Beilinson's conjectures I

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1 Deligne's conjecture

As we saw, Deligne made a conjecture for varieties (actually at the level of motives) for the special values of L-function. If $X/\mathbf{Q}$ is a smooth projective variety, we can form the L-function $L(H^i X, s)$, and Deligne's conjecture concerns $L(H^i X, q)$ for certain "critical" a.

There is a functional equation relating

$$L(s) \leftrightarrow L(i+1-s).$$

Set $p = i+1$. By the functional equation, we can assume that $q \geq p/2$. For most of the talk we'll be assuming that actually $q \gg p/2$. The reason is that we usually want to avoid $q = p/2$, which is the "central point" (the value here encompasses $L(E,1)$ for an elliptic curve E). The point $q = p/2 + 1/2$ is the "right of center" point (and its values encompass $\zeta(1)$).

Deligne's conjecture says that in the "critical case"

$$L(H^i X, q) \sim_{\mathbf{Q}} \text{a period determinant.}$$

This period determinant is for a map

$$(2\pi i)^q H^i_B(X, \mathbf{Z})^{F_\infty = (-1)^q} \to H^i_{\mathrm{dR}}(X)/F^q H^i_{\mathrm{dR}}(X). \tag{1.1}$$

Here

- H^i_B is the singular cohomology of the complex points,
- F_∞ is the the automorphism on $X(\mathbf{C})$ induced by conjugation.

This is map beteen $\mathbf{C}$-vector spaces with $\mathbf{Q}$-structures, so it makes sense to take its determinant.

*Notes taken by Tony Feng

Example 1.1. We'll work through the statement of the conjecture for an elliptic curve. The map (1.1) is

$$(2\pi i)H^1(E(\mathbf{C}))^- \to H^1_{\mathrm{dR}}(E)/H^0(E,\Omega^1).$$

We need to choose bases for the $\mathbf{Q}$-structures on either side. On the right side, we have $H^0(E,\Omega^1) = \mathbf{Q}\cdot\omega$. Extend this to a basis $\langle \omega, \eta\rangle$. For the left side, we actually choose a *dual* basis $\{\gamma^+, \gamma^-\}$ splitting the *homology* as

$$H_1(E(\mathbf{C}), \mathbf{Q}) \cong \mathbf{Q}\gamma^+ \oplus \mathbf{Q}\gamma^-$$

where γ^+ and γ^- are eigenvectors for F_∞.

The period matrix is

$$\Omega = \begin{pmatrix} \int_{\gamma^+}\omega & \int_{\gamma^-}\omega \\ \int_{\gamma^+}\eta & \int_{\gamma^-}\eta \end{pmatrix}.$$

Let $\{\gamma_*^+, \gamma_*^-\}$ be the dual bases in $H^1(E,\mathbf{Q})$ to $\{\gamma^+,\gamma^-\}$. The period will be obtained by writing $\gamma_*^- = a\omega + b\eta$, and then taking the η component. In terms of the matrix Ω, we have

$$(a,b)\Omega = (0,1).$$

Then

$$(a,b) = (0,1)\Omega^{-1} = (0,1)\begin{pmatrix} \int_{\gamma^-}\eta & -\int_{\gamma^-}\omega \\ -\int_{\gamma^+}\eta & \int_{\gamma^+}\omega \end{pmatrix}\frac{1}{\det\Omega}$$

and in particular

$$b = \frac{1}{2\pi i}\int_{\gamma^+}\omega.$$

What is the meaning of criticality? It basically guarantees that (1.1) is an isomorphism, which is necessary to make sense of the determinant.

Under our assumptions (that q is away from the central point), (1.1) is always injective. We're going to construct a "missing piece" of the picture:

- A *motivic cohomology group* $H^p_{\mathcal{M}}(X, \mathbf{Q}(q))$.
- A *Deligne cohomology group* $H^p_{\mathcal{D}}(X,\mathbf{Q}(q))$ which is basically the cokernel of (1.1).
- A regulator map

  $$H^p_{\mathcal{M}}(X,\mathbf{Q}(q)) \otimes \mathbf{C} \xrightarrow{\sim} H^p_{\mathcal{D}}(X,\mathbf{Q}(q)) \otimes \mathbf{C}$$

  which is an isomorphism.

The shape of the refined conjecture will be that the determinant of the map (1.1) augmented by Deligne cohomology is the special value.

2 Motivic cohomology

2.1 Overview of properties

We will define motivic cohomology later, but first we explain some of the important properties that we're looking for.

- For every p, q we have $X \rightsquigarrow H^p_{\mathcal{M}}(X, \mathbf{Q}(q))$. This is a contravariant functor from smooth varieties $X/\mathbf{Q}$ to (conjecturally finite-dimensional) $\mathbf{Q}$-vector spaces.
- There is a comparison map

$$H^p_{\mathcal{M}}(X, \mathbf{Q}(q)) \to H^p_{\text{ét}}(X, \mathbf{Q}_\ell(q)).$$

- There is a map to Deligne cohomology:

$$H^p_{\mathcal{M}}(X, \mathbf{Q}(q)) \to H^p_{\mathcal{D}}(X, \mathbf{Q}(q))$$

  which vanishes when $p > 2q$ and conjecturally for $q \leq 0$.
- We have $H^{2q}_{\mathcal{M}}(X, \mathbf{Q}(q)) = CH^q(X) \otimes \mathbf{Q}$.

Remark 2.1. The finite-dimensionality is only over $\mathbf{Q}$; over $\mathbf{C}$, Chow groups can be huge!

Remark 2.2. A particula motivic cohomology group can be identified with a (rational) Chow group. Under this identification, the map to étale cohomology is the usual cycle class map.

Here is a "table of values" for motivic cohomology.

	H^0	H^1	H^2	H^3	H^4
$\mathbf{Q}(0)$	$\mathbf{Q}$	0	0	0	0
$\mathbf{Q}(1)$	0	$H^0(\mathcal{O}_X^*)$	$\text{Pic}(X) \otimes \mathbf{Q}$	0	0
$\mathbf{Q}(2)$	?	?	?		$CH^2(X) \otimes \mathbf{Q}$

What are the mysterious entries, intuitively? At one extreme motivic cohomology is the Chow group. The motivic cohomology group immediately to the left describes "relation in the Chow group". The group to the left of that describes "higher order relatons", i.e. "relations among relations".

	H^0	H^1	H^2	H^3	H^4
$\mathbf{Q}(0)$	$\mathbf{Q}$	0	0	0	0
$\mathbf{Q}(1)$	0	$H^0(\mathcal{O}_X^*)$	$\text{Pic}(X) \otimes \mathbf{Q}$	0	0
$\mathbf{Q}(2)$	...	...	higher relations	"relations for CH^2"	$CH^2(X) \otimes \mathbf{Q}$

Example 2.3. For an elliptic curve E, Beilinson's conjecture predicts $L(H^1 E, 1)$ in terms of $H^2_{\mathcal{M}}(E, \mathbf{Q}(1))$. The latter is the Chow group of E, which has to do with rational points on E.

On the étale version, there is a spectral sequence

$$H^i(G_{\mathbf{Q}}, H^j_{\text{ét}}(E_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell(1))) \implies H^2_{\text{ét}}(E, \mathbf{Q}_\ell(1)).$$

Usually we think of the L-function as being defined in terms of $H^1(G_{\mathbf{Q}}, H^j_{\text{ét}}(E_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell(1)))$, but from this perspective it is just one "part" of $H^2_{\text{ét}}(E, \mathbf{Q}_\ell(1))$.

2.2 The definition

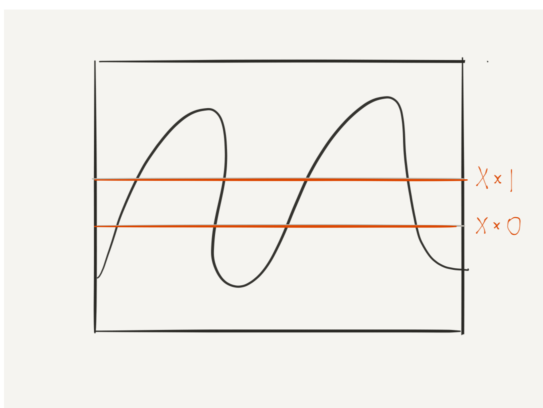
Let's first talk about the Chow group: $CH^p(X)$ is the free abelian group on codimension p cycles of X, modulo the relations generated by $\mathrm{Div}(f)$ for f a meromorphic function on Y of codimension $p-1$.

$$CH^p(X) = \frac{\mathbf{Z}\langle \text{codim. } p \text{ cycles} \subset X\rangle}{\langle \mathrm{Div}(f) \mid f \text{ on } Y \text{ of codim } p-1 \subset X\rangle}.$$

We want to write down a complex in which $CH^p(X)$ is the final cohomology term. Bloch defined a notion of "higher Chow group". One starts with $C^p(X)$, the free abelian group of cycles of codimenson p in X. Then one has

$$C^p(X \times \mathbf{A}^1) \rightrightarrows C^p(X).$$

The two maps are $Z \in C^p(X \times \mathbf{A}^1) \mapsto Z \cap X \times \{0\}$ and $Z \cap X \times \{1\}$.

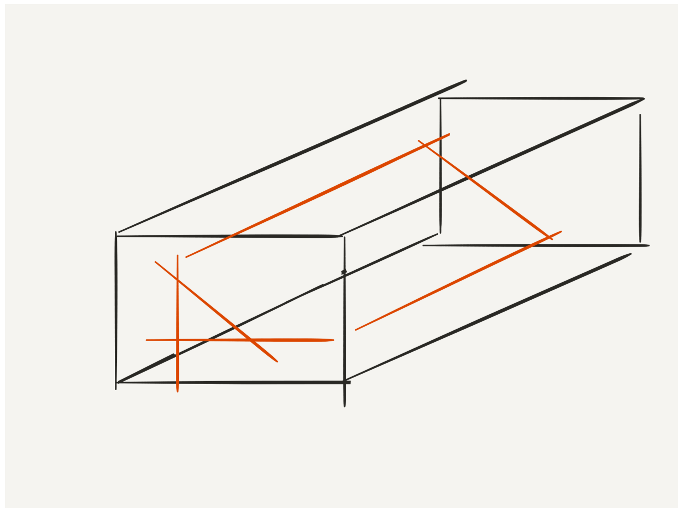


Here by $C^p(X \times \mathbf{A}^1)$ we really mean the subgroup of cycles intersecting the slices $X \times \{0\}$ and $X \times \{1\}$ transversally. That is part of what makes this definition impossible to compute with.

Next we have cycles on $X \times \mathbf{A}^2$. There are three different maps corresponding to the three different intersections.

$$C^p(X \times \mathbf{A}^2) \substack{\rightarrow\\\rightarrow\\\rightarrow} C^p(X \times \mathbf{A}^1) \rightrightarrows C^p(X).$$

(Again, we are only working with cycles intersect everything transversely.)



Definition 2.4. We define the *higher Chow groups* $CH^p(X, \bullet)$ to be the cohomology of the complex

$$\dots C^p(X \times \mathbf{A}^2) \rightrightarrows C^p(X \times \mathbf{A}^1) \rightrightarrows C^p(X).$$

The motivic cohomology group are just re-indexing of the higher Chow groups.

Definition 2.5. We define the *motivic cohomology groups* by

$$H^{2p-q}_{\mathcal{M}}(X, \mathbf{Z}(p)) = CH^p(X, q).$$

Voevodsky gave another definition of motivic cohomology which is easier to work with, but not so intuitive.

2.3 Examples

We'll look at the following examples:

- $L(H^1E, 1)$, which corresponds to $H^2_{\mathcal{M}}(E, \mathbf{Q}(1))$.
- For S a surface $L(H^2S, 2)$, which corresponds to $HM^3(S, \mathbf{Q}(2))$.
- For C a curve of genus g $L(H^1C, 2)$, which corresponds to $H^2_{\mathcal{M}}(C, \mathbf{Q}(2))$.
- $\zeta(3)$, which corresponds to $H^1_{\mathcal{M}}(\text{Spec }\mathbf{Q}, \mathbf{Q}(3))$.

To summarize:

L-function	i	p	q	Motivic Cohomology	Special value
$L(H^1E, 1)$	1	2	1	$H^2_{\mathcal{M}}(E, \mathbf{Q}(1))$	center
$L(H^2S, 2)$	2	3	2	$H^3_{\mathcal{M}}(S, \mathbf{Q}(2))$	right of center
$L(H^1C, 2)$	1	2	2	$H^2_{\mathcal{M}}(C, \mathbf{Q}(2))$	far right
$\zeta(3)$	0	1	3	$H^1_{\mathcal{M}}(\text{Spec }\mathbf{Q}, \mathbf{Q}(3))$	far right

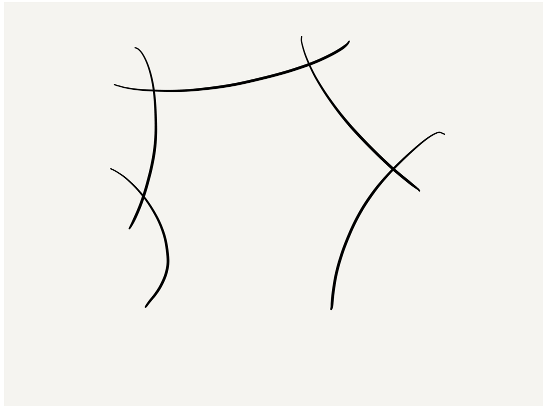


Figure 1: A concrete element of $H^3_{\mathcal{M}}(S, \mathbf{Q}(2))$ can be represented by a chain of curves, with functions on the curve such that the zeros and poles cancel out.

Example 2.6. Let's consider the motivic cohomology group $H^3_{\mathcal{M}}(S, \mathbf{Q}(2))$. This has a presentation with generators $\sum(D_i, f_i)$ where D_i is a divisor and f_i is a (meromorphic) function on D_i such that $\sum \mathrm{Div}(f_i) = 0$, modulo some relations that we'll describe later.

$$H^3_{\mathcal{M}}(S, \mathbf{Q}(2)) = \frac{\{\sum(D_i, f_i) \colon \sum \mathrm{Div}(f_i) = 0\}}{???} \tag{2.1}$$

We are trying to describe $H^3_{\mathcal{M}}(S, \mathbf{Q}(2)) = CH^2(S, 1)$. A pair (D, f) defines a codimension 2 cycle on $S \times \mathbf{A}^1$, namely the graph of f.

What are the relations? They should come from codimension two cycles on $S \times \mathbf{A}^2$. One obvious way to produce such is to take the graph of a function $X \to \mathbf{A}^2$. If you unwind the meaning of such relations, you see that in (2.1) you need to quotient by the tame symbol $\{F, G\}_{\text{tame}}$ when $F, G \in \mathcal{M}(S)^*$ (non-zero meromorphic function on S). Here,

$$\{F, G\}_{\text{tame}} = \sum_{D \text{ pole or zero of } F \text{ or } G} \left(D, \frac{F^{v_D(G)}}{G^{v_D(F)}} (-1)^{v_D(F) v_D(G)} \right).$$

It's easy to exhibit elements of motivic cohomology (see Figure 2.6), but it's hard to show that an element is non-zero. Basically the only way to do so is to compute its regulator (and show it's non-zero).

Étale realization. We want to construct a map

$$H^3_{\mathcal{M}}(S, \mathbf{Q}(2)) \to H^3_{\text{ét}}(S, \mathbf{Q}_\ell(2)).$$

Away from the zeros and poles, f_i gives a class in $H^1(D_i - |\operatorname{Div}(f)|, \mathbf{Q}_\ell(1))$. We can take that class and push it into the surface. The condition on cancellation of zeros and poles comes in trying to extend f_i to the union of the D_i. (Actually, it extends not to D_i but to cohomology with coefficients in some shriek pullback, which you can then push forward.)

If $Y \subset S$ were a smooth curve, you would have a long exact sequence

$$H^1_{\text{ét}}(Y, \mathbf{Q}_\ell(1)) \to H^3_{\text{ét}}(S, \mathbf{Q}_\ell(2)) \to H^3_{\text{ét}}(S - Y, \mathbf{Q}_\ell(2)) \xrightarrow{+1} .$$

So morally the étale realizationis the pushforward of the class of $f_i \in H^1(D_i, \mathbf{Q}_\ell(1))$.

Example 2.7. Now let's talk about the motivic cohomology group $H^2_{\mathcal{M}}(C, \mathbf{Q}(2))$. Here the presentation is the kernel of all the tame symbols on $\mathbf{Q}(C)^* \otimes \mathbf{Q}(C)^*$, quotiented out by $\langle f \otimes (1-f) \colon f \in \mathbf{Q}(C)^* \rangle$:

$$H^2_{\mathcal{M}}(C, \mathbf{Q}(2)) = \frac{\ker(\{F, G\}_p \text{ for all } p) \subset \mathbf{Q}(C)^* \otimes \mathbf{Q}(C)^*}{\langle f \otimes (1-f) \colon f \in \mathbf{Q}(C)^* \rangle}.$$

Here the tame symbol is defined by

$$\{F, G\}_p = \left(\frac{F^{v_p(G)}}{G^{v_p(F)}} (-1)^{v_p(F) v_p(G)} \right) (p).$$

Étale realization. We want to make a map

$$H^2_{\mathcal{M}}(C, \mathbf{Q}(2)) \to H^2_{\text{ét}}(C, \mathbf{Q}(2)).$$

Let $U = C - \mathrm{Div}(F, G)$. Since F and G are non-vanishing on U, they give classes $[F]$ and $[G] \in H^1_{\text{ét}}(U, \mathbf{Q}_\ell(1))$. After all, $H^1_{\text{ét}}(U, \mathbf{Q}_\ell(1))$ classifies étale covers of U with Galois group μ_{ℓ^n}, and you get such a cover of U by taking the ℓ^n roots of F or G. Then the cup product

$$[F] \smile [G] \in H^2_{\text{ét}}(U, \mathbf{Q}_\ell(2)).$$

We want to extend this over C. We haven't yet used the fact that this is in the kernel of the tame symbol maps, but that fact implies that you can lift this class (by a certain long exact sequence relating $H^2_{\text{ét}}(U)$ and $H^2_{\text{ét}}(C)$.

3 Some Remarks

3.1 K-theory

For X smooth and quasiprojective, we have an isomorphism

$$\bigoplus CH^p(X) \otimes \mathbf{Q} \xrightarrow{\sim} K_0(X) \otimes \mathbf{Q}.$$

In fact there are natural maps in both directions. The map $\leftarrow$ is the Chern character and the map $\to$ is $Z \mapsto [\mathcal{O}_Z]$ (on a smooth X, any coherent sheaf admits resolution by vector bundles, hence has a class in K_0). These are essentially inverse maps.

The K-theory side is much nicer. For instance, when defining the product structure there are no transversality issues to worry about (as in intersection theory).

In general, we have an isomorphism

$$\bigoplus CH^p(X, q) \otimes \mathbf{Q} \xrightarrow{\sim} K_q(X).$$

This was Beilinson's original definition of motivic cohomology, using K_q as defined by Quillen.

3.2 Motives

We want our conjecture to work replacing X by a motive. What difficulties are there in stating our conjecture for motives?

One is that we haven't defined motivic cohomology. If you try to define the Chow group, you run into the problem that projectors are up to homological equivalence, and hence aren't well-defined projectors on Chow (since that is defined up to the stronger notion of rational equivalence).

If we have two homologically equivalent motives, then we have $CH^* \supset CH^*_{\text{hom-triv}}$. If f, g have the same cohomology class, then they (thought of as correspondences) induce the same map on $CH^*/CH^*_{\text{hom-triv}}$.

We need a conjecture to make this well-defined. The essence of the conjecture is that there exists a filtration (of rings) on which homologically trivial correspondences are well-defined on the associated graded.

Conjecture 3.1 (Beilinson-Bloch)**.** *We can extend $CH^*_{hom\text{-}triv}$ to a filtration*

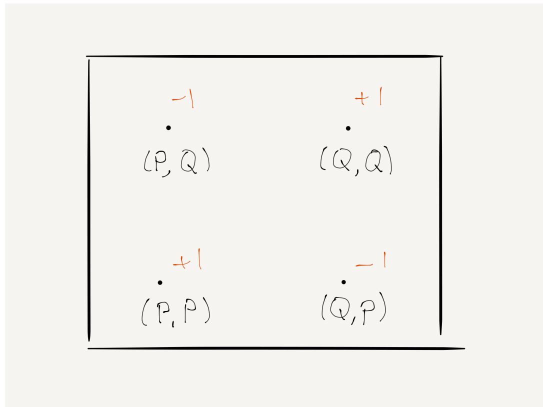
$$\underbrace{F^0CH^*}_{=CH^*} \supset \underbrace{F^1CH^*}_{CH^*_{hom\text{-}triv}} \supset \dots$$

such that $F^pF^q \subset F^{p+q}$, and which is stable by f_*, f^*.

Also, over $\mathbf{Q}$ we have $F^2 = 0$.

This implies that one can make sense of $\mathrm{gr}^\bullet CH^*$ for a motive and homologically trivial correspondence are well-defined on the associated graded.

Example 3.2. For C a curve over $\mathbf{Q}$ of genus 2 and $P, Q \in C(Q)$, the cycle $-(P, Q) + (Q, Q) - (Q, P) + (P, P)$ with sign $-1, 1, -1, 1$ should be trivial in $CH^2(C \times C)$, because it's $(C \times \{P\} - C \times \{Q\}) \cdot (\{P\} \times C - \{Q\} \times C)$:



I don't think this has ever been checked.

Example 3.3. Let X be an abelian variety. Consider $D \in CH^1(X) = \mathrm{Pic}(X)$. Take $x \in X(\mathbf{Q})$. Suppose D is ample or something, so $D \in F^0(CH^1(X))$. We claim that $t_x^* D - D \in F^1 CH^1(X)$. The reason is because we can look at $[t_x^* D \times \{1\}] - [D \times \{0\}]$ on $X \times X$, which is homologically trivial because it is the difference of two "vertical classes", whose push-forward is $t_x^* D - D$.

The theorem of the square says that $(t_x^* - 1)(t_y^* - 1)D = 0$, so we are seeing that the "operator" $t_x^* - 1$ moves D down through the filtration.

4 Deligne cohomology

4.1 Motivation

Let's start with Chern classes of line bundles. Let X be a smooth projective variety over $\mathbf{C}$ and L a line bundle on X. Then we have a first Chern class $c_1(L) \in H^2(X, \mathbf{Z}) \cap F^1 H^2(X, \mathbf{C})$, and this first step of the Hodge filtration maps to $H^1(X, \Omega^1)$. So given a line bundle we should be able to make a class in $H^1(X, \Omega^1)$, and L will be classified by this map $H^1(X, \mathcal{O}_X^*) \to H^1(\Omega^1)$ which sends locally sends $f \mapsto \frac{df}{f}$.

So we've constructed a "topological Chern class" and also a "de Rham version". The idea of Deligne cohomology is to make a class that *simultaneously* gives "topological" and "de Rham" versions.

The Deligne cohomology group will be denoted $H^p_{\mathcal{D}}(X, \mathbf{Z}(q))$. This is morally "singular cohomology classes valued in $(2\pi i)^q \mathbf{Z}$ which also lie in the qth step of the Hodge filtration". For example, a line bundle L will give a class in $H^2_{\mathcal{D}}(X, \mathbf{Z}(1))$.

4.2 Definition

We define $H^p_{\mathcal{D}}(X, \mathbf{Z}(q))$ to be the pth hypercohomology group, on X, of the complex

$$(2\pi i)^q \mathbf{Z} \to \mathcal{O} \to \Omega^1 \to \ldots \to \Omega^{q-1}. \tag{4.1}$$

If

$$\begin{array}{cccccccccc} \mathcal{F} = & 0 & \longrightarrow & (2\pi i)^q\mathbf{Z} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \ldots \\ & & & \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{G} = & 0 & \longrightarrow & \mathcal{O} & \longrightarrow & \Omega^1 & \longrightarrow & \Omega^2 & \longrightarrow & \ldots \end{array}$$

there's a long exact sequence

$$\ldots \to H^p_{\mathcal{D}}(X, \mathbf{Z}(q)) \to H^p(X, \mathcal{F}) \to H^p(X, \mathcal{G}) \to \ldots$$

and

$$\begin{aligned} H^p(X, \mathcal{F}) &= H^p(X, (2\pi i)^q \mathbf{Z}) \\ H^p(X, \mathcal{G}) &= H^p(X, \mathbf{C})/F^q H^p. \end{aligned}$$

So to first approximation you should think of Deligne cohomology as the kernel of this map, but actually there are some other things in the long exact sequence.

4.3 Examples

What is $H^1_{\mathcal{D}}(X, \mathbf{Z}(1))$? For $\mathbf{Z}(1)$, the complex (4.1) is quasi-isomorphic to $\mathcal{O}^*$, by the diagram

$$\begin{array}{ccc} 2\pi i\mathbf{Z} & \longrightarrow & \mathcal{O} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}^* \end{array}$$

induced by the exponential short exact sequence

$$0 \to 2\pi i \mathbf{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

so $H^1_{\mathcal{D}}(X, \mathbf{Z}(1)) \cong H^0(X, \mathcal{O}^*)$.

You can also think of this as cohomology classes in $H^1(X, \mathbf{Z}(1))$ that become trivial in $H^1(X, \mathcal{O})$. Imagine $X = U \cap V$. From the Cech perspective, an element of $H^1(X, \mathbf{Z}(1))$ is a map $f\colon U \cap V \to \mathbf{Z}(1)$. To say that this becomes trivial in $\mathcal{O}$ is to say that $f = f_U - f_V$ where $f_V \in \Gamma(U, \mathcal{O}_U)$ and $f_v \in \Gamma(V, \mathcal{O}_V)$. The corresponding invertible function in $H^0(X, \mathcal{O}^*)$ is obtained by gluing $\exp(f_U)$ and $\exp(f_V)$.

This is the prototypical way to write down examples: you write down a class in $H^1(X, \mathbf{Z}(1))$ and exhibit a reason why it's trivial.

Next, we consider $H^2_{\mathcal{D}}(X, \mathbf{Z}(1)) \subset H^1(X, \mathcal{O}^*) = \mathrm{Pic}(X)$. By definition, $H^2_{\mathcal{D}}(X, \mathbf{Z}(2))$ is the cohomology of

$$(2\pi i)^2 \mathbf{Z} \to \mathcal{O} \to \Omega^1$$

which is quasi-isomorphic to the complex $\mathcal{O}^* \to \Omega^1$ sending $f \mapsto df/f$:

$$\begin{array}{ccccc} (2\pi i)^2\mathbf{Z} & \longrightarrow & \mathcal{O} & \longrightarrow & \Omega^1 \\ & & \downarrow & & \downarrow \\ & & \mathcal{O}^* & \longrightarrow & \Omega^1 \\ & & f & \longmapsto & \frac{df}{f} \end{array}$$

Again, writing down a class in $H^2_{\mathcal{D}}(X, \mathbf{Z}(1))$ consists of writing down an element of $H^1(\mathcal{O}^*)$ plus a certificate that it becomes trivial in $H^1(\mathcal{O}^*)$. Here, a class in $H^1(\Omega^1)$ can be thought of as a cocycle $f_{U\cap V} \in \Gamma(U \cap V, \mathcal{O}^*)$. The condition for it to vanish in $H^1(\Omega^1)$ is that there exist ω_U, ω_V such that

$$\omega_U - \omega_V = \frac{df_{U\cap V}}{f_{U\cap V}}.$$

This means that the connections $d+\omega_U$ and $d+\omega_V$ patch together to give a connection on line bundle. So in this case we see that $H^2_{\mathcal{D}}(X, \mathbf{Z}(2))$ classifies line bundles *plus* a holomorphic connection.