat cohomology group to the points of a certain unipotent k-group, whose point count is then related to its dimension.

(iii) The main new feature arising at p is the presence of the term $q^{\alpha(X)}$, which of course plays no role in the proof above for the prime-to-p result. This is accomplished by relating the Frobenius action on the flat cohomology groups to certain Witt vector and crystalline cohomology groups, and then proving a general lemma (Lemma 7.2 in [M], to be applied to the Witt vector cohomology) about modules over the ring $W[V]$, where W is the ring of Witt vectors over k and $Vw = w^{\sigma^{-1}}V$ for $w \in W$ (where $\sigma : W \to W$ is the Frobenius map). (The element V is a version of the Verschiebung.)

One then relates the Witt vector cohomology groups back to the coherent cohomology groups of $\mathcal{O}_X$ to finish the proof.

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