Borel's proof of Beilinson's Conjecture for number fields

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As we have discussed, Beilinson conjectureed that an L-value is a regulator on motivic cohomology. The only case in which anything is known is for ζ_F for F an imaginary quadratic field. By this we mean that Borel gave a relation between the values of ζ_F at integers and regulators on $K_*\mathcal{O}_F$. (Some further work is required to identify this with Beilinson's conjecture, i.e. that the Borel regulator really is the Beilinson regulator in this case.)

1 Hopf algebras

1.1 Cohomology of compact Lie groups

Recall that for the unitary group U_n, the cohomology ring is an exterior algebra on odd generators:

$$H^*(U_n) = \mathbf{C}[e_1, e_3, e_5, \ldots, e_{2n-1}].$$

We will need this later so we review why this is true.

Example 1.1. Let $A = \mathbf{Q}[G]$ for a finite group G. This is a Hopf algebra. Let's not fuss about the general definition of a Hopf algebra; suffice it to say that it reflect extra structure, which we can see because we can tensor and dualize representations of A. The ability to tensor is explained by a *coproduct*

$$\Delta \colon A \to A \otimes A$$

sending $g \mapsto g \otimes g$. The ability to dualize is explained by a map $A \to A$ sending $g \mapsto g^{-1}$.

The dual algebra $A^\vee = \text{Functions}(G, \mathbf{Q})$ is also a Hopf algebra because the dual of the coproduct is a product structure, and the dual of a product is a coproduct structure.

Note that A is a non-commutative algebra, but has commutative coproduct. On the other side, the product is commutative and the coproduct is non-commutative.

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If G is a compact Lie group, then $H^*(G, \mathbf{C})$ is a Hopf algebra. The coproduct structure comes from the multiplication map

$$m \colon G \times G \to G.$$

Also $H_*(G, \mathbf{C})$ is a Hopf algebra, with product coming from $m \colon G \times G \to G$ and coproduct induced by

$$\Delta \colon G \to G \times G$$

sending $g \mapsto (g, g)$.

Proposition 1.2. *If G is compact connected then $H_*(G, \mathbf{C})$ is commutative and $H^*(G, \mathbf{C})$ is co-commutative.*

Proof. Fix a Riemannian metric on G which is left and right invariant. By de Rham theory,

$$H^*(G, \mathbf{C}) \cong \{\text{harmonic forms on } G\}.$$

We claim that a harmonic form on a compact group must automatically be invariant; denote the invariant forms by $\mathrm{Inv}(G)$. Why? Translation by G can only act trivially on cohomology, but there is a unique harmonic representative for each de Rham cohomology class. This shows that harmonic forms are contained in $\mathrm{Inv}(G)$.

We next claim that in fact this is an equality: $\{\text{harmonic forms on } G\} = \mathrm{Inv}(G)$. How can we see this? We want to show that if $\omega \in \mathrm{Inv}(G)$ then $d\omega = 0$. Well, an invariant form is determined by its value at the identity, so it suffices to study the Lie algebra. The effect of inversion $i(x) = x^{-1}$ on the group acts by -1 on the Lie algebra. Therefore, it acts by $(-1)^p$ on forms of degree p. Since d is equivariant for it, we have

$$(-1)^{p+1} d\omega = i^* d\omega = d i^* \omega = (-1)^p \omega$$

which shows that $d\omega = 0$. Applying the same argument to $*\omega$ shows that ω is both closed and coclosed, hence harmonic.

Now consider the commutative diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{xy} & G \\ {\scriptstyle i \times i}\downarrow & & \downarrow{\scriptstyle i} \\ G \times G & \xrightarrow{yx} & G \end{array}$$

Since we have identified the cohomology with invariant forms, we know that i acts as $(-1)^{\deg}$ on cohomology, this shows the graded commutativity of the Hopf algebra structure. □

Theorem 1.3 (Milnor, Moore)**.** *Suppose $A = \bigoplus_{i \geq 0} A_i$ is a graded commutative, graded cocommutative Hopf algebra over a field k in characteristic 0 (so $A_0 = k$). Then A is free on the subspace $\mathrm{Prim}(A)$, which is $\{a \in A \colon \Delta a = a \otimes 1 + 1 \otimes a\}$.*

In particular, if A is finite-dimensional then there can be no generators in even degree, so A must be a free exterior algebra on elements in odd degree. This explains why the cohomology of a compact Lie group must be free on odd generators.

Example 1.4. If A were in even degree then Spec A would be a commutative algebraic group G. Then G is a product of $\mathbf{G}_a$'s and $\mathbf{G}_m$'s but we claim that in fact $G \cong \mathbf{G}_a^N$. This is because the graded structure on A implies that there is a $\mathbf{G}_m$-action on G, and there cannot be a $\mathbf{G}_m$-action on the toral part. So $G \cong \mathbf{G}_a^N$. Therefore, Prim(A) can be identified with $\{\chi \colon G \to \mathbf{G}_a\}$.

1.2 Invariant forms

Let's go back to $\mathrm{Inv}(U_n)$. These are completely determined by what happens at the identity, and they have to be conjugacy-invariant there. In other words, they are alternating j-forms $\bigwedge^j \mathfrak{u}_n \to \mathbf{C}$ which are invariant under conjugation by U_n. This is the same as alternating forms $\bigwedge^j M_n(\mathbf{C}) \to \mathbf{C}$ which are invariant by $\mathrm{GL}_n \mathbf{C}$.

Here is one such form: $X \mapsto \mathrm{Tr}(X)$. More generally, for any k we can consider $\mathrm{Tr}(X_1, \ldots, X_k)$ - this is a conjugacy-invariant form, but it is not alternating. To amend that, we anti-symmetrize it; but that is automatically 0 if k is even. So the candidate forms are $v_k(X_1, \ldots, X_k) =$ anti-symmetrization of $\mathrm{Tr}(X_1, \ldots, X_k)$. It turns out that these are generators for $\mathrm{Inv}(U_n)$.

Why are they primitive? Consider for instance $v_3 = \mathrm{Tr}(XYZ) - \mathrm{Tr}(XZY)$. Pull back this form via the multiplication map

$$U_n \times U_n \xrightarrow{x,y \mapsto xy} U_n.$$

Unfortunately it is not clear that the pullback is still an invariant form. One would like to argue this by saying that the diagram

$$\begin{array}{ccc} U_n \times U_n & \xrightarrow{m} & U_n \\ {\scriptstyle t_x \times t_y}\downarrow & & \downarrow{\scriptstyle t_{xy}} \\ U_n \times U_n & \longrightarrow & U_n \end{array}$$

commutes, but actually it doesn't (because of the non-commutativity of U_n).

To fix this, consider embedding into a larger group

$$\begin{array}{ccc} U_n \times U_n & \longrightarrow & U_n \\ {\scriptstyle I \times I}\downarrow & & \downarrow{\scriptstyle I} \\ U_{2n} \times U_{2n} & \longrightarrow & U_{2n} \end{array}$$

with the embedding $I \colon U_n \hookrightarrow U_{2n}$ being

$$g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}.$$

This is conjugate to the embedding

$$J\colon g \mapsto \begin{pmatrix} 1 & \\ & g \end{pmatrix}.$$

So the diagram above is homotopic to

$$\begin{array}{ccc} U_n \times U_n & \longrightarrow & U_n \\ {\scriptstyle I\times J}\downarrow & & \downarrow{\scriptstyle I} \\ U_{2n} \times U_{2n} & \longrightarrow & U_{2n} \end{array}$$

Since the forms v_n on U_n are the restriction of the analogous U_{2n}-invariantforms on U_{2n}, this shows that their pullbacks to $U_n \times U_n$ are in fact invariant.

2 K-theory of number fields

Let R be any commutative ring. Consider

$$\mathrm{GL}_\infty(R) := \varinjlim \mathrm{GL}_N(R).$$

Then $H_*(\mathrm{GL}_\infty R, \mathbf{Q})$ is a graded commutative cocommutative Hopf algebra. The point is that it has a product, via the construction

$$\mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_{2n}.$$

There is a map

$$K_i R \to \mathrm{Prim}(H_i \,\mathrm{GL}_\infty(R), \mathbf{Q}).$$

For $i > 0$, this is an isomorphism after tensoring with $\mathbf{Q}$, at least when $R = \mathcal{O}_F$. So for our purposes, we can think of K-theory as this piece of group homology.

Remark 2.1. The (group) homology of

$$\ldots \mathrm{GL}_N \mathcal{O}_F \hookrightarrow \mathrm{GL}_{N+1} \mathcal{O}_F \hookrightarrow$$

stabilizes. Quillen proved this to show that K-theory stabilizes, hence the K-theory of rings of integers is finitely generated.

Let F be an imaginary quadratic field and $\mathcal{O}$ is the ring of integers. Borel showed that

$$(K_i\mathcal{O})_{\mathbf{Q}} = \begin{cases} \mathbf{Q} & i \text{ odd } \geq 3 \\ 0 & \text{other } i > 0 \end{cases}$$

Also, there is a natural map

$$(K_i\mathcal{O})_{\mathbf{Q}} \to \mathbf{R}$$

called the *Borel regulator*. (So called because K_1 is the unit group, and for $i = 1$ this specializes to the classical regulator.) Borel prove that the image is $\sim \mathbf{Q} \cdot \zeta_F(\frac{i+1}{2})$. These zeta values arise as the volumes of $\mathrm{SL}_n \mathcal{O}_F \backslash \mathrm{SL}_n \mathbf{C}$.

How do we compute $H^*(\mathrm{SL}_N \mathcal{O}, \mathbf{C})$? We have an action of $\mathrm{SL}_N \mathcal{O}$ on $\mathrm{SL}_n \mathbf{C}/\mathrm{SU}(n)$, which is a contractible symmetric space S_n of unitary forms. Then

$$H^*(\mathrm{SL}_n \mathcal{O}, \mathbf{C}) = H^*(S_n/\mathrm{SL}_n \mathcal{O}, \mathbf{C}).$$

This S_n comes with an invariant metric, and for it $X_n := S_n/\mathrm{SL}_n \mathcal{O}$ has finite volume (but it not compact). (This was the reason for switching from GL_n to SL_n.) Consider $\mathrm{Inv}^*(S_n)$, the space of invariant forms on S_n. This sits inside differential forms on X_n.

Borel showed that the inclusion of Inv^j (meaning left $\mathrm{SL}_n \mathbf{C}$-invariant j-forms on $\mathrm{SL}_n \mathbf{C}/\mathrm{SU}_n$) into differential forms on X induces an isomorphism in H^* when $j \leq n/4$. As before, $\mathrm{Inv}^j \subset \mathrm{Harm}^j$. The harmonic forms sit inside differential forms on X_n. We'd like to use Hodge theory to go backwards, but here we can't because our X isn't compact. What Borel does is to consider an intermediate space of forms of "moderate growth". More precisely, replace X_n by some truncation, and impose growth conditions on forms so that they lie L^2.

The inclusion $\mathrm{Inv}^j \subset \mathrm{Harm}^j$ is only an equality in low degree. Since $S_n = \mathrm{SL}_n \mathbf{C}/\mathrm{SU}_n$, an invariant j-form in $\mathrm{Inv}^j(S_n)$ is a function $\bigwedge^j T_e \to \mathbf{C}$ which is invariant by SU_n, where $T_e = \mathfrak{sl}_n\mathbf{C}/\mathfrak{su}_n$. Since $\mathfrak{su}_n$ is a real structure of $\mathfrak{sl}_n\mathbf{C}$, we can identify this with $i\mathfrak{su}_n$. Thus we are looking at $\{\bigwedge^j(i\mathfrak{su}_n) \to \mathbf{C}\}^{\mathrm{SU}_n}$, and since trace is no longer informative this is $\mathbf{C}[\nu_3, \nu_5, \ldots, \nu_{2n-1}]$.

In conclusion, we have found that in low degree $H^*(\mathrm{SL}_n \mathcal{O}, \mathbf{C})$ is free exterior in degrees $3, 5, \ldots$. Therefore $H^*(\mathrm{SL}_\infty \mathcal{O}, \mathbf{C})$ is $\mathbf{C}[e_3, e_5, e_7, \ldots]$. So $\mathrm{Prim}\, H^j(\mathrm{SL}_\infty \mathcal{O}, \mathbf{C}) = \mathbf{C}$ for $j = 3, 5, \ldots$.

We emphasize that the generators here are naturally indexed by the corresponding cohomology classes of the unitary group. 1

The same discussion applies for homology.

3 The Borel Regulator

Now we come to the second part, which is the regulator map

$$(K_i\mathcal{O})_{\mathbf{Q}} \to \mathbf{R}.$$

We have a map

$$\mathrm{Prim}\, H_j(\mathrm{SL}_\infty \mathcal{O}_F, \mathbf{Q}) \xrightarrow{\int \nu_j} \mathbf{C}$$

where ν_j is regarded as a cohomology class via

$$\nu_j \to \mathrm{Inv}^j \to H^j(\mathrm{SL}_n \mathcal{O}, \mathbf{C}).$$

What is the image? We need to construct some explicit classes on the LHS. Where can we get explicit cycles? If G is a group over F embedded in SL_N, then we get a map from the locally symmetric space for G to the locally symmetric space for $\mathrm{SL}_n =: X_N$. We can then push forward the fundamental class for the locally symmetric space attached to G.

Example 3.1. Let D be a division algebra of rank 4 over F. Fixing an order $\mathcal{O}_D$, consider the norm-1 units $\mathcal{O}_D^{(1)}$. We can embed this in $\mathrm{SL}_n \mathcal{O}_F$ by its action on D. This gives a map of symmetric spaces,

$$\pi \colon \mathcal{O}_D^{(1)} \backslash \mathrm{SL}_2 \mathbf{C} / \mathrm{SU}_2 \to X_N.$$

The left hand side is 3-dimensional; let the fundamental class be μ. Then the image of the Borel regulator for $i = 3$ contains

$$\langle \pi_* \mu, \nu_3 \rangle = \int_{\mathcal{O}_D^{(1)} \backslash \mathrm{SL}_2 \mathbf{C} / \mathrm{SU}_2} \pi^* \nu_3.$$

This is basically $\zeta_F(2)$, up to factors of π.

Now suppose D is a divison algebra of rank 9 over F. Then $\dim \mathcal{O}_D^{(1)} \backslash \mathrm{SL}_2 \mathbf{C} / \mathrm{SU}_2 = 8 = 3 + 5$. The image of the Borel regulator for $i = 9$ contains

$$\langle \pi_* \mu, \nu_3 \wedge \nu_5 \rangle = \zeta_F(2) \zeta_F(3).$$

Let e_3, e_5 be generators for $\operatorname{Prim} H_i(\mathrm{SL}_\infty)$. Then $\pi_* \mu = e_3 e_5 + \dots$ with the extra stuff being higher-degree. (This happens in every degree by primitivity). So

$$\langle \pi_* \mu, \nu_3 \wedge \nu_5 \rangle = \langle e_3, \nu_3 \rangle \langle e_5, \nu_5 \rangle.$$

We computed the first term to be $\zeta_F(2)$, and the product was computed to be $\zeta_F(2)\zeta_F(3)$, so we can deduce that $\langle e_5, \nu_5 \rangle \sim \zeta_F(3)$.