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Linking the conjectures of Artin-Tate and Birch-Swinnerton-Dyer

LINKING THE CONJECTURES OF ARTIN–TATE
AND BIRCH–SWINNERTON-DYER

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1. Introduction

In 1965 Birch and Swinnerton-Dyer suggested a conjecture about
the arithmetic of elliptic curves defined over Q. During the next few
years various people gradually extended this conjecture to abelian
varieties of any dimension, and defined over any global field. In final
form (for objects associated to a given abelian variety) it describes
the behaviour near 1 of the L-series in terms of the Mordell–Weil and
Shafarevich–Tate groups, two arithmetic objects.

To some algebraic surfaces equipped with additional structure one
can naturally associate abelian varieties defined over function fields;
Artin noticed that in this case the Brauer group of the surface is
closely related to the Shafarevich–Tate group. This observation led
him and Tate to try to rephrase the original conjecture in terms
related to the surface. As Tate explained [25], they were led to make a
geometric conjecture; of course they expected the equivalence of
their conjecture and the arithmetic one of Birch and Swinnerton-
Dyer. The main result of this paper is Theorem 6.1, which establishes
the equivalence (under mild restrictions on the surface).

In the remainder of this section we introduce our notation and state
the conjectures. Global properties of these surfaces equipped with
additional structure are explored in Section 2 and again, in terms of
the Néron–Severi group, in section 4. Local properties are studied in
Section 3. An explicit calculation of an intersection matrix in Section
5 links the local and global properties. In the final section a study of
the ζ-function of the surface by means of the Weil conjectures is

combined with the explicit form of this matrix to prove the equivalence of the conjectures.

NOTATION. Through this paper unless otherwise indicated we use the following symbols as indicated here:

k = a finite field of characteristic p, with q elements

ℓ = any prime distinct from p;

V = a nonsingular geometrically irreducible one-dimensional k-scheme;

N_v = the number of elements of the finite field $k(v)$;

K = the function field $k(V)$;

X_K = a smooth proper geometrically irreducible curve defined over K;

A = the Jacobian of the curve X_K;

$[G]$ = the order of the finite group G;

$\bar{S} = S \times_k \bar{k}$, the base extension of a k-scheme to the algebraic closure $\bar{k}$ of k;

F = the geometric Frobenius morphism on such a scheme $\bar{S}$, defined as identity on (k-rational) closed points; qth-power on structure sheaves

φ = arithmetic Frobenius, the lifting to $\bar{S}$ of the qth-power map on $\bar{k}$

$H_\ell^i(S) = \varprojlim H^i(\bar{S}, \; \mathbb{Z}/\ell^n \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)$, the ith ℓ-adic cohomology group;

$P_i(S, T) = \det(1 - \mathsf{F}_i T)$, the characteristic polynomial for the action induced by F on $H_\ell^i(S)$;

G_m or G_X = the sheaf multiplicative group (on the scheme X).

Sheaves and cohomology groups are, unless otherwise indicated, taken in the étale sense.

1. *The conjectures.* Both the arithmetic and geometric conjectures are stated carefully in [25]. Here we run through them quickly to establish more notation.

Let A be an abelian variety of dimension d defined over the global field K. At almost all places v of K, A has good reductions to abelian varieties A_v defined over the finite residue fields $k(v)$. At any such place write the characteristic polynomial for the action of the (arithmetic) Frobenius element φ, the generator of $G(\bar{k}/k(v))$, on the Tate module $T_\ell(A_v) = \varprojlim {}_{\ell^n} A_v(k(v))$ as

$$P_v(A, T) = \det(1 - \varphi T).$$

This polynomial, known to have integral coefficients independent of the choice of ℓ, has complex reciprocal roots $\{\alpha_1, \ldots, \alpha_{2d}\}$ of absolute value $(N_v)^{1/2}$ and characterized by the equality

$$[A_v(k_n)] = \prod (1 - \alpha_i^n) \qquad\qquad 1 \le i \le 2d$$

where k_n is the extension field of $k(v)$ of degree n.

Up to multiplication by a nonzero element of K there is a unique nonvanishing top order holomorphic differential form ω on A. For all places v of K – even for places where A has bad reduction – we can define an integral:

$$\int_{A(K_v)} |\omega|_v \mu_v^d$$

where, to explain the symbols, K_v denotes the completion of K at v, $|\ |_v$ is the v-adic valuation on K_v, and μ_v is the Haar measure on K_v giving measure 1 to the ring of v-integers. (Since $K = k(V)$ all places v of K are discrete.)

Finally, when μ_v is normalized as above for all v, $\Pi \mu_v$ defines a Haar measure on the adele ring A_K. We write $|\mu|$ for the volume of the quotient group A_K/K of A_K by the discrete subring K.

Let S be a finite set of places v of K containing all "bad" places v, at which either the abelian variety A has bad reduction, or else the reduced differential form ω_v is not nonzero, nonpolar. Birch and Swinnerton-Dyer have associated to A an L-series:

$$L_S^*(A, s) = |\mu|^d \Big/ \Big(\prod_{v \notin S} P_v(A, N_v^{-s}) \prod_{v \in S} \int_{A(K_v)} |\omega|_v \mu_v^d \Big).$$

The asymptotic behaviour of $L_S^*(A, s)$ as $s \to 1$ is independent of the choice of S, since for v good, both $P_v(A, N_v^{-1})$ and the integral over $A(K_v)$ equal $[A_v(k(v))]/(N_v)^d$.

By the Mordell–Weil Theorem the group $A(K)$ is finitely generated; we always write a basis for $A(K)$ modulo torsion as $\{\alpha_1, \ldots, \alpha_r\}$, where of course r is the rank of $A(K)$ modulo torsion. For A self-dual (e.g. a Jacobian) Néron and Tate have defined a height pairing from $(A(K)/\text{tor}) \times (A(K)/\text{tor}) \to \mathbf{R}$;

$$|\det\langle \alpha_i, \alpha_j \rangle| \qquad\qquad 1 \le i, j \le r$$

denotes the absolute value of the determinant of this pairing. The last ingredient is the Shafarevich–Tate group $Ш(A, K)$, the set of everywhere locally trivial principal homogeneous spaces for A; equivalently, we can also define it by

$$Ш(A, K) = \bigcap_{v \in |V|} \ker(H^1(K, A) \to H^1(K_v, A))$$

where K_v is the ordinary henselization of K at $v \in |V|$. Now we state the arithmetic conjecture of Birch and Swinnerton–Dyer, for a self-dual abelian variety A:

BSD(A, K): The group $Ш(A, K)$ is finite. The L-series $L^*_S(A, s)$ has a zero of order exactly $r = \operatorname{rank} A(K)$ at $s = 1$. As $s \to 1$,

$$L^*_S(A, s) \sim (s - 1)^r \frac{[Ш(A, K)] |\det\langle \alpha_i, \alpha_j \rangle|}{[A(K)_{\text{tor}}]^2}$$

To state the geometric conjecture, consider X a smooth surface defined over k. The Néron–Severi group $NS(X)$ of algebraic equivalence classes of divisors on X is, by the Theorem of the Base [12], known to be finitely generated as abelian group. We write $\rho(X) = \rho$ for its rank and $\{D_1, \ldots, D_\rho\}$ for a basis of $NS(X)$ modulo torsion.

$$|D_i \cdot D_j| \qquad\qquad 1 \leq i, j \leq \rho(X)$$

denotes the absolute value of the determinant of the intersection matrix.

Pic°_X, the connected component of identity of the Picard scheme of X, is an abelian variety defined over k (and $Pic^{\circ}_X(\bar{k}) = \operatorname{Pic}^{\circ}(X)$ is the set of linear equivalence classes of divisors algebraically equivalent to zero on X). Write

$$\alpha(X) = \chi(X, 0_X) - 1 + \dim_k Pic^{\circ}_X$$

and define the (cohomological) Brauer group of X by

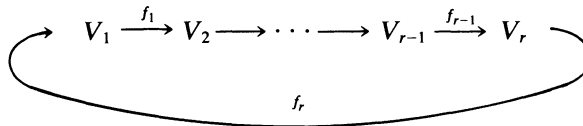$$\operatorname{Br}(X, k) = H^2(X, \mathbb{G}_m)$$

(Artin has shown that for X smooth of dimension ≤ 2, $\operatorname{Br}(X, k)$ is a birational invariant.) We now state the geometric conjecture of Artin and Tate:

AT(X, k): $\operatorname{Br}(X, k)$ is a finite group. The polynomial $P_2(X, T)$ has

q^{-1} as a root of multiplicity exactly $\rho(X)$; *as* $s \to 1$,

$$P_2(X, q^{-s}) \sim (1 - q^{(1-s)})^{\rho(X)} \frac{[\mathrm{Br}(X, k)]|D_i \cdot D_j|}{q^{\alpha(X)}[\mathrm{NS}(X)_{\mathrm{tor}}]^2}$$

2. *A lemma*. To calculate characteristic polynomials we frequently quote the following lemma of linear algebra, presented here for want of a better place. Define (not in the standard fashion) the characteristic polynomial of an endomorphism φ of S, a finite-dimensional vector space over an arbitrary field E, as $P(\varphi, T) = \det(1 - \varphi T)$ for φ considered as a matrix. Consider $\{V_1, \ldots, V_r\}$ a set of isomorphic finite-dimensional E-vector spaces with a collection of E-linear maps $f_i : V_i \to V_{i+1}$ cyclically permuting the V_i:

The f_i induce an endomorphism $f = \bigoplus f_i$ of $V = \bigoplus V_i$ and f^r induces endomorphisms F_i of each V_i. The following pleasant result holds.

LEMMA 1.1: *Let* $f = \bigoplus f_i$ *and* $F = \bigoplus F_i = f^r$ *be as described. Let* $P(f, T)$ *denote the characteristic polynomial of* f *acting on* V, *and* $Q(F_i, T)$ *that of* F_i *acting on* V_i. *The polynomial* $Q(F_i, T)$ *is independent of choice of* i, *and*

$$P(f, t) = Q(F_i, t^r).$$

PROOF: Since formation of characteristic polynomial is unaffected by base field extension, assume k algebraically closed. Choose a basis $\mathscr{B}^1$ of V_1. Choose bases $\mathscr{B}^2$ of V_2, $\mathscr{B}^3$ of $V_3, \ldots$, and finally $\mathscr{B}^r$ of V_r seriatim, so that the matrices of f_1 relative to $\mathscr{B}^1$ and $\mathscr{B}^2$, of f_2 relative to $\mathscr{B}^2$ and $\mathscr{B}^3, \ldots$, and finally of f_{r-1} relative to $\mathscr{B}^{r-1}$ and $\mathscr{B}^r$ are upper triangular. With respect to the basis $\{\mathscr{B}^1, \mathscr{B}^2, \ldots, \mathscr{B}^r\}$ of $V = \bigoplus V_i$:

$$P(f, t) = \det \begin{vmatrix} I & 0 & \cdot & \cdot & \cdot & 0 & -tT_r \\ -tT_1 & I & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & -tT_2 & I & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & -tT_3 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & I & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -tT_{r-1} & I \end{vmatrix}$$

where $T_1, T_2, \ldots, T_{r-1}$ are upper triangular. Row reduction changes this to an upper triangular matrix with $r - 1$ blocks I on the diagonal, and a last diagonal block

$$I - t^r T_{r-1} T_{r-2} \ldots T_1 T_r = I - t' F_r;$$

this matrix has as its determinant $Q(F_r, t')$. A different sequence of row and column reductions (which cannot change the determinant) gives as ith diagonal block $I - t' F_i$, with all other blocks I. ■

This lemma will usually be applied to the ℓ-adic cohomology groups. When X is a k-irreducible variety which splits over $\bar{k}$ into r (conjugate) irreducible varieties, by choosing notation carefully we can write

$$\bar{X} = \sum X_i \qquad\qquad\qquad 1 \le i \le r$$

where $F(X_1) = X_2, \ldots, F(X_r) = X_1$. By functoriality F induces maps on cohomology $F_i : H^j(X_i) \to H^j(X_{i-1})$. Using the lemma on the map $\bigoplus F_i$ gives the corollary

COROLLARY: *Let X_j denote any fixed component of the cycle $\bar{X}$. For all i,*

$$P_i(X, T) = P_i(X_j, T').$$

2. Global properties of fibrations

1. *Definitions.* To relate the Artin–Tate conjecture for a smooth surface to the Birch–Swinnerton-Dyer conjecture, we need to furnish the surface with additional structure.

DEFINITION: A *fibration* (defined over the field k) is a triple (X, V, f) of objects (defined over k) with
 (1) V a complete (smooth and geometrically irreducible by our notation) curve;
 (2) X a smooth surface; and
 (3) $f : X \to V$ a proper flat morphism with generic fiber X_K a curve (smooth, proper, and geometrically irreducible) defined over $K = k(V)$.

Further, we assume that f is cohomologically flat (in dimension zero) as well.

Given a fibration (X, V, f), a k-rational (Weil) divisor D on X is mapped by f either to a divisor on V or onto all of V, with some multiplicity $m(D)$. Set $\alpha = \gcd\{m(D)\}$ as D runs through all the k-rational divisors on X. We say that the fibration (X, V, f) has *degree* α; we choose once and for all a fixed k-rational divisor Ω of degree $m(\Omega) = \alpha$ consisting only of "horizontal" components (we think of Ω as an "α-fold section"). The fibration is said to be *of genus* d if X_K, considered as curve over K, has genus d, or equivalently [4] if for all complete fibers X_v,

$$d = \dim_{k(v)} H^1(X_v, \mathcal{O}_{X_v}).$$

REMARKS: 1. $\alpha = \gcd\{$degrees of cycles on $X_K(K)\}$. If f admits a section, α is 1.

2. In fact we can compute α as $(\Omega \cdot X_\mu)$, the intersection degree of Ω and any complete vertical geometric fiber of f. Immediately, each closed fiber X_v must, as a curve defined over the residue field $k(v)$, be of multiplicity dividing α (see Section 3 for a definition of multiplicity).

3. The condition of cohomological flatness (in dimension zero), which can be restated as the condition that universally

$$f_* \mathcal{O}_X \simeq \mathcal{O}_V,$$

allows us to use Raynaud's precise numerical results, stated in Section 5. Raynaud gives conditions [22; §7.2.1] which show that for a flat proper morphism $f : X \to V$ admitting a section (or even with all fibers of multiplicity one) cohomological flatness (in dimension zero) is equivalent to the single equality

$$f_* \mathcal{O}_X = \mathcal{O}_V.$$

And this equality is guaranteed by the first sentence of (3) in the definition.

Rather than starting with the surface X, we could have adopted a different attitude, beginning with k, V, and a curve X_K (of course smooth, etc.) defined over $K = k(V)$.

DEFINITION: A triple (X, V, f) is a *model for* X_K *over* V (defined over k) if X is a two-dimensional k-scheme and $f: X \to V$ is a proper flat morphism (defined over k) with generic fiber X_K. The model is *smooth* or *regular* if X is, and (*strictly*) *local* if V is $\mathrm{Spec}(R)$ for R a (strictly) henselian ring.

2. *Cohomology.* In studying a fibration we play off the local properties – typified by the structure of the degenerate fibers, studied in the next section – against the global properties, principally expressed in terms of the Néron-Severi group and the ζ-function. Much global information is provided cohomologically by the Leray Spectral Sequence:

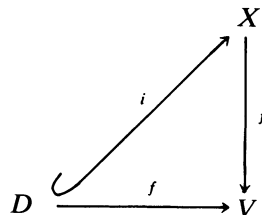$$H^p(V, R^q f_* \mathbf{G}_{m,X}) \;\Rightarrow\; H^{p+q}(X, \mathbf{G}_{m,X})$$

The vanishing theorem of Artin cited here makes the spectral sequence quite simple to use (Artin's results come from [9, Br III], where Theorem 2.1 is numbered as 3.2, and Theorem 2.3 can be extracted from the fourth section of the article).

THEOREM 2.1 (Artin): *Let* S *and* V *be locally Noetherian regular schemes, and let* $f: X \to V$ *be a proper flat morphism with all its fibers curves. Assume* V *is also a curve and that all the local rings of* V *are Japanese. Then for all* $q > 1$,

$$R^q f_* \mathbf{G}_{m,X} = 0.$$

PROPOSITION 2.2: *Let* $f: X \to V$ *be a proper flat morphism from a surface onto a curve (all defined over* k) *for which* $f_* \mathcal{O}_X = \mathcal{O}_V$, *and assume that there is a* k-*rational divisor* $D \subset X$ *such that* $f|: D \to V$ *is a flat surjective* n-*to*-1 *covering. Then the kernel of each edge homomorphism* $H^p(V, \mathbf{G}_V) \to H^p(X, \mathbf{G}_X)$ *is contained in the kernel of* H^p (*n*th-*power*) *acting on* $H^p(V, \mathbf{G}_V)$.

PROOF: In the diagram

combine the edge homomorphisms for f and i to get a map

$$H^p(V, G_V) \xrightarrow{f^*} H^p(X, G_X) \xrightarrow{i^*} H^p(D, G_D).$$

As $D \to V$ is flat and surjective of degree n, $\mathcal{O}_D$ is an $\mathcal{O}_V$-module locally free of rank n. The norm map $N : \mathcal{O}_D^* \to \mathcal{O}_V^*$ induces nth-power on $\mathcal{O}_V^*$. In other words, the long composition

$$H^p(V, G_V) \xrightarrow{f^*} H^p(X, G_X) \xrightarrow{i^*} H^p(D, G_D) \xrightarrow{H^p(N)} H^p(V, G_V)$$

is just H^p (nth-power). ∎

When (X, V, f) is a fibration, one relates the groups $\mathrm{Br}(X, k)$ and $\mathrm{III}(A, K)$, where A denotes the Jacobian of the generic fiber, by comparing both with $H^1(V, R^1 f_* G_X)$. By Proposition 2.2, since $\mathrm{Br}(V)$ is zero (because V is a complete curve over a finite field) $\mathrm{Br}(X)$ and $H^1(V, R^1 f_* G_X)$ differ only by a finite group. In [9, Br III, §4] it is proven that $H^1(V, R^1 f_* G_X)$ and $\mathrm{III}(A, K)$ also differ only by finite groups. Consequently, we have Theorem 2.3.

THEOREM 2.3 (Artin): *Let (X, V, f) be a k-fibration with A the Jacobian of generic fiber. Finiteness of $\mathrm{Br}(X, k)$ and of $\mathrm{III}(A, K)$ are equivalent. If (X, V, f) has degree 1, in fact the two groups are isomorphic.*

3. *Néron models.* For the moment, k will denote either a finite or an algebraically closed field, and V will be a (nonsingular geometrically irreducible) k-scheme of dimension 1 with function field K. Write the generic fiber of a V-scheme $S \to V$ as $S_K \to K$. (At least when V is either a complete curve over k a finite field or $\mathrm{Spec}(D)$ for a discrete valuation ring D) Néron has shown that any abelian variety A defined over K has a *minimal model* $A_V \to V$, a smooth commutative group scheme over V which represents, on the category of smooth V-schemes, the functor:

$$T \rightsquigarrow \hom_{\mathrm{Spec}\, K}(T_K, A).$$

We write A_V° for the open subscheme of A_V characterized by having for its fiber at the closed point v of V the connected component of identity $(A_v)^\circ$ of the v-fiber of the group scheme A_V.

Consider a regular model (X, V, f) of a generic curve X_K of genus d. A, the Jacobian of X_K, is an abelian variety of dimension d defined over K; each fiber A_v of the Néron model is an algebraic group of dimension d also, and so fiber by fiber the space of invariant differential d-forms on A_V is one dimensional; these patch to give a line bundle on V.

LEMMA 2.4: *Let (X, V, f) be a regular model for the generic curve X_K of genus d, and let ω denote the line bundle on V whose fiber ω_v is the space of invariant differential d-forms on A_v°. $R^1 f_* \mathcal{O}_X$ is a locally free sheaf of rank d, and*

$$\omega^{-1} = \Lambda^d (R^1 f_* \mathcal{O}_X).$$

If (X, V, f) is a fibration the line bundle ω has

$$\deg(\omega) = d \cdot \chi(V, \mathcal{O}_V) - \chi(V, R^1 f_* \mathcal{O}_X).$$

PROOF: The stalk at a geometric point of V is given by

$$(R^1 f_* \mathcal{O}_X)_\mu = \varinjlim H^1(U \times_V X, \mathcal{O}_X)$$

where the limit is taken over μ-punctured spaces U étale over V. Since $\mathcal{O}_X$ is quasi-coherent, we can calculate using Zariski cohomology rather than the étale cohomology. But for the Zariski cohomology, we known that for all geometric points μ of V, the groups $H^2(X_\mu, \mathcal{O}_{X_\mu})$ are zero, and that the groups $H^1(X_\mu, \mathcal{O}_{X_m})$ as $\bar{k}$-vector spaces all have dimension d. By the base-change theorem for Zariski cohomology [cf. 19, lecture 7.3°] the sheaf $R^1 f_* \mathcal{O}_X$ is locally free of rank d.

By the Riemann–Roch Theorem, a locally free sheaf E of rank d on a complete curve V induces a line bundle $\Lambda^d E$ on V satisfying

$$\chi(V, E) = \deg(\Lambda^d E) + d \cdot \chi(V, \mathcal{O}_V).$$

As $R^1 f_* \mathcal{O}_X$ is isomorphic to the tangent space at the zero-section to $Pic_{X/V}$, $\Lambda^d R^1 f_* \mathcal{O}_X$ is dual to ω. ∎

3. Degenerate fibers

1. *Structure of the fibers.* The arithmetic properties of a smooth surface X which is part of the fibration (X, V, f) depend on the structures of the degenerate fibers of X. Here we localize to study the fibers one by one.

DEFINITION: Let (X, V, f) be a fibration and $V(v) \hookrightarrow V$ be the inclusion of the ordinary henselization of V at v. The base-extended model $(X(v), V(v), f(v) = f)$ is the *associated local model* (for X_K) at v; for base extension over the inclusion of a strict henselization $V(\mu) \hookrightarrow V$ at a geometric point μ above v we refer to the associated *strictly* local model.

The following standard notation will apply to any (degenerate) closed fiber X_s of any smooth model (X, V, f). We write the closed fiber X_s, which is a curve defined over $k(s)$, as a sum of $k(s)$-irreducible components with multiplicities

$$X_s = \sum p_a X_a \qquad\qquad 1 \le a \le h_s$$

The integer $d_s = \gcd\{p_a\}$ is the *multiplicity* of the fiber X_s and we call the fiber *non-reduced* if $d_s > 1$. If so, writing $p_a = d_s m_a$ we define a reduced divisor by

$$R_s = \sum m_a X_a \quad ; \quad X_s = d_s R_s$$

and R_s is the least positive rational multiple of the complete fiber X_s. The $k(s)$-components X_a are in general not absolutely irreducible. If, over $\overline{k(s)}$, X_a splits as a cycle of q_a distinct irreducible components

$$X_a \times_{k(s)} \overline{k(s)} = \sum X_{aj} \qquad\qquad 1 \le j \le q_a$$

we say that X_a has *separable fiber multiplicity* q_a. (A sum over a always denotes $k(s)$-components, which are not necessarily absolutely irreducible; sums over j denote completely split – e.g. geometric – components.) For τ a geometric point over s, we see that the geometric fiber X_τ has $n_s = \sum p_a q_a$ components, multiplicities

counted. On the base-extended scheme $\bar{X}$ the base-extension of the closed fiber X_s, as the disjoint union of $\deg(s)$ geometric fibers, has a total of $n_s \cdot \deg(s)$ components. We associate to the closed fiber an anonymous integer defined by the equality

$$d_s \Delta_s = \gcd\{p_a q_a\}$$

which expresses how much of the structure of the fiber on the base-extended surface $\bar{X}$ was not apparent over k.

PROPOSITION 3.1: *Any k-rational divisor D on X intersects the closed fiber X_s with total intersection multiplicity divisible by $d_s \Delta_s$. If (X, V, f) is a fibration of degree 1, each closed (or geometric) fiber has multiplicity 1. If f admits a section σ, each fiber has a component of multiplicity 1.*

PROOF: By definition D intersects the closed fiber X_s with total intersection multiplicity $(D \cdot \overline{X_s}) = \Sigma(D \cdot X_\tau)$ as τ runs through the geometric points above s. $(D \cdot X_\tau) = (D \cdot \Sigma\, p_a \Sigma\, X_{aj}) = \Sigma\, p_a (D \cdot \Sigma\, X_{aj})$. For each a, the components X_{aj} are conjugate over k; if D intersects one of the X_{aj} it must intersect X_{aj} for each j, $1 \le j \le q_a$. We see $(D \cdot \Sigma\, X_{aj}) = q_a(D \cdot X_{aj})$ for any particular choice of j. This shows that $(D \cdot X_\tau)$ and so $(D \cdot X_s)$ is divisible by $d_s \Delta_s = \gcd\{p_a q_a\}$. "Degree 1" means that $(\Omega \cdot X_\tau) = 1$, so $d_s \Delta_s$ is 1, and X_s has multiplicity 1. If there is a section, $\sigma(V)$ hits each closed fiber in a single point and with multiplicity 1. This point must be on a component of multiplicity 1. ∎

Normalization allows us to read off the precise structure of certain étale sheaves on a (completely split) fiber X_s.

LEMMA 3.2: *Let X_s be a completely split fiber. Write $(X_s)_{\text{red}} = \bigcup X_j$ as a union of distinct absolutely irreducible components and denote by $\tilde{X}_j$ the normalization of the (reduced) component X_j. Write $\pi : \tilde{X}_s = \amalg\, \tilde{X}_j \to (X_s)_{\text{red}}$ for the obvious map. Let $\mathcal{F}$ denote either $\mathbf{G}_m$ or μ_{ℓ^n}, considered as étale sheaf on X_s, and $\tilde{\mathcal{F}}$ be the corresponding sheaf on $\tilde{X}_s$. Then*

$$0 \to \mathcal{F} \to \pi_* \tilde{\mathcal{F}} \to \mathcal{C} \to 0$$

is an exact sequence of sheaves with the cokernel $\mathcal{C}$ a skyscraper sheaf

concentrated on the singular points of X_s. *If* $k(s)$ *is (separably) algebraically closed the higher cohomology of* $\mathscr{C}$ *vanishes.*

PROOF: At a nonsingular point Q of X_s, $\mathscr{T}$ and $\pi_* \tilde{\mathscr{T}}$ have the same stalk since near Q, $\tilde{X}_j \simeq X_j$. This shows that $\mathscr{C}$ is a skyscraper concentrated on the singular points of the fiber; $\mathscr{C} = \bigoplus i_* \mathscr{C}_P$. For $k(s)$ algebraically closed (e.g. in the strictly local case) each singular point P is Spec k for k an algebraically closed field. As inclusion of closed point is a finite morphism, we see that

$$H^i(X_s, \mathscr{C}) = H^i(X_s, \bigoplus i_* \mathscr{C}_P) \simeq \bigoplus H^i(P, \mathscr{C}_P).$$

For $i > 0$, this last group is zero since étale cohomology over spec k is Galois cohomology, which is zero as k is algebraically closed. ∎

2. *ζ-functions.* Following Deligne [5], for X an algebraic k-scheme and $x \in |X|$ a closed point of X, write N_x for the number of elements in the finite field $k(x)$; this number happens to be $q^{\deg(x)}$. Define

$$\zeta(X, s) = \prod (1 - (N_x)^{-s})^{-1} \qquad\qquad x \in |X|.$$

As this product depends only on the underlying set of closed points of X, $\zeta(X, s) = \zeta(X_{\mathrm{red}}, s)$ and for a fibration (X, V, f), $\zeta(X, s) = \Pi \zeta(X_v, s)$, the product over $v \in |V|$. Write $q^{-s} = T$; since N_x is just $T^{\deg(x)}$, we can rewrite $\zeta(X, s)$ as $Z(X, T)$. In this form the Weil conjectures provide a rational expression, valid even for X singular,

$$Z(X, T) = \frac{P_1(X, T) \cdots P_{2n-1}(X, T)}{P_0(X, T) \cdots P_{2n}(X, T)}$$

where $P_i(X, T)$ denotes the characteristic polynomial for the action of F on the ℓ-adic cohomology group $H^i_\ell(X)$, and $n = \dim X$. $P_i(X, T)$ is known to have integral coefficients independent of the choice of $\ell \neq p$, provided that X is smooth and proper. Whenever X is smooth Poincaré duality provides a perfect pairing for all i, $0 \le i \le 2n$,

$$H^i(X)(d) \otimes H^{2n-i}(X)(n - d) \to \mathbb{Q}_\ell$$

which we can use to calculate the polynomials $P_i(X, T)$: recall that the maps induced by F and φ on the ℓ-adic cohomology groups are

(cohomological) inverses. Since by definition φ acts on $H^i_\ell(X)(d)$ as $q^d\varphi$ does on $H^i_\ell(X)$, F acts on $H^i_\ell(X)$ as $q^d\mathsf{F}$ does on $H^i_\ell(X)(d)$. Calculating for the action induced on $H^i_\ell(X)(d)$, we get

$$P_i(X, T) = \det(1 - q^d\mathsf{F}\, T).$$

PROPOSITION 3.3: *The ζ-function of the complete curve X_v of genus d defined over the finite field $k(v)$ and with $(X_v)_{\mathrm{red}} = \bigcup X_a$ is*

$$Z(X_v, T) = \frac{P_1(X_v, T)}{(1 - T) \cdot \Pi(1 - (N_v T))^{q_a}} \qquad 1 \le a \le h_v$$

If we write A°_v for the connected component of identity of the Picard scheme of X_v we get the precise value

$$P_1(X_v, N_v^{-1}) = [A^\circ_v(k(v))]/(N_v)^d.$$

PROOF: Without changing the ζ-function we replace X_v by $X_{v\,\mathrm{red}}$. Write this reduced curve still as X_v, and write q for N_v.

$$P_0(X_v, T) = \det(1 - \mathsf{F}_0 T).$$

Since F acts trivially on $H^0_\ell(X_v) = \mathbb{Q}_\ell$, we conclude that $P_0(X_v, T) = (1 - T)$.

Recall that $P_1(X_v, T) = \det(1 - q\mathsf{F}_1 T)$ for the action of F_1 on $H^1_\ell(X_v)(1)$. By the Riemann Hypothesis all the inverse roots of F_1 have absolute value $q^{1/2}$, so we calculate that $P_1(X_v, T)$ is also $\det(1 - \varphi_1 T)$ for the action induced on $H^1_\ell(X_v)(1)$ by φ. By Kummer Theory, $H^1_\ell(X_v)(1)$ is just $T_\ell(Pic_{\bar{X}_v}(\overline{k(v)})) = T_\ell(A^0_v(\overline{k(v)}))$ via the canonical identification of $H^1(\bar{X}_v, \mathsf{G}_m)$ with $Pic_{X_v}(\overline{k(v)})$. Since A°_v is a smooth connected group scheme over $k(v)$ a field of $q = N_v$ elements, a lemma of Milne [15, page 182] shows that the claimed value of $P_1(X_v, T) = \det(1 - \varphi_1 T)$ is correct.

For $P_2(X_v, T)$ we need to work harder. Base extension over $k(v) \hookrightarrow \overline{k(v)}$ replaces X_v by $\bar{X}_v = \bigcup\bigcup X_{aj}$. Write $\tilde{X}_{aj} \to X_{aj}$ for the normalizations. By Lemma 3.2 applied to $\bar{X}_v$ and the sheaf μ_{ℓ^n} we see that

$$H^2(\bar{X}_v, \mu_{\ell^n}) = \bigoplus_a \bigoplus_j H^2(\tilde{X}_{aj}, \mu_{\ell^n}).$$

Taking the projective limit over n and tensoring with $\mathbb{Q}_\ell$, find

$$H^2_\ell(X_v)(1) = \bigoplus\bigoplus H^2(\tilde{X}_{aj})(1).$$

By Poincaré duality on the smooth schemes $\bar{X}_{aj}$ we see that

$$H^2_c(X_v)(1) \cong \bigoplus \bigoplus H^0(\bar{X}_{aj}) \simeq \bigoplus_a (\bigoplus_j \mathbf{Q}_\ell).$$

F acts by permuting cyclically the components in each $\bigoplus_j \mathbf{Q}_\ell$, and F^{q_a} acts as identity. Lemma 1.1 tells us that $P_2(X_v, T)$ is $\Pi(1 - (qT)^{q_a})$, as claimed, with the product for $1 \le a \le h_v$. ∎

4. The Néron–Severi group

Henceforth, unless otherwise stated, (X, V, f) is a fibration of degree α with Ω the chosen k-rational divisor of degree α; A is the Jacobian of the generic fiber X_K; and B is the K/k trace of A, as explained below.

. 1. *The group of divisors.* Since X is a smooth surface all its local rings are regular, hence factorial; this shows that on X (or $\bar{X}$) the notions of Cartier (locally principal) and Weil (codimension one) divisors coincide. We identify a k-rational divisor D on X with its base extension $\bar{D}$ on $\bar{X}$; via this identification we view Div X as the subgroup of k-rational divisors in the divisor group Div $\bar{X}$. Notions of algebraic and linear equivalence are always taken with reference to $\bar{X}$. The map f^* defines injections of the groups Div V and Div $\bar{V}$ of divisors on V and $\bar{V}$ into Div X, Div $\bar{X}$. We view the groups Div V, etc. as subgroups of Div X, etc.

By specifying generating sets of divisors on $\bar{X}$ we define several subgroups of Div $\bar{X}$. Namely, $\text{Div}_{\text{vert}}(\bar{X})$ is generated by the irreducible curves C on $\bar{X}$ for which $\bar{f}(C)$ is a single point; $\text{Div}_{\text{hor}}(\bar{X})$ by the irreducible curves C for which $\bar{f}: C \to \bar{V}$ is surjective; $\text{Div}_0(\bar{X})$ by those C which intersect each complete vertical fiber with total intersection multiplicity zero; and $\text{Div}^\circ(\bar{X})$ by those C which are algebraically equivalent to zero on $\bar{X}$. (The image in Div $\bar{X}$ of) $\text{Div}^\circ(\bar{V})$, the set of divisors on $\bar{V}$ algebraically equivalent to zero, is the collection of all divisors of the form $\Sigma a_v X_v$ with the X_v complete fibers and the a_v integers with $\Sigma a_v = 0$. We also define $\text{Div}_0(\bar{V})$ to be the set of divisors on $\bar{X}$ which can be expressed as a rational linear combination of complete fibers. Intersecting with Div X these subgroups define similar subgroups of Div X; e.g. $\text{Div}_{\text{vert}}(X)$ is generated by the $k(v)$-irreducible components of the various fibers X_v. Reading these subgroups modulo $D_\ell(\bar{X})$, the group of divisors linearly equivalent to zero, we deduce subgroups of Pic $\bar{X}$ and Pic X; e.g. $\text{Pic}_{\text{vert}}(X)$ is the

group of divisor classes representable by divisors in $\text{Div}_{\text{vert}}(X)$. Working modulo $\text{Div}^\circ(\bar{X})$ we define subgroups $\text{NS}_{\text{vert}}(X)$ and $\text{NS}_0(X)$ of $\text{NS}(X)$ the group of algebraic equivalence classes of divisors on X.

LEMMA 4.1: *Write $\mathscr{D}$ for the subgroup $\text{Div}_{\text{vert}}(X) \oplus \mathbb{Z}\Omega$ of $\text{Div } X$. We have a filtration of $\text{Div } X$ by subgroups*

$$0 \to f^* \text{ Div } V \hookrightarrow \text{Div}_{\text{vert}}(X) \hookrightarrow \mathscr{D} \hookrightarrow \text{Div } X$$

PROOF: Identifying Cartier and Weil divisors on $\bar{X}$ we see that $\text{Div } \bar{X} = \text{Div}_{\text{vert}}(\bar{X}) \oplus \text{Div}_{\text{hor}}(\bar{X})$. Since f, and therefore $\bar{f}$, is defined over k, this is a direct sum decomposition of $G(\bar{k}/k)$-modules. Galois-invariant groups give the filtration. ∎

We read this filtration modulo algebraic equivalence to arrive at a similar one for $NS(X)$. To do so we need to know the quotient groups, which we can analyze by putting the divisors in a canonical form. Any vertical divisor F can be written as a sum

$$F = \sum F_v$$

of divisors each supported on a single complete vertical fiber. Any divisor E supported on X_v can be written uniquely as a $\mathbb{Z}$-linear combination

$$E = rX_v + \sum s_a X_{va} \qquad 0 \le s_1 < p_1 = m_1 d_v$$

(Such a statement is clear for the base-extended divisors $\bar{E}$ and $\bar{X}_v$, and descends by Galois-invariance.) Any $G \in \text{Div } X$ intersects the generic fiber with degree $(G \cdot X_K) = \alpha \cdot b(G)$ a multiple of α; by replacing G by $G - b(G)\Omega$, we can force G into $\text{Div}_0(X)$.

LEMMA 4.2: *Write D_ℓ for the subgroup of $\text{Div } X$ of divisors linearly equivalent to zero. Then (via the map $\bar{\tau}$ defined in the proof)*

$$\text{Div } X/(\mathscr{D} + D_\ell) \simeq A(K).$$

PROOF: As we saw above, $\text{Div } X/\mathbb{Z} \cdot \Omega \simeq \text{Div}_0(X)$ so by an elementary isomorphism theorem

$$\text{Div } X/(\mathscr{D} + D_\ell) \simeq \text{Div}_0(X)/(\text{Div}_{\text{vert}}(X) + D_\ell).$$

Define a map $\tau : \mathrm{Div}_0(X) \to A(K)$ by $\tau(D) = [D_K]$, the linear equivalence class as divisor on X_K of the degree-zero divisor $D_K = D \cap X_K$. Notice that $\tau(D) = \tau(E) \Leftrightarrow D_K \sim E_K$. By regularity of the surface X this holds $\Leftrightarrow D \sim E + W$ for some vertical divisor W. We have shown $\bar{\tau}$ injective; surjectivity is obvious, as if $\delta = [D_K] \in A(K)$, $\delta = \tau(\bar{D}_K)$ for $\bar{D}_K \in \mathrm{Div}_0(X)$ the Zariski closure on X of D_K. ∎

LEMMA 4.3: $\mathrm{Div}^\circ(X) \cap \mathrm{Div}_{\mathrm{vert}}(X) = \mathrm{Div}^\circ(V)$.

PROOF: The inclusion $\mathrm{Div}^\circ(V) \subset \mathrm{Div}^\circ(X) \cap \mathrm{Div}_{\mathrm{vert}}(X)$ is clear. Say that the vertical divisor W is in $\mathrm{Div}^\circ(X)$. Then for all v and a we have $0 = (W \cdot X_{va})$, which implies by negative semi-definiteness of the intersection matrices of partial fiber components that $W = \Sigma\, s_v X_v$ is a rational linear combination of complete fibers, i.e. $W \in \mathrm{Div}_\mathbf{Q}(V)$. Also $0 = (W \cdot \Omega)$, which shows that $\Sigma s_v = 0$. Fix a particular degenerate fiber X_0. Modifying W by divisors in $\mathrm{Div}^\circ(V)$ of the form $X_v - X_0$, we may assume that $W = \Sigma\, r_v X_v$, $0 \le v \le h$, is a rational linear combination of the degenerate fibers with $0 \le r_v < 1$ for all $v > 0$. Consider

$$\pi : \tilde{X} \to X \otimes_V \Omega \to X$$

where $\tilde{X}$ is a desingularization of $X \otimes_V \Omega$ achieved by blowings-up, with no contraction of exceptional fibers of the first kind. Since $W \approx 0$, $\pi^* W = \pi^*(\Sigma\, r_v X_v)$ is also equivalent to zero. If $0 < r_v$, $|r_v X_v| = |X_v|$. Because of this, for $0 < r_v < 1$, $|\pi^*(r_v X_v)| = |\pi^*(X_v)| = |\Sigma\, \tilde{X}_w|$, the sum of fibers at points $w \in \Omega$ over v. The divisor $\pi^* W$ is then a rational linear combination of the degenerate fibers $\tilde{X}_w$; in fact because $\tilde{X} \to \Omega$ admits a section (already $X \times_V \Omega \to \Omega$ does) by Proposition 3.1 we see that the divisor $\pi^* W$ is an integral linear combination of the $\tilde{X}_w$.

Consider a component C of multiplicity p in X_v. $\pi^* C = \Sigma\, \tilde{C}_w$; $\tilde{C}_w$ occurs with multiplicity p in each fiber $\tilde{X}_w$ over X_v. If $0 \le r_v < 1$, in the divisor $W = \Sigma\, r_v X_v$, C appears with multiplicity $a = r_v p$, and we see that $0 \le a < p$. In the divisor $\pi^* W$, the lifted component C_w occurs in the fiber $\tilde{X}_w$ with the multiplicity a. For $\pi^* W$ to contain $\tilde{X}_w$ with integral multiplicity, p must divide a; a must be zero, whence $r_v = 0$. This shows that for $0 < v \le h$, $r_v = 0$. By the sum formula, $r_0 = 0$ also. In other words, W was in $\mathrm{Div}^\circ(V)$. ∎

2. *The K/k trace.* Given a field k – arbitrary for the moment – we say that the extension of fields K/k is *primary* if the algebraic closure of k inside K is purely inseparable over k. This happens if K is a function

field with field of constants k. Given a primary extension K/k and an abelian variety A defined over K, we consider triples (J, A, j) of an abelian variety J defined over k, the given K-abelian variety A, and a map defined over K

$$j : J \to A.$$

With the obvious morphisms, such triples form the category of *k-abelian varieties over A*.

DEFINITION: A *K/k trace* of A is a final object (B, A, τ) of this category, provided that $\tau : B(\bar{k}) \to A(K\bar{k})$ has finite kernel.

By Chow's theorem that an abelian variety contains no moving family of subvarieties, one sees that the K/k trace always exists and is unique [11, §8.3]. Further it is independent of ground field in the sense that for E/k an extension independent of K/k, $\mathrm{Tr}_{K/k}A = \mathrm{Tr}_{EK/K}A$ for A considered as abelian variety over EK. Writing this trace as B, we also have $A(K) \cap \tau(B(E)) = \tau(B(k))$. [12, §5.1, Prop. 2].

PROPOSITION 4.4: *The abelian variety $A = \mathrm{Jac}(X_K)$ has for K/k trace (an abelian variety which is purely inseparably isogenous to) the quotient abelian variety $\mathrm{Pic}\,{}^{\circ}_{X}/\mathrm{Pic}\,{}^{\circ}_{V}$.*

PROOF: By stability of trace under base-field extension we may work over $\bar{k}$ on the base-extended varieties $\bar{X}$, $\bar{V}$. (Sloppily, we still write them as X, V.) The map $\tau : \mathrm{Div}_0(X) \to A(K\bar{k})$ in the proof of Lemma 4.2 defines (by Lemma 4.3) an injection of groups $\mathrm{Div}^{\circ}(X)/(\mathrm{Div}^{\circ}(V) + D_\ell) \hookrightarrow A(K\bar{k})$. By the universal mapping property of the Picard variety, τ defines a morphism of abelian varieties from $\mathrm{Pic}\,{}^{\circ}_{X}$ into A, which clearly factors through a morphism

$$\lambda : \mathrm{Pic}\,{}^{\circ}_{X}/\mathrm{Pic}\,{}^{\circ}_{V} \to A.$$

Notice that λ induces an injection on $\bar{k}$-valued points. We also define a map in the other direction. By regularity of X,

$$\pi : A(K\bar{k}) \to \mathrm{Div}\, X/(\mathrm{Div}_{\mathrm{vert}}(X) + D_\ell) \simeq \mathrm{Pic}\, X/\mathrm{Pic}_{\mathrm{vert}}X$$

is well-defined by taking $\delta = [D_\beta] \in A(K\bar{k})$ to the class represented by $\bar{D}_\beta$, the Zariski closure on X of the X_K-divisor D_β.

Now let (J, A, j) be a $\bar{k}$-abelian variety over A. Write c for the map defined on $J(\bar{k})$ by $\pi \circ j$. Under specialization we get for each $v \in |V|$ a

morphism

$$j_v : J \xrightarrow{\ j\ } A \xrightarrow{\ \text{spzn}\ } A_v$$

of J into the algebraic group scheme $A_v = Pic_{X_v}$; by connectedness j_v maps J into the connected component of identity A_v°, whose $\bar{k}$-valued points correspond to divisor classes on X_v having degree 0 as divisor on each partial fiber component. Since $j_v(\alpha) = [c(\alpha) \cap X_v]$, for any point α in $J(\bar{k})$ we see that c actually maps $J(\bar{k})$ into $\text{Pic}_* X/(\text{Pic}_{\text{vert}}(X) \cap \text{Pic}_* X)$, where $\text{Pic}_* X$ denotes the set of divisor classes intersecting each partial fiber component with degree 0. But $\text{Pic}_{\text{vert}}(X) \cap \text{Pic}_* X$ is just $\text{Pic}_0(V)$, the image under $\text{Div } X \to \text{Pic } X$ of $\text{Div}_0(V)$. We have defined a map

$$c : J(\bar{k}) \longrightarrow \frac{\text{Pic}_* X}{\text{Pic}_0(V)} \subset \frac{\text{Pic } X}{\text{Pic}_0(V)}.$$

Consider the diagram below, in which the middle row (degree-2 terms of the Leray Spectral Sequence for $f_* G_m$) is exact because $\text{Br } V$ is zero for a complete curve V over an algebraically closed field, and the cokernel in the middle column is, by the Theorem of the Base [12] a finitely generated abelian group.

$$
\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 \longrightarrow & \text{Pic}^\circ V & \hookrightarrow & \text{Pic}^\circ X & \longrightarrow & \text{Pic}^\circ X/\text{Pic}^\circ V & \longrightarrow 0 \\
& \uparrow & & \uparrow & & & \\
0 \longrightarrow & \text{Pic } V & \xrightarrow{f^*} & \text{Pic } X & \longrightarrow & H^0(V, R^1 f_* G_m) = \text{Pic}(X/V) & \longrightarrow 0 \\
& \updownarrow \downarrow \text{deg} & & \downarrow q & & & \\
0 \longrightarrow & \mathbb{Z} & \xrightarrow{i} & \mathbb{Z}^\gamma/\mathcal{H} & \longrightarrow & \mathscr{C} & \longrightarrow 0 \\
& \downarrow & & \downarrow & & & \\
& 0 & & 0 & & &
\end{array}
$$

The map $i : \mathbb{Z} \to \mathbb{Z}^\gamma/\mathcal{H}$ is injective. In fact, choosing any fixed point $v \in |V|$, we see that the (dotted) map $n \mapsto [n \cdot v] \in \text{Pic } V$ splits the first column. For $i(n) = 0$ we would need $q(f^*[n \cdot v]) = 0$, or that $f^*[n \cdot v] \in \text{Pic}^\circ X$. That would imply that $0 = (\Omega \cdot n X_v) = n\alpha$, so $n = 0$. By the snake lemma we get the exact sequence of cokernels

$$0 \to \text{Pic}^\circ X/\text{Pic}^\circ V \to \text{Pic}(X/V) \simeq \text{Pic } X/\text{Pic } V \to \mathscr{C} \to 0$$

with $\mathscr{C}$ a discrete group. Since the map Pic X/Pic $V \to$ Pic X/Pic$_0(V)$ is surjective and, by Lemma 4.3, Pic$^\circ$ $X \cap$ Pic$_0(V) =$ Pic$^\circ$ V we get a similar exact sequence

$$0 \to \text{Pic}^\circ X/\text{Pic}^\circ V \to \text{Pic } X/\text{Pic}_0(V) \to \mathscr{C}' \to 0$$

with another discrete group $\mathscr{C}'$. As the group $J(\bar{k})$ is divisible, c maps $J(\bar{k})$ into Pic$^\circ$ X/Pic$^\circ$ $V = (Pic^\circ_X/Pic^\circ_V)(\bar{k})$.

Apply all this to (B, A, ρ), a K/k trace of A. By definition of trace, there is a morphism $\alpha : Pic^\circ_X/Pic^\circ_V \to B$ for which $\rho \circ \alpha = \lambda$. Writing, e.g., $\bar{\varphi}$ for the map induced on $\bar{k}$-valued points by a morphism of abelian varieties φ, we see that $\bar{\lambda} = \bar{\rho} \circ \bar{\alpha}$. Since $\bar{\lambda}$ is injective so is $\bar{\alpha}$; i.e. α is purely inseparable. By the argument above, the map $\bar{\rho} : B(\bar{k}) \to A(K\bar{k})$ factors through $\bar{\lambda}$, i.e. there is a map $\bar{\beta} : B(\bar{k}) \to (Pic^\circ_X/Pic^\circ_V)(\bar{k})$ for which $\bar{\rho} = \bar{\lambda} \circ \bar{\beta}$. This shows that $\bar{\rho} = \bar{\rho} \circ \bar{\alpha} \circ \bar{\beta}$, i.e. that $\bar{1}_B - \bar{\alpha} \circ \bar{\beta}$ maps $B(\bar{k})$ into the finite group ker $\bar{\rho}$. By divisibility of $B(\bar{k})$ we see that $\bar{1}_B - \bar{\alpha} \circ \bar{\beta}$ is the zero map. Thus $\bar{\alpha}$ is also surjective, and we conclude that α is an isogeny. ■

3. *The structure of* NS(X). Recall that NS(X) denotes the group of algebraic equivalence classes of divisors on X; NS$_{\text{vert}}(X)$ is the image in NS(X) of Div$_{\text{vert}}(X) \subset$ Div X.

PROPOSITION 4.5: *Write $\mathscr{N}$ for* NS$_{\text{vert}}(X) \oplus \mathbb{Z} \cdot \Omega$. NS($X$) *has a filtration by subgroups*

$$0 \to f^* \text{NS}(V) \hookrightarrow \text{NS}_{\text{vert}}(X) \hookrightarrow \mathscr{N} \hookrightarrow \text{NS}(X)$$

with respective quotients $\mathbb{Z}$, Q, $\mathbb{Z}$, *and* $A(K)/B(k)$, *where Q is the group of partial vertical fibers,*

$$Q = \bigoplus_v \frac{\bigoplus_a \mathbb{Z} X_{va}}{\mathbb{Z} X_v}$$

By the Mordell–Weil Theorem $A(K)$ is finitely generated as an abelian group because K is finitely generated over the prime field. As B is an abelian variety defined over a finite field, $B(k)$ is a finite group.

COROLLARY: NS(X) *is a finitely generated abelian group of rank* $\rho(X) = 2 + r + \Sigma(h_v - 1)$, *where $r =$ rank of $A(K)$ and h_v is the number of $k(v)$-rational components of the fiber X_v.*

PROOF OF PROPOSITION 4.5: We break up the filtration of Lemma 4.1 into short exact sequences which we read modulo algebraic equivalence.

(1) f^* Div $V/\approx \ \simeq \ \mathbb{Z}$.

We prove that for $\mathscr{E}$ a divisor on V, $f^*\mathscr{E} \approx 0$ on X if and only if $\mathscr{E} \approx 0$ on V. But $f^*\mathscr{E} \approx 0 \Rightarrow (f^*\mathscr{E} \cdot \Omega) = 0 \Rightarrow \alpha \deg \mathscr{E} = 0 \Rightarrow \deg \mathscr{E} = 0$, i.e. $\mathscr{E} \approx 0$. By pullback, $\mathscr{E} \approx 0 \Rightarrow f^*\mathscr{E} \approx 0$.

(2) The following exact sequence holds:

$$0 \to \mathbb{Z} = f^* \operatorname{NS}(V) \hookrightarrow \operatorname{NS}_{\mathrm{vert}}(X) \to Q \to 0.$$

In fact our favorite group isomorphism shows

$$\frac{\operatorname{NS}_{\mathrm{vert}}(X)}{f^* \operatorname{NS}(V)} \simeq \frac{\operatorname{Div}_{\mathrm{vert}}(X)}{(f^* \operatorname{Div} V + \operatorname{Div}^\circ X) \cap \operatorname{Div}_{\mathrm{vert}}(X)}.$$

By Lemma 4.3 the denominator is f^* Div $V + f^*$ Div$^\circ$ $V = f^*$ Div V, so the cokernel is as claimed.

(3) $\mathscr{N}$, the image in $\operatorname{NS}(X)$ of $\operatorname{Div}_{\mathrm{vert}}(X) \oplus \mathbb{Z}\Omega$, is the direct sum $\operatorname{NS}_{\mathrm{vert}}(X) \oplus \mathbb{Z}\Omega$.

In fact, for $W \in \operatorname{Div}_{\mathrm{vert}}(X)$, $W + n\Omega \approx 0$ on $X \Rightarrow ((W + n\Omega) \cdot X_K) = 0$. But this degree is $n\alpha = n(\Omega \cdot X_K)$, so $n = 0$ and already $W \approx 0$.

(4) In the exact sequence

$$0 \to \mathscr{N} \hookrightarrow \operatorname{NS}(X) \to \operatorname{Cokernel} \to 0$$

the cokernel is exactly $A(K)/B(k)$.

By the isomorphism theorem

$$\frac{\operatorname{NS}(X)}{\mathscr{N}} \simeq \frac{\operatorname{Div} X}{\operatorname{Div}_{\mathrm{vert}}(X) + \mathbb{Z}\Omega + \operatorname{Div}^\circ X} \simeq \frac{\operatorname{Div}_0(X)}{\operatorname{Div}_{\mathrm{vert}}(X) + \operatorname{Div}^\circ X + D_\ell}$$

since $D_\ell \subset \operatorname{Div}^\circ(X)$. This last group is

$$\frac{\operatorname{Div}_0(X)/(\operatorname{Div}_{\mathrm{vert}}(X) + D_\ell)}{\operatorname{Div}^\circ(X)/(\operatorname{Div}_{\mathrm{vert}}(X) + D_\ell)} \simeq \frac{A(K)}{B(k)}$$

by the results of Lemma 4.2 and Proposition 4.4. ■

As a complement we can give, up to an explicit finite index, a set of generators of $\operatorname{NS}(X)$ modulo torsion. For $\alpha \in A(K)$ we will say that a divisor $\mathscr{A}$ on X represents α if in $A(K)$, $\alpha = [\mathscr{A} \cap X_K]$.

PROPOSITION 4.6: *Let r be the rank of $A(K)$ and choose divisors $\mathscr{A}_1, \ldots, \mathscr{A}_r$ representing $\alpha_1, \ldots, \alpha_r$ a basis of $A(K)$ modulo torsion. Let F be any complete fiber of f. Write $\mathscr{N}'$ for the subgroup of $\mathscr{N}$ generated by F, Ω, and $\{X_{va}$ for all v, and, for each v, all $a > 1\}$. Then $\mathscr{N}'$ is a free subgroup of $\mathscr{N}$, and $\mathscr{N}'$ and $\{\mathscr{A}_i\}$ generate a $\mathbb{Z}$-submodule of finite index*

$$\frac{[A(K)_{\text{tor}}]}{[B(k)]} \cdot \frac{\Pi \, d_v m_{v1}}{[\text{NS}(X)_{\text{tor}}]}$$

in $\text{NS}(X)$ modulo torsion.

PROOF: The fiber F is a generator of $f^* \text{NS}(V) \simeq \mathbb{Z}$. To show freedom for $\mathscr{N}'$ it suffices to show that if $D = aF + b\Omega + \Sigma_v \Sigma_{a>1} c_{va} X_{va} \approx 0$ then a, b, and all the c_{va} are zero. Let G be any complete fiber. $D \approx 0 \Rightarrow (D \cdot G) = 0$. But $(D \cdot G) = b(\Omega \cdot G) = b\alpha$, so $b = 0$. Next fix a particular degenerate fiber X_v. There is a partial fiber component X_{vr} intersecting $\Sigma_a c_{va} X_{va}$ with strictly negative multiplicity since the sum is not a rational multiple of the fiber X_v. But $(D \cdot X_{vr}) = ((\Sigma_a c_{va} X_{va}) \cdot X_{vr})$, a negative number, a contradiction unless the c_{va} are all zero. We have shown that D must be just aF. Intersecting with Ω, we see a is also zero.

Define another subgroup $\mathscr{N}''$ by the equation $\mathscr{N}'' \oplus \mathbb{Z}\Omega = \mathscr{N}'$. Then

$$\frac{\mathscr{N}}{\mathscr{N}'} \simeq \frac{\text{NS}_{\text{vert}}(X)}{\mathscr{N}''} \simeq \frac{\text{NS}_{\text{vert}}(X)/f^* \text{NS}(V)}{\mathscr{N}''/f^* \text{NS}(V)} \simeq \frac{Q}{\mathscr{N}''/f^* \text{NS}(V)}$$

where, as earlier, Q is the group of partial fibers. Since $\mathscr{N}''$ misses the components X_{v1}, $\mathscr{N}''/f^* \text{NS}(V)$ has index $\Pi \, d_v m_{v1}$ in Q.

Define a new group A' by exactness of row A of the following diagram; row B is exact by Proposition 4.5.

A. $0 \longrightarrow \mathscr{N}' \longrightarrow \text{NS}(X) \longrightarrow A' \longrightarrow 0$

$\qquad\qquad\qquad \uparrow \qquad\qquad \| \qquad\qquad \downarrow$

B. $0 \longrightarrow \mathscr{N} \longrightarrow \text{NS}(X) \longrightarrow A(K)/B(k) \longrightarrow 0$

By the snake lemma we find that A' is an extension of $A(K)/B(k)$ by the finite group $\mathscr{N}/\mathscr{N}'$. Consequently the basis $\alpha_1, \ldots, \alpha_r$ of $A(K)$ modulo torsion is also a basis of A' modulo torsion. Applying

Tor($*$, $\mathbb{Q}/\mathbb{Z}$) to row A gives the long exact sequence

$$0 \to \mathcal{N}'_{\text{tor}} \to \text{NS}(X)_{\text{tor}} \to A'_{\text{tor}}$$

$$\to \mathcal{N}' \otimes \mathbb{Q}/\mathbb{Z} \to \text{NS}(X) \otimes \mathbb{Q}/\mathbb{Z} \to A' \otimes \mathbb{Q}/\mathbb{Z} \to 0,$$

which demonstrates that $\mathcal{N}'$ and $\{\mathcal{A}_i\}$ generate a submodule of $\text{NS}(X)$ modulo torsion of index

$$\frac{[A'_{\text{tor}}]}{[\text{NS}(X)_{\text{tor}}]} = \frac{\Pi\, d_v m_{v1}}{[\text{NS}(X)_{\text{tor}}]} \cdot \frac{[A(K)_{\text{tor}}]}{[B(K)]}$$

where the value $[A'_{\text{tor}}]$ comes from the diagram. ∎

5. The intersection matrix

Given any set $\{F_1, \ldots, F_r\}$ of divisors on X, we write $(F_1, \ldots, F_r)$ for the absolute value of the determinant of the associated intersection matrix. In this section, we will approximate $(D_1, \ldots, D_{\rho(X)})$ for a basis $\{D_1, \ldots, D_{\rho(X)}\}$ of $\text{NS}(X)$ modulo torsion. To do this, we interpret Proposition 4.6 as providing a set of approximate generators of $\text{NS}(X)$ modulo torsion. Throughout, unless otherwise specified, $|X_{va} \cdot X_{vb}|$ is to be understood as $(X_{v2}, \ldots, X_{vh})$.

1. *The product formulation.* Call a divisor $D \in \text{Div}_0(X)$ *good* if it intersects each $k(v)$-irreducible component of each closed fiber X_v with total multiplicity 0. Tate has found a characterization of the height pairing on $A(K)$ modulo torsion:

Let A, B be good divisors on X representing α, $\beta \in A(K)$ modulo torsion. Then $\langle \alpha, \beta \rangle = -(A \cdot B) \log q$.

PROPOSITION 5.1: We have the equality

$$(\Omega; F; \mathcal{A}_1, \ldots, \mathcal{A}_r; \{X_{va}\}) = \alpha^2 \frac{|\det\langle \alpha_i, \alpha_j \rangle|}{(\log q)^r} \cdot \prod_v |X_{va} \cdot X_{vb}|$$

PROOF: Set $d = \text{lcm}\,\{|X_{va} \cdot X_{vb}|\}$ as v ranges over the degenerate fibers. The divisors $\mathcal{A}_i$ representing the basis elements α_i of $A(K)$ modulo torsion are all in $\text{Div}_0(X)$. In fact we can modify the multiples $d\mathcal{A}_i$ by vertical divisors to get good divisors $\mathcal{A}'_i$ working fiber by fiber. To make $d\mathcal{A}_i$ good for the fiber X_v, since $\mathcal{A}_i$ is in

$\mathrm{Div}_0(X)$ we merely need to solve the equations

$$\left(\left(d\mathscr{A}_i + \sum_{a>1} b_a X_{va}\right) \cdot X_{vj}\right) = 0 \qquad\qquad 1 < j \le h_v$$

By Cramer's rule this system has an integral solution for each bad v because d_v divides d. We find that

$$(\Omega; F; \{\mathscr{A}_i'\}; \{X_{va}\}) = (\Omega; F; \{d\mathscr{A}_i\}; \{X_{va}\}) = d^{2r}(\Omega; F; \{\mathscr{A}_i\}; \{X_{va}\}).$$

Since the $\mathscr{A}_i'$ were all chosen good, the intersection matrix $(\Omega; F; \{\mathscr{A}_i'\}; \{X_{va}\})$ is of the form in the diagram, where empty regions stand for zero matrices.

	Ω	F	$\{\mathscr{A}_i'\}$	$\{X_{va}\}$	$\cdots$	$\{X_{za}\}$
Ω	Ω^2	α	*******	*****		*****
F	α	0				
$\{\mathscr{A}_i'\}$	$****$		$(\mathscr{A}_i' \cdot \mathscr{A}_j')$			
$\{X_{va}\}$	$****$			$(X_{va} \cdot X_{vb})$		
$\vdots$					$\ddots$	
$\{X_{za}\}$	$****$					$(X_{za} \cdot X_{zb})$

By Tate's criterion, $(\mathscr{A}_i' \cdot \mathscr{A}_j') = \langle d\alpha_i, d\alpha_j\rangle/\log q$. This matrix then has absolute value of its determinant

$$\alpha^2 |(\mathscr{A}_i', \mathscr{A}_j')| \cdot \prod_v |X_{va} \cdot X_{vb}| = d^{2r}\alpha^2 \frac{|\det\langle \alpha_i, \alpha_j\rangle|}{(\log q)^r} \prod_v |X_{va} \cdot X_{vb}|$$

whence the lemma. ∎

2. *Relation to Néron models.* For the moment let (X, V, f) be a local model for X_K over V, with A_V the Néron model over V for A, the Jacobian of X_K. The following theorem [22; §8.1.2] links the structure of the closed fiber X_v of X with that of the fiber A_v of the Néron model at $v \in |V|$.

THEOREM 5.2 (Raynaud): *Let $(\bar{X}, \bar{V}, \bar{f})$ be a strictly local model with closed fiber $X_s = d_s R_s = \Sigma\, p_j X_j$, $1 \leq j \leq n$.*

(1) *There is a non-exact complex of abelian groups*

$$0 \to D_0(\bar{X}) \xrightarrow{i} D(\bar{X}) \xrightarrow{\alpha} \mathbb{Z}^n \xrightarrow{\beta} \mathbb{Z} \to 0$$

where the maps are defined as

$$i: \ X_s \mapsto \sum p_j X_j;$$

$$\alpha: \ E \mapsto (X_1 \cdot E), (X_2 \cdot E), \dots, (X_n \cdot E); \text{ and}$$

$$\beta: \ (a_1, \dots, a_n) \mapsto \sum a_j p_j \text{ is "total degree".}$$

$D(\bar{X})$ *denotes the group of divisors supported on the closed fiber, and* $D_0(\bar{X})$ *is the subgroup of principal divisors in* $D(\bar{X})$.

(2) $\ker \alpha / \operatorname{im} i \simeq \mathbb{Z}/d_s$.

(3) *If f is cohomologically flat (in dimension zero), $\ker \beta / \operatorname{im} \alpha \simeq A_s/A_s^0$.*

By negative semi-definiteness of the intersection matrix of partial fiber components it is clear that $\ker \alpha = \mathbb{Z} \cdot R_s$ and that we can express $\operatorname{im} \alpha$ by the exact sequence

(Im) $0 \to \mathbb{Z} \cdot R_s \to \bigoplus_j \mathbb{Z} X_j \to \operatorname{im} \alpha \to 0$.

We restate (3) of this theorem in a form more convenient for later use.

THEOREM 5.2′: *Let $\bar{V}$ be a strictly local scheme with closed point s. To a strictly local model $(\bar{X}, \bar{V}, \bar{f})$ is associated an exact sequence of abelian groups*

$(*_s)$ $0 \to \operatorname{im} \alpha_s \to \ker \beta_s \to A_s/A_s^0 \to 0$

which expresses the structure of the (group of $\bar{k}$-rational points on

the) *fiber A_s of the Néron model in terms of the closed fiber X_s, provided that the map $\bar{f}$ is cohomologically flat (in dimension zero*).

Now return to the k-fibration (X, V, f) for which $A = \mathrm{Jac}(X_K)$ has Néron model A_V over V. To study the fiber A_v of A_V at v we can restrict our attention to the associated local model $(X(v), V(v), f)$ and to the associated strictly semi-local model $(X(v) \times_k \bar{k}, V(v) \times_k \bar{k}, \bar{f})$. This semi-local model has $\deg(v)$ (disjoint) closed fibers, which are just the geometric fibers X_μ of $(\bar{X}, \bar{V}, \bar{f})$ lying above the closed fiber X_v. For each (geometric) fiber X_μ there is an exact sequence $(*_\mu)$ and thus we get an associated sequence for the strictly semi-local model

$$(\oplus *_\mu) \qquad 0 \to \bigoplus \mathrm{im}\,\alpha_\mu \to \bigoplus \ker \beta_\mu \to \bigoplus A_\mu / A_\mu^\circ \to 0.$$

Write the maps $\bigoplus \alpha_\mu$, $\bigoplus \beta_\mu$ as $\bar{\alpha}$, $\bar{\beta}$. Since $G(\bar{k}/k)$ permutes the set of $\bar{k}$-irreducible components of $X_v \times_k \bar{k}$ it acts on $\mathrm{im}\,\bar{\alpha}$, $\ker \bar{\beta}$. Write $\mathrm{im}\,\alpha$, $\ker \beta$ for $G(\bar{k}/k)$-fixed elements.

PROPOSITION 5.3: $[A_v : A_v^\circ] = [\ker \beta : \mathrm{im}\,\alpha] \cdot \delta_v$ *where δ_v is some integer dividing Δ_v.*

PROOF: Taking Galois cohomology of $(\oplus *_\mu)$ gives an exact sequence

$$0 \to \mathrm{im}\,\alpha \hookrightarrow \ker \beta \to H^0(G, \bigoplus A_\mu / A_\mu^0) \to C \to 0$$

of groups for some subgroup C of $H^1(G, \mathrm{im}\,\bar{\alpha})$. The lemma follows by calculations that $H^1(G, \mathrm{im}\,\bar{\alpha}) \simeq Z/\Delta_v$ and $H^0(G, \bigoplus A_\mu / A_\mu^0) \simeq A_v / A_v^0$; both calculations are made from the Hochschild–Serre Spectral Sequence.

Consider a Galois module of a particular type: for H a subgroup of the abelian group G and M an H-module, we define a G-module $\bigoplus M$, the direct sum extended over G/H, by having G/H act merely by permuting components, and having H act on each component by its original action on M. Such a G-module is *induced* by the H-module M. For a G-module induced by the H-module M, Shapiro's lemma holds: $H^p(G, \bigoplus M) \simeq H^p(H, M)$. In fact in the Hochschild–Serre Spectral Sequence $H^p(G/H, H^q(H, \bigoplus M)) \Rightarrow H^{p+q}(G, \bigoplus M)$ the groups $H^q(H, \bigoplus M) \simeq \bigoplus H^q(H, M)$ are coinduced as G/H-modules: G/H merely permutes the components. But then the spectral sequence degenerates to $H^0(G/H, \bigoplus H^q(H, M)) \simeq H^q(G, \bigoplus M)$, and the first group is just $H^q(H, M)$.

All the Galois modules we deal with are induced. Write G' for $G(\bar{k}/k(v))$. $\bigoplus A_\mu/A_\mu^\circ$ is induced by the G'-module A_μ/A_μ°; im $\bar{\alpha}$ is induced by the G'-module im α_μ; and after rewriting the exact sequence of G'-modules (Im) above as

(Im) $0 \to \mathbb{Z} R_\mu \xrightarrow{\varphi_\mu} \bigoplus_a \bigoplus_j \mathbb{Z} X_{aj} = \bigoplus \mathscr{X}_a \to \text{im } \alpha_\mu \to 0,$

we recognize the G'-module $\bigoplus \mathscr{X}_a$ as a direct sum of modules induced by G_a-modules for various subgroups G_a of G'. We now compute:

(1) $H^0(G, \bigoplus A_\mu/A_\mu^\circ) \simeq A_v/A_v^\circ:$

By Shapiro's lemma, $H^0(G, \bigoplus A_\mu/A_\mu^\circ)$ is $H^0(G', A_\mu/A_\mu^\circ)$. By Lang's Theorem, this is $A_\mu/A_\mu^\circ(k(v))$, isomorphic by Lemma 5.6 to the group of components A_v/A_v°.

(2) $H^1(G, \text{im } \bar{\alpha}) = \mathbb{Z}/\Delta_v:$

First, $H^1(G, \text{im } \bar{\alpha}) \simeq H^1(G', \text{im } \alpha_\mu)$. Take cohomology of the exact sequence (Im). $H^1(G', \bigoplus \mathscr{X}_a) \simeq \bigoplus H^1(G_a, \mathbb{Z} X_{va}) = 0$. Similarly, $H^2(G', \mathbb{Z} R_\mu) = H^2(G', \mathbb{Z})$ is known to be $\mathbb{Q}/\mathbb{Z}$, and $H^2(G', \bigoplus \mathscr{X}_a) \simeq \bigoplus_a \mathbb{Q}/\mathbb{Z}$. The map $H^2(\varphi_\mu): \mathbb{Q}/\mathbb{Z} \to \bigoplus_a \mathbb{Q}/\mathbb{Z}$ is given by

$$\varphi_\mu : n \mapsto (n \cdot m_1 q_1, \ldots, n \cdot m_h q_h)$$

since G_a has index q_a in G' and φ_μ takes R_μ to $(m_1 X_1, \ldots, m_h X_h)$. $H^1(G', \text{im } \alpha_\mu)$, the kernel, is $\mathbb{Z}/\gcd\{m_a q_a\} = \mathbb{Z}/\Delta_v$ by definition.   ∎

In the next lemma, the explicit form of the map $\bar{\alpha}$ provides a numerical value for $[\ker \beta : \text{im } \alpha]$.

LEMMA 5.4: $[\ker \beta : \text{im } \alpha]$ and $|X_{va} \cdot X_{vb}|$ satisfy

$$[\ker \beta : \text{im } \alpha] \cdot \frac{\Pi_a q_a \deg(v)}{\deg(v)} = \frac{\Delta_v}{(m_{v1})^2} |x_{va} \cdot X_{vb}|.$$

PROOF: Write part of Raynaud's complex as

$(\#_\mu)$ $0 \to \text{im } \alpha_\mu \hookrightarrow \mathbb{Z}^n \xrightarrow{(1/d_\mu)\beta_\mu} \mathbb{Z} \to 0$

where the last map, $(a_1, \ldots, a_n) \mapsto \Sigma a_j m_j$, is surjective by definition of d_μ. Summing as usual over the geometric fibers of the semi-local

model and using (Im) to express im α_μ, we get the complex of $G(\bar{k}/k)$-modules

$$0 \to \bigoplus \frac{\bigoplus_a \bigoplus_i \mathbb{Z} X_{\mu aj}}{\mathbb{Z} R_\mu} \to \mathbb{Z}^n \xrightarrow{(1/d_v)\beta} \mathbb{Z} \to 0$$

where the non-indexed sums are over the fibers X_μ of the semi-local model. Galois fixed-part yields the complex (5.1) below. To extract information on [ker β : im α] notice that $D' = \bigoplus_{a>1} \mathbb{Z} X_{va}$, the free $\mathbb{Z}$-module generated by all the $k(v)$-components of X_v except the first one, has index m_{v1} in im α.

(5.1)
$$0 \longrightarrow \text{im } \alpha = \frac{\bigoplus_a \mathbb{Z} X_{va}}{\mathbb{Z} R_v} \overset{\alpha}{\hookrightarrow} \mathbb{Z}^h \xrightarrow{\beta/d_v} \mathbb{Z} \longrightarrow ?$$

$$\uparrow \qquad\qquad \downarrow \gamma \qquad \|$$

$$D' = \bigoplus_{a>1} \mathbb{Z} X_{va} \overset{\delta}{\longrightarrow} \mathbb{Z}^h \overset{\pi_1}{\longrightarrow} \mathbb{Z} \longrightarrow ?$$

The map α, induced by $\bar{\alpha}$ on $(\text{im } \bar{\alpha})^G$, has matrix

$$\left(\frac{1}{q_a \cdot \deg(v)} X_{va} \cdot X_{vb} \right) \qquad 1 \le a, b \le h_v$$

and β, induced by $\bar{\beta}$, is the map $(e_1, \ldots, e_h) \mapsto \Sigma\, p_a q_a e_a$. The map γ is just a change of coordinates converting β/d_v to projection on the first coordinate, and fixing all other coordinates. Commutativity of the diagram forces δ to have $h \times (h-1)$ matrix

$$\begin{pmatrix} 0 & \cdot & \cdot & 0 \\ & \frac{1}{q_a \deg(v)} X_{va} X_{vb} & \end{pmatrix} \qquad 1 < a, b \le h_v$$

By linear algebra,

(5.2) $$[\ker \pi_1 : \text{im } D'] = \left| \frac{1}{q_a \deg(v)} X_{va} \cdot X_{vb} \right| \qquad 1 < a, b \le h_v$$

Chasing around the commutative square, the index is also

(5.3) $$[\ker \pi_1 : \text{im } D'] = m_{v1}[\ker \beta : \text{im } \alpha] \frac{m_{v1} q_1}{\Delta_v}$$

since the vector $(0, b_2, \ldots, b_h) \in \ker \pi_1$ is in the image of the map $\gamma \Leftrightarrow$

there is some b_1 with $0 = \Sigma\, m_a q_a b_a$, $1 \le a \le h_v$. This happens if and only if $m_1 q_1$ divides the sum $\Sigma\, m_a q_a b_a$, $1 < a \in h_v$. Combine (5.2) and (5.3) to prove the lemma. ∎

Combining Lemma 5.4 with Propositions 5.1 and 5.3 we can give a more arithmetic interpretation to the determinant of the matrix.

PROPOSITION 5.5: *The intersection determinant satisfies*

$$(\Omega; F; \{\mathscr{A}_i\}; \{X_{va}\}) = \alpha^2 \frac{|\det\langle \alpha_i, \alpha_j\rangle|}{(\log q)^r} \prod_v |X_{va} \cdot X_{vb}|$$

where for some integer ϵ_v dividing Δ_v^2,

$$\frac{\epsilon_v}{(m_{v1})^2} |X_{va} \cdot X_{vb}|_{1 < a,b} = [A_v : A_v^0] \frac{\Pi_a\, q_a\, \deg v}{\deg v} \qquad 1 \le a \le h_v$$

LEMMA 5.6: *Let A be an algebraic group defined over k which splits over $\bar{k}$ as*

$$A \times_k \bar{k} = A_1 \times \cdots \times A_r$$

a product of transitively conjugate algebraic groups, defined over k_r, by necessity. Then $[A(k)] = [A_i(k_r)]$ for any choice of i.

PROOF: $A(k_r) \simeq A_1(k_r) \times \cdots \times A_r(k_r)$; $A(k)$ is the fixed set under (arithmetic) Frobenius. Yet if $(P_1, \ldots, P_r)$ is a point of $A_1(k_r) \times \cdots \times A_r(k_r)$, $\varphi(P_1, \ldots, P_r) = (P_r^{(q)}, P_1^{(q)}, \ldots, P_{r-1}^{(q)})$ where $P^{(q)}$ denotes the point whose coordinates are qth-powers of those of P, and the numbering of the A_i has been arranged so that φ takes A_1 into A_2, etc. ∎
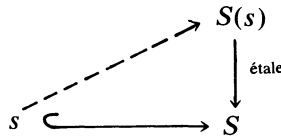
6. The main Theorem

1. *Deduction from the stronger result.* After all this preparation we will prove the equivalence of the arithmetic and geometric conjectures. We actually prove that the equivalence amounts to an equality between orders of groups.

THEOREM 6.1: *Let k be a finite field, K a function field in one variable over k with complete nonsingular (geometrically irreducible)*

*model V. Let X be a smooth surface defined over k and equipped with
a proper, flat, cohomologically flat (in dimension zero) k-rational map
$f: X \to V$. If the generic fiber X_K of f (as curve defined over K) is
nonsingular and geometrically irreducible, and contains a K-rational
cycle of degree 1, then the conjectures AT(X, k) and BSD(A, K) are
equivalent, where A denotes the K-abelian variety* $\mathrm{Jac}(X_K)$.

Before stating the stronger version, we need a definition. We
consider a geometric property of S an algebraic F-scheme for an
arbitrary field F. The property is said to hold *locally for the F-étale
topology at s*, a closed point of S, if there is an s-punctured scheme
$S(s)$ étale over S

$$
\begin{array}{ccc}
 & & S(s) \\
 & \nearrow & \Big\downarrow \text{\footnotesize étale} \\
s \;\lhook\joinrel\longrightarrow & & S
\end{array}
$$

for which the property holds. Of course, a reasonable property holds
locally for the F-étale topology exactly when it holds for the ordinary
henselization.

LEMMA 6.2: *Let* (X, V, f) *be a k-fibration; f is of degree δ locally
for the k-étale topology at v exactly when* $d_v\Delta_v = \delta$.

PROOF: Consider the strictly local model $(X(\mu), V(\mu), f(\mu))$ at a
geometric point μ above v. If the closed fiber $X_v = \Sigma p_a X_a$, the
geometric fiber (of the strictly local model) is $X_\mu = \Sigma p_a \Sigma X_{aj}$. Ray-
naud has shown [22, §7.1.1] the existence of a horizontal divisor D_{aj}
on $X(\mu)$ intersecting the closed fiber X_μ only on the component X_{aj}
and with $(D_{aj} \cdot X_{aj}) = p_a$. Consequently the strictly local model has
degree divisible by $d_v = \gcd\{p_a\}$. On the other hand, a horizontal
divisor D on $X(\mu)$ is base-extended from a divisor defined locally for
the k-étale topology if and only if it is $G(\bar{k}/k)$-invariant. By the proof
of Proposition 3.1, in G-invariant form Raynaud's result shows the
existence at v of locally k-étale divisors D_a of degrees $p_a q_a$, whence
there is a locally k-étale divisor at v of degree $d_v\Delta_v$. By Proposition
3.1, we see that X is locally of degree exactly $d_v\Delta_v$. ∎

THEOREM 6.3: *Let k, K, X, and V be as in Theorem 6.1. Assume
that the fibration* (X, V, f) *is of degree α and that locally everywhere
for the k-étale topology it is of degree 1. Then the equivalence, above,*

of the Artin–Tate and Birch–Swinnerton-Dyer conjectures holds exactly when the equality

$$\alpha^2[\mathrm{Br}(X, k)] = [\text{Ш}(A, K)]$$

holds between the orders of the groups.

To deduce Theorem 6.1 from this result, just notice that the Zariski closure on X of the given K-rational cycle on X_K of degree 1 is a k-rational divisor of degree 1 on X, so $\alpha = 1$ and (X, V, f) is locally everywhere of degree 1. By Theorem 6.3 we need only check the equality $[\mathrm{Br}(X, k)] = [\text{Ш}(A, K)]$ which follows by Artin's Theorem 2.3.

2. *The Proof.* Collected here are five results easily combined to prove Theorem 6.3. Of these, one depends purely on the structure of the degenerate fibers, two are global statements about the surface, and two relate to the structure of the Néron–Severi group. Proof of the two as-yet-unproven results follow that of the theorem.

PROPOSITION 6.4: *Let S denote any finite set of closed points of V containing all v such that X_v is degenerate or, for the given invariant differential form ω on A_K, the reduced form ω_v is not non-zero non-polar. Write B for the K/k trace of A. Then*

$$(6.1) \qquad P_2(X, q^{-s}) = (1 - q^{1-s})^2 P_1(B, q^{-s}) P_1(B, q^{1-s}) \frac{L_S^*(A, s)}{|\mu|^d}$$

$$\cdot \prod_{v \in S} \frac{\int_{A(K_v)} |\omega_v| \mu_v^d \cdot \Pi_a (1 - N_v^{q_a(1-s)})}{P_1(X_v, N_v^{-s}) \cdot (1 - N_v^{1-s})}.$$

PROPOSITION 4.6: *The divisors Ω; F any complete vertical fiber; $\{\mathscr{A}_i\}$; and $\{X_{va}$ for $a > 1\}$ generate a free submodule of $\mathrm{NS}(X)$ modulo torsion of finite index*

$$\frac{[A(K)_{\mathrm{tor}}]}{[B(k)]} \cdot \frac{\Pi \, d_v m_{v1}}{[\mathrm{NS}(X)_{\mathrm{tor}}]}$$

and rank $\rho(X) = 2 + r + \Sigma (h_v - 1)$ *inside that group.*

PROPOSITION 3.3: *The ζ-function of the closed fiber X_v is*

$$Z(X_v, T) = \frac{P_1(X_v, T)}{(1 - T) \cdot \Pi(1 - (N_v T)^{q_a})}$$

with the product taken for $1 \le a \le h_v$. Writing A_v^0 for the connected component of identity of Pic_{X_v} the precise value holds:

$$P_1(X_v, N_v^{-1}) = [A_v^0(k(v))]/(N_v)^d.$$

PROPOSITION 5.5: *The intersection determinant satisfies*

$$(\Omega; F; \{\mathscr{A}_i\}; \{X_{va}\}) = \alpha^2 \frac{|\det\langle a_i, \alpha_j\rangle|}{(\log q)^r} \prod_v |X_{va} \cdot X_{vb}|$$

where for some integer ϵ_v diving Δ_v^2,

$$\frac{\epsilon_v}{(m_{v1})^2}|X_{va} \cdot X_{vb}|_{1<a,b} = [A_v : A_v^0]\frac{\Pi_a q_a \deg v}{\deg v} \qquad 1 \le a \le h_v$$

PROPOSITION 6.5: *Write $\alpha(X) = \chi(X, \mathcal{O}_X) - 1 + \dim Pic_X^0$, and write b for $\dim B$. Then*

$$|\mu|^d q^{b + \deg \omega} = q^{\alpha(X)}.$$

PROOF OF THEOREM 6.3: Proposition 6.4 gives an explicit formula for $P_2(X, q^{-s})$. To prove the theorem, we merely need to let s approach 1, combine terms, and interpret the various factors. As $s \to 1$, BSD(A, K) tells us that

(6.1.1) $\qquad L_S^*(A, s) \sim (s - 1)^r \frac{[\text{III}(A, K)]}{[A(K)_{\text{tor}}]^2}|\det\langle \alpha_i, \alpha_j\rangle|$

As s tends to 1, in (6.1) we can combine the factors involving $(s - 1)$, $(1 - q^{1-s})$, and $(1 - N_v^{q_a(1-s)})$ to get

(6.1.2) $\qquad (1 - q^{1-s})^{2+r+\Sigma(h_v-1)} \cdot \frac{1}{(\log q)^r} \prod_{v \in S} \frac{\Pi q_a \deg(v)}{\deg(v)}$

since $(s - 1) \log q \sim (1 - q^{1-s})$ and an N_v-factor $\sim q_a \deg(v) (1 - q^{1-s})$.

Because B is an abelian variety of dimension b defined over k, we can interpret $P_1(B, T)$ – cf. the proof of Proposition 3.3 – as the characteristic polynomial of *arithmetic* Frobenius operating on $T_\ell(B(\bar{k})) \otimes_{Z_\ell} \mathbf{Q}_\ell$. This interpretation yields the functional equation

$$P_1(B, T) = q^b T^{2b} P_1(B, (qT)^{-1}).$$

Substituting $T = 1$, we find that

$$(6.1.3) \qquad P_1(B, q^{-1})P_1(B, 1) = P_1(B, 1)^2/q^b = [B(k)]^2/q^b.$$

Combining equations (6.1) and (6.1.1)–(6.1.3), and the "rank" statement of Proposition 4.6, we find that as $s \to 1$,

$$(6.2) \qquad P_2(X, q^{-s}) \sim (1 - q^{1-s})\rho^{(X)} \frac{[B(k)]^2}{|\mu|^d q^b} \frac{[\text{Ш}(A, K)]}{[A(K)_{\text{tor}}]^2} \frac{|\det\langle\alpha_i, \alpha_j\rangle|}{(\log q)^r}$$

$$\cdot \prod_{v \in S} \frac{\int_{A(K_v)} |\omega|_v \mu_v^d \cdot \Pi \, q_a \, \deg(v)}{P_1(X_v, N_v^{-1}) \deg(v)}$$

In the final product factor, the factors $P_1(X_v, N_v^{-1})$ almost cancel with the integrals. Precisely, for ω' (possibly different from ω) a differential form with nonzero nonpolar reduction at v,

$$\int_{A(K_v)} |\omega|_v \mu_v^d = \left|\frac{\omega}{\omega'}\right|_v \int_{A(K_v)} |\omega'|_v \mu_v^d$$

$$= \left|\frac{\omega}{\omega'}\right|_v [A(K_v):A^0(K_v)][A^0(K_v):A^1(K_v)] \int_{A^1(K_v)} |\omega'|_v \mu_v^d$$

where $A^0(K_v)$, $A^1(K_v)$ are the subgroups of $A(K_v)$ reducing to nonsingular points, or the origin, on A_v. Under the uniformization of an abelian variety via local parameters at the origin $A^1(K_v)$ corresponds to the space m_v^d; also $A^0(K_v)/A^1(K_v)$ isomorphically reduces to $A_v^0(k(v))$, as does $A(K_v)/A^0(K_v)$ to the group A_v/A_v^0 of connected components of A_v. By Proposition 3.3,

$$(6.2.1) \qquad \int_{A(K_v)} |\omega|_v \mu_v^d / P_1(X_v, N_v^{-1}) = \left|\frac{\omega}{\omega'}\right|_v [A_v : A_v^0].$$

Using (6.2.1) we can rewrite the last two factors of (6.2) as

$$(6.2.2) \qquad \frac{1}{q^{\deg \omega}} \frac{|\det\langle\alpha_i, \alpha_j\rangle|}{(\log q)^r} \prod_{v \in S} [A_v : A_v^0] \frac{\Pi \, q_a \, \deg(v)}{\deg(v)}$$

By Proposition 5.5, this is – except for the q-power factor –

$$(6.2.3) \qquad \frac{1}{\alpha^2}(\Omega; F; \{\mathscr{A}_i\}; \{X_{va}\}) \prod_{v \in S} \frac{\epsilon_v}{(m_{v1})^2}.$$

Substitute (6.2.2) and (6.2.3) into (6.2) and use Proposition 6.5 to find

$$(6.3) \qquad P_2(X, q^{-s}) \sim (1 - q^{1-s})^{\rho(X)} \frac{[\text{III}(A, K)][B(k)]^2}{q^{\alpha(X)}[A(K)_{\text{tor}}]^2}$$

$$\cdot \frac{(\Omega; F; \{\mathcal{A}_i\}; \{X_{va}\})}{\alpha^2} \prod_{v \in S} \frac{\epsilon_v}{(m_{v1})^2}$$

By linear algebra, $(\Omega; F; \{\mathcal{A}_i\}; \{X_{va}\})/|D_i \cdot D_j|$ is the square of the index in Proposition 4.6. Using this to eliminate from (6.3) assorted terms, we come up with yet another expression

$$(6.4) \qquad P_2(X, q^{-s}) \sim (1 - q^{1-s})^{\rho(X)} \frac{[\text{III}(A, K)]}{q^{\alpha(X)}} \frac{|D_i \cdot D_j|}{[NS(X)_{\text{tor}}]^2} \frac{\Pi \, d_v^2 \epsilon_v}{\alpha^2}.$$

Comparing (6.4) with the statement of the Artin–Tate conjecture, we have reduced the equivalence of conjectures to the equality

$$[\text{III}(A, K)] \cdot \prod d_v^2 \epsilon_v = \alpha^2 [\text{Br}(X, k)].$$

The "locally everywhere of degree 1" assumption guarantees, via Lemma 6.2, that for all v, $d_v \Delta_v = 1$. Since ϵ_v divides Δ_v^2, the theorem is proved. ∎

3. The leftover proofs

PROPOSITION 6.4: *The equality below holds*:

$$P_2(X, q^{-s}) = (1 - q^{1-s})^2 P_1(B, q^{-s}) P_1(B, q^{1-s}) \frac{L_S^*(A, s)}{|\mu|^d}$$

$$\cdot \prod_{v \in S} \frac{\int_{A(K_v)} |\omega_v| \mu_v^d \cdot \Pi_a (1 - N_v^{q_a(1-s)})}{P_1(X_v, N_v^{-s}) \cdot (1 - N_v^{1-s})}.$$

PROOF: We start by writing

$$\zeta(X, s) = \prod \zeta(X_v, s) \qquad\qquad v \in |V|$$

By the formalism of the Weil Conjectures, since X is smooth we can

(using Poincaré Duality) write this as

$$(6.5) \quad \frac{P_1(X, q^{-s})P_1(X, q^{1-s})}{(1-q^{-s})P_2(X, q^{-s})(1-q^{2-s})} = \prod \frac{P_1(X_v, N_v^{-s})}{(1-N_v^{-s})\cdot\prod (1-N_v^{q_a(1-s)})}$$

where the explicit form of the right side of the equation comes from Proposition 3.3. Grouping factors on the right, we can write

$$\zeta(V, s) = \prod (1 - N_v^{-s})^{-1} \qquad\qquad v \in |V|$$

and similarly for $\zeta(V, s-1)$. Since V is also smooth, Poincaré Duality allows us to rewrite (6.5) as

$$(6.6) \quad \text{LHS} = \frac{P_1(V, q^{-s})}{(1-q^{-s})(1-q^{1-s})} \cdot \frac{P_1(V, q^{1-s})}{(1-q^{1-s})(1-q^{2-s})} \cdot \prod_{v\in S} P_1(X_v, N_v^{-s})$$

$$\cdot \prod_{v\in S} \frac{P_1(X_v, N_v^{-s})(1-N_v^{1-s})}{\prod_a (1-N_v^{q_a(1-s)})}$$

Solving (6.6) for $P_2(X, q^{-s})$ and noticing that $P_1(X, T)/P_1(V, T) = P_1(B, T)$ by Proposition 4.4, we reach

$$(6.7) \quad P_2(X, q^{-s}) = (1-q^{1-s})^2 P_1(B, q^{-s})P_1(B, q^{1-s}) \frac{1}{\prod_{v\notin S} P_1(X_v, N_v^{-s})}$$

$$\cdot \prod_{v\in S} \frac{\prod_a (1-N_v^{q_a(1-s)})}{P_1(X_v, N_v^{-s})(1-N_v^{1-s})}$$

In both (6.6) and (6.7) we have split the product-factor into "good" and "bad" parts. By definition,

$$(6.8) \quad L_S^*(A, s) = \frac{|\mu|^d}{\prod_{v\notin S} P_1(X, N_v^{-s}) \prod_{v\in S} \int_{A(K_v)} |\omega_v| \mu_v^d}$$

since, as in the proof of Proposition 3.3, $P_1(X_v, T)$ can be interpreted as the characteristic polynomial of arithmetic Frobenius acting on A_v whenever X_v is nondegenerate. Substituting (4) into (3) proves the proposition. ∎

PROPOSITION 6.5: $|\mu|^d q^{b+\deg \omega} = q^{\alpha(X)}$.

PROOF: Use the Leray Spectral Sequence for $f_* \mathcal{O}_X$. Recall that for étale sheaves lifted from quasi-coherent (Zariski) sheaves, $H^i_{\text{ét}} = H^i_{\text{Zar}}$.

This shows that $H^0(V, R^2f_*\mathcal{O}_X)$ is zero since each stalk of $R^2f_*\mathcal{O}_X$ is zero; also $H^2(V, \mathcal{O}_V)$ and $H^3(V, \mathcal{O}_V)$ are both zero. The spectral sequence then gives the exact sequences

$$H^0(V, \mathcal{O}_V) \simeq H^0(X, \mathcal{O}_X)$$

$$0 \to H^1(V, \mathcal{O}_V) \to H^1(X, \mathcal{O}_X) \to H^0(V, R^1f_*\mathcal{O}_X) \to 0$$

$$0 \to H^2(X, \mathcal{O}_X) \simeq H^1(V, R^1f_*\mathcal{O}_X) \to 0$$

Combine these to get $\chi(X, \mathcal{O}_X) = \chi(V, \mathcal{O}_V) - \chi(V, R^1f_*\mathcal{O}_X)$. By Lemma 2.4, $\chi(V, R^1f_*\mathcal{O}_X) = -\deg \omega + d \cdot \chi(V, \mathcal{O}_V)$; these two equalities, together with the fact that V has genus g, give

$$\chi(X, \mathcal{O}_X) = (1 - d)(1 - g) + \deg \omega.$$

Proposition 4.4 shows that $g + b = \dim Pic_X^0$. It is well known that for the particular choice made for μ, $|\mu| = q^{g-1}$. Therefore

$$|\mu|^d q^b q^{\deg \omega} = q^{d(g-1)+b+\deg \omega}$$

which exponent is just $\chi(X, \mathcal{O}_X) - 1 + \dim Pic_X^0$. ∎

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