Bloch-Kato Conjecture: Baby Version

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1 The Global Bloch-Kato Selmer Group

In this talk K be a *global* field and $G_K := \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group over K. We let V be a p-adic representation of G_K (finite-dimensional over $\mathbf{Q}_p$), and Σ a finite set of places of K *containing* p *and* ∞, outside which V is unramified. So we can and will view V as a representation of $G_{K,\Sigma}$ - for technical reasons, it is at times important to work with that rather than the full G_K!

We are going to define a *global Bloch-Kato Selmer group*

$$H^1_f(G_K, V) \subset H^1(G_K, V)$$

which we think of as being, in the same spirit as the classical Selmer group, cut out by local conditions. More precisely, for each place v of K we have a restriction map $H^1(G_K, V) \to H^1(G_{K_v}, V)$ and we have defined a local Bloch-Kato Selmer group $H^1_f(G_{K_v}, V) \subset H^1(G_{K_v}, V)$.

Definition 1.1. We define the *Bloch-Kato Selmer group* $H^1_f(G_K, V)$ to be the subspace of elements of $H^1(G_K, V)$ that land in $H^1_f(G_{K_v}, V) \subset H^1(G_{K_v}, V)$ under the local resriction maps:

$$\begin{tikzcd} H^1_f(G_K, V) \arrow[r, hook] \arrow[d] & H^1(G_K, V) \arrow[d] \\ \prod_v H^1_f(G_{K_v}, V) \arrow[r, hook] & \prod_v H^1(G_{K_v}, V) \end{tikzcd}$$

1.1 Recap of local groups

Let's remind you what the local Bloch-Kato Selmer groups were. We defined:

- For $v \nmid p$,
$$H^1_f(G_{K_v}, V) := \ker\left(H^1(G_{K_v}, V) \to H^1(I_{K_v}, V)\right)$$
where $I_{K_v} \subset G_{K_v}$ is the inertia subgroup.

- For $v \mid p$,

$$H^1_f(G_{K_v}, V) := \ker\big(H^1(G_{K_v}, V) \to H^1(G_{K_v}, V \otimes B_{\mathrm{crys}})\big)$$

An important property of these definitions was that they were "self-dual" under $V \mapsto V^*(1)$:

Proposition 1.2. *Under the perfect pairing between $H^1(G_{K_v}, V)$ and $H^1(G_{K_v}, V^*(1))$, the subspaces $H^1_f(G_{K_v}, V)$ and $H^1_f(G_{K_v}, V^*(1))$ are orthogonal.*

Recall our favorite examples from the previous talk.

Example 1.3. If A/K is an abelian variety and $V = V_p(A)$, then under the Kummer map

$$A(K_v) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \to H^1(G_{K_v}, V)$$

we have

$$A(K_v) \xrightarrow{\sim} H^1_f(G_{K_v}, V).$$

In particular, this implies that the *global* Bloch-Kato Selmer group for V as described above coincides with the classical p-adic Selmer group for A, so in particular

$$\dim_{\mathbf{Q}_p} H^1_f(G_K, V) = \operatorname{rank}_{\mathbf{Z}} A(K)$$

if $Ш_A$ is finite.

Example 1.4. If $V = \mathbf{Q}_p(1)$, then under the Kummer map

$$\widehat{K_v}^{\times} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \xrightarrow{\sim} H^1(G_{K_v}, V)$$

we have

$$\widehat{\mathcal{O}_{K_v}^{\times}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \xrightarrow{\sim} H^1_f(G_{K_v}, V).$$

This basically implies that the global Kummer map induces an isomorphism

$$\widehat{\mathcal{O}_K^{\times}} \otimes_{\mathbf{Z}} \mathbf{Q}_p \xrightarrow{\sim} H^1_f(G_K, \mathbf{Q}_p).$$

I say "basically" because there are some annoying technical issues in dealing with the full absolute Galois group G_K; in this case the fix is easy (see Proposition 2.12 of Bellaiche's notes for a discussion). This is interesting - while the structure of local units is relatively elementary to see, the structure of global units is subtle (Dirichlet's unit theorem).

We computed that

$$\dim_{\mathbf{Q}_p} H^1_f(G_{K_v}, V) = \begin{cases} 0 & v \nmid p \\ [K_v : \mathbf{Q}_p] & v \mid p \end{cases}$$

using the local dimension formulas - the slogans were "dimension of invariants" if $v \nmid p$ and "dimension of invariants plus $[K_v : \mathbf{Q}_p]$ times the number of positive Hodge-Tate weights" if $v \mid p$.

1.2 Global Galois cohomology

The Bloch-Kato conjecture predicts that for a representation V coming from geometry, we should have

$$\dim_{\mathbf{Q}_p} H^1_f(G_K, V) - \dim_{\mathbf{Q}_p} H^0(G_K, V) = \mathrm{ord}_{s=0} L(V, s).$$

In particular, it would be useful to have a dimension formula for $H^1_f(G_K, V)$. Last time we described some robust dimension formulas for the local Bloch-Kato Selmer groups, which were built from the theorems on local Galois cohomology (the trio of cohomological dimension 2, duality, and Euler-Poincaré characteristic formula). What about global analogues of these results?

There are analogues for Galois cohomology of global field. The Galois cohomology on p-adic vector spaces should vanish again in degree greater than 2. Also, one has an analogue of the Euler characteristic formula:

Proposition 1.5 (Global Euler characteristic formula)**.** *We have*

$$\begin{aligned}\dim H^0(G_{K,\Sigma}, V) - \dim H^1(G_{K,\Sigma}, V) + &\dim H^2(G_{K,\Sigma}, V) \\ &= \sum_{v\mid\infty} H^0(G_v, V) - [K : \mathbf{Q}] \dim V.\end{aligned}$$

(We won't use this result; it is simply stated for context.) In general, it is very difficult to determine $H^2(G_{K,\Sigma}, V)$ (apparently computing $\dim H^2(G_{K,\Sigma}, \mathbf{Q}_p)$ is an open problem for most K) so this formula can only be used to give a lower bound for $H^1(G_{K,\Sigma}, V)$.

The duality results however, are a fair bit messier than in the local case. They are packaged under various results with the heading "Poitou-Tate", but the entirety of this package is confusing. We will state just state the results that we need.

Proposition 1.6. *Let $i = 0, 1,$ or 2. In the duality between $\prod_{v\in\Sigma} H^1(G_v, V)$ and $\prod_{v\in\Sigma} H^1(G_v, V^*(1))$ the images of $H^1(G_{K,\Sigma}, V)$ and $H^1(G_{K,\Sigma}, V^*(1))$ are orthogonal to each other.*

The second definition requires a bit of a digression on the formalism of "Selmer structures", which abstract the setup we've been working in.

Definition 1.7. A *Selmer structure* on V is a collection of subspaces $\mathcal{L} = (L_v \subset H^1(G_K, V))$ for each v such that $L_v = H^1_{\mathrm{ur}}(G_K, V)$ for almost all v. The *Selmer group* attached to V is the subspace of $H^1(G_K, V)$ consisting of elements landing in L_v under each restriction map $H^1(G_K, V) \to H^1(G_{K_v}, V)$:

$$\begin{tikzcd} H^1_{\mathcal{L}}(G_K, V) \arrow[r, hook] \arrow[d] & H^1(G_K, V) \arrow[d] \\ \prod_v L_v \arrow[r, hook] & \prod_v H^1_{\mathcal{L}}(G_{K_v}, V) \end{tikzcd}$$

The Bloch-Kato Selmer group is obtained by taking the Selmer structure ($L_v = H^1_f(G_{K_v}, V)$. (At infinite places $H^1(G_{K_v}, V) = 0$, so this is only possible choice). However, one can arrive at variants of the conjecture by different choices:

- For $v \nmid p$, the only Selmer structures we'll ever use are 0, $H^1_{\mathrm{ur}}(G_{K_v}, V)$ or $H^1(G_{K_v}, V)$.
- For $v \mid p$, there are other Selmer structures obtained by using different period rings in place of B_{crys}. We'll touch on this later.

Given a Selmer structure $\mathcal{L} = (L_v)$ for V, we obtain a *dual Selmer structure* $\mathcal{L}^\perp = (L_v^\perp)$ for $V^*(1)$.

Proposition 1.8. *Let $\mathcal{L}$ be a Selmer structure on V and $\mathcal{L}^\perp$ the dual Selmer structure on $V^*(1)$. Then we have*

$$\begin{aligned} h^1_{\mathcal{L}}(G_K, V) - h^0(G_K, V) = h^1_{\mathcal{L}^\perp}(G_K, V^*(1)) - h^0(G_K, V^*(1)) \\ + \sum_v (\dim L_v - h^0(G_{K_v}, V)) \end{aligned}$$

1.3 The global Bloch-Kato Selmer group

From Proposition 1.8 we can easily derive a dimension "formula" for the global Bloch-Kato Selmer group. We just need to collect together the facts:

- The Bloch-Kato Selmer structure is "self-dual" in the sense that $H^1_f(G_{K_v}, V)^\perp = H^1_f(G_{K_v}, V^*(1))$,
- ($\ell \neq p$ dimension formula) for $v \nmid p$, we have

  $$h^1_f(G_{K_v}, V) = h^0(G_{K_v}, V)$$

- ($\ell = p$ dimension formula) for $v \mid p$, we have

  $$h^1_f(G_{K_v}, V) - h^0(G_{K_v}, V) = [K_v : \mathbf{Q}_p]\#\{\text{positive Hodge-Tate weights for } V \text{ at } v\}.$$

  From now on let's denote $d_v^+ := [K_v : \mathbf{Q}_p]\#\{\text{positive Hodge-Tate weights for } V \text{ at } v\}$.

- For $v \mid \infty$, we have $h^1_f(G_K, V) = 0$. The term $h^0(G_{K_v}, V)$ is only interesting if v is a real place, in which case the Galois group $G_{K_v} \cong \mathbf{Z}/2$ is generated by complex conjugation, which splits V into $+1$ and -1 eigenspaces: $V = V^+ \oplus V^-$, and $h^0(G_{K_v}, V)$ is the dimension of V^+.

Combining these ingredients with Proposition 1.8, we find

Proposition 1.9. *Let $\mathcal{L}$ be a Selmer structure on V and $\mathcal{L}^\perp$ the dual Selmer structure on $V^*(1)$. Then we have*

$$\begin{aligned} h^1_f(G_K, V) - h^0(G_K, V) = h^1_f(G_K, V^*(1)) - h^0(G_K, V^*(1)) \\ + \sum_{v|p} d_v^+(V) - \sum_{v|\infty} h^0(G_{K_v}, V) \end{aligned}$$

Example 1.10. We saw in Example 1.4 that for $V = \mathbf{Q}_p(1)$, we have

$$h^1_f(G_K, \mathbf{Q}_p(1)) = \operatorname{rank}_{\mathbf{Z}} \mathcal{O}_K^\times.$$

We know that the latter should be $r_1 + r_2 - 1$ by Dirichlet's unit theorem, but let's *derive* this using our dimensional formula. Proposition 1.9 applied to $\mathbf{Q}_p$ tells us that

$$h^1_f(G_K, \mathbf{Q}_p) - 1 = h^1_f(G_K, \mathbf{Q}_p(1)) + \sum_{v|p} d_v^+(\mathbf{Q}_p) - \sum_{v|\infty} h^0(G_{K_v}, \mathbf{Q}_p)$$

To digest each term, note:

- Complex conjugation acts trivially on $\mathbf{Q}_p$, so the contribution from the infinite places is
$$-\sum_{v|\infty} h^0(G_{K_v}, \mathbf{Q}_p) = -(r_1 + r_2).$$
- At all the finite place $v \mid p$ the Hodge-Tate weight of $\mathbf{Q}_p$ is 0, which is not positive, so the contribution is
$$\sum_{v|p} d_v^+(\mathbf{Q}_p) = 0.$$

Therefore, we end up with

$$h^1_f(G_K, \mathbf{Q}_p) - 1 = h^1_f(G_K, \mathbf{Q}_p(1)) - (r_1 + r_2).$$

So it suffices to show that $h^1_f(G_K, \mathbf{Q}_p) = 0$. Well, let's try to identify the subspace

$$H^1_f(G_K, \mathbf{Q}_p) \subset H^1(G_K, \mathbf{Q}_p) = \operatorname{Hom}_{\text{cts}}(G_K, \mathbf{Q}_p).$$

By global class field theory,

$$\operatorname{Hom}_{\text{cts}}(G_K, \mathbf{Q}_p) = \operatorname{Hom}_{\text{cts}}(\widehat{K^\times \backslash \mathbf{A}_K^\times}, \mathbf{Q}_p)$$

The restriction map $H^1(G_K, \mathbf{Q}_p) \to H^1(G_{K_v}, \mathbf{Q}_p)$ simply corresponds to the restriction map

$$\operatorname{Hom}_{\text{cts}}(G_K, \mathbf{Q}_p) \to \operatorname{Hom}_{\text{cts}}(G_{K_v}, \mathbf{Q}_p)$$

and by local-global compatibility of class field theory is the same as the restriction map

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{cts}}(G_K, \mathbf{Q}_p) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\mathrm{cts}}(G_{K_v}, \mathbf{Q}_p) \arrow[d, equal] \\
\mathrm{Hom}_{\mathrm{cts}}(\widehat{K^\times \backslash \mathbf{A}_K^\times}, \mathbf{Q}_p) \arrow[r] & \mathrm{Hom}_{\mathrm{cts}}(\widehat{K^\times}, \mathbf{Q}_p)
\end{tikzcd}$$

Now, recall that the local Bloch-Kato Selmer group $H^1_f(G_{K_v}, \mathbf{Q}_p)$ was one-dimensional in all cases, the corresponding homomorphism $K_v^\times \to \mathbf{Q}_p$ being generated by the valuation. The upshot is that the restriction must kill $\mathcal{O}_V^\times$, so

$$H^1_f(G_K, \mathbf{Q}_p) = \mathrm{Hom}(K^\times \backslash \mathbf{A}_K^\times / \prod_v \mathcal{O}_v^\times, \mathbf{Q}_p).$$

But $K^\times \backslash \mathbf{A}_K^\times / \prod_v \mathcal{O}_v^\times$ is precisely the class group of K, and is therefore finite! So this is indeed 0.

2 The Baby Bloch-Kato Conjecture

We now restrict our attention to p-adic representations V *coming from geometry*, i.e. appearing as a subquotient of $H^n(X_{\text{ét}}, \mathbf{Q}_p)$ for some smooth proper variety X/K.

Conjecture 2.1 (Baby Bloch-Kato Conjecture)**.** *For such a V, we have*

$$h^1_f(G_K, V) - h^0(G_K, V) = \mathrm{ord}_{s=0} L(V, s).$$

Let's briefly recall the definition of the L-function $L(V, s)$. It is assembled via an Euler product.

- For a finite place v of K not dividing p,

  $$L_v(V, s) = \det(\mathrm{Id} - \mathrm{Frob}_v\, q_v^{-s}, V^{I_v})^{-1}$$

  where Frob_v is the *geometric* Frobenius.

- For a finite place v dividing p,

  $$L_v(V, s) = \det(\mathrm{Id} - \phi^{f_v} q_v^{-s}, D_{\mathrm{crys}}(V|_{G_v}))^{-1}$$

  where $q = p^{f_v}$ and ϕ is the "Frobenius" which exists on B_{crys} (and hence on D_{crys}).

- For finite places, the L-factor is a product of Gamma functions times normalization factors.

The L-function $L(V,s)$ is a product of the Euler factors for finite v; the completed L-function $\Lambda(V,s)$ is the product of $L(V,s)$ with the infinite Euler factors. The L-function conjecturally admits a functional equation, the shape of which is perhaps most memorable in the form

$$\Lambda(V^*(1),-s) = \epsilon(V,s)\cdot\Lambda(V,s).$$

Example 2.2. For an abelian variety A and $V = V_p(A)$, we know that $V^*(1) = V_p(\widehat{A})$. Since $h^0(G_K,V) = 0$, the conjecture predicts (conditional on the finiteness of $Ш_A$)

$$\operatorname{rank}_{\mathbf{Z}} \widehat{A}(K) = h^1_f(G_K, V_p(\widehat{A})) = \operatorname{ord}_{s=0} L(V,s) = \operatorname{ord}_{s=0} L(V_p(\widehat{A}),s).$$

Since $V_p(\widehat{A}) = V^*(1)$, we have

$$\operatorname{ord}_{s=0} L(V_p(\widehat{A}),s) = \operatorname{ord}_{s=1} L(V^*,s) = L(A,s).$$

Example 2.3. For $V = \mathbf{Q}_p$, the conjecture predicts

$$h^1_f(G_K,\mathbf{Q}_p(1)) = \operatorname{ord}_{s=0} L(V,s) \sim \operatorname{ord}_{s=0}\zeta_K(s)$$

where ζ_K is the Dedekind zeta function of K. Indeed, the left hand side is r_1+r_2-1 as computed earier, and the right hand side is also r_1+r_2-1 by the classical analytic theory.

Obviously, this is going to be extremely difficult to prove. However, one might ask about basic stability properties, such as if the Bloch-Kato conjecture is consistent with the usual natural operations on the two sides. For instance:

Proposition 2.4. *The Bloch-Kato conjecture is stable under induction.*

The proof is a triviality, using that both sides of the conjecture are stable under induction.

Proposition 2.5. *Subject to a conjecture about the symmetry of Hodge-Tate weights, the Bloch-Kato conjecture is consistent with the functional equation.*

The proof just involves some careful accounting, using the dimensional formula Proposition 1.9 on the algebraic side and the functional equation (and description of the infinite factors) on the analytic side. This accounting reduces to a certain equality, which is the "conjectural" symmetry described above.

3 Variants of the Bloch-Kato conjecture

We're going to formulate variants of the Bloch-Kato conjecture for variants of the Bloch-Kato Selmer group. The relationship between these variants is analogous to the relationship between the S-units $\mathcal{O}_{K,S}^\times$ and $\mathcal{O}_K^\times$.

3.1 More local groups

Let's resurrect part of the table from last time.

Property	**Period Ring**	**Structure**	**ℓ-adic analogue**
de Rham	B_{dR}	Filtration	pot. unipotent on inertia
Crystalline	B_{crys}	Frobenius ϕ	unramified

Definition 3.1. Let V be a p-adic representation. We define

$$H^1_g(G_{K_v}, V) = \ker\left(H^1(G_{K_v} V) \to H^1(G_{K_v}, V \otimes B_{\mathrm{dR}})\right).$$

Recall that B_{crys} had a Frobenius operator ϕ. We define

$$H^1_e(G_{K_v}, V) = \ker\left(H^1(G_{K_v}, V) \to H^1(G_{K_v}, V \otimes B_{\mathrm{crys}}^{\phi=1})\right).$$

Since obviously $B_{\mathrm{crys}}^{\phi=1} \subset B_{\mathrm{crys}} \subset B_{\mathrm{dR}}$, we have

$$H^1_e(G_{K_v}, V) \subset H^1_f(G_{K_v}, V) \subset H^1_g(G_{K_v}, V).$$

Recall that we thought of $H^1_f(G_{K_v} V)$ as related to "representations coming from geometry with good reduction" since B_{crys} is the period ring for this class. Just as crystalline is the p-adic analogue of the ℓ-adic notion of unramified, we think of H^1_f as the p-adic analogue of the ℓ-adic H^1_{ur}.

Where do H^1_g and H^1_e fit into this picture? Well, H^1_g is associated to B_{dR}, which is related to *any* "representations coming from geometry", which by a theorem of Berger is "potentially semi-stable". In the ℓ-adic setting, *any* representation coming from geometry is automatically potentially semi-stable by Grothendieck's monodromy theorem. So the analogue of H^1_g in the ℓ-adic setting is the full H^1. The group H^1_e turns out to be analogous to 0.

Definition 3.2. If $v \nmid p$, then we *define* $H^1_g(G_{K_v}, V) := H^1(G_{K_v}, V)$ and $H^1_e(G_{K_v}, V) = 0$.

Proposition 3.3 (Second fundamental exact sequence)**.** *There is an exact sequence*

$$0 \to \mathbf{Q}_p \to B_{\mathrm{crys}}^{\phi=1} \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \to 0.$$

Corollary 3.4 (Bloch-Kato exponential)**.** *There is a natural surjective map*

$$D_{\mathrm{dR}}(V)/D_{\mathrm{dR}}^+(V) \to H^1_e(G_{K_v}, V)$$

with kernel $D_{\mathrm{crys}}(V)^{\phi=1}/V^{G_{K_v}}$.

In the case where $V = V_p(A)$ for an abelian variety A, this can be identified with the classical exponential map from an open subgroup of $\mathrm{Lie}(A)$ to A.

Corollary 3.5 (Dimension formula for H^1_e)**.** *If V is de Rham then we have*

$$\dim H^1_e(G_K, V) = \dim D_{\mathrm{dR}}(V)/D^+_{\mathrm{dR}}(V) + \dim H^0(G_K, V) - \dim D_{\mathrm{crys}}(V)^{\phi=1}.$$

Proposition 3.6. *If V is de Rham, then for the (perfect) pairing between $H^1(G_K, V)$ and $H^1(G_K, V^*(1))$ the orthogonal of $H^1_e(G_K, V)$ is $H^1_g(G_K, V^*(1))$.*

The proof is not too difficult, but involves more playing around with the period rings than seems reasonable to do here, given that we did not even attempt to define them.

Corollary 3.7 (Dimension formula for H^1_g)**.** *If V is de Rham, then we have*

$$\dim H^1_g(G_K, V) = \dim D_{\mathrm{dR}}(V)/D^+_{\mathrm{dR}}(V) + \dim H^0(G_K, V) + \dim D_{\mathrm{crys}}(V^*(1))^{\phi=1}.$$

Proof. Combine Proposition 3.6 with the dimension formula for H^1_e and the local Euler characteristic formula for H^1. □

Example 3.8. Let's compute all the dimensions for the Tate twists $\mathbf{Q}_p(n)$. The only interesting question is: what is $\dim D_{\mathrm{crys}}(\mathbf{Q}_p(n))^{\phi=1}$? Since we have said nothing about the definition of B_{crys} or ϕ we can't answer this question rigorously, so let's settle for a heuristic understanding. The operator ϕ is supposed to correspond to the Frobenius operator on de Rham cohomology. The de Rham cohomology "associated to" $\mathbf{Q}_p(1)$ should be $H^1_{\mathrm{dR}}(\mathbf{G}_m)$. So how does Frobenius act on $\mathbf{G}_m$? Basically by raising to the pth power, so ϕ acts on $D_{\mathrm{crys}}(\mathbf{Q}_p(1))$ by multiplication by p, and hence on $D_{\mathrm{crys}}(\mathbf{Q}_p(n))$ by multiplication by p^n. Therefore, $\dim D_{\mathrm{crys}}(\mathbf{Q}_p(n)^*(1))^{\phi=1}$ is only non-zero when $n = 1$.

The table below summarizes the dimensions of $\mathbf{Q}_p(n)$ for the different groups.

n	$H^1(\mathbf{Q}_p(n))$	$H^1_f(\mathbf{Q}_p(n))$	$H^1_e(\mathbf{Q}_p(n))$	$H^1_g(\mathbf{Q}_p(n))$
$n < 0$	$[K : \mathbf{Q}_p]$	0	0	0
$n = 0$	$[K : \mathbf{Q}_p] + 1$	1	0	1
$n = 1$	$[K : \mathbf{Q}_p] + 1$	$[K : \mathbf{Q}_p]$	$[K : \mathbf{Q}_p]$	$[K : \mathbf{Q}_p] + 1$
$n > 1$	$[K : \mathbf{Q}_p]$	$[K : \mathbf{Q}_p]$	$[K : \mathbf{Q}_p]$	$[K : \mathbf{Q}_p]$

3.2 The S-Selmer group

Definition 3.9. Let S be a finite set of finite places of K. We define $H^1_{f,S}(G_K, V)$ to be the Selmer group associated with the Selmer structure (L_v) where

- If $v \notin S$, then $L_v = H^1_f(G_{K_v}, V)$.
- If $v \in S$ does not divide p, then $L_v = H^1(G_{K_v}, V)$.
- If $v \in S$ divides p, then $L_v = H^1_f(G_{K_v}, V)$.

Example 3.10. It is simple to extend the above arguments to show that

$$h^1_{f,S}(G_K, \mathbf{Q}_p(1)) = \operatorname{rank}_{\mathbf{Z}} \mathcal{O}^\times_{K,S}.$$

We just have to show that for all v, the local versions coincide

$$H^1_{f,S}(G_{K_v}, \mathbf{Q}_p(1)) = \mathcal{O}^\times_{K_v,S}.$$

This is only new for $v \in S$, in which case the right side becomes $K_v^\times$. This in turn is only new when $v \mid p$. But in this case the dimension formula shows that $H^1_g(G_{K_v}, \mathbf{Q}_p(1)) = H^1(G_{K_v}, \mathbf{Q}_p(1))$, so again reduces to the classical theory of the Kummer isomorphism.

Definition 3.11. We define $H^1_g(G_K, V) = \varinjlim_S H^1_{f,S}(G_K, V)$. In other words, it is the subspace of $H^1(G_K, V)$ consisting of elements whose image under every local restriction map lands in $H^1_g(G_{K_v}, V)$, and in $H^1_f(G_{K_v}, V)$ for all but finitely many v.

3.3 The S-Bloch-Kato conjecture

As you may haved guessed, the S-version of the Bloch-Kato conjecture is

$$h^1_{f,S}(G_K, V^*(1)) - h^0(G_K, V^*(1)) = \operatorname{ord}_{s=0} L_S(V, s).$$

4 Exploring the Bloch-Kato Conjecture

In these sections, we will discuss "second-order" aspects of the Bloch-Kato conjecture which connect to other themes in number theory and algebraic geometry.

4.1 A motivic interpretation

Grothendieck conjectured the existence of a category $\mathcal{M}_K$ of *pure motives*, which is the "universal cohomology theory" for smooth property varieties in the sense that it should possess realization functors factoring the various cohomology theories attached to smooth proper varieties over K.

$$\begin{tikzcd} & & \{G_K\text{-rep.}/\mathbf{Q}_p\} \\ \{\text{smooth proper } X/K\} \arrow[r] \arrow[urr, "H^*_{\text{ét}}(-)"] \arrow[drr, "H^*_{\mathrm{dR}}(-)"'] & \mathcal{M}_K \arrow[ur, dotted, "\mathrm{Real}_{\text{ét}}"] \arrow[dr, dotted, "\mathrm{Real}_{\mathrm{dR}}"] & \\ & & \{\text{filtered vector spaces}/K\} \end{tikzcd}$$

He further conjectured that the category $\mathcal{M}$ should appear as a full subcategory in a category $\mathcal{MM}$ of *mixed motives*, representing the universal cohomology theory for *all* varieties over K, which possesses realization functors extending those above

$$\begin{tikzcd}
 & & \{G_K\text{-rep.}/\mathbf{Q}_p\} \\
\{\text{varieties } X/K\} \arrow[r] \arrow[urr, "H^*_{\mathrm{dR}}(-)"] \arrow[drr, "H^*_{\text{ét}}(-)"'] & \mathcal{MM}_K \arrow[ur, dotted, "\mathrm{Real}_{\text{ét}}"'] \arrow[dr, dotted, "\mathrm{Real}_{\mathrm{dR}}"] & \\
 & & \{\text{filtered vector spaces}/K\}
\end{tikzcd}$$

Now, a cohomology class $\xi \in H^1_g(G_K, V)$ represents an extension

$$1 \to V \to W \to \mathbf{Q}_p \to 0$$

such that W is de Rham at all places $v \mid p$ and unramified at all but finitely many $v \nmid p$. We have discussed that representations coming from geometry satisfy exactly the preceding properties. The Fontaine-Mazur conjecture predicts that for *irreducible* representations, these local conditions are the *only* obstructions to a p-adic representation coming from a global variety:

Conjecture 4.1 (Fontaine-Mazur)**.** *If V is de Rham at all places dividing p and unramified at all but finitely many places, then it is a subquotient of $H^n(X, \mathbf{Q}_p)$ for some smooth proper variety X/K.*

Now, the extension under consideration is clearly not irreducible, but it does satisfy the necessary local conditions. Therefore, in spirit of the Fontaine-Mazur conjecture we expect that the extension class W should come from some (not necesssarily smooth proper) variety, and hence be the realization of some *mixed motive*.

Conjecture 4.2. *The étale realization functor induces an isomorphism*

$$\mathrm{Ext}^1_{\mathcal{MM}_K}(\mathbf{Q}_p, V) \cong H^1_g(G_K, V)$$

where $\mathbf{Q}_p$ is the object whose realizaton is $\mathbf{Q}_p$ and V is the object whose realization if V.

This conjecture tells us that we can interpret $H^1_g(G_K, V)$ as the group of extensions in the category of mixed motives, and the Bloch-Kato conjecture predicts a description in terms of L-functions. (Last time we discussed how this category of mixed motives is expected to have cohomological dimension one, which was an underlying intuition for the orthogonality of the groups $H^1_f(G_{K_v}, V)$ and $H^1_f(G_{K_v}, V^*(1))$ under the cup product.)

4.2 Relation to Grothendieck's yoga of weights

With this motivic interpretation in hand, we can test the predictions coming from Bloch-Kato with conjectured properties of mixed motives.

Definition 4.3. We say that V has *weight* w if for all places v outside a finite set, the characteristic polynomial of Frob_v on V is algebraic, with eigenvalues having absolute value $q_v^{w/2}$ under all complex embeddings.

Example 4.4. If $V = H^n(X, \mathbf{Q}_p) = H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ then V has weight n, thanks to Deligne's proof of the Weil conjectures.

In particular, $\mathbf{Q}_p(i) = H^2(\mathbf{P}^1, \mathbf{Q}_p)^\vee$ has weight $-2i$.

For instance, Grothendieck's "yoga of weights" emphasizes that motivic weights should go *up* for extensions of pure motives in the category $\mathcal{MM}_K$.

Can we see this prediction using the Bloch-Kato conjecture? In the spirit of viewing it as a formula for the dimension of the space of extensions in terms of L-functions, let's change gears and see what information we can extract from the L-function.

Simple analytic estimates show that if V has weight w, then the Euler product for $L(V, s)$ should converge for $\operatorname{Rep} s > 1 + w/2$. Sketch of proof: one writes

$$L_v(V, s) = \prod (1 - \alpha_{i,v} q_v^{-s})$$

where $|\alpha_{i,v}| = q_v^{-w/2}$. This converges if and only if

$$\sum_v \sum_i \alpha_{i,v} q_v^{-s}$$

converges, and we can majorize the latter by

$$\sum_v \sum_i q^{-w/2} q_v^{-\operatorname{Rep} s}.$$

The major contribution comes from those v of degree 1 (so $q_v = p$), where one uses the usual facts about convergence of the sum of reciprocals of primes.

Therefore, if V is pure of weight w then the poles of $L(V, s)$ lie on the line $\operatorname{Rep} s = w/2$, so the Euler product visibly has no zeros for $\operatorname{Rep} s > 1 + w/2$. In particular, suppose that V is pure of weight $w \geq 0$, and also (for simplicity irreducible). Then $V^*(1)$ is pure of weight $-w-2$, so $L(V, s)$ is non-vanishing for $\operatorname{Rep} s > -w/2 \leq 0$. It is simple to use the dimension formula for h^1_g to show that in this case, $h^1_f(G_K, V) = h^1_g(G_K, V)$ (there is a slight wrinkle if $V = \mathbf{Q}_p$ is the trivial representation, in which $L(V, s)$ has order of vanishing -1. However, the h^0 term compensates for this). Therefore, the Bloch-Kato conjecture predicts:

Prediction 4.5. *If V is pure of non-negative weight, then*

$$h^1_g(G_K, V) = 0.$$

By the other Bloch-Kato conjecture, this predicts that $\operatorname{Ext}^1_{\mathcal{MM}_K}(\mathbf{Q}_p, V) = 0$, harmonizing with Grothendieck's yoga of weights.

4.3 Relation to Fontaine-Mazur

If V is pure of any weight w, then $\mathrm{End}(V) = V^* \otimes V$ is pure of weight $w - w = 0$, so a special case of Prediction 4.5 is:

Prediction 4.6. *If V is pure, then*

$$h^1_g(G_K, \mathrm{End}(V)) = 0.$$

To put this context, we give an extremely brief introduction to the deformation theory of Galois representations. For a Galois representation $V/\mathbf{F}_q$, Mazur introduced the idea of lifting V to a representation over a bigger ring. More precisely, he considered the functor Def_V on the category of local Artin $W(\mathbf{F}_q)$-algebras defined by

$$\mathrm{Def}_V(A) = \{\text{deformations of } \rho \text{ to } A \cong$$

In concrete terms, this functor parametrizes lifts of a representation $\rho\colon G_K \to \mathrm{GL}_n(\mathbf{F}_q)$ to $\rho_A\colon G_K \to \mathrm{GL}_n(A)$:

$$\begin{tikzcd} & \mathrm{GL}_n(A) \arrow[d] \\ G_K \arrow[r] \arrow[ur, dotted] & \mathrm{GL}_n(\mathbf{F}_q) \end{tikzcd}$$

This functor turns out to be pro-representable by a "universal deformation ring", which we could think of as cutting out the moduli space of such deformations. In particular, the tangent space of this deformation functor is

$$\mathrm{Def}_V(\mathbf{F}_q[\epsilon]/\epsilon^2) = \{\text{deformations of } \rho \text{ to } \mathbf{F}_q[\epsilon]/\epsilon^2\}/\cong$$

and it is a standard lemma in the theory that

$$\mathrm{Def}_V(\mathbf{F}_q[\epsilon]/\epsilon^2) = H^1(G_K, \mathrm{End}(V)).$$

To sketch how this goes, a deformation of ρ to $\mathbf{F}_q[\epsilon]/\epsilon^2$ is a G_K-module M over $\mathbf{F}_q[\epsilon]/\epsilon^2$ such that $M/\epsilon \cong V$. Then we have a short exact sequence (of Galois modules over $\mathbf{F}_q$)

$$0 \to \epsilon M \to M \to V \to 0.$$

Then ϵM is naturally a Galois module over $\mathbf{F}_q$, which is isomorphic to V. (In terms of matrices, the Galois action on M is given by matrices with entries of the form $a + b\epsilon$, and the action of V is obtained by setting $\epsilon = 0$. But ϵ also kills ϵM, so the Galois action on M is the same.) So in fact $\mathrm{Def}_V(\mathbf{F}_q[\epsilon]/\epsilon^2)$ is simply the group of extensions

$$0 \to V \xrightarrow{\alpha} M \xrightarrow{\beta} V \to 0.$$

(To recover the action of ϵ on M from an extension, simply apply $\alpha\beta$.)

Taking a leap of faith, we can believe that there should be an analogous deformation functor for V a p-adic representation of G_K. Then we should similarly have $\mathrm{Def}_V(\mathbf{F}_q[\epsilon]/\epsilon^2) = H^1(G_K, \mathrm{End}(V))$. What does the subspace $H^1_g(G_K, \mathrm{End}(V))$ describe? An extension in $H^1_g(G_K, \mathrm{End}(V))$ splits after extending coefficients to B_{dR}, which implies that it is de Rham. (More generally, if $W \cong V \oplus V'$ is a sum of two de Rham representations, then $D_{\mathrm{dR}}(W) \cong D_{\mathrm{dR}}(V) \oplus D_{\mathrm{dR}}(V')$ so W is admissible if V and V' are.) Conversely, if W is de Rham then we can choose a splitting on de Rham cohomology and tensor up to B_{dR} to obtain a Galois-equivariant splitting.

The upshot is that $H^1_g(G_K, \mathrm{End}(V))$ describes (heuristically) the tangent subspace of deformations that remain de Rham at $v \mid p$ and unramified at almost all $v \nmid p$. But by the Fontaine-Mazur conjecture, all such representations come from global varieties X/K, of which there are only countably many, and in particular not enough to form one-parameter families. Therefore, Predicton 4.6 can be viewed as an infinitesmal version of the Fontaine-Mazur conjecture.

4.4 Symmetry of Hodge-Tate weights

Definition 4.7. For V a p-adic representation of G_K, we let $m_k(V|_{G_v}$ be the multiplicity of the Hodge-Tate weight k for $V|_{G_v}$. We define the *total multiplicity* of k as a Hodge-Tate weight to be

$$m_k(V) = \sum_{v|p} [K_V : \mathbf{Q}_p] m_k(V|_{G_v}).$$

Conjecture 4.8. *Let V be a p-adic representation coming from geometry. Let w be the motivic weight of V. Then we have for all k,*

$$m_k = m_{w-k}.$$

This should be thought of analogous to the classical symmetry of the Hodge diamond

$$h^p(X, \Omega^q) = h^q(X, \Omega^p).$$

For the full $H^n(X, \mathbf{Q}_p)$, we know that

$$H^n(X, \mathbf{Q}_p) \otimes \mathbf{C}_p = \bigoplus_{i+j=n} H^i(X, \Omega^j_{\mathbf{C}_p})(-j)$$

has weight n, so in this case Prediction 4.8 specializes to

$$h^i(X, \Omega^j) = h^j(X, \Omega^i).$$

(Although we no longer have complex conjugation to thank for this symmetry, it should still follow from Serre duality and the Lefschetz Hyperplane Theorem for étale cohomology.)

Proposition 4.9. *Prediction 4.5 implies Conjecture 4.8.*

Proof. We give the proof in several steps.

1. It is almost immediate that the motivic weight doesn't change under induction or restriction. Therefore, we can induce from G_K to $G_{\mathbf{Q}}$ to reduce the conjecture to the case $K = \mathbf{Q}$. This is simpler, because there is only one place over p to worry about.

2. However, we next want to *restrict* to the absolute Galois group of a quadratic imaginary field over which p splits into vv'. By the compatibility with restriction, it suffices to treat this case. Let $k_1 \leq \ldots k_d$ be the Hodge-Tate weights at v and $k_1' \geq k_2' \geq \ldots \geq k_d'$ be the Hodge-Tate weights at v'. We'll show that $k_i + k_i' = w$.

3. It suffices to prove that $k_1 + k_1' = w$. Indeed, applying this to each exterior power $\bigwedge^i V$, we get the system
$$\begin{aligned} k_1 + k_1' &= w \\ (k_1 + k_2) + (k_1' + k_2') &= 2w \\ &\vdots \\ (k_1 + \ldots + k_d) + (k_1' + \ldots k_d') &= dw. \end{aligned}$$

4. There is an admissible character χ of G_K with Hodge-Tate weight a at v and b at v', by global class field theory. It suffices to define it locally on each $K_w^\times$, and check that it is trivial on $K^\times \subset \mathbf{A}_K^\times$. We choose it to be unramified at $w \neq v, v'$, sending a uniformizer ϖ_w to q_w^{-1}, and equal to the ath and bth powers of the cyclotomic characters at v and v'. Then by design the image of a global $(x, x, \ldots) \in K^\times$ is $\operatorname{Nm}(x)^{-1} \operatorname{Nm}(x) = 1$.

   Twisting $V \mapsto V \otimes \chi$ takes $w \mapsto w + a + b$ and $k_i \mapsto k_i + a$, $k_i' \mapsto k_i' + b$, so the conjecture for V is equivalent to that for $V \otimes \chi$. In particular, it suffices to study the case $w = 0$.

5. Now under the assumption that $w = 0$ and $\dim V = d$, we claim that
$$\begin{aligned} \sum_{k<0} m_k(V) &\leq d \\ \sum_{k>0} m_k(V) &\leq d \\ \sum_{k\leq 0} m_k(V) &\geq d \\ \sum_{k\geq 0} m_k(V) &\geq d. \end{aligned}$$

The latter two inequalities obviously follow from the first two. In turn, the first is equivalent to the second by applying $V \mapsto V^*(1)$. To see the first one, recall the dimension formula

$$\begin{aligned} h^1_f(G_K, V) - h^0(G_K, V) = h^1_f(G_K, V^*(1)) - h^0(G_K, V^*(1)) \\ + \sum_{v|p} d^+_v(V) - \sum_{v|\infty} h^0(G_{K_v}, V) \end{aligned}$$

This $\sum_{v|p} d^+_v(V)$ is exactly what we are calling $\sum_{k>0} m_k(V)$. Since V has weight 0, Prediction 4.5 implies that $h^1_f(G_K, V) - h^0(G_K, V) = 0$. Then

$$\sum_{k>0} m_k(V) = h^0(G_K, V^*(1)) - \ldots \leq \dim V^*(1) = d.$$

6. Suppos $k_1 + k'_1 = a$. If $a = 0$ then we are done, otherwise $a < 0$ or $a > 0$. By dualizing if necessary, we may assume that $a > 0$. We claim that $a = 1$. Indeed, we can twist by a character of motivic weight 0 so that $k'_1 = 1$. Then $k_1 = a - 1 \geq 0$. By the previous part, $k_1 = 0$ or else $k_1 + \ldots + k_d + k'_1 > d$.

7. We see that $a = -1, 0$, or 1 for all V. But applying this to $V \otimes V$ and using the same reasoning, we conclude that $a \neq \pm 1$.

□

4.5 Partial Results

The Bloch-Kato conjecture amounts to two inequalities

$$\begin{aligned} h^1_f(G_K, V^*(1)) \geq \operatorname{ord}_{s=0} L(V, s) + h^0(G_K, V^*(1)) \\ h^1_f(G_K, V^*(1)) \leq \operatorname{ord}_{s=0} L(V, s) + h^0(G_K, V^*(1)) \end{aligned}$$

If V is pure of motivic weight $w \neq 2$, then $h^0(G_K, V^*(1)) = 0$ so the lower bound is trivial as long as $w \neq 2$ and the L-function is non-vanishing.

There is progress on Bloch-Kato in a limited number of cases.

- The conjecture is known for $V = \mathbf{Q}_p(n)$ for all number fields K. The lower bound is trivial because $\mathbf{Q}_p(n)$ has motivic weight $w = -2n \neq 1$. We know the analytic side of Bloch-Kato from the theory of the Dedekind zeta function, so the upper bound follows the knowledge of Galois cohomology furnished by Borel's computation of K-theory for number fields together with a theorem of Soulé relating K-theory and Galois cohomology.

- For f a modular eigenform of level $\Gamma_1(N)$ of even weight $k = 2k'$, there is an attached Galois representation $V_p(f)$ of motivic weight $2k' - 1$. The upper bound for $V_p(f)$ was shown by Kato using Euler systems. The lower bound is always true for motivic weight $w = 2k' - 1 - 2n \neq 0$, i.e. $n \neq k'$. Alas, for an elliptic curve modularity tells us that $V_p(E) = V_p(f)(1)$ for a weight 2 eigenform f, so this misses the most interesting case.

- The lower bound can sometimes be attacked by automorphic methods, all descending from ideas employed by Ribet in the proof of the converse to Herbrand's theorem. For example, work of Bellaïche-Chenevier/Skinner-Urban constructs a non-zero extension in $H^1_f(G_K, V_p(f)(k'))$ when $\mathrm{ord}_{s=0} L(V, s) \geq 1$.