Motives and Deligne's Conjecture

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January 13, 2016

1 Introduction

Deligne's conjecture [Del] asserts that the values of certain L-functions are equal, up to rational multiples, to the determinants of certain period matrices. The conjectures are most easily stated for projective, smooth varieties over number fields. Such a variety has an L-function whose local factors encode Frobenius eigenvalues. The isomorphism between de Rham and Betti cohomologies gives a period map; since the two cohomology theories each come with a canonical rational structure, the determinant of the period map is well-defined up to rational multiple. Deligne's conjecture, then, asserts the equality of a particular special L-function value with this determinant (subject to a constraint called "criticality" on the Hodge structure of the variety).

In fact, Deligne's conjecture holds in the context of general "motives." Roughly speaking, a motive is a piece of the cohomology of a variety. In this general context, Deligne's conjecture applies to varieties over a number field, modular forms (pieces of the cohomology of modular curves), and Artin L-functions (associated to zero-dimensional motives).

This talk will be structured as follows. First, we will make a precise statement of Deligne's conjecture for smooth projective varieties. This will require a bit of Hodge theory. Next, we will give a precise definition (one of many) of the category of motives (over a field k with coefficients in a number field E). Finally we will state Deligne's conjecture for motives.

We will save various examples of the conjecture for next week.

2 Deligne's Conjecture for Varieties

Let X be a smooth projective variety over $\mathbb{Q}$. We attach the following structures to X:

1. The *Betti cohomology* $H^*_B(X)$ with coefficients in $\mathbb{Q}$ of X (or more precisely the analytification of its base change $X_{\mathbb{C}}$ to $\mathbb{C}$).

2. The *de Rham cohomology* $H^*_{dR}(X)$ with coefficients in $\mathbb{Q}$, or its complexification $H^*_{dR}(X) \otimes \mathbb{C}$. The complexification admits a Hodge decomposition

$$H^n_{dR}(X) \otimes \mathbb{C} = \oplus H^{pq}(X),$$

where the sum ranges over pairs of nonnegative integers (p, q) with $p + q = n$. Additionally, we have a canonical isomorphism

$$H^n_{dR}(X) \otimes \mathbb{C} \equiv H^n_B(X) \otimes \mathbb{C}.$$

This complex vector space thus comes equipped with two real structures, induced from the rational structures $H^n_{dR}(X)$ and $H^n_B(X)$. These two real structures give rise to two "complex conjugations" (i.e. conjugate-linear involutions), which we denote ι_{dR} and ι_B, respectively. One verifies that these two involutions commute and their product is F_∞, which can be described concretely as follows: Writing $X_{\mathbb{C}}$ as $X \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\mathbb{C}$, we see that complex conjugation on $\operatorname{Spec}\mathbb{C}$ induces an involution ι_∞ on $X_{\mathbb{C}}$. Then F_∞ is the involution on the Betti cohomology of $X_{\mathbb{C}}$ induced from ι_∞.

The involution F_∞ exchanges the Hodge factors H^{pq} and H^{qp}.

3. The ℓ-adic étale cohomology $H^*_\ell(X)$, for all rational primes ℓ. This is a finite-dimensional vector space over $\mathbb{Q}_p$.

   It comes equipped with a Frobenius endomorphism F_p for all p outside a finite set S (which we may take to be the places at which X has bad reduction) and not equal to ℓ. We may consider for any pair (p,ℓ) the characteristic polynomial of F_p acting on $H^n_\ell(X)$. It turns out that this polynomial always has coefficients in $\mathbb{Q}$; and furthermore if we fix $p \notin S$ and allow ℓ to vary (distinct from p), the polynomial is independent of ℓ. We will denote by

$$\det(1 - F_p t)$$

   the determinant evaluated on any permissible $H^*_\ell(X)$, with the understanding that the coefficients are rational.

   For the finitely many $p \in S$, we may consider instead the inertial invariants $(H^*_\ell(X))^{I_p}$. On this subspace the "Frobenius action" is well-defined, and one shows again that its characteristic polynomial has rational coefficients and is independent of ℓ, provided ℓ is chosen prime to p. Hence we may again consider the quantity

$$\det(1 - F_p t),$$

   where F_p is understood to be the restriction of Frobenius to the inertial invariants.

We now introduce the two quantities which are related by Deligne's conjecture.

2.1 The L-function

Recall that X was a smooth projective variety over $\mathbb{Q}$. Let's agree to consider the comohomology of X in some dimension i.

The L-function attached to X is defined as an Euler product with factors coming from the finite and infinite places of $\mathbb{Q}$. For all finite primes p we define

$$L^i_p(s,X) = \left(\det(1 - F_p p^{-s})\right)^{-1}.$$

(The right-hand quantity was defined in the previous section; recall that for bad primes p it is necessary to restrict to the inertial invariants of the étale cohomology of X. Implicit in our notation is the choice of index i: the right-hand side refers to the ℓ-adic H^i.)

For the infinite place of $\mathbb{Q}$ we will define $L_\infty(s,X)$ to be a product of gamma functions. Recall the notation (standard in number theory)

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$$

$$\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

We decompose the Hodge structure associated to the Betti cohomology of X into subspaces minimal for the Hodge decomposition and the involution F_∞. Specifically, consider spaces of the following forms: for $p \neq q$, we take the union of a one-dimensional subspace of H^{pq} and its F_∞-image in H^{qp}. For $p = q$ we take F_∞-eigenspaces in H^{pp}. Then $H^n(X)$ decomposes as a sum of such subspaces.

To each such subspace we assign a gamma factor as follows. For a two-dimensional subspace of $H^{pq} \oplus H^{qp}$, where $p \neq q$, we may take $p < q$, and we assign to this subspace the factor

$$\Gamma_{\mathbb{C}}(s-p).$$

For a one-dimensional eigenspace for F_∞ in H^{pp}, with eigenvalue $(-1)^{p+\epsilon}$, where ϵ is either 0 or 1, we take as our gamma factor

$$\Gamma_{\mathbb{R}}(s-p+\epsilon).$$

We define $L_\infty(s,X)$ to be the product of these factors, over all the subspaces in the decomposition of $H^*_B(X)$, and $L(s,X)$ the product over all places, finite and infinite, of $\mathbb{Q}$. Implicit in this definition is the statement, conjectural in general, that this L-function extends meromorphically to the complex plane. (The product converges for s with large real part.)

At this point we should mention an issue which will arise when we generalize to motives: a motive, like a variety, will have a Hodge structure, which comes equipped with the Hodge decomposition into subspaces H^{pq}, and (if the base field is real) an involution F_∞. But in general the indices p and q may be arbitrary integers (i.e. possibly negative). The definition just stated carries over verbatim.

As an easy exercise, we may ask when this $L(s,X)$ has a trivial pole at $s = 0$. Here a "trivial pole" is a pole coming from the gamma factors L_∞. One finds that if the Hodge structure has any nontrivial H^{pq} with p and q nonnegative and distinct, then L has such a pole. Furthermore, if there is on some H^{pp}, with $p \geq 0$, the involution F_∞ has $+1$ as an eigenvalue, then there is a pole. Otherwise, $L_\infty(s,X)$ has no pole at $s = 0$.

We are interested in the value $L(0,X)$. If L has a pole (or a zero) at $s = 0$ then the question is not interesting. (One could ask for the leading term of $L(s,X)$ in this case. We will not.) We say that X is *critical* if there is not a trivial pole or zero at $s = 0$. For varieties, it is sufficient to require that $H^{pq} = 0$ for p and q distinct, and that F_∞ act as -1 on each H^{pp}.

Even though we haven't yet defined motives, this seems like a good time to explain the criticality condition in this more general context.

There is a conjectural functional equation, for any motive M, relating $L(s,M)$ to $L(1-s,M^*)$, where M^* is the dual of M. For the moment it is enough to know that duality interacts with the Hodge structure as follows: $H^{pq}(M)$ is dual to $H^{-p,-q}(M^*)$. Also, F_∞ has the same eigenvalues on $H^{pq}(M)$ and $H^{-p,-q}(M^*)$.

In general one says that M is *critical* if neither $L_\infty(s,M)$ nor $L_\infty(1-s,M^*)$ has a pole at $s = 0$. It is a straightforward exercise to show that criticality is equivalent to the following conditions.

1. For $p \geq 0$, the involution L_∞ acts as -1 on H^{pp}.

2. For $p < 0$, the involution L_∞ acts as $+1$ on H^{pp}.

3. If $p \neq q$ and $\min(p,q) \geq 0$, then $H^{pq} = 0$.

4. If $p \neq q$ and $\max(p,q) < 0$, then $H^{pq} = 0$.

Deligne's conjecture will be stated only for critical varieties and motives.

By the way, criticality is not as stringent a condition as it first appears. First, if we only consider the i-th cohomology, then we only need the conditions to be satisfied for the Hodge factors H^{pq} with $p+q=i$. More importantly, the theory of motives offers a Tate twist by which we can translate indices (p,q) to $(p-a,q-a)$. If $i=2a-1$ is odd, then after twisting the motive becomes critical automatically; if $i=2a$ is even, then one needs only that L_∞ acts either as $+1$ or as -1 on all of H^{aa}. Even if L_∞ has both of ± 1 as eigenvalues on H^{aa} it will often be possible, using the theory of motives, to isolate some components of H^{aa} which are contained in one eigenspace or the other.

2.2 The Period Matrix

Suppose from now on that X is some variety whose i-th cohomology is critical (perhaps after a Tate twist – this will not affect the exposition).

Consider the following two spaces, which are naturally quotients of $H^i_{DR}(X)$ according to the Hodge filtration: set

$$H^-_{DR}(X) = \oplus_{p<0} H^{pq}(X)$$
$$H^+_{DR}(X) = \oplus_{q\geq 0} H^{pq}(X),$$

where the direct sums are taken over pairs (p,q) with $p+q=i$. Criticality has the following implication: if i is odd, then H^-_{DR} and H^+_{DR} coincide, and are of half the dimension of $H^i_{DR}(X)$ by conjugation. If $i=2a$ is even, then H^+_{DR} decomposes as the direct sum of H^-_{DR} and H^{aa}; and the latter is an eigenspace (either $+1$ or -1) for F_∞. Again, we have

$$\dim H^\pm_{DR} = \frac{1}{2}(\dim H_{DR} \pm \dim H^{aa}).$$

Decompose $H_B(X)$ into $+1$ and -1 eigenspaces for F_∞, which we call $H^+_B(X)$ and $H^-_B(X)$, respectively. Using the fact that F_∞ interchanges H^{pq} and H^{qp}, we see that the dimensions of $H^+_B(X)$ and $H^-_B(X)$ are the same as the dimensions of $H^+_{DR}(X)$ and $H^-_{DR}(X)$, taken in one order or the other (depending on the eigenvalue of F_∞ on H^{aa}).

Suppose for the sake of exposition that the aforementioned eigenvalue is $+1$. Then (again using what we know about F_∞ and the Hodge decomposition) the canonical Betti - de Rham comparison isomorphism induces a map on filtered parts

$$I^+ : H^+_B(X) \otimes \mathbb{C} \equiv H^+_{DR}(X) \otimes \mathbb{C}.$$

Now both sides of this isomorphism started out life as $\mathbb{Q}$-vector spaces. Choose a $\mathbb{Q}$-basis for each side and let

$$c^+ = \det I^+.$$

Deligne's conjecture (for varieties) asserts that $L^i(s,X)$ is a rational multiple of c^+.

Concretely one can make this very explicit as follows. Let (γ_i) be a basis for the Betti *homology* dual to H^+_B, consisting of topological cycles. Let (ω_j) be a basis for the rational differentials H^+_{DR}. Then c^+ is the determinant of the matrix $(\int_{\gamma_i}\omega_j)_{i,j}$. In general it takes some work to make this precise because one has to integrate algebraic de Rham classes over Betti cycles, which is not as simple as it looks. This will not concern us.

2.3 Tate Twists

Since Tate twists are so important let's see what they do.

On the cohomology of the Tate motive $H^2(T)$ the p-Frobenius acts by multiplication by p. Hence, tensoring with T multiplies Frobenius eigenvalues by p. Thus, the local L-factor at finite places shifts by one:

$$L_p(s, M(1)) = \left(\det(1 - F_p p^{-s})|_{M(1)}\right)^{-1} = \left(\det(1 - F_p p^{1-s})|_M\right)^{-1} = L_p(s-1, M).$$

On the level of the period map, all the periods multiply by the period of the Tate motive, which is $2\pi i$. (This is most easily seen by viewing the Tate motive as the cohomology of $\mathbb{G}_m$, something which is not possible in the formalism we have set up. Alternatively it can be verified by a calculation on $\mathbb{P}^1$.)

2.4 The Conjecture for Elliptic Curves

We will check that for elliptic curves over $\mathbb{Q}$, Deligne's conjecture is compatible with the conjecture of Birch and Swinnerton-Dyer.

Let E be an elliptic curve over $\mathbb{Q}$. Let $L(E, s) = L^1(E, s)$ be the L-function associated to the first cohomology of E, as defined above. The BSD conjecture, discussed at length in our seminar already, implies the following: if the group of $\mathbb{Q}$-rational points of E has nonzero rank, then $L(E, 1) = 0$; otherwise, we have the formula

$$L(E, 1) = \frac{(\Pi_p c_p)\Omega_E Ш_E R_E}{|E(\mathbb{Q})|^2}.$$

We won't define all the symbols above. We note simply that in this context $R_E = 1$, all the c_p's are rational, $Ш_E$ is rational (provided it's finite, which is only conjectured) and the denominator is evidently rational as well. Hence, up to rational multiple, $L(E, 1)$ is conjectured to be equal to

$$\Omega_E = \int_\gamma \omega.$$

Here ω is a particular choice of rational global differential on E, called the Néron differential; and γ is the real cycle on E.

To apply Deligne's conjecture we need to make M (the motive consisting of the H^1 part of E) critical. As is, M has Hodge components $H^{1,0}$ and $H^{0,1}$; so the Tate twist $M(-1)$ is critical. Now the Tate twist has the effect of shifting the L-function as we have defined it:

$$L(M(-1), s) = L(M, s+1).$$

Thus, Deligne's conjecture predicts the value of

$$L(M(-1), 0) = L(M, 1).$$

The Tate twist also has the effect of multiplying all the periods by $2\pi i$. So, since M has period Ω_E, Deligne predicts that $L(M, 1)$ is a rational multiple of $2\pi i\Omega_E$.

Deligne's L-function differs from the L-function appearing in BSD by a gamma factor whose value at 1 is exactly $2\pi i$.

3 Motives with Coefficients in $\mathbb{Q}$

To make a general statement of Deligne's conjecture we will need the language of motives. A motive will turn out to be a sort of cohomological "piece" of a variety. Every motive will have Betti cohomology, de Rham cohomology, ℓ-adic cohomology and so forth. Every variety will be a motive (or more precisely there will be a functor from varieties to motives). There will also be a motive corresponding to "the H^1 of an elliptic curve," for instance. More importantly for applications, modular forms will have a motivic interpretation, since they live naturally in the cohomology of the modular curves. So will Artin L-functions, which will arise from zero-dimensional motives, i.e. finite field extensions.

The idea behind motives is to pick out "pieces of cohomology" *geometrically.* For inspiration, we may consider the algebra of endomorphisms of an abelian variety, or the Hecke algebra for a modular form. In fact it's enough to consider the rational algebra and forget the integral structure. We know that these algebras record much of the same information as the cohomology, yet they are defined "geometrically" without reference to cohomology. The connection is provided by the fact that an endomorphism of an abelian variety, or a Hecke correspondence, induces an endomorphism on cohomology. Provided the cohomology theory has characteristic-zero coefficients, one may pass to the rational endomorphism or Hecke algebra, and again obtain an action on the cohomology.

Our definition of the category of motives will include the following ingredients. We start with a notion of "correspondence" with rational coefficients, which will generalize the endomorphism and Hecke algebras just mentioned. We will introduce new objects ("parts of varieties") picked out by projectors in the algebra of correspondences. Finally, we will define a sort of motivic Tate twist and formally introduce an inverse twist. Crucially, all the usual cohomology theories are defined on the category of motives.

Fix a number field k. We will define the category of *motives over k with coefficients in* $\mathbb{Q}$. Consider first the category whose objects are smooth projective varieties (if we ever say "variety" we will mean "smooth projective variety") over k, and whose morphisms are defined as follows: for any two objects X and Y, we take $\mathrm{Hom}(X, Y)$ to be the set of algebraic cycles on $X \times Y$, taken up to rational equivalence, with coefficients in $\mathbb{Q}$. Such cycles are referred to as "correspondences." Any element of $\mathrm{Hom}(X, Y)$ should induce a map on cohomology $H^*(Y) \to H^*(X)$, for any "reasonable" cohomology theory (including all those mentioned in these notes – but see the warning below). Also note that the group of cycles comes filtered by dimension, and this filtration (correctly indexed) plays well with the degrees on cohomology, in a way which the reader may make precise.

Some words about correspondences. A function $X \to Y$ is a special case of a correspondence, by means of its graph in $X \times Y$. The language of correspondences also makes precise the notion of a "multivalued function from X to Y" – a correspondence finite and flat over X. These multivalued functions account for most of the correspondences from X to Y coming from cycles of the same dimension as X, which in turn are exactly those correspondences that do not shift the dimension of cohomology.

Some authors only consider cycles of the same dimension as X in the definition of a motive. Others replace "rational equivalence" with a different equivalence relation. It should be mentioned that, with different definitions, much of the theory of motives is conjectural.

Warning 3.1. *As a matter of fact it's not quite true that an element of* $\mathrm{Hom}(X, Y)$ *induces a map on cohomology. The problem is that our correspondences are defined only up to rational equivalence, and thus its action on cohomology might not be well-defined.*

One defines in an evident way the *composition* of two correspondences, by analogy with the composition of functions. One verifies that this composition makes $\mathrm{Hom}(X, X)$ into a $\mathbb{Q}$-algebra. (This is a matter of checking associativity of composition, and so forth. It's all straightforward.)

One would like to make the category of motives into an abelian category. To do this we need to produce kernels and images for all correspondences. Consider, as an example, $\mathbb{P}^1$, which has nontrivial cohomology in H^0 and H^2 (for whatever cohomology theory you like). The cubing map f on $\mathbb{P}^1$ induces the identity on H^0 and multiplication by 3 on H^2. Hence, if i denotes the identity map on $\mathbb{P}^1$, then $\ker(f - 3i)$ should be some object whose only cohomology is in H^2. There is no such variety, so we'll have to introduce some objects like this.

In fact the standard approach to motives is slightly different. One introduces only objects corresponding to projectors, i.e. elements $p \in \mathrm{Hom}(X, X)$ satisfying $p^2 = p$. Formally one may denote such an object (X, p). One defines $\mathrm{Hom}((X, p), (Y, q))$ as the set of $f \in \mathrm{Hom}(X, Y)$ which factor through both p and q. The object (X, p) turns out to be the image of p acting on X, while $(X, i_X - p)$ is its kernel. We call this the category of *effective motives*.

We would like to know that all the cohomology theories extend in the natural way to this new category. One might hope that this could be accomplished as a purely formal matter: to define the cohomology of (X, p) one would simply consider the action of p on the cohomology of X; this action would itself be a projector, and one would define the cohomology of (X, p) to be its image. The problem arises because of the warning above. There is a workaround, described in [Jan], which allows one to define cohomology on this new category. It should be mentioned that this relies on the "standard conjectures." Specifically one needs to assume that the "H^i-part" of any variety X belongs to our category of motives, i.e. that it can be picked out by algebraic cycles.

In general this construction produces what is known as a Karoubian category, which need not be abelian. However, with the definition of motive we have chosen (in particular, using numerical equivalence as the equivalence relation) we do indeed arrive at an abelian category. See [Jan] for a short proof. The essential idea is a finite-dimensionality... in fact that the space of cycles (with rational coefficients) up to rational equivalence is finite dimensional.

Consider again the example of $\mathbb{P}^1$. We want to construct a projector whose "image" has H^2 but no H^0; a natural candidate would be a correspondence that induces the identity on $H^2(\mathbb{P}^1)$ but the zero map on $H^0(\mathbb{P}^1)$. It is easy to construct such a map from the cubing map mentioned above, which suggests how finite-dimensionality hypotheses might be useful for constructing projectors in general. More cleanly, let p be the correspondence $\{e\} \times \mathbb{P}^1 \subseteq \mathbb{P}^1 \times \mathbb{P}^1$. Then one may verify that $p^2 = p$ and p induces the desired maps on cohomology.

The pair $(\mathbb{P}^1, p)$ is known as the Tate motive T. It will be very important in the future. Let's record some of its properties. Its only cohomology (for any cohomology theory you like) occurs in H^2, and $H^2(T)$ has dimension 1. In the ℓ-adic theory, $H^2(T) = \mathbb{Q}_\ell(1)$ is a Tate twist of $\mathbb{Q}_\ell$. In Hodge theory, it is $H^{1,1}(T)$ which is nontrivial.

There is a notion of tensor products on effective motives, induced from the product $(X, Y) \mapsto X \times Y$ on varieties over k. Tensoring with T has the effect of shifting all cohomology up two dimensions. On étale cohomology every cohomology group is hit by a Tate twist. In Hodge theory, H^{pq} becomes $H^{p+1,q+1}$. The dimensions of cohomology do not change. We denote $X(1) = X \otimes T$. It can be shown that $X \mapsto X(1)$ is fully faithful [Man]. This functor is known as the Tate twist.

The category of motives is formed from the category of effective motives by inverting the Tate twist, that is, by formally adjoining motives $X(-n)$, for every effective motive X and integer n. The motives $X(-n)$ have cohomology in all the cohomology theories we are discussing, defined in the obvious way.

For any curve one can produce a motive whose only cohomology is the H^1 of the curve.

Similarly, there are motives corresponding to modular forms, roughly because modular forms can be cut out by elements of the Hecke algebra.

3.1 Deligne's Conjecture for Motives with Coefficients in $\mathbb{Q}$

For motives over $\mathbb{Q}$, the statement of the conjecture given above carries over verbatim. Indeed, any motive has ℓ-adic cohomology, from which one can define the L-function; and it has Betti and de Rham cohomologies (with Hodge structures and F_∞), related by a period isomorphism.

Over a different number field, one applies Weil's restriction of scalars to arrive at a statement over $\mathbb{Q}$. To make this rigorous one would have to verify that every step in our definition of motives is compatible with restriction of scalars. We won't do that here, but it is worth mentioning what happens on the L-function side.

Let M be a motive over a number field k, and let $\mathrm{Res}_{k/\mathbb{Q}} M$ denote its Weil restriction. (One may without harm restrict attention to the special case of H^i of a variety.) The L-function of M is defined as follows. There is one local factor for every place ν of k, coming from the map $G_{K_\nu} \to G_K$, as in the rational case. For each infinite place τ of k we consider the complexified de Rham cohomology

$$H_{DR}(X) \otimes_{k,\tau} \mathbb{C}.$$

If τ is real then we decompose this, as above, into spaces minimal for the projectors defining the Hodge filtration and L_∞. On the other hand if τ is complex there is no L_∞; instead, we simply decompose it into one-dimensional subspaces in a way which respects the Hodge filtration. In this case we associate to a one-dimensional subspace of H^{pq} the gamma factor

$$\Gamma_{\mathbb{C}}(s-p),$$

with the assumption that $p \leq q$. The product of all these, taken over all τ, is denoted $L(s,M)$.

The L-function so defined is related to the Weil restriction by the identity

$$L(s,M) = L(s, \mathrm{Res}_{k/\mathbb{Q}} M),$$

where the right-hand side is the previously defined L-function of a motive over $\mathbb{Q}$. It is this quantity which appears on the L-function side of Deligne's conjecture over arbitrary number fields.

3.2 Another example: Elliptic Curve over a Number Field

Let E be an elliptic curve over a number field K, and let M be the motive that has only the H^1 of E.

Deligne's conjecture applies to the Weil restriction of scalars $A = \mathrm{Res}_{K/\mathbb{Q}} E$.

First, here's a quick, sloppy review of Weil restriction. If $d = [K : \mathbb{Q}]$ then the Weyl restriction of a variety over K is a variety over $\mathbb{Q}$ of d times the original dimension. So, the Weil restriction of our elliptic curve will be a variety – an abelian variety, as it turns out – over $\mathbb{Q}$.

Let's suppose for concreteness that our elliptic curve is cut out by a single equation f in two variables x and y. (Weierstrass form is fine.) The polynomial f has coefficients in K. Now we're going to do something analogous to taking real and imaginary parts (for the extension $\mathbb{C}/\mathbb{R}$) to view a geometric object over $\mathbb{C}$ as an object over $\mathbb{R}$ with twice the dimension: Take a basis $e_1, \ldots, e_d$ for K regarded as a $\mathbb{Q}$-vector space; then any $t \in K$ can be expressed in terms of its "e_i parts" as

$$t = t_1 e_1 + \cdots t_d e_d.$$

Now we can express the polynomial f in terms of this basis; that is, if we write

$$x = x_1 e_1 + \cdots + x_d e_d$$

and

$$y = y_1 e_1 + \cdots + y_d e_d$$

then computing formally in K as a vector space over $\mathbb{Q}$, we find

$$f(x, y) = f_1(x_1, y_1, \ldots, x_d, y_d) e_1 + \cdots + f_d(x_1, y_1, \ldots, x_d, y_d) e_d.$$

Setting the right-hand side equal to zero gives d equations in the $2d$ variables x_i and y_i; the equations $(f_1, \ldots, f_d)$ have rational coefficients, and we may regard the result as a d-dimensional variety over $\mathbb{Q}$.

By construction we have

$$(\operatorname{Res}_{K/\mathbb{Q}} E)(\mathbb{Q}) = E(K),$$

and in fact it's easy to verify the more general functoriality

$$(\operatorname{Res}_{K/\mathbb{Q}} E)(R) = E(K \otimes_{\mathbb{Q}} R),$$

for any $\mathbb{Q}$-algebra R.

In particular, if R is an algebraically closed field, then $K \otimes_{\mathbb{Q}} R$ is isomorphic to d copies of R, with the isomorphism given by the d embeddings σ_i of K into R. In this case, the abelian variety $(\operatorname{Res}_{K/\mathbb{Q}} E)$ is isomorphic over R to the product of the d elliptic curves E^{σ_i} over R arising by base change from E/K via the d different embeddings. In particular this is the case over $\mathbb{C}$.

Deligne's conjecture predicts the critical L-value associatied to the Weil restriction of E; since L-functions are invariant under Weil restriction this is the same as the critical L-value of E. According to BSD for number fields we expect to see a product of periods – the d periods of E via the d embeddings of K in $\mathbb{C}$ – and the square root of the discriminant of E. Our goal is to see where those come from in terms of the Deligne conjecture.

So, let's compute periods. To simplify the exposition we'll suppose that K is totally real; and we'll work with H^1.

Betti cohomology: For the Betti cohomology (even the rational Betti cohomology) we pass to $\mathbb{C}$, where A (the Weil restriction of E) decomposes as the product of the E^{σ_i}. Each E^{σ_i} has in its Betti homology a one-dimensional conjugation-invariant subspace, generated by the "real cycle" γ_i on E^{σ_i}. These d real cycles are a rational basis for the Betti homology.

The de Rham cohomology is more interesting. Weil restriction applies to global differentials as well, taking global differentials on E/K to global differentials on $A/\mathbb{Q}$. This fills out the d-dimensional space (vector space over $\mathbb{Q}$) of global differentials on A. Specifically, in terms of our basis e_i for K over $\mathbb{Q}$, we may write down the global differentials

$$\omega_j = e_j \omega$$

on E, where $\omega = dx/y$ in Weierstrass form, say. These are of course the same differential up to scaling on E/K, but when restricted to $A/\mathbb{Q}$ they give independent differentials, which we see as follows: Consider again the base change of A to $\mathbb{C}$ (any algebraically closed field will do, here), which decomposes as the sum of elliptic curves E^{σ_i}. Then on the i-th component of A, the differential ω_j restricts to

$$\sigma_i(e_j)\omega.$$

It follows that the ω_j are independent on A.

Now the period matrix with respect to our chosen rational bases has (i, j)-entry equal to

$$\sigma_i(e_j) \int_{\gamma_i} \omega.$$

The determinant of this matrix is equal to

$$\left(\Pi_i \int_{\gamma_i} \omega\right) \det\left(\sigma_i(e_j)\right),$$

and the determinant on the right is, up to a rational multiple, the discriminant of $K/\mathbb{Q}$.

4 Motives with Coefficients

To state Deligne's conjecture in full generality we need the notion of motives with coefficients in a number field E. This E is not to be confused with the base field k. Motives with coefficients arise from modular forms (Hecke eigenforms) whose coefficients generate a number field E; or from abelian varieties with nontrivial endomorphism rings. There are two equivalent definitions of motives with coefficients, as we now explain.

On one hand we may modify our previous definition as follows: for our initial category of varieties with correspondences as morphisms, we initially took for morphisms the cycles on $X \times Y$, with rational equivalence; we could plausibly write this as

$$\mathrm{Hom}(X, Y) = \mathrm{Cyc}(X \times Y) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Instead of this, we consider cycles with coefficients in the field E, namely

$$\mathrm{Hom}(X, Y) = \mathrm{Cyc}(X \times Y) \otimes_{\mathbb{Z}} \mathbb{E}.$$

Then we proceed as before, adjoining objects for every projector, and then formally inverting the Tate twist functor. Call the resulting category $\mathcal{M}^1_E$.

On the other hand, we may define a motive with coefficients in E as the following data: a motive M with coefficients in $\mathbb{Q}$, and an E-module action on M, that is to say, a map $E \to \mathrm{End}\, M$. Call the resulting category $\mathcal{M}^2_E$.

Consider for concreteness an elliptic curve M with complex multiplication by the imaginary quadratic field E. (We won't distinguish between the curve and its associated motive.) Here the second formulation is straightforward: E acts on M through endomorphisms.

In the first formulation, we have

$$\mathrm{Hom}(M, M) = E \otimes E$$

(in any of the categories introduced in our definition of motive). Now $E \otimes E$ is the direct sum of two copies of E; in particular, multiplication induces projection

$$E \otimes E$$

onto one of the direct factors. If e denotes the corresponding idempotent, then the pair (M, e) is an effective motive with coefficients in E.

In light of this example, we now show how to pass between the two categories of motives with coefficients in an arbitrary number field E. The two definitions agree on $E = \mathbb{Q}$, where we write $\mathcal{M}_{\mathbb{Q}}$ for the category of motives.

First we pass from $\mathcal{M}_E^1$ to $\mathcal{M}_E^2$. There is a natural functor $\mathcal{M}_{\mathbb{Q}} \to \mathcal{M}_E^1$, which we denote $M \mapsto M_E$. It is enough to define our functor $M_E^1 \to M_E^2$ on elements of the form $M_E \in M_E^1$. Such an element gets mapped to $M \otimes E \in M_E^2$. This $M \otimes E$ is the direct sum of $[E : \mathbb{Q}]$ copies of M; if for every N we write

$$\mathrm{Hom}(N, M \otimes E) = \mathrm{Hom}(N, M) \otimes E,$$

then $\mathrm{End}(M \otimes E)$ acquires a natural E-module structure.

For the opposite construction, suppose given some $M \in \mathcal{M}_E^2$. Then $\mathrm{End}\, M$ has an E-module structure, so $\mathrm{End}\, M_E$ has a structure of $E \otimes E$-module. Let e be the idempotent of $E \otimes E$ corresponding to the natural projection to E; then our functor maps M to (M_E, e).

4.1 Cohomology for Motives with Coefficients

The generalization of our three cohomology theories (étale, Betti, de Rham with Hodge structure) to motives with coefficients requires some explanation. We will use the characterization of motives with coefficients as motives equipped with an E-module structure.

1. The rational Betti cohomology $H_B(M)$. This comes with a structure of E-vector space. The involution F_∞ (defined when the base field is $\mathbb{Q}$, or for every real place of k) is E-linear, so we may regard its eigenspaces $H_B^{\pm}(X)$ as E-vector spaces.

   Next consider the complex Betti cohomology $H_B(M) \otimes_{\mathbb{Q}} \mathbb{C}$. This inherits a structure of free module over $E \otimes_{\mathbb{Q}} \mathbb{C}$, which is isomorphic to

   $$\oplus_\sigma \mathbb{C},$$

   where the sum is taken over all embeddings $\sigma : E \to \mathbb{C}$.

   Hence we get a decomposition of $H_B(M) \otimes_{\mathbb{Q}} \mathbb{C}$ into pieces

   $$H_B(\sigma, M) = (H_B(M) \otimes_{\mathbb{Q}} \mathbb{C}) \otimes_{E \otimes_{\mathbb{Q}} \mathbb{C}, \sigma} \mathbb{C} = H_B(M) \otimes_{E,\sigma} \mathbb{C}.$$

2. The same holds for the complexified de Rham cohomology $H_{DR}(M)$. We get

   $$H_{DR}(M) \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_\sigma H_{DR}(\sigma, M),$$

   where

   $$H_{DR}(\sigma, M) = H_{DR}(M) \otimes_{E,\sigma} \mathbb{C}.$$

All these de Rham cohomologies come equipped with the Hodge filtration; since the Hodge filtration can be characterized algebraically before the passage to $\mathbb{C}$, we know that the dimensions of the filtered parts are independent of σ. After passage to $\mathbb{C}$ the filtrations split and become Hodge decompositions, with the dimension of $H^{pq}(\sigma, M)$ still independent of σ. Similarly one checks that the multiplicities of the eigenvalues ± 1 of F_∞ on $H^{aa}(\sigma, M)$ in case the dimension $2a$ of cohomology is even and τ is a real place of k (or $k = \mathbb{Q}$) are independent of σ. Hence the definition of criticality carries over and is independent of σ. Henceforth we assume that M is critical.

We define $H^{\pm}_{DR}(M)$ as a sum of components of the Hodge decomposition, as above: one is the sum of H^{pq} over $p < 0$, the other of $p \geq 0$; chosen in such a way that $H^+_{DR}(M)$ and $H^+_B(M)$ have the same dimension as vector spaces over E.

Then we get a map

$$I^+ : H^+_{DR}(M) \otimes_{\mathbb{Q}} \mathbb{C} \to H^+_B(M)$$

whose determinant c^+ is one side of the Deligne conjecture.

It is worth noting that c^+ is an element of $E \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_\sigma \mathbb{C}$. In concrete terms, it is a tuple of period determinants, one arising from each embedding of E in $\mathbb{C}$.

3. The ℓ-adic cohomology $H_\ell(M)$. This, again, inherits a structure of E-module, hence also a structure of E_ℓ-module, where E_ℓ is the completion of E at ℓ. We may decompose it as a product of $H_\lambda(M)$ as λ ranges over the places of E above ℓ. The minimal polynomials of Frobenius (defined as usual) have coefficients a priori in E_λ; they turn out again to lie in E and be independent of ℓ. This defines Euler factors L_λ at finite places; their coefficients lie in E.

   For every embedding σ of E in $\mathbb{C}$ we obtain a different choice of L-function; hence we should more canonically consider our L-function to take values in $E \otimes \mathbb{C}$.

   The factor at infinity L_∞ is defined as usual. It follows from our remarks on the Hodge structure that the gamma factor is independent of the choice of embedding σ.

   In any event, we have now defined an L-function $L * (s, M)$, holomorphic on some right half-plane, with values in $E \otimes \mathbb{C}$. As usual, it is conjectured that this function continues meromorphically to the entire complex plane and satisfies a functional equation.

4.2 Deligne's Conjecture: General Statement

We are now ready to state Deligne's conjecture in full generality.

Conjecture 4.1. *Let M be a motive over a number field k, with coefficients in a number field E, and suppose M is critical. Let $c^+ \in E \otimes \mathbb{C}$ be the determinant of the period map, which we recall is defined up to multiplication by E^*. Then we have*

$$L(M, 0) = ec^+,$$

for some $e \in E$.

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