L-Functions and Tamagawa Numbers of Motives

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dedicated to A. Grothendieck

0. Introduction

The notion of a motif was first defined and studied by A. Grothendieck, and this paper is an attempt to understand some of the implications of his ideas for arithmetic. We will formulate a conjecture on the values at integer points of L-functions associated to motives. Conjectures due to Deligne and Beilinson express these values "modulo $\mathbf{Q}^*$ multiples" in terms of archimedean period or regulator integrals. Our aim is to remove the $\mathbf{Q}^*$ ambiguity by defining what are in fact Tamagawa numbers for motives. The essential technical tool for this is the Fontaine-Messing theory of p-adic cohomology. As evidence for our Tamagawa number conjecture, we show that it is compatible with isogeny, and we include strong results due to one of us (Kato) for the Riemann zeta function and for elliptic curves with complex multiplication.

To recall how the Tamagawa numbers of algebraic groups are related to special values of L-functions, we consider the example $SL_{n,\mathbf{Q}}$. The Tamagawa number, which is 1 in this case, is defined to be the volume of the adelic points modulo global points

$$SL_n(\mathbf{Q})\backslash \prod_{p\leq\infty}{}' SL_n(\mathbf{Q}_p) = SL_n(\mathbf{Q})\backslash SL_n(\mathbb{A}_{\mathbf{Q}})$$

with respect to a canonical measure on $SL_n(\mathbb{A}_{\mathbf{Q}})$, the Tamagawa measure. To define this, choose an isomorphism (det=highest exterior power)

$$\det_{\mathbf{Q}}(sl_n(\mathbf{Q})) \cong \mathbf{Q}. \tag{0.1}$$

Associated to this choice, one can define Haar measures μ_p on $SL_n(\mathbf{Q}_p)$ for $p \leq \infty$, and a measure μ on $SL_n(\mathbb{A}_{\mathbf{Q}})$. The product formula implies

*Partially funded by the NSF.

P. Cartier et al. (eds.), *The Grothendieck Festschrift*

that μ is independent of the choice in (0.1). To actually do the calculation, note the strong approximation theorem gives

$$\mu(SL_n(\mathbf{Q})\backslash SL_n(\mathbb{A}_{\mathbf{Q}})) = \mu_\infty(SL_n(\mathbf{Z})\backslash SL_n(\mathbf{R}) \prod_{p<\infty} \mu_p(SL_n(\mathbf{Z}_p))$$

If (0.1) is chosen so $\det_{\mathbf{Z}}(\mathrm{sl}_n(\mathbf{Z})) \cong \mathbf{Z}$, the μ_p satisfy

$$\mu_p(SL_n(\mathbf{Z}_p)) = \prod_{i=2}^{n}(1-p^{-i})$$

so the fact that the Tamagawa number is 1 is equivalent to

$$\prod_{i=2}^{n}\zeta(i) = \mu_\infty(SL_n(\mathbf{Z})\backslash SL_n(\mathbf{R}))$$

where ζ is the Riemann zeta function.

We turn now to the case of a motif. Broadly speaking, a motif M is a sutiable direct factor of $H^r(X, \mathbf{Z}(n))$ for some universal cohomlogy theory for smooth, complete varieties X. Suppose X is defined over $\mathbf{Q}$, so one has an associated L-function $L(M,s)$. Deligne and Beilinson have formulated conjectures sbout the special value $L(M,0)$ **[De2]**, **[Be1]**. To fix ideas, suppose $M = H^r(X,\mathbf{Z}(n))$. We will always assume $n \geq (r+1)/2$ (i.e., the weight of $M \leq -1$), but for purposes of this introduction let us assume M has weight < -2. The intermediate jacobian

$$A(\mathbf{C}) = H^r(X(\mathbf{C}),\mathbf{C})/\{H^r(X,\mathbf{Z}(n)) + F^n H^r(X(\mathbf{C}),\mathbf{C})\}$$

has a real structure given by invariants of conjugation acting simultaneously on the topological space $X(\mathbf{C})$ and on the coefficients $\mathbf{C}$. Moreover, the canonical identification

$$H^r(X(\mathbf{C}),\mathbf{C}) \cong H^r_{DR}(X/\mathbf{Q}) \otimes \mathbf{C}$$

defines a $\mathbf{Q}$-structure on the tangent space $H^r(X(\mathbf{C}),\mathbf{C})/F^nH^r(X(\mathbf{C}),\mathbf{C})$ to $A(\mathbf{C})$. The choice of an isomorphism

$$\det(H^r_{DR}(X/\mathbf{Q})/F^nH^r_{DR}(X/\mathbf{Q})) \xrightarrow{\sim} \mathbf{Q} \tag{0.2}$$

thus determines a Haar measure ω_∞ on $A(\mathbf{C})$. In the cases considered by Deligne, $A(\mathbf{R})$ is compact, and his conjecture reads

$$L(M,0) \in \omega_\infty(A(\mathbf{R}))\cdot\mathbf{Q},$$

i.e. $L(M,0)$ is a rational multiple of the volume of $A(\mathbf{R})$. Deligne has also pointed out [De5] that Beilinson's conjectures can be formulated similarly. In the Beilinson case, $A(\mathbf{R})$ is not compact, but the image of the regulator map on a suitable motivic cohomology group defines a subgroup one might call $A(\mathbf{Z}) \subset A(\mathbf{R})$. Then Beilinson conjectures that $A(\mathbf{Z}) \subset A(\mathbf{R})$ is discrete and co-compact, and

$$L(M,0) \in \omega_\infty(A(\mathbf{R})/A(\mathbf{Z})) \cdot \mathbf{Q}^*.$$

Notice that ω_∞ depends up to a rational number on the non-canonical choice we made in (0.2), so the $\mathbf{Q}$ or $\mathbf{Q}^*$ ambiguity appears necessary. (However, when $n > r$, so $F^n H^r(X(\mathbf{C}),\mathbf{C}) = (0)$, a polarization on $H^r_{DR}(X/\mathbf{Q})$ determines a preferred isomorphism (0.2).) We will define abelian groups $A(\mathbf{Q}_p)$ for all p, with Haar measure μ_p (depending in the expected way on the choice of (0.2)) such that for almost all p,

$$\mu_p(A(\mathbf{Q}_p)) = P_p(M,1),$$

where (ignoring factors at bad primes)

$$L(M,s) = \prod P_p(M,p^{-s})^{-1}.$$

We also define (more precisely, we postulate) an abelian group $A(\mathbf{Q})$ mapping to all the $A(\mathbf{Q}_p), p \leq \infty$. The properties we need essentially follow from Beilinson's conjectures for motivic cohomology, however, and one can hope that $A(\mathbf{Q})$ is isomorphic to the relevant motivic cohomology (essentially $H^{r+1}_{\mathcal{M}}(X,\mathbf{Z}(n))$, although one has to be careful about bad fibres). The Tamagawa number is defined by

$$\operatorname{Tam}(M) = \mu\Big(\prod_{p\leq\infty} A(\mathbf{Q}_p)/A(\mathbf{Q})\Big)$$

Just as in the algebraic group case, this is canonically defined independent of any choices. Roughly speaking, the statement that $\operatorname{Tam}(M)$ is a rational number is equivalent to the Beilinson and Deligne conjectures. In fact, if the motivic cohomology term $A(\mathbf{Q})$ is defined using a regular model of X over $\mathbf{Z}$, then the Beilinson conjecture says $L(M,0) \in \mu_\infty(A(\mathbf{R})/A(\mathbf{Q}))\cdot\mathbf{Q}^*$. Since $\mu_p(A(\mathbf{Q}_p)) = L_p(M,0)^{-1}$ for almost all p, and since the Euler product expression for $L(m,0)$ converges in weights ≤ -3, we get the assertion.

Clearly it would be more elegant to work with the variety over $\mathbf{Q}$ rather than over $\mathbf{Z}$. We do this, denoting the corresponding groups $B(\mathbf{Q}_p)$ and

$B(\mathbf{Q})$. One might think of $A(\mathbf{Q}_p) = B(\mathbf{Z}_p)$ and $A(\mathbf{Q}) = B(\mathbf{Z})$. The quotient $B(\mathbf{Q}_p)/A(\mathbf{Q}_p)$ must be discrete and should have rank equal to the order of local zero of $P_p(M^*(1), u)$ at $u = 1$. Unfortunately, our description of it relies on conjectural properties of motivic cohomology. We therefore prefer to emphasize the A-theory, where at least the local properties are actually proved.

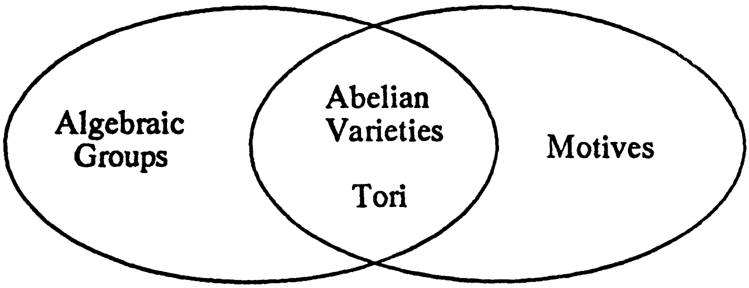
We define a group $\text{Ш}(M)$, which we conjecture to be finite, and we propose a conjectural Tamagawa number formula

$$\text{Tam}(M) = \#H^0(\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}), M^* \otimes \mathbf{Q}/\mathbf{Z}(1)) \cdot (\#\text{Ш}(M))^{-1}$$

which is precisely analogous to the formula of Ono [On2] for tori. We prove that the truth of this formula is isogeny invariant in a suitable sense.

The outline of this paper is as follows. §1 is a summary of some of the "linear algebra" associated to the rings B_{crys} and B_{DR} of Fontaine. §2 contains a comparison result between Fontaine-Messing theory and the Coates-Wiles homomorphism which is the key to results on the Riemann zeta function proved in §6. Strictly speaking this section could be posponed until after §4, but we have put it earlier because it gives some feeling for the deeper, non-formal aspects of the problem. §3 studies the cohomology of the local galois groups with values in the Tate module T associated to the motive. In particular, the subgroup $A(\mathbf{Q}_p) = H^1_f(\mathbf{Q}_p, T) \subset H^1(\mathbf{Q}_p, T)$ which generalizes the notion of the image of the $\mathbf{Q}_p$-points of an abelian variety under the boundary map of the Kummer sequence, is introduced and the exponential map, which generalizes the p-adic exponential for a p-divisible group and enables us define the local Tamagawa measure, is defined. §4 is devoted to local volume calculations. It contains the proof that $\mu_p(A(\mathbf{Q}_p)) = P_p(M, 1)$ for almost all p, as well as a much more precise result for the motif $\mathbf{Z}(n)$ associated to the Riemann zeta function. §5 contains the global conjectures. The main result here is the invariance of our conjecture under isogeny. Finally, §6 and 7 contain the arguments for the Riemann zeta function and for CM elliptic curves.

Here, then, is a picture of the land of Tamagawa numbers.



The reader should look at it closely. Anyone who perceives a larger "third oval" containing the other two and explaining their relation, must contact us immediately.

We would like to thank Professors Deligne, Fontaine, Jannsen, Messing, Perrin-Riou, and Soulé for helpful conversations on subjects related to this paper.

1. The rings B_{crys} and B_{DR}

The purpose of this section is to summarize properties of the ring B_{crys} which we will use. Much of the discussion is for the convenience of the reader and duplicates material in [**Fo**] and [**FM**].

Let K be a complete characteristic 0 discrete valuation field with valuation ring Λ and residue field $k = \Lambda/\mathfrak{m}$ perfect of characteristic $p > 0$. Let $\overline{K}$ be the algebraic closure of K, $A = \overline{\Lambda}$ the integral closure of Λ in $\overline{K}$, and write $A_n = A/p^n A$.

Let $W_n(A_1)$ be the ring of p-Witt vectors of length n on A_1. Elements in $W_n(A_1)$ are represented in Witt coordinates by n-tuples $(a_0, \dots, a_{n-1})$ of elements of A_1. There is a surjective ring homomorphism

$$\begin{aligned} &\Theta_n : W_n(A_1) \twoheadrightarrow A_n \\ &\Theta_n(a_0, \dots, a_{n-1}) = \hat{a}_0^{p^n} + p\hat{a}_1^{p^{n-1}} + \cdots + p^{n-1}\hat{a}_{n-1} \end{aligned} \tag{1.1}$$

where $\hat{a}$ is any lifting of a. Define

$$\begin{aligned} &\vartheta_n : W_{n+1}(A_1) \to W_n(A_1), \\ &\vartheta_n(a_0, \dots, a_n) = (a_0^p, \dots, a_{n-1}^p). \end{aligned} \tag{1.2}$$

One checks that

$$\Theta_n \circ \vartheta_n \equiv \Theta_{n+1} \mod p^n. \tag{1.3}$$

Let $\varprojlim A$ be the multiplicative monoid which is the inverse limit of copies of A under the p-th power map. Reduction mod p induces an isomorphism of multiplicative monoids, $\varprojlim A \cong R \stackrel{\text{def}}{=} \varprojlim A_1$. Note that R is a ring. One can identify $\varprojlim_{\vartheta} W_n(A_1)$ with $W(R) = \varprojlim_{v} W_n(R)$ where $v_n : W_{n+1}(R) \to W_n(R)$ is the projection $v_n(a_0, \dots, a_n) = (a_0, \dots, a_{n-1})$. The Teichmuller map $[\,] : A_1 \to W_n(A_1)$ (resp. $R \to W(R)$) is given by $[a] = (a, 0, 0, \dots, 0)$.

It is compatible with multiplication. The diagram

$$
(1.4)\qquad
\begin{array}{ccccc}
R & \xrightarrow{[\,]} & W(R) & \longrightarrow & W_n(A_1) \\
\Vert & & & & \downarrow{\scriptstyle \Theta_n} \\
\varprojlim A & \longrightarrow & A & \longrightarrow & A_n
\end{array}
$$

commutes. Finally, W and W_n have a frobenius operator f (induced by the p-th power operator on the argument ring when that ring has characteristic p). The image of $[]$ is the set of elements x in $W(R)$ such that $f(x) = x^p$. We will fix a generator ε of $\mathbf{Z}_p(1)$ with $\varepsilon \stackrel{\text{def}}{=} [1, \zeta_1, \zeta_2, \ldots] \in \varprojlim A$, where $\zeta_{i+1}^p = \zeta_i$. It follows from (1.4) above that the image e_n of ε in $W_n(A_1)$ lies in $1 + \mathrm{Ker}(\Theta_n)$, and $f(e_n) = e_n^p$.

Following [FM], we define

$$B_n = W_n^{PD}(A_1); B_\infty = \varprojlim B_n$$

(limit with respect to maps $B_{n+1} \to B_n$ induced by

$$\vartheta_n : W_{n+1}(A_1) \to W_n(A_1).)$$

to be the $W_n(A_1)$-algebras obtained by adjoining divided powers to the ideals $\mathrm{Ker}(\Theta_n)$ in a universal way compatible with the divided powers on the ideal (p). Θ_n extends to surjections

$$
(1.6)\qquad
\begin{gathered}
\Theta_n : B_n \twoheadrightarrow A_n; \mathrm{Ker}(\Theta_n) \stackrel{\text{def}}{=} J_n \\
\Theta_\infty : B_\infty \twoheadrightarrow \widehat{A} \stackrel{\text{def}}{=} \varprojlim A_n
\end{gathered}
$$

(limit with respect to $\bmod p^n : A_{n+1} \to A_n$).

The rings B_n are linked with the crystalline cohomology via the theorem of Fontaine and Messing [FM]

$$(1.7)\qquad B_n \cong \Gamma((\mathrm{Spec}(A_n))_{\mathrm{crys}}, \mathcal{O}_{\mathrm{crys}}).$$

The frobenius f on $W_n(A_1)$ satisfies $f(\mathrm{Ker}(\Theta_n)) \subset \mathrm{Ker}(\Theta_n) + (p)$, so there is an induced frobenius f on the divided envelope B_n. (This also follows directly from (1.7)). In addition, the Galois group $G = \mathrm{Gal}(\overline{K}/K)$ acts on B_n.

For $x \in J_n$, the divided power structure gives us for every $N \geq 1$ an element $\gamma_N(x) \in J_n$ having all the natural properties of $x^N/N!$. In particular, $\log(1+x) \in J_n$ is defined for $n \leq \infty$. We define

$$t_n = \log(e_n) \in J_n; t = t_\infty = \log(e_\infty) \in J_\infty. \tag{1.8}$$

We can think of t as $\log([\varepsilon])$ in a certain $W(R)$-algebra. Note that $f(t_n) = p \cdot t_n$. The assignments $\zeta_n \mapsto t_n; \varepsilon \mapsto t$ give G-equivariant inclusions

$$\mu_{p^n} \hookrightarrow J_n; \mathbf{Z}_p(1) \hookrightarrow J_\infty. \tag{1.9}$$

Here is some further notation we will need:

$$B^+_{\text{crys}} = B_\infty \otimes \mathbf{Q}; J_{\mathbf{Q}} = J_\infty \otimes \mathbf{Q}; B_{\text{crys}} = B^+_{\text{crys}}[t^{-1}]; \tag{1.10}$$

$J_n^{[r]}$ = r-th divided power of J_n (generated by products $\prod \gamma_{N_i}(x_i)$ with $\sum N_i \geq r$.)

The completion $\mathbf{C}_p$ of $\overline{K}$ is related to B^+_{crys} via

$$0 \to J_{\mathbf{Q}} \to B^+_{\text{crys}} \to \mathbf{C}_p \to 0. \tag{1.11}$$

Writing K_0 for the quotient field of $W(k)$, one has

$$H^0(G, B^+_{\text{crys}}) = H^0(G, B_{\text{crys}}) \cong K_0. \tag{1.12}$$

The above constructions also lead to exact sequences for $r \geq 0$

$$0 \to \mathbf{Q}_p(r) \to J_{\mathbf{Q}}^{[r]} \xrightarrow{1-p^{-r}f} B^+_{\text{crys}} \to 0. \tag{1.13}$$

In a sense, B_{crys} is the ring of all p-adic periods associated to varieties with good reduction at p. In order to consider also bad reduction primes, we need the larger ring

$$B^+_{DR} = \varprojlim_r B^+_{\text{crys}}/J_{\mathbf{Q}}^{[r]} \tag{1.14}$$

B^+_{DR} is a complete discrete valuation ring with residue field $\mathbf{C}_p$ and maximal ideal generated by the image of the element t described above. We write

$$B_{DR} = B^+_{DR}[t^{-1}] \tag{1.15}$$

for the corresponding discretely valued field. As G-module,

$$(B_{DR})^i/(B_{DR})^{i+1} \cong \mathbf{C}_p(i); \quad -\infty < i < \infty. \tag{1.15.1}$$

Also

$$H^0(G, B_{DR}^+) \cong H^0(G, B_{DR}) \cong K. \tag{1.16}$$

Proposition 1.17. *The following sequences are exact*

$$0 \to \mathbf{Q}_p \xrightarrow{\alpha} B_{\mathrm{crys}}^{f=1} \oplus B_{DR}^+ \xrightarrow{\beta} B_{DR} \longrightarrow 0 \tag{1.17.1}$$

where $\alpha(x) = (x, x)$ *and* $\beta(x, y) = x - y$*;*

$$0 \longrightarrow \mathbf{Q}_p \xrightarrow{\alpha} B_{\mathrm{crys}} \oplus B_{DR}^+ \xrightarrow{\beta} B_{\mathrm{crys}} \oplus B_{DR} \longrightarrow 0 \tag{1.17.2}$$

where $\alpha(x) = (x, x)$ *and* $\beta(x, y) = (x - f(x), x - y)$.

Proof. We have $B_{\mathrm{crys}}^{f=1} \cap B_{DR}^+ = \mathbf{Q}_p$ by ([Fo], (4.12) and (4.13)). Also $1 - f : B_{\mathrm{crys}} \to B_{\mathrm{crys}}$ is surjective because $B_{\mathrm{crys}} \supset J_{\mathbf{Q}}^{[r]} \cdot t^{-r}$ for $r \geq 0$, and $1 - p^{-r} f : J_{\mathbf{Q}}^{[r]} \twoheadrightarrow B_{\mathrm{crys}}^+$ by ([FM], 2.1). It remains to prove that $B_{\mathrm{crys}}^{f=1} + B_{DR}^+ = B_{DR}$.

Lemma 1.17.3. *Fix* $r \geq 1$. *Then* $B_{DR}^+ \subset t \cdot B_{DR}^+ + \{a \in B_{\mathrm{crys}} \mid f(a) = p^r \cdot a\}$.

Note the lemma implies $t^{-r} \cdot B_{DR}^+ \subset B_{DR}^+ + B_{\mathrm{crys}}^{f=1}$ as claimed.

Proof of (1.17.3). The ring $\widehat{A} \subset \mathbf{C}_p$ is is defined in (1.6). Taking limits with respect to the p-th power map, we get $\varprojlim \widehat{A} \cong \varprojlim A \cong R$ so there is a Teichmuller map making the diagram

$$\begin{array}{ccc} \varprojlim \widehat{A} & \xrightarrow{[\,]} & B_\infty \\ \downarrow & & \downarrow \\ \widehat{A} & \longrightarrow & \mathbf{C}_p \end{array}$$

commute. If $a = (a_0, \ldots) \in \varprojlim \widehat{A}$ satisfies $a_0 \in 1 + p \cdot \widehat{A}$, then $[a] \in 1 + ((p) + J_\infty)$, so $\log([a]) \in B_\infty$ is defined and maps to $\log(a_0) \in \mathbf{C}_p$. Also $f(\log([a])) = p \cdot \log([a])$, so

$$f\{\log([a^{(1)}]) \cdot \ldots \cdot \log([a^{(r)}])\} = p^r \cdot \log([a^{(1)}]) \cdot \ldots \cdot \log([a^{(r)}]).$$

Since $\mathbf{C}_p$ is generated as a $\mathbf{Q}_p$-vector space by expressions $\log(a^{(1)}) \cdot \ldots \cdot \log(a^{(r)})$, we find $B_{\mathrm{crys}}^+ \subset J_{\mathbf{Q}} + \{a \in B_{\mathrm{crys}} \mid f(a) = p^r \cdot a\}$. Since $B_{DR}^+/t \cdot B_{DR}^+ \cong B_{\mathrm{crys}}^+/J_{\mathbf{Q}}$, the lemma is proved. Q.E.D.

This completes the proof of Proposition (1.17). Q.E.D.

Remarks 1.18. We will need to consider continuous Galois cohomology for sequences obtained from the sequences in (1.17) by tensoring with continuous representations on free $\mathbf{Z}_p$-modules of finite rank. For more about the topology on the rings B_{crys} and B_{DR} the reader is referred to [Fo]. Let us, however, at least sketch e.g. how one sees that the sequence (1.17.1) leads to a long exact sequence of continuous cohomology. Since $B_{DR} = \varinjlim t^{-n} B_{DR}^{+}$ it will suffice to show the map $\beta_n : (t^{-n}B_{\text{crys}}^{+})^{f=1} \oplus B_{DR}^{+} \twoheadrightarrow t^{-n}B_{DR}^{+}$ admits a continuous (but not G-equivariant!) splitting. For this, one reduces easily to showing $\overline{\beta} : (t^{-n}B_{\text{crys}}^{+})^{f=1} \twoheadrightarrow t^{-n}B_{DR}^{+}/B_{DR}^{+}$ admits a continuous section. The group on the right has a subgroup $\overline{V}$ consisting of the image mod B_{DR}^{+} of the group $V \subset (t^{-n}B_{\text{crys}}^{+})^{f=1}$ of all expressions of the form

$$\sum_{1\leq j\leq n} \prod_{i\leq j} \log([a^{i,j}]) \cdot t^{-j} \tag{1.18.1}$$

where $a^{i,j} = (a_k^{i,j})_{k=0,1,2,\dots}$ with $a_0^{i,j} \in 1 + p \cdot \widehat{A}$ and $(a_k^{i,j})^p = (a_{k-1}^{i,j})$. Note that such an expression makes sense also in $B_{\text{crys}}^{f=1}$. We have, with notation as above

$$[a^{i,j}] \in 1 + pW(R) + \operatorname{Ker}(\Theta),$$

so $\log([a^{i,j}]) \in p \cdot B_\infty + J_\infty$. It follows that $V \subset t^{-n}B_\infty$. One sees easily that $\bigcap p^m\overline{V} = (0)$, and the $p^m\overline{V}$ form a basis of neighborhoods of $t^{-n}B_{DR}^{+}/B_{DR}^{+}$. Since $\overline{V}$ is separated, it is free over $\mathbf{Z}_p$ so the surjection $V \twoheadrightarrow \overline{V}$ splits. To check that a splitting of this map induces a continuous map

$$t^{-n}B_{DR}^{+}/B_{DR}^{+} = \overline{V} \otimes \mathbf{Q} \to V \otimes \mathbf{Q} \subset (t^{-n}B_{\text{crys}}^{+})^{f=1}$$

it suffices to show that the $p^mV \to 0$ in $(t^{-n}B_{\text{crys}}^{+})^{f=1}$, but this follows because $p^m t^{-n} B_\infty \to 0$ in B_{crys} as $m \to \infty$.

2. The Coates-Wiles homomorphism and Fontaine-Messing theory

The Coates-Wiles homomorphism has an important role in the local theory of Cyclotomic fields [Wa]. The aim of of this section is to derive a formula (2.1.1) describing the relation between the Coates-Wiles homomorphism and the Fontaine-Messing theory of p-adic periods [FM]. The result will be used in §6 to study the Tamagawa number of the motif $\mathbf{Z}(r)$.

For the remainder of this section, K denotes an unramified finite extension of $\mathbf{Q}_p$ with ring of integrs $\mathcal{O}_K$. Let

$$G = \operatorname{Gal}(K(\zeta_{p^\infty})/K)$$

be the Galois group of the p-cyclotomic extension, and let

$$P = \mathrm{Gal}(K(\zeta_{p^\infty})^{\mathrm{ab}}/K(\zeta_{p^\infty}))$$
$$U = \varprojlim_n (\mathcal{O}_K[\zeta_{p^n}]^*)$$

where $K(\zeta_{p^\infty})^{\mathrm{ab}}$ is the maximal abelian extension of $K(\zeta_{p^\infty})$ and the $\varprojlim$ is taken with respect to the norms. We regard P and U as G-modules in the natural way. By the local class field theory, U is identified with the inertia subgroup of P.

The Coates-Wiles homomorphism is a canonical continuous G-homomorphism

$$\phi^r_{CW} : U \to K(r) = K \otimes \mathbf{Q}(r)$$

defined for $r \geq 1$.

Next we denote the continuous Galois cohomology $H^q(\mathrm{Gal}(\overline{K}/K),)$ by $H^q(K, \ )$. Since

$$P/U \cong \hat{\mathbf{Z}} = \varprojlim_n \mathbf{Z}/n\mathbf{Z}$$

with the trivial action of G, and since

$$H^q(G, \mathbf{Q}_p(r)) = (0) \quad \text{for } q, r \geq 1$$

we have for $r \geq 1$

$$\begin{aligned} H^1(K, K(r)) &\cong H^0(G, H^1(K(\zeta_{p^\infty}), K(r))) \\ &\cong \mathrm{Hom}_G(P, K(r)) \cong \mathrm{Hom}_G(U, K(r)). \end{aligned}$$

We regard ϕ^r_{CW} as an element of $H^1(K, K(r))$ via these isomorphisms.

The main result of this section is:

Theorem 2.1. *Let K be a finite unramified extension of $\mathbf{Q}_p$ for p an odd prime, and take $r \geq 1$. Then the boundary map from (1.13)*

$$\partial^r : K = H^0(K, B^+_{\mathrm{crys}}) \to H^1(K, \mathbf{Q}_p(r))$$

satisfies

$$\partial^r(a) = T(a \cdot \phi^r_{CW})/(r-1)!$$

where $T : H^1(K, K(r)) \to H^1(K, \mathbf{Q}_p(r))$ is the trace for $K/\mathbf{Q}_p$. In particular, if $K = \mathbf{Q}_p$, we have

$$\phi^r_{CW} = (r-1)! \cdot \partial^r(1). \tag{2.1.1}$$

The above result will be deduced from an explicit reciprocity law for the motif $\mathbf{Z}(r)$ given in Theorem 2.6. Here we regard the classical explicit reciprocity law as the explicit reciprocity law for $\mathbf{Z}(1)$ or for $\mathbf{G}_m$. We expect there will be explicit reciprocity laws for elliptic curves with CM.

We now recall Coleman's definition of the Coates-Wiles homomorphism [Co2]. Consider the power series ring

$$\mathcal{R} = \mathcal{O}_K[[T]]; \quad z = 1 + T.$$

It will be convenient to regard $\mathcal{R}$ as the ring of functions on the formal multiplicative group $\widehat{G}_m$ over $\mathcal{O}_K$ with the canonical function z. Let for $a \in \mathbf{Z}_p$

$$\sigma_a \text{ (resp. } f, \text{ resp. } \varphi) : \mathcal{R} \to \mathcal{R}$$

be the unique ring homomorphism characterized by the property that the restriction of σ_a (resp. f, resp. φ) to $\mathcal{O}_K$ is the identity map (resp. the frobenius of $\mathcal{O}_K$, resp. the frobenius of $\mathcal{O}_K$), and

$$\sigma_a(z) = z^a \text{ (resp. } f(z) = z^p, \text{ resp. } \varphi(z) = z).$$

Here

$$z^a = \sum_{n=0}^{\infty} (n!)^{-1} \Big(\prod_{i=0}^{n-1} (a - i) \Big) (z-1)^n \in \mathcal{R}.$$

In other words, σ_a comes from the a-th power map $\widehat{\mathbf{G}}_m \to \widehat{\mathbf{G}}_m$. Since σ_p and f are finite flat maps of degree p, there are norm and trace maps

$$N_p, T_p (\text{resp. } N_f, T_f) : \mathcal{R} \to \mathcal{R}$$

satisfying

$$N_p \circ \sigma_p(x) = x^p; \quad T_p \circ \sigma_p(x) = px;$$
$$N_f \circ f(x) = x^p; \quad T_f \circ f(x) = px; \quad \forall x \in \mathcal{R}.$$

We define the action of $G = \mathrm{Gal}(\mathbf{Q}_p(\zeta_{p^\infty})/\mathbf{Q}_p)$ on $\mathcal{R}$ as follows. The action of G on $\mathbf{Z}_p(1)$ defines an isomorphism $\kappa : G \cong \mathbf{Z}_p^*$. The action of $\kappa^{-1}(a)$ on $\mathcal{R}$ is by σ_a. All the above operations on $\mathcal{R}$ commute with this G-action. Fix a generator $\nu = (\zeta_{p^n})_{n=0,1,\ldots}$ for $\mathbf{Z}_p(1)$. Define $\rho_n : \mathcal{R} \to \mathcal{O}_K[\zeta_{p^n}]$ for $n \geq 1$ to be the unique $\mathcal{O}_K$ homomorphism such that $\rho_n(z) = \zeta_{p^n}$. Then ρ_n is also compatible with the G-action,and we have commutative diagrams

(2.1.1)
$$\begin{array}{ccccccc}
\mathcal{R} & \xrightarrow{\sigma_p} & \mathcal{R} & \quad & \mathcal{R}^* & \xrightarrow{N_p} & R^* \\
{\scriptstyle \rho_n}\downarrow & & \downarrow{\scriptstyle \rho_{n+1}} & & {\scriptstyle \rho_{n+1}}\downarrow & & \downarrow{\scriptstyle \rho_n} \\
\mathcal{O}_K[\zeta_{p^{n+1}}] & \hookrightarrow & \mathcal{O}_K[\zeta_{p^{n+1}}] & & \mathcal{O}_K[\zeta_{p^{n+1}}]^* & \xrightarrow{\text{norm}} & \mathcal{O}_K[\zeta_{p^n}]^*.
\end{array}$$

Theorem 2.2. (Coleman) *There exists an isomorphism of G-modules*

$$U \cong \{a \in \mathcal{R}^* \mid N_f(a) = a\}; \quad u = \varprojlim u_n \mapsto g_u$$

where g_u is characterized as the unique element of $\mathcal{R}^*$ such that

$$\rho_n(\varphi^{-n}(g_u)) = u_n$$

for any $n \geq 1$.

Write*

$$t = \log(z) = \sum_{i=1}^{\infty}(-1)^{i-1}T^i/i; \quad T = z - 1,$$

and consider the inclusion $\mathcal{R} \subset K[[t]] \cong K[[T]]$. Define the homomorphisms $\phi^r : U \to K$ $(r \geq 0)$ by

$$\log(g_u) = \sum_{n=1}^{\infty} \phi^r(u)t^r/r! \in K[[t]].$$

The map

$$\phi^r_{CW} : U \to K(r); \quad u \mapsto \phi^r(u) \otimes \nu^{\otimes r}$$

is compatible with the G-action and is independent of the choice of ν generating $\mathbf{Z}_p(1)$. We call these maps Coates-Wiles homomorphisms

Let

$$\widehat{\Omega}^1_{\mathcal{R}} = \varprojlim_n \Omega^1_{\mathcal{R}/\mathbf{Z}}/p^n\Omega^1_{\mathcal{R}/\mathbf{Z}},$$

a free $\mathcal{R}$ module of rank one with standard basis dz/z. Define

$$T_p : \widehat{\Omega}^1_{\mathcal{R}} \to \widehat{\Omega}^1_{\mathcal{R}}; \quad T_p(h \cdot dz/z) = p^{-1}T_p(h) \cdot dz/z.$$

(Note $T_p(\mathcal{R}) \subset p\mathcal{R}$.) We have

$$\phi^r_{CW}(u)/(r-1)! = \text{Res}(t^{-r}dg_u/g_u) \otimes \nu^{\otimes r}; \text{ for } u \in U, \tag{2.3}$$

where Res is the residue for the field of normal power series $K((t))$. Since $T = \exp(t) - 1$, we get

$$\text{Res}(t^{-r}\widehat{\Omega}^1_{\mathcal{R}}) \subset ((r-1)!)^{-1}.\mathcal{O}_K. \tag{2.4}$$

---

* The referee points out that there is a natural homomorphism $\nu : \mathcal{R} \to \mathcal{R}$ where R is as in §1 such that $\nu(z) = \varepsilon$. Thus t in §2 =log(z) while t in §1 =log$([\nu(z)])$. For $u = \lim u_n$ in U, $\nu(g_u) = (a_k)$ with $a_k = \lim u_{n+k}^{p^n}$.

We rewrite theorem (2.1) in the form of the explicit reciprocity law for the motif $\mathbf{Z}(r)$. Consider the cup product pairing

$$H^1(K,\mathbf{Q}_p(r)) \times (\varprojlim_n H^1(K(\zeta_{p^n}),\mathbf{Z}/p^n(1)\otimes\mathbf{Q}) \to \\ \to \varprojlim_n H^2(K(\zeta_{p^n}),\mathbf{Z}/p^n(r+1))\otimes\mathbf{Q} \cong \mathbf{Q}_p(r),$$

where the inverse limits are taken with respect to the norm maps, and the last isomorphism is induced by the canonical isomorphism

$$H^2(K(\zeta_{p^n}),\mathbf{Z}/p^n(r+1)) \cong \mathbf{Z}/p^n(r).$$

Theorem (2.1) is equivalent to the assertion for $a \in K$ and $u \in U$ that the above pairing sends

$$(2.5)\qquad (\partial^r(a),u) \in H^1(K,\mathbf{Q}_p(r)) \times \varprojlim H^1(K(\zeta_{p^n}),\mathbf{Z}/p^n(1)) \\ \mapsto \mathrm{Tr}_{K/\mathbf{Q}_p} \mathrm{Res}(at^{-r}dg_u/g_u)\otimes \nu^{\otimes r}.$$

Here we denote the image of u in $\varprojlim H^1(K(\zeta_{p^n}),\mathbf{Z}/p^n(1))$ under the canonical Kummer isomorphism $K(\zeta_{p^n})^\times/K(\zeta_{p^n})^{\times n} \cong H^1(K(\zeta_{p^n}),\mathbf{Z}/p^n(1))$ by the same letter u.

In Theorem (2.6) below, we prove the $\bmod p^n$ version of this result for every $n \geq 1$. For this, we need an integral version of sequence (1.13), proved in ([FM] III, 1.1, and III 5.2). Write $[x]$ for the integer part of a real number x, and define

$$c = c(r) = \sum_{i=0}^{\infty}[r(p-1)^{-1}p^{-i}]; \quad J_\infty^{(r)} = \{x \in J_\infty^{[r]} \mid f(x) \in p^r B_\infty\}$$

Then the sequence

$$(2.5.1)\qquad 0 \to p^{-c}\mathbf{Z}_p(r) \to J_\infty^{(r)} \xrightarrow{1-p^{-r}f} B_\infty \to 0$$

is exact. For $m \geq c(r)$, let

$$\partial^r_{n,m} : \mathcal{O}_K \to H^1(K,\mathbf{Z}/p^n(r))$$

be the composite

$$\mathcal{O}_K \to H^0(K,B_\infty) \xrightarrow{\partial} H^1(K,p^{-c}\mathbf{Z}_p(r)) \xrightarrow{p^m} \\ H^1(K,\mathbf{Z}_p(r)) \to H^1(K,\mathbf{Z}/p^n(r)),$$

where ∂ is the boundary map from (2.5.1). Thus

$$\partial^r = p^{-m}(\mathbf{Q} \otimes \varprojlim_n \partial^r_{n,m})$$

The explicit reciprocity law is the following:

Theorem 2.6. *Let K be an unramified finite extension of $\mathbf{Q}_p$, and let $\mathcal{R}$, ν be as above. Then for $r, n \geq 1$ and $m \geq c(r+1)$, there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{O}_K \times \mathcal{R}^\times & \xrightarrow{(a,b)\mapsto a\cdot db/b} & \hat{\Omega}^1_{\mathcal{R}} \\ {\scriptstyle (\partial^r_{n,m},\rho_n)}\downarrow & & \downarrow{\scriptstyle \varepsilon^r_{n,m}} \\ H^1(K, \mathbf{Z}/p^n(r)) \times H^1(K(\zeta_{p^n}), \mathbf{Z}/p^n(1)) & \longrightarrow & H^2(K(\zeta_{p^n}), \mathbf{Z}/p^n(r+1)) \\ & & \wr\| \\ & & \mathbf{Z}/p^n(r) \end{array}$$

where

$$\varepsilon^r_{n,m}(\omega) \equiv p^m \operatorname{Tr}_{K/\mathbf{Q}_p} \operatorname{Res}(t^{-r} T^n_p(\omega)) \otimes \nu^{\otimes r} \mod p^n.$$

Note $c(r+1) \geq c(r-1) \geq \operatorname{ord}_p((r-1)!)$ so the expression on the right is integral by (2.4). If $r < p-2$, we can take $m = 0$ since $c(p-2) = 0$. It is probable that the hypothesis on m in (2.6) can be weakened to $m \geq c(r)$.

Corollary 2.6.1. *For $u = (u_n)_n \in U$, the lower pairing in the above diagram sends $(\partial^r_{n,c(r)}(a), u_n)$ to $p^{c(r)} \cdot \operatorname{Tr}_{K/\mathbf{Q}_p} \operatorname{Res}(at^{-r} \cdot dg_u/g_u) \otimes \nu^{\otimes r}$ mod p^n.*

Note that the corollary is an easy consequence of the theorem. Indeed, (2.6) shows that the image of $(\partial^r_{n,m}(a), u_n)$ in $\mathbf{Z}/p^n(r)$ for $m \geq c(r+1)$ is $\varepsilon^r_{n,m}(a \cdot d(\varphi^{-n}(g_n))/\varphi^{-n}(g_n))$. Since $N_f(g_u) = g_u$, $T_f(dg_u/g_u) = dg_u/g_u$ and hence $T^n_p(d(\varphi^{-n}(g_n))/\varphi^{-n}(g_n)) = dg_u/g_u$. We can replace m by $c(r)$ by passing to the inverse limit.

Note (2.1) is deduced from (2.6.1) as in (2.5).

To prove (2.6), we must use some of the more difficult aspects of the theory of [FM]. We need the exact sequence of sheaves

$$0 \to S^r_n \to \underline{\underline{J}}^{(r)}_n \xrightarrow{1-p^{-r}f} \underline{\underline{\mathcal{O}}}^{\text{crys}}_n \longrightarrow 0 \tag{2.7}$$

on the small syntomic site over $\mathrm{Spec}(\mathcal{R})$ or that over $\mathrm{Spec}(\mathcal{O}_K[\zeta_{p^n}])$. Here $\underline{\underline{\mathcal{O}}}_n^{\mathrm{crys}}$ is defined so that $\underline{\underline{\mathcal{O}}}_n^{\mathrm{crys}}(X')$ for a syntomic scheme X' over X is the ring of global sections of the structural sheaf of the crystalline site on $X' \otimes \mathbf{Z}/p^n\mathbf{Z}$. If we denote by $\underline{\underline{\mathcal{O}}}_n$ the sheaf

$$\underline{\underline{\mathcal{O}}}_n(X') = \Gamma(X' \otimes \mathbf{Z}/p^n\mathbf{Z}, \underline{\underline{\mathcal{O}}}),$$

and by $\underline{\underline{J}}_n^{[r]} \subset \underline{\underline{\mathcal{O}}}_n^{\mathrm{crys}}$ the r-th divided power of the ideal

$$\mathrm{Ker}(\underline{\underline{\mathcal{O}}}^{\mathrm{crys}} \to \underline{\underline{\mathcal{O}}}_n)$$

then $\underline{\underline{J}}^{(r)}$ is defined by

$$\underline{\underline{J}}_n^{(r)} = \mathrm{Image}\Big(\{a \in \underline{\underline{J}}_{n+r}^{[r]} \mid f(a) \in p^r \cdot \underline{\underline{\mathcal{O}}}_{n+r}^{\mathrm{crys}}\} \to \underline{\underline{J}}_n^{[r]}\Big).$$

(The sheaf $\underline{\underline{J}}_n^{(r)}$ is not introduced in [**FM**] but the authors learned its definition from Fontaine and Messing. The exactness of (2.7) follows from [**FM**] III 1.1.)

We also need the canonical homomorphism

$$(2.9) \qquad \alpha = \alpha_n^r : H^q(\mathrm{Spec}(\mathcal{O}_K[\zeta_{p^n}])_{\mathrm{syn}}, S_n^r) \to H^q(K(\zeta_{p^n}), \mathbf{Z}/p^r(r))$$

of [**FM**] III, 5. The maps α_n^r for various r have the following compatibility with the product structures, by the construction in *op. cit.*. Let $c(r)$ be as above, and define $\alpha_{n,m}^r = p^{m-c(r)}\alpha_n^r$ for $m \geq c(r)$. Then, if $m \geq c(r)$, $m' \geq c(r')$, and $m + m' \geq c(r + r')$, we have

$$\alpha_{n,m}^r(x) \cdot \alpha_{n',m'}^r(y) = \alpha_{n,m+m'}^{r+r'}(xy)$$

for the products

$$S_n^r \times S_n^{r'} \to S_n^{r+r'}; \mathbf{Z}/p^n(r) \times \mathbf{Z}/p^n(r') \to \mathbf{Z}/p^n(r + r').$$

Now consider the commutative diagram

$$
\begin{array}{ccc}
\mathcal{O}_K \times \mathcal{R}^\times & \xrightarrow[\;1\;]{(a,b)\mapsto a\, db/b} & \Omega^1_{\mathcal{R}} \\
\downarrow & & \downarrow \\
H^0(\mathrm{Spec}(\mathcal{R})_{syn}, \underline{\underline{\mathcal{O}}}^{\mathrm{crys}}_n) \times H^1(\mathrm{Spec}(\mathcal{R})_{syn}, S^1_n) & \longrightarrow & H^1(\mathrm{Spec}(\mathcal{R})_{syn}, \underline{\underline{\mathcal{O}}}^{\mathrm{crys}}_n) \\
\downarrow{\scriptstyle \rho_n} & & \downarrow{\scriptstyle \rho_n} \\
H^0(\mathrm{Spec}(\mathcal{O}_K[\zeta_{p^n}])_{syn}, \underline{\underline{\mathcal{O}}}^{\mathrm{crys}}_n & \longrightarrow & H^1(\mathrm{Spec}(\mathcal{O}_K[\zeta_{p^n}])_{syn}, \underline{\underline{\mathcal{O}}}^{\mathrm{crys}}_n) \\
\times & & \\
H^1(\mathrm{Spec}(\mathcal{O}_K[\zeta_{p^n}])_{syn}, S^1_n) & 2 & \\
\downarrow & & \downarrow \\
H^1(\mathrm{Spec}(\mathcal{O}_K[\zeta_{p^n}])_{syn}, S^r_n) & \longrightarrow & H^2(\mathrm{Spec}(\mathcal{O}_K[\zeta_{p^n}])_{syn}, S^{r+1}_n) \\
\times & & \\
H^1(\mathrm{Spec}(\mathcal{O}_K[\zeta_{p^n}])_{syn}, S^1_n) & & \\
\downarrow{\scriptstyle (\alpha^r_{n,m}, \alpha^1_{n,0})} & & \downarrow{\scriptstyle \alpha^{r+1}_{n,m}} \\
H^1(K(\zeta_{p^n}), \mathbf{Z}/p^n(r)) \times H^1(K(\zeta_{p^n}), \mathbf{Z}/p^n(1)) & \longrightarrow & H^2(K(\zeta_{p^n}), \mathbf{Z}/p^n(r+1)) \\
& & \downarrow \\
& & \mathbf{Z}/p^n(r)
\end{array}
$$

Here the vertical arrows of the square (1) (resp. (2)) are defined via the isomorphism

$$H^q(\mathrm{Spec}(\mathcal{R})_{\mathrm{syn}}, \underline{\underline{\mathcal{O}}}^{\mathrm{crys}}_n) \cong H^q(\widehat{\Omega}^{\cdot}_{\mathcal{R}/\mathbf{Z}} \otimes \mathbf{Z}/p^n\mathbf{Z})$$

([**FM**]) and the first Chern class map

$$\mathcal{R}^\times \to H^1(\mathrm{Spec}(\mathcal{R})_{syn}, S^1_n)$$

[**Gros**] (resp. by the exact sequences

(2.11) $$0 \to S_n^i \to \underline{\underline{J}}_n^{(i)} \xrightarrow{1-p^{-i}f} \underline{\underline{\mathcal{O}}}^{\mathrm{crys}} \to 0 \quad (i = r, r+1)).$$

In the diagram, m is any integer such that $m \geq c(r+1)$. The composite of all vertical arrows on the left-hand side is equal to

$$(a, b) \mapsto (\partial_{n,m}^r(a), \rho_n(b)).$$

Let

$$\delta_{n,m}^r : \widehat{\Omega}_{\mathcal{R}}^1 \to \mathbf{Z}/p^n(r)$$

be the composite of all vertical arrows on the right side. Now (2.6) is reduced to the

Key Lemma 2.12. $\delta_{n,m}^r = \varepsilon_{n,m}^r$.

Proof. In the case $r = 1$ this interpretation of the usual explicit reciprocity law for Hilbert symbols via Fontaine-Messing theory was proven in [**Ka**]. We shall reduce (2.12) to this case. We extend $\delta_{n,m}^r$ to a homomorphism

$$\widetilde{\delta}_{n,m}^r : \widehat{\Omega}_{\mathcal{R}}^1 \to \mathbf{Z}_p(r)$$

defined as the inverse limit over i of

(2.12.1) $$\widehat{\Omega}_{\mathcal{R}}^1 \to \mathbf{Z}/p^{n+i}(r); \quad \omega \mapsto \delta_{n+i,m}^r((p^{-1}\sigma_p)^i(\omega))$$

where $p^{-1}\sigma_p(h \cdot dz/z) = \sigma_p(h) \cdot dz/z$. The compatibility of the homomorphisms (2.12.1) with the reduction maps $\mathbf{Z}/p^{n+i+1}(r) \to \mathbf{Z}/p^{n+i}(r)$ follows because by (2.1.1), the injection "p"; $\mathbf{Z}/p^{n+i}(r) \to \mathbf{Z}/p^{n+i+1}(r)$ sends $\delta_{n+i,m}^r((p^{-1}\sigma_p)^i(\omega))$ to

$$\delta_{n+i+1,m}(\sigma_p(p^{-1}\sigma_p)^i(\omega)) = p \cdot \delta_{n+i+1,m}^r((p^{-1}\sigma_p)^{i+1}(\omega))$$

Let

$$\widetilde{\delta}_n^r = p^{-m}(\mathbf{Q}_p \otimes \delta_{n,m}^r) : \widetilde{\Omega}_{\mathcal{R}}^1 \to \mathbf{Q}_p(r),$$

which is independent of m. The desired equation (2.12) follows from

$$\widetilde{\delta}_n^r(\omega) = \mathrm{Tr}_{K/\mathbf{Q}_p} \mathrm{Res}(t^{-r}T_p^n(\omega))$$

for $\omega \in \widehat{\Omega}_{\mathcal{R}}^1$. Using Lemmas 2.13 and 2.14 below, this formula in turn is reduced to the case $r = 1$ where it is proven in [Ka].

Lemma 2.13. *Let $e \gg 0$ be a sufficiently large integer. Then:*
(i) *δ^r_n annihilates $(z^{p^n}-1)^e \cdot \widehat{\Omega}^1_{\mathcal{R}} \otimes \mathbf{Q}$.*
(ii) *Let $\gamma \in \mathcal{R} \otimes \mathbf{Q}$ be such that*

$$\gamma = \sum_{i=1}^{e-1} (-1)^{i-1}(z^{p^n}-1)^i/i \mod (z^{p^n}-1)^{r+1}\mathcal{R} \otimes \mathbf{Q}$$

Then if $r \geq 2$, the following diagram commutes

$$\begin{array}{cccc} \omega & \widehat{\Omega}^1_{\mathcal{R}} & \xrightarrow{\tilde{\delta}^{r-1}_n} & \mathbf{Q}_p(r-1) \\ \downarrow & \downarrow & & \downarrow{\scriptstyle \otimes\nu} \\ \gamma\omega & \widehat{\Omega}^1_{\mathcal{R}} & \xrightarrow{\tilde{\delta}^{r}_n} & \mathbf{Q}_p(r). \end{array}$$

Lemma 2.14. *If $h : \widehat{\Omega}^1_{\mathcal{R}} \otimes \mathbf{Q} \to \mathbf{Q}_p(r)$ is a $\mathbf{Q}_p[G]$-homomorphism annihilating $(z^{p^n}-1)^{r-1}\widehat{\Omega}^1_{\mathcal{R}} \otimes \mathbf{Q}$, then $h = 0$.*

Note $\tilde{\delta}^r_n$ and $\omega \mapsto \mathrm{Tr}_{K/\mathbf{Q}_p} \mathrm{Res}(t^{-r}T^n_p(\omega))$ are $\mathbf{Q}_p[G]$-homomorphisms. By (2.13) and by induction on r, the difference of these two annihilates $(z^{p^n}-1)\widehat{\Omega}^1_{\mathcal{R}} \otimes \mathbf{Q}$. We first prove (2.13) with $e = r+2$. (It will follow from (2.12) that the lemma is true with $e = r$.) We show that the image of $(z^{p^n}-1)^e\widehat{\Omega}^1_{\mathcal{R}}$ in $H^2(\mathrm{Spec}(\mathcal{O}_K[\zeta_{p^n}])_{syn}, S^{r+1}_n)$ under the composition of right-hand vertical arrows from diagram (2.10) is zero, from which (2.13))(i) follows.

Regard z^{p^n} as an element of $H^0(\mathrm{Spec}(\mathcal{O}_K[\zeta_{p^n}])_{syn}, \underline{\underline{\mathcal{O}}}^{syn}_n)$. It suffices to show there exists a homomorphism of sheaves on $\mathrm{Spec}(\mathcal{O}_K[\zeta_{p^n}])_{syn}, \lambda : \underline{\underline{\mathcal{O}}}^{\mathrm{crys}}_n \to \underline{\underline{J}}^{(r+1)}_n$ such that $(1-p^{-r-1}f) \circ \lambda$ is multiplication by $(z^{p^n}-1)^e$ on $\underline{\underline{\mathcal{O}}}^{\mathrm{crys}}_n$.

Let $h = (z^{p^n}-1)/(z^{p^{n-1}}-1)\mathcal{R}$, so $(h) = \mathrm{Ker}\,\rho_n$. We regard h as a section of $\underline{\underline{J}}^{(1)}_n$ on $\mathrm{Spec}(\mathcal{O}_K[\zeta_{p^n}])_{syn}$. We have

$$(p^{-1}f)(z^{p^n}-1) = (z^{p^n}-1) \cdot (p^{-1}f)(h).$$

This implies, for a local section x of $\mathcal{O}^{\mathrm{crys}}_n$ and for $e \geq r+1$, we have

$$(p^{-r-1}f)((z^{p^n}-1)^e x) = p^{e-r-1}(z^{p^n}-1)^e((p^{-1}f)(h))^e f(x).$$

It follows, for $e \geq r+2$, the sum

$$\lambda(x) \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} (p^{-r-1}f)^i((z^{p^n}-1)^e x)$$

is a finite sum and belongs to $\underline{\underline{J}}_n^{(e)} \subset \underline{\underline{J}}_n^{(r+1)}$. Clearly,

$$(1-p^{-r-1}f)\lambda(x) = (z^{p^n}-1)^e x.$$

Next, (2.13)(ii) follows from the commutative diagram

$$\begin{array}{ccccc}
\widehat{\Omega}^1_{\mathcal{R}} & \longrightarrow & H^2(\mathrm{Spec}(\mathcal{O}_K[\zeta_{p^n}])_{syn}, S_n^r) & \xrightarrow{\alpha^r_{n,m}} & H^2(K(\zeta_{p^n}), \mathbf{Z}/p^n(r)) \\
\downarrow{\scriptstyle (e-1)!\gamma} & & \downarrow{\scriptstyle (e-1)!\cdot\log(z^{p^n})} & & \downarrow{\scriptstyle (e-1)!\cdot\nu} \\
\widehat{\Omega}^1_{\mathcal{R}} & \longrightarrow & H^2(\mathrm{Spec}(\mathcal{O}_K[\zeta_{p^n}])_{syn}, S_n^{r+1} & \xrightarrow{\alpha^{r+1}_{n,m}} & H^2(K(\zeta_{p^n}), \mathbf{Z}/p^n(r+1))
\end{array}$$

for $m \geq c(r+1)$. The commutativity of the left-hand square follows from the proof of (2.13)(i), and the right-hand square commutes because

$$\alpha^1_{n,0}(\log(z^{p^n})) \equiv \nu \mod p^n$$

[FM].

Finally, we prove (2.14). For $i \geq 0$, $(z^{p^n}-1)^i \widehat{\Omega}^1_{\mathcal{R}} \otimes \mathbf{Q}$ is stable under the action of G, and there exists an isomorphism of $K[G]$-modules

$$\begin{aligned}
K[\mathbf{Z}/p^n\mathbf{Z}(1)](i+1) &\cong (z^{p^n}-1)^i \widehat{\Omega}^1_{\mathcal{R}} / (z^{p^n}-1)^{i+1} \widehat{\Omega}^1_{\mathcal{R}} \\
[a\nu \mod p^n] \otimes \nu^{\otimes i+1} &\mapsto z^a (z^{p^n}-1)^i \cdot dz/z \quad (a \in \mathbf{Z}).
\end{aligned}$$

Since the action of G on $K[\mathbf{Z}/p^n\mathbf{Z}(1)]$ factors through a finite quotient of G, we have

$$\mathrm{Hom}_{\mathbf{Q}_p[G]}(K[\mathbf{Z}/p^n\mathbf{Z}(1)](i), \mathbf{Q}_p(r)) = (0) \text{ if } i \neq r,$$

which proves (2.14).

This completes the proof of (2.6). Q.E.D.

3. H^1 of local Galois representations

In this section K denotes a finite extension of $\mathbf{Q}_p$. Given a motif M over K, we need some p-adic space with a measure on it in order to construct the Tamagawa measure. In the example of an abelian variety A, this space is $A(K)$, the space of K-p;oints of A, and it embeds in $H^1(K,T)$ where $T = \varprojlim_n {}_nA(\overline{K})$ is the Tate module, and the map is the inverse limit of boundary maps from exact sequences

$$0 \longrightarrow {}_nA(\overline{K}) \xrightarrow{n} A(\overline{K}) \longrightarrow A(\overline{K}) \longrightarrow 0.$$

We can interpret $A(K)$ motivically as the group of extensions defined over K of the dual abelian variety A^* by $\mathbf{G}_m$, but we don't really understand the category of mixed motives well enough to say much about this group of extension when A is replaced by a more general motive M. On the other hand, the motivic interpretation of T via étale cohomology generalizes easily. Our problem becomes how to identify the subgroup $A(K) \subset H^1(K,T)$, together with theTamagawa measure on it, in some fashion which can be generalized to other motives.

By using the rings B_{crys} and B_{DR} of Fontaine as modified in [FM] (cf. §1) we shall define for any $\hat{\mathbf{Z}}$-module T of finite rank, with a continuous action of $\text{Gal}(\overline{K}/K)$, subgroups

$$(3.1) \qquad H^1_e(K,T) \subset H^1_f(K,T) \subset H^1_g(K,T) \subset H^1(K,T)$$

called the exponential part, the finite part, and the geometric part. In particular

$$(3.2) \qquad H^1_g(K,T) = \text{Ker}(H^1(K,T) \rightarrow H^1(K, B_{DR} \otimes_{\hat{\mathbf{Z}}} T)).$$

For the Tate module T of an abelian variety A, these three subgroups coincide and equal the image of $A(K)$ in $H^1(K,T)$. As a consequence, we remark that for an abelian variety

$$(3.3) \quad H^1(K, A(\overline{K})) \cong H^1(K, T \otimes \mathbf{Q}/\mathbf{Z})/H^1_*(K,T) \otimes \mathbf{Q}/\mathbf{Z} \quad (* = e, f, g).$$

This group is needed to define $\text{Ш}(A)$, the Tate-Shafarevich group, whose order enters into the Tamagawa number conjecture (equivalent by [Bl1] to the Birch and Swinnerton-Dyer conjecture) for A. When T is the Tate module of a general motif, the groups

$$(3.4) \quad H^1_*(K,T) \quad \text{and} \quad H^1(K, T \otimes \mathbf{Q}/\mathbf{Z})/H^1_*(K,T) \otimes \mathbf{Q}/\mathbf{Z} \quad (* = e, f, g)$$

will play analogous roles.

We shall also prove a duality result (3.8) for these groups, generalizing Tate duality

$$A(K) \cong \text{ Pontryagin dual of } H^1(K, A^*(\overline{K})).$$

The notion of the motivic part of H^1 has been considered independently by a number of other authors.

Let k be the residue field of K, and let K_0 be the quotient field of the Witt vectors $W(k)$. $K_0 \subset K$ is the maximum unramified subfield of K. For a finite dimensional $\mathbf{Q}_p$-vector space V endowed with a continuous $\mathrm{Gal}(\overline{K}, K)$-action, Fontaine defines

$$\text{(3.5)} \quad \mathrm{Crys}(V) \overset{def}{=} H^0(K, B_{\mathrm{crys}} \otimes V); \quad DR(V) \overset{def}{=} H^0(K, B_{DR} \otimes V).$$

Then $\mathrm{Crys}(V)$ is a K_0-vector space with a frobenius

$$f = (f \text{ on } B_{\mathrm{crys}}) \otimes (\mathrm{id.\ on\ } V).$$

On the other hand, $DR(V)$ is a K-vector space with a decreasing filtration

$$\text{(3.6)} \qquad DR(V)^i = H^0(K, B^i_{DR} \otimes V) \quad (i \in \mathbf{Z}),$$

and $K \otimes_{K_0} \mathrm{Crys}(V) \hookrightarrow DR(V)$. One has ([Fo], (5.1))

$$\dim_{K_0}(\mathrm{Crys}(V)) \leq \dim_K(DR(V)) \leq \dim_{\mathbf{Q}_p}(V).$$

If $\dim_{K_0}(\mathrm{Crys}(V)) = \dim_{\mathbf{Q}_p}(V)$ (resp. if $\dim_K(DR(V)) = \dim_{\mathbf{Q}_p}(V)$) V is said to be a crystalline (resp. a de Rham) representation of $\mathrm{Gal}(\overline{K}/K)$.

For a prime number ℓ and a $\mathbf{Q}_p$-vector space V of finite dimension endowed with a continuous action of $\mathrm{Gal}(\overline{K}/K)$ we define $\mathbf{Q}_\ell$-subspaces

$$\text{(3.7)} \qquad H^1_e(K,V) \subset H^1_f(K,V) \subset H^1_g(K,V) \subset H^1(K,V)$$

as follows. If $\ell \neq p$, let

$$\text{(3.7.1)} \qquad \begin{cases} H^1_e(K,V) = (0); H^1_g(K,V) = H^1(K,V); \\ H^1_f(K,V) = \mathrm{Ker}(H^1(K,V) \to H^1(K_{nr},V)), \end{cases}$$

where K_{nr} is the maximal unramified extension of K. If $\ell = p$, let

$$\text{(3.7.2)} \qquad \begin{cases} H^1_e(K,V) = \mathrm{Ker}(H^1(K,V) \to H^1(K, B^{f=1}_{\mathrm{crys}} \otimes V)) \\ H^1_f(K,V) = \mathrm{Ker}(H^1(K,V) \to H^1(K, B_{\mathrm{crys}} \otimes V)) \\ H^1_g(K,V) = \mathrm{Ker}(H^1(K,V) \to H^1(K, B_{DR} \otimes V)). \end{cases}$$

For a prime number ℓ and a free $\mathbf{Z}_\ell$-module T of finite rank endowed with a continuous action of $\mathrm{Gal}(\overline{K}/K)$-action, we define
(3.7.3)

$$H^1_*(K,T) \stackrel{def}{=} \iota^{-1}(H^1_*(K,T\otimes\mathbf{Q})); \quad \iota : H^1(K,T) \to H^1(K,T\otimes\mathbf{Q})$$
$$(* = e,f,g).$$

Thus $H^1_*(K,T)$ always contains the torsion subgroup of $H^1(K,T)$. If T is a free $\widehat{\mathbf{Z}}$-module of finite rank with continuous $\mathrm{Gal}(\overline{K}/K)$-action, let $T_\ell = T\otimes_{\widehat{\mathbf{Z}}}\mathbf{Z}_\ell$. Define

$$(3.7.4) \qquad H^1_*(K,T) \stackrel{def}{=} \prod H^1_*(K,T_\ell) \quad (* = e,f,g).$$

The following remark is easily proved and gives some insight into the groups H^1_*. Let V be as above, and let $\alpha \in H^1(K,V)$. α corresponds to an extension

$$0 \longrightarrow V \longrightarrow E \longrightarrow \mathbf{Q}_\ell \longrightarrow 0.$$

Assume $\ell \neq p$ (resp. $\ell = p$, resp. $\ell = p$) and that V is an unramified (resp. crystalline, resp. de Rham) representation. Then E is an unramified (resp. crystalline, resp. de Rham) representation if and only if $\alpha \in H^1_f(K,V)$ (resp. $\alpha \in H^1_f(K,V)$, resp. $\alpha \in H^1_g(K,V)$).

Proposition 3.8. *Let ℓ be a prime number, and let V (resp. T) be a finite-dimensional $\mathbf{Q}_p$-vector space (resp. free $\widehat{\mathbf{Z}}$-module of finite rank) with a continuous Galois action. If $\ell = p$, we assume V (resp. $T\otimes\mathbf{Q}$) is a de Rham representation. Let $V^* = Hom_{\mathbf{Q}_\ell}(V,\mathbf{Q}_\ell)$ (resp.$T^* = Hom_{\mathbf{Z}_\ell}(T,\mathbf{Z}_\ell)$). Then in the perfect pairing*

$$H^1(K,V)\times H^1(K,V^*(1))\times H^2(K,\mathbf{Q}_\ell(1)) \longrightarrow^{\cong} \mathbf{Q}_\ell$$

resp.

$$H^1(K,T)\times H^1(K,T^*\otimes\mathbf{Q}_\ell/\mathbf{Z}_\ell(1)) \to H^2(K,\mathbf{Q}_\ell/\mathbf{Z}_\ell(1)) \cong \mathbf{Q}_\ell/\mathbf{Z}_\ell,$$

$H^1_e(K,V)$ and $H^1_g(K,V^*(1))$ (resp. $H^1_e(K,T)$ and $H^1_g(K,T^*(1)\otimes\mathbf{Q}_\ell/\mathbf{Z}_\ell)$ are the exact annihilators of each other. The same statement holds with e replaced by g and g by e and also when e and g are both replaced by f.

Proof. The assertion for T follows easily from that for V. Furthermore,the case $\ell \neq p$ is easy, so we consider only the case of V with $\ell = p$. Recall (1.14) $B^+_{DR} \subset B_{DR}$ is the discrete valuation ring.

Lemma 3.8.1. *Let V be a de Rham representation of* $\mathrm{Gal}(\overline{K}/K)$. *Then* $H^1(K, B_{DR}^+ \otimes V) \to H^1(K, B_{DR} \otimes V)$ *is injective.*

Proof of 3.8.1. Consider the exact sequence

(3.8.2)

$$0 \to H^0(K, B_{DR}^+ \otimes V) \to H^1(K, B_{DR} \otimes V) \to H^0(K, B_{DR}/B_{DR}^+ \otimes V).$$

We have

$$\begin{aligned}\dim_K(DR(V)) &= \dim_K H^0(K, B_{DR} \otimes V) \\ &\underset{(1)}{\leq} \dim_K H^0(K, B_{DR}^+ \otimes V) + \dim_K H^0(K, B_{DR}/B_{DR}^+ \otimes V) \\ &\underset{(2)}{\leq} \sum_{i \in \mathbf{Z}} \dim_K H^0(K, \mathbf{C}_p(i) \otimes V) \underset{(3)}{\leq} \dim_{\mathbf{Q}_p}(V),\end{aligned}$$

where (2) follows from $B_{DR}^i/B_{DR}^{i+1} \cong \mathbf{C}_p(i)$ (1.15.1), and (3) is a result of Tate [**Ta2**]. Since V is assumed de Rham, all these inequalities are in fact equalities. In particular, (1) is an equality, proving the lemma. Q.E.D.

Remark 3.8.3. The same argument shows

$$H^1(K, B_{DR}^i \otimes V) \to H^1(K, B_{DR}^{i-j} \otimes V)$$

is injective for any $i \in \mathbf{Z}$ and any j with $0 \leq j \leq \infty$.

Corollary 3.8.4 *Let V be a de Rham representation as above. Then there is a commutative diagram of exact sequences:*

$$\begin{array}{ccccccccc} 0 \to & H^0(K,V) & \xrightarrow{\alpha} & \mathrm{Crys}(v)^{f=1} \oplus DR(V)^0 & \xrightarrow{\beta} & DR(V) & \to & H^1_e(K,V) & \to 0 \\ & \| & & \cap & & \downarrow{\scriptstyle (0,id)} & & \cap & \\ 0 \to & H^0(K,V) & \xrightarrow{\alpha} & \mathrm{Crys}(V) \oplus DR(V)^0 & \xrightarrow{\gamma} & \mathrm{Crys}(V) \oplus DR(V) & \to & H^1_f(K,V) & \to 0 \end{array}$$

Here $\mathrm{Crys}(V)^{f=1} = \{a \in \mathrm{Crys}(V) \mid f(a) = a\}$, $\alpha(x) = (x,x)$, $\beta(x,y) = x - y$, *and* $\gamma(x,y) = (x - f(x), x - y)$. *In particular,*

$$\dim_{\mathbf{Q}_p} H^1_f(K,V) = \dim_{\mathbf{Q}_p}(DR(V)/DR(V)^0) + \dim_{\mathbf{Q}_p} H^0(K,V),$$

and

$$H^1_f(K,V)/H^1_e(K,V) \cong \mathrm{Crys}(V)/(1-f)\,\mathrm{Crys}(V).$$

Proof. The exact sequences are obtained by tensoring the sequences (1.17) with V and taking Galois cohomology, using Lemma 3.8.1.

We now consider the assertion of (3.8) for $H^1_f(K,V)$ and $H^1_f(K,V^*(1))$ in the case $\ell = p$. First we show these two groups annihilate each other. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Crys}(V^*(1)) \oplus DR(V^*(1)) & \xrightarrow{\delta} & H^1(K,V^*(1)) \\ {\scriptstyle (1)}\downarrow & & {\scriptstyle (2)}\downarrow \\ H^1(K, B_{\mathrm{crys}} \otimes V \otimes V^*(1)) \oplus H^1(K, B_{DR} \otimes V \otimes V^*(1)) & \xrightarrow{\varepsilon} & H^2(K, V \otimes V^*(1)) \\ & & \downarrow \\ & & \mathbf{Q}_p \end{array}$$

where (1) and (2) are given by cup product with an element $\alpha \in H^1(K,V)$ and δ and ε are the connecting maps of the exact sequences (1.17.2) $\otimes V^*(1)$ and (1.17.2)$\otimes V \otimes V^*(1)$, respectively. If $\alpha \in H^1_f(K,V)$, the map (1) is zero since (1) depends only on the image of α in $H^1(K, B_{\mathrm{crys}} \otimes V)$. It follows in this case that (2) is zero on the image $H^1_f(K,V^*(1))$ of δ.

To see that $H^1_f(K,V)$ and $H^1_f(K,V^*(1))$ are exact annihilators, it is enough to show

$$\text{(3.8.5)} \qquad \dim_{\mathbf{Q}_p} H^1_f(K,V) + \dim_{\mathbf{Q}_p} H^1_f(K,V^*(1)) = \dim_{\mathbf{Q}_p} H^1(K,V).$$

Fontaine has shown in [Fo] that V de Rham implies $V^*(1)$ de Rham, so we can apply the dimension formula in (3.8.4) together with local duality to these two representations

$$\begin{aligned} &\dim_{\mathbf{Q}_p} H^1_f(K,V) + \dim_{\mathbf{Q}_p} H^1_f(K,V^*(1)) = \dim_{\mathbf{Q}_p}(DR(V)/DR(V)^0) \\ &+ \dim_{\mathbf{Q}_p}(DR(V^*(1))/DR(V^*(1))^0) + \dim_{\mathbf{Q}_p} H^0(K,V) + \dim_{\mathbf{Q}_p} H^2(K,V). \end{aligned}$$

One knows

$$\sum (-1)^i \dim_{\mathbf{Q}_p} H^i(K,V) = -[K:\mathbf{Q}_p] \dim_{\mathbf{Q}_p} V \qquad \text{([Se],II(5.7))}$$
$$\dim_K DR(V)^0 + \dim_K DR(V^*(1))^0 = \dim_K DR(V) \quad \text{[Fo]}$$

The desired equation (3.8.5) follows. Finally we prove the assertion of (3.8) for the groups $H^1_e(K,V)$ and $H^1_g(K,V^*(1))$ when $\ell = p$. Consider

the commutative diagram
(3.8.6)

$$\begin{array}{ccccc} H^1(K,V)\times DR(V^*(1)) & & \xrightarrow{(id.,\delta)} & & H^1(K,V)\times H^1(K,V^*(1)) \\ \downarrow & & & & \downarrow \\ H^1(K,B_{DR}\otimes V)\times DR(V^*(1)) & \xrightarrow{\cup} & H^1(K,B_{DR}(1)) & \xrightarrow{\varepsilon} & H^2(K,\mathbf{Q}_p(1))\cong \mathbf{Q}_p \end{array}$$

where δ and ε are connecting homomorphisms of the exact sequences (1.17.1) $\otimes V^*(1)$ and (1.17.1)$\times \mathbf{Q}_p(1)$ respectively, and $\cup$ is the cup product. Simple manipulations reduce us to showing that the composition $\varepsilon \circ \cup$ along the bottom is a perfect pairing.

To this end, recall we have defined (1.8) an element $t \in B_{\text{crys}}^{f=p}$ whose image in B_{DR}^+ generates the maximal ideal and on which Galois acts via the cyclotomic character. Multiplication by t therefore identifies the sequence obtained from (1.17.1) by tensoring with $\mathbf{Q}_p(1)$ with the top line of the following diagram, where all maps are the natural ones:
(3.8.7)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Q}_p(1) & \longrightarrow & B_{\text{crys}}^{f=p}\oplus B_{DR}^1 & \longrightarrow & B_{DR} & \longrightarrow & 0 \\ & & \| & & \cup & & \cup & & \\ 0 & \longrightarrow & \mathbf{Q}_p(1) & \longrightarrow & (B_{\text{crys}}^{f=p}\cap B_{DR}^+)\oplus B_{DR}^1 & \longrightarrow & B_{DR}^+ & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle pr_1} & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{Q}_p(1) & \longrightarrow & B_{\text{crys}}^{f=p}\cap B_{DR}^+ & \longrightarrow & \mathbf{C}_p & \longrightarrow & 0. \end{array}$$

We will show the analogous pairing
(3.8.8)

$$H^1(K,\mathbf{C}_p\otimes V)\times H^0(K,\mathbf{C}_p\otimes V^*)\xrightarrow{\cup} H^1(K,\mathbf{C}_p)\xrightarrow{\gamma} H^2(K,\mathbf{Q}_p(1)\cong \mathbf{Q}_p$$

is perfect, where γ is defined using the bottom row of (3.8.7). Assuming this for a moment, to see that the bottom line of (3.8.6) is perfect we tensor (3.8.7) with V and V^* and identify $DR(V^*(1))\cong DR(V^*)$ (with a shift of Hodge filtration) via multiplication by t. The pairing

$$\begin{array}{ccc} H^1(K,B_{DR}^+\otimes V)\times DR(V^*(1))^{-1} & \longrightarrow & H^2(K,\mathbf{Q}_p(1))\cong \mathbf{Q}_p \\ \| & & \\ DR(V^*)^0 & & \end{array}$$

then factors through (3.8.8). Since, by (3.8.1), $DR(V^*)^0 \twoheadrightarrow H^0(K,\mathbf{C}_p\otimes V^*)$, it follows that any element in $H^1(K,B_{DR}^+\otimes V)$ which is orthogonal to $DR(V^*(1))^{-1}$ lies in $H^1(K, B^1_{DR} \otimes V) \cong H^1(K, B^+_{DR} \otimes V(1))$. We can replace V by $V(1)$ and iterate this argument to get the null space in $H^1(K, B^+_{DR} \otimes V)$ contained in $\cap H^1(K, B^i_{DR} \otimes V)$. Replacing V by $V(n)$ for $n > 0$, we get a similar result for the part of the null space contained in $H^1(K, B^n_{DR} \otimes V)$. But $B^i_{DR}/B^{i+1}_{DR} \cong \mathbf{C}_p(i)$ and $H^1(K, \mathbf{C}_p \otimes V(i)) = (0)$ for all but finitely many i. (Indeed, de Rham representations have a Hodge-Tate decomposition $\mathbf{C}_p \otimes V \cong \mathbf{C}_p(n_i)$ and by a result of Tate, $H^i(K, \mathbf{C}_p(j))$ has dimension 1 if $j = 0$ and $i = 0$ or 1, and is zero otherwise.) Using convergence properties of continuous cohomology, it follows that

$$H^1(K, B^i_{DR} \otimes V) = \begin{cases} (0) & i \gg 0 \\ H^1(K, B_{DR} \otimes V) & i \ll 0. \end{cases}$$

This implies that the null space in $H^1(K, B_{DR} \otimes V)$ is trivial. The proof that the null space in $DR(V^*)$ is trivial is similar, recalling that

$$H^1(K, B^n_{DR} \otimes V) \twoheadrightarrow H^1(K, \mathbf{C}_p(n) \otimes V)$$

by (1.15.1).

It remains to show the pairing (3.8.8) is perfect. Since V has a Hodge-Tate decomposition we reduce immediately to the case $V = \mathbf{Q}_p(n)$. Both cohomology groups are trivial,and there is nothing to prove unless $n = 0$. Finally therefore, everything boils down to showing the boundary map $\gamma : H^1(K, \mathbf{C}_p) \to H^2(K, \mathbf{Q}_p(1)) \cong \mathbf{Q}_p$ is non-zero. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{Z}_p(1) & \longrightarrow & \varprojlim \overline{K}^* & \xrightarrow{pr_1} & \overline{K}^* & \longrightarrow & 0 \\
 & & \| & & \cup & & \cup & & \\
0 & \longrightarrow & \mathbf{Z}_p(1) & \longrightarrow & \varprojlim (1+pA)^{\times p^{-n}} & \longrightarrow & (1+pA)^{\times} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \log[\,]} & & \downarrow{\scriptstyle \log} & & \\
0 & \longrightarrow & \mathbf{Q}_p(1) & \longrightarrow & B^{f=p}_{\mathrm{crys}} \cap B^+_{DR} & \longrightarrow & \mathbf{C}_p & \longrightarrow & 0.
\end{array}$$

where $A \subset \overline{K}$ is the ring of integers, the inverse limits are with respect to the p-th power map, and log[] denotes the log of the Teichmuller representative (§1). If we identify $H^1(K, \mathbf{Q}_p(1)) \cong (\varprojlim K^*/K^{*p^n}) \otimes \mathbf{Q}$, it follows immediately that the boundary map $\partial : K = H^0(K, \mathbf{C}_p) \to H^1(K, \mathbf{Q}_p(1))$ is the p-adic exponential. Thus, the composition

$$H^1(K, \mathbf{Q}_p) \to H^1(K, \mathbf{C}_p) \to H^2(K, \mathbf{Q}_p(1))$$

is cup product with exp(1). This is non-zero since the pairing

$$H^1(K, \mathbf{Q}_p) \otimes H^1(K, \mathbf{Q}_p(1)) \to H^2(K, \mathbf{Q}_p(1))$$

is perfect. This completes the proof of (3.8). Q.E.D.

Example 3.9. Consider the case $V = \mathbf{Q}_p(r)$. By (3.8) and (3.8.4) we have the following table for the dimensions of $H^1_*(K, \mathbf{Q}_p(r))$ for $* = e, f, g$.

r	$H^1_e(K, \mathbf{Q}_p(r))$	$H^1_f(K, \mathbf{Q}_p(r))$	$H^1_g(K, \mathbf{Q}_p(r))$	$H^1(K, \mathbf{Q}_p(r))$
$r < 0$	0	0	0	$[K : \mathbf{Q}_p]$
$r = 0$	0	1	1	$[K : \mathbf{Q}_p] + 1$
$r = 1$	$[K : \mathbf{Q}_p]$	$[K : \mathbf{Q}_p]$	$[K : \mathbf{Q}_p] + 1$	$[K : \mathbf{Q}_p] + 1$
$r > 1$	$[K : \mathbf{Q}_p]$	$[K : \mathbf{Q}_p]$	$[K : \mathbf{Q}_p]$	$[K : \mathbf{Q}_p]$

For $r = 0$, $H^1_f(K, \mathbf{Q}_p) \subset \mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), \mathbf{Q}_p)$ are the unramified homomorphisms. For $r = 1$, $H^1_f(K, \mathbf{Q}_p(1))$ is the image of the exponential map

$$(\mathcal{O}_K^\times) \otimes \mathbf{Q} \to (\varprojlim K^*/K^{*p^n}) \otimes \mathbf{Q} \cong H^1(K, \mathbf{Q}_p(1)).$$

The following exponential map plays an important role in later sections.

Definition 3.10. Let V be a de Rham representation of $\mathrm{Gal}(\overline{K}/K)$. We define the exponential map

$$DR(V)/DR(V)^0 \to H^1_e(K, V)$$

to be the connecting homomorphism in (3.8.4). It is surjective with kernel $\mathrm{Crys}(V)^{f=1}/H^0(K, V)$.

Example 3.10.1. Let G be a commutative formal Lie group of finite height over $\mathcal{O}_K$. Write T for the p-adic Tate module of G, and let $V = T \otimes \mathbf{Q}$. Then V is a de Rham representation, and $DR(V)/DR(V)^0$ is identified with the tangent space of G_K ([Fo],§6). We claim that the exponential map (3.10) coincides with the classical one, i.e. that the following diagram commutes;

$$\begin{array}{ccc} \tan(G_K) & \xrightarrow{\exp} & G(\mathcal{O}_K) \otimes \mathbf{Q} \\ \| & & \downarrow \partial \\ DR(V)/DR(V)^0 & \xrightarrow{\exp} & H^1(K, V), \end{array}$$

where the upper (resp. lower) exp is the exponential map in the classical sense (resp. in (3.10)), and ∂ is defined by the boundary maps of the Kummer sequences ($A =$ integral closure of $\mathcal{O}_K$)

$$0 \to T/p^nT \to G(A) \xrightarrow{p^n} G(A) \to 0.$$

To see this, for $\chi \in T^*(1)$, consider the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T & \longrightarrow & \varprojlim G(A) & \longrightarrow & G(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \chi} & & \\
0 & \longrightarrow & \mathbf{Z}_p(1) & \longrightarrow & \varprojlim A^\times & \longrightarrow & A^\times & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \log[\ ]} & & \downarrow{\scriptstyle \log} & & \\
0 & \longrightarrow & \mathbf{Q}_p(1) & \longrightarrow & B_{\text{crys}}^{f=p} \cap B_{DR}^+ & \longrightarrow & \mathbf{C}_p & \longrightarrow & 0 \\
 & & \| & & \cap & & \cap & & \\
0 & \longrightarrow & \mathbf{Q}_p(1) & \longrightarrow & B_{\text{crys}}^{f=1}(1) & \longrightarrow & (B_{DR}/B_{DR}^+)(1) & \longrightarrow & 0.
\end{array}$$

(Note χ corresponds to a homomorphism $G \to \widehat{\mathbf{G}}_m$ defined over A.) We get, therefore, a map

$$T^*(1) \to \text{Hom}(\text{"top sequence"}, \text{"bottom sequence"}).$$

i.e., a map "top sequence" $\to$ "bottom sequence"$\otimes T(-1)$. Taking cohomology yields a commutative square

$$\begin{array}{ccc}
G(\mathcal{O}_K) & \xrightarrow{\partial} & H^1(K,T) \\
\downarrow & & \downarrow \\
DR(V)/DR(V)^0 & \xrightarrow{\exp} & H^1(K,V).
\end{array}$$

By the construction, the left vertical arrow of this diagram is the classical logarithm when we identify $DR(V)/DR(V)^0$ with the tangent space of G_K. This proves (3.10.2) commutative.

It follows from (3.8.4) and the duality (3.8) that in this case

$$H_f^1(K,V) = H_e^1(K,V); \ \text{rank Hom}(\widehat{\mathbf{G}}_m, G) = \dim_{\mathbf{Q}_p} H_g^1(K,V)/H_f^1(K,V).$$

Example 3.11. Let A be an abelian variety over K. Write $T = \prod T_\ell$ for the (total) Tate module. Then V_p is a de Rham representation [Fo] and the diagram

$$\begin{array}{ccc} \tan(A) & \xrightarrow{\exp} & A(K)\otimes\mathbf{Q} \\ \wr\| & & \wr\| \quad \partial \\ DR(V_p)/DR(V_p)^0 & \xrightarrow{\exp} & H^1_e(K,V_p) \end{array} \tag{3.11.1}$$

commutes, with $\tan(A)$ the tangent space of A at the origin, upper exp the classical one, lower exp as in (3.10), and ∂ from the Kummer sequence. Moreover,

$$H^1_e(K,T) = H^1_f(K,T) = H^1_g(K,T). \tag{3.11.2}$$

Indeed, for (3.11.1), we can replace K by a finite extension so the Neron model has semi-stable reduction and then apply the previous example. For (3.11.2), it suffices by duality to show one of the equalities, e.g. $H^1_g = H^1_f$, and we may replace T by V. The p-part follows from (3.8.4) because $\mathrm{Crys}(V)^{f=1} = (0)$. Indeed, the piece of Crys(V) of slope 0 can be identified with a part of the Weil cohomology of the abelian variety quotient of the special fibre, and the frobenius has no eigenvalues 1 by the Weil conjectures. For the non p-part, it suffices to note

$$H^1(\mathrm{Gal}(K^{nr}/K), H^0(\mathrm{Gal}(\overline{K}/K^{nr}), V)) = (0)$$

because $A(K)$ has no divisible elements.

4. Volumes and L-functions; the local situation

In this section, we consider a relation between local L-functions and measures on Galois cohomology groups. We denote by K a finite extension of $\mathbf{Q}_p$, by K_{nr} the maximal unramified extension of K, and by K_0 the maximal unramified subfield of K over $\mathbf{Q}_p$.

For a prime number ℓ and a $\mathbf{Q}_\ell$-vector space V of finite dimension endowed with a continuous action of $\mathrm{Gal}(\overline{K}/K)$, put

$$P(V,u) = \begin{cases} \det_{\mathbf{Q}_p}(1 - f_K u : H^0(K_{nr}, V)) \in \mathbf{Q}_\ell[u] & \text{if } \ell \neq p \\ \det_{K_0}(1 - f_K u : \mathrm{Crys}(V)) \in K_0[u] & \text{if } \ell = p \end{cases}$$

Here if $\ell \neq p$, f_K denotes the action of an element of $\mathrm{Gal}(\overline{K}/K)$ which acts on $\mathbf{Z}_\ell(-1)$ by $p^{[K_0:\mathbf{Q}_p]}$. If $\ell = p$, f_K denotes the K_0-linear map $f^{[K_0\,\mathbf{Q}_p]}$. We call $P(V,u)^{-1}$ the local L-function attached to V. The aim of this section is to prove Theorems 4.1 and 4.2 below.

Theorem 4.1. *Let ℓ and V be as above, and assume $P(V,1) \neq 0$.*

(i) *Assume $\ell \neq p$. Then $(0) = H^1_e(K,V) = H^1_f(K,V)$. If V is unramified and T is a Gal($\overline{K}/K$)-stable $\mathbf{Z}_\ell$-sublattice in V, then*

$$\# H^1_f(K,T) = |P(V,1)|_\ell^{-1}$$

where $|\ |_\ell$ is the normalized absolute value on $\mathbf{Q}_\ell$.

(ii) *Assume $\ell = p$ and V is a de Rham representation. Then*

$$DR(V)/DR(V)^0 \xrightarrow[\cong]{\exp} H^1_e(K,V) = H^1_f(K,V)$$

(iii) *Assume $\ell = p$, K is unramified over $\mathbf{Q}_p$, V is a crystalline representation, and the following conditions (*) holds:*
(*) There exists $i \leq 0$ and $j \geq 1$ with $j - i < p$ such that $DR(V)^i = DR(V)$ and $DR(V)^j = (0)$.
Let $D \subset \mathrm{Crys}(V) = DR(V)$ be a strongly divisible lattice (i.e., a finitely generated $\mathcal{O}_K$-submodule of $DR(V)$ such that

$$D = \sum p^{-i} f(D^i);\ with\ D^i = D \cap DR(V)^i).$$

Let $T \subset V$ be the Galois stable sublattice constructed from D in[**FL**] *(cf. below for a review). Then*

$$\mu(H^1_f(K,T)) = |P(V,1)|_p^{-1}$$

where μ is the Haar measure of $H^1_f(K,V)$ induced from the Haar measure of D/D^0 having total measure 1 via the exponential map. Here $|\ |_p$ is the absolute value on $K = K_0$ such that $|p|_p = p^{-1}$.

Theorem 4.2. *Let K be an unramified finite extension of $\mathbf{Q}_p$, p an odd prime, and let $r \geq 2$. Consider the Haar measure μ on $H^1(K, \hat{\mathbf{Z}}(r))$ induced by the canonical Haar measure on K via the exponential map*

$$\exp : K \underset{\sim}{\longrightarrow} H^1(K, \hat{\mathbf{Z}}(r)) \otimes \mathbf{Q}.$$

Then

$$\mu(H^1(K, \hat{\mathbf{Z}}(r))) = (1 - q^{-r})|(r-1)!|_K \cdot \# H^0(K, \mathbf{Q}_p/\mathbf{Z}_p(1-r)),$$

where q is the order of the residue field of K and $| \ |_K$ is the normalized absolute value of K.

Parts (i) and (ii) of Theorem 4.1 are straightforward, using the fact that $H^1_f(T_\ell)$ is the coinvariants of frobenius for an unramified representation when $\ell \neq p$, and $H^1_f(V_p)/H^1_e(V_p)$ has rank equal to the K_0-rank of $\mathrm{Crys}(V_p)^{f=1}$ by (3.8.4). Note this latter group is zero since $P(V,1) \neq 0$. The proof of (iii) is more delicate. It depends on the theory of Fontaine-Laffaille [**FL**] reviewed below.

A filtered Dieudonné module over $\mathcal{O}_K$ is an $\mathcal{O}_K$-module D of finite type endowed with:
(a) a decreasing filtration $(d^i)_{i\in\mathbf{Z}}$ where the D^i are direct factors of D;
(b) a family of frobenius linear maps $f_i : D^i \to D$ satisfying

$$D^i = D \text{ for } i << 0; D^i = (0) \text{ for } i >> 0,$$
$$f_i|_{D^{i+1}} = p \cdot f_{i+1} \text{ for any } i,$$
$$D = \sum_{i\in\mathbf{Z}} f_i(D^i).$$

The important and somewhat surprising fact is that the category of filtred Dieudonné modules over $\mathcal{O}_K$ is abelian.

To a filtered Dieudonné module D satisfying the conditioned

(*) $$\exists i,j \in \mathbf{Z} \text{ such that } D^i = D, D^j = (0), \text{ and } j - i < p,$$

one can associated naturally a $\mathbf{Z}_p$-module $T(D)$ of finite type endowed with a continuous action of $\mathrm{Gal}(\overline{K}/K)$, as follows. (The definition of $T(D)$ introduced here is the modified one of Fontaine-Messing [**FM**], and differs slightly from the original definition in [**FL**], but the results of that paper used below continue to hold with the same proofs.)

If $D^1 = (0)$ and $D^{2-p} = D$, let

$$T(D) = \mathrm{Ker}(1 - f : \mathrm{Fil}^0(B_\infty \otimes_{\mathcal{O}_K} D) \to B_\infty \otimes_{\mathcal{O}_K} D).$$

Here

$$\mathrm{Fil}^0(B_\infty \otimes D) = \sum J^{[i]}_\infty \otimes D^{-i} \subset B_\infty \otimes D,$$

and f is the unique homomorphism such that

$$f(x \otimes y) = p^{-i} f(x) \otimes f_{-i}(y); \quad 0 \leq i < p, x \in J^{[i]}_\infty, y \in D^{-i}.$$

(Note $f(J_\infty^{[i]}) \subset p^i B_\infty$ for $i < p$ [FM].) To handle shifts in the filtration, define the Tate twist $D_{(r)}$ for $r \in \mathbf{Z}$ by

$$(D_{(r)})^i = D^{i+r}; f_i \text{ for } D_{(r)} = f_{r+i} \text{ for } D.$$

Then, more generally for a filtered Dieudonné module D satisfying the condition (*), take $r \in \mathbf{Z}$ such that $(D_{(r)})^1 = (0)$ and $(D_{(r)})^{2-p} = D_{(r)}$. Define

$$T(D) = T(D_{(r)})(-r),$$

where the $(-r)$ is the Tate twist as a galois module. Then $T(D)$ is independent of the choice of r as above. Indeed, if $r, r' \in \mathbf{Z}$ satisfy the above condition and $r \leq r'$, the map from $T(D)$ defined using r to $T(D)$ defined using r' given by $x \mapsto (t^{r'-r}x) \otimes t^{\otimes(r-r')}$ with t as in (1.8) is bijective.

Theorem 4.3. (Fontaine-Laffaille) *The functor $D \mapsto T(D)$ is exact and fully faithful for D's satisfying (*) with fixed i and j. It commutes with Tate twists and preserves ranks as well as lengths for objects of finite length. Moreover, $T(D) \otimes \mathbf{Q}$ is a crystalline representation of $Gal(\overline{K}/K)$ and the canonical map $B_{\mathrm{crys}} \otimes_{\mathbf{Z}_p} T(D) \to B_{\mathrm{crys}} \otimes_{\mathcal{O}_K} D$ is a bijection whose inverse induces and isomorphism*

$$D \otimes_{\mathcal{O}_K} K \xrightarrow{\sim} \mathrm{Crys}(T(D) \otimes \mathbf{Q}) = DR(T(D) \otimes \mathbf{Q})$$

which preserves the frobenius filtrations. (Here the frobenius on $D \otimes K$ means $p^i f_i$ for $i \ll 0$. This is independent of i.)

Theorem 4.1(iii) follows from Lemma 4.5 below.

Lemma 4.4. *For a filtered Dieudonné module D, let*

$$h^0(D) = Ker(1 - f_0 : D^0 \to D), h^1(D) = \mathrm{Coker}(1 - f_0 : D^0 \to D)$$
$$h^i(D) = (0) \text{ for } i \geq 2.$$

Let 1_{FD} be the "unit filtered Dieudonné module" defined by

$$(1_{FD})^i = \begin{cases} \mathcal{O}_K & \text{for } i \leq 0 \\ 0 & \text{for } i > 0 \end{cases}$$

and f_i on $(1_{FD})^i$ for $i \leq 0$ is p^{-i} times the usual frobenius. Then

$$h^i(D) \cong Ext^i(1_{FD}, D) \text{ for all } i.$$

Proof. *The family $\{h^i\}_{i\geq 0}$ is a cohomological functor and we see easily that $H^0(D) \cong Hom(1_{FD}, D)$. To prove the lemma, it therefore suffices to show that h^1 is effacable, i.e that for any D and any $x \in h^1(D)$, there exists an injection $D \hookrightarrow E$ such that the image of x in $h^1(E)$ is zero. Take a representative $\widetilde{x} \in D$ of $x \in D/(1-f_0)D^0$, and define E by*

$$E = D \oplus \mathcal{O}_K, E^i = D^i \text{ for } i > 0; E^i = D^i \oplus \mathcal{O}_K(\widetilde{x}, 1) \text{ for } i \leq 0,$$
$$f_i(\widetilde{x}, 1) = p^{-i}(0,1) \text{ for } i \leq 0.$$

Then we have an exact sequence $0 \to D \to E \to 1_{FD} \to 0$, and the boundary map $h^0(1_{FD}) \to h^1(D)$ sends 1 to x. Hence x dies in $h^1(E)$, proving the lemma.

Lemma 4.5. *Let D be a filtered Dieudonné module. Assume there exist $i \leq 0$ and $j \geq 1$ such that $j - i < p$, $D^i = D$, and $D^j = (0)$, i.e. that D and 1_{FD} satisfy condition (*) above for common i and j. Let $T = T(D)$, $V = T \otimes \mathbf{Q}$, and consider the canonical maps*

$$\theta_k : h^k(D) \cong Ext^k(1_{FD}, D) \to Ext^k_{\mathrm{Gal}}(\mathbf{Z}_p, T) = H^k(K, T).$$

Then
(a) *θ_0 is an isomorphism and θ_1 is an injection.*
(b) *We have a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 \to H^0(D)\otimes\mathbf{Q} \to & D^0\otimes\mathbf{Q} & \xrightarrow{1-f_0} & D\otimes\mathbf{Q} & \to h^1(D)\otimes\mathbf{Q} \to 0 \\ \theta_0\downarrow & a\downarrow & & b\downarrow & \theta_1\downarrow \\ 0 \to H^0(K,V) \to & \mathrm{Crys}(V)\oplus DR(V)^0 & \to & \mathrm{Crys}(V)\oplus DR(V) & \to H^1(K,V) \end{array}$$

where $a(x) = (x,x)$, $b(x) = (x,0)$ with the identification

$$D \otimes \mathbf{Q} = \mathrm{Crys}(V) = DR(V),$$

and the lower row is that of (3.8.4). In particular, θ_1 induces an isomorphism

$$h^1(D) \otimes \mathbf{Q} \cong H^1_e(K, V),$$

and $\exp : DR(V)/DR(V)^0 \to H^1(K,V)$ *coincides with the composite*

$$(D/D^0)\otimes\mathbf{Q} \xrightarrow[1-f]{} (D/(1-f_0)D^0)\otimes\mathbf{Q} = h^1(D)\otimes\mathbf{Q} \xrightarrow[\theta_1]{} H^1(K,V).$$

(c) *If, moreover, D is torsion free, then θ_1 induces an isomorphism*

$$h^1(D) \cong H^1_e(K,T) = Ker(H^1(K,T) \to H^1(K,V)/H^1_e(K,V)).$$

Proof. *The assertions in (a) are a formal consequence of the full faithfulness in (4.3). Let r be such that $(D_{(r)})^1 = (0)$ and $(D_{(r)})^{2-p} = D_{(r)}$. There is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \to V & \to \mathrm{Fil}^0(B_\infty \otimes D_{(r)}) \otimes \mathbf{Q}_p(-r) & \xrightarrow{c} & B_\infty \otimes D_{(r)} \otimes \mathbf{Q}_p(-r) & \to 0 \\ & \| & \downarrow a & & \downarrow b & \\ 0 & \to V & \to (B_{\mathrm{crys}} \otimes V) \oplus (B^+_{DR} \otimes V) & \to & (B_{\mathrm{crys}} \otimes V) \oplus (B_{DR} \otimes V) & \to 0, \end{array}$$

with $c = (1 - f) \otimes id$. (Surjectivity of c follows from [FM] (2.3), and the lower line is (1.17.2)$\otimes V$.) The assertions in (b) are straightforward from this. Finally, (c) is equivalent to showing the cokernel of $h^1(D) \to H^1(K,T)$ is torsion free. This follows from the diagram

$$\begin{array}{ccccccc} h^0(D/pD) & \longrightarrow & h^1(D) & \xrightarrow{p} & h^1(D) & \longrightarrow & H^1(D/pD) \\ \| \wr & & \cap & & \cap & & \cap \\ H^0(K,T/pT) & \longrightarrow & H^1(K,T) & \longrightarrow & H^1(K,T) & \longrightarrow & H^1(K,T/pT). \end{array}$$

We turn now to the proof of Theorem 4.2. The non-p-part is easy, so we consider only the p-part. It is convenient to work in the derived category C of the category of abelian groups. For a morphism $h : X \to Y$ in C such that the kernel and cokernel of $H^i(h) : H^i(X) \to H^i(Y)$ are zero for almost all i, and are finite for all i, write

$$\chi(h) = \prod_{i\in\mathbf{Z}} (\#(\mathrm{Coker}\, H^i(h)) \cdot \#(Ker H^i(h))^{-1})^{(-1)^i}. \tag{4.6}$$

For an object X of C such that $H^i(X)$ is zero for almost all i and finite for all i, write

$$\chi(X) = \prod (\# H^i(X))^{(-1)^i}.$$

Let C′ be the category obtained from C by inverting morphisms in C satisfying the above condition for h. For objects X, Y of C, we write $X \to Y$ for a morphism in C and $X \dashrightarrow Y$ for a morphism in C′. Note the definition *of $\chi(h)$ extends naturally to any isomorphism in C'. With these notations we have*

(4.6.1)
$$\begin{aligned}&\mu(H^1(K,\mathbf{Z}_p(r)))\cdot\#(H^0(K,\mathbf{Q}_p/\mathbf{Z}_p(1-r)))^{-1}\\&=\mu(H^1(K,\mathbf{Z}_p(r)))\cdot\prod_{i\neq 1}\#(H^i(K,\mathbf{Z}_p(r)))^{(-1)^i}\quad\textit{Tate duality}\\&=\chi(\mathcal{O}_K\overset{\exp}{\dashrightarrow}R\Gamma(K,\mathbf{Z}_p(r))[1]).\end{aligned}$$

Our aim is to prove this number is equal to

$$|1-q^{-r}|_p^{-1}\cdot|(r-1)!|_K.$$

Let $G=Gal(K(\zeta_{p^\infty})/K)$, $P=Gal(K(\zeta_{p^\infty})^{ab}/K(\zeta_{p^\infty}))$, $U=\varprojlim\mathcal{O}_K[\zeta_{p^n}]^\times$ as in §2. Consider the morphisms

$$\begin{aligned}R\Gamma(K,\mathbf{Z}_p(r))[1]&=R\Gamma(G,R\Gamma(K(\zeta_{p^\infty}),\mathbf{Z}_p(r)))[1]\\&\xrightarrow{a}R\Gamma(G,Hom(P,\mathbf{Z}_p(r)))\\&\xrightarrow{b}R\Gamma(G,Hom(U,\mathbf{Z}_p(r)))\\&\overset{c}{\dashrightarrow}R\Gamma(G,Hom(\mathcal{O}_K[[G]],\mathbf{Z}_p(r)))=Hom_{\mathbf{Z}_p}(\mathcal{O}_K,\mathbf{Z}_p(r)),\end{aligned}$$

where a is defined because

$$H^q(K(\zeta_{p^\infty}),\mathbf{Z}_p(r))=\begin{cases}\mathbf{Z}_p(r) & q=0\\ Hom(P,\mathbf{Z}_p(r)) & q=1\\ 0 & q\geq 2,\end{cases}$$

and $\chi(a)=\chi(R\Gamma(G,\mathbf{Z}_p(r)))$. Also b is defined by the inclusion $U\subset P$, so

$$\chi(b)=\chi(R\Gamma(G,Hom(P/U,\mathbf{Z}_p(r))))^{-1}=\chi(R\Gamma(G,\mathbf{Z}_p(r)))^{-1}.$$

*Finally, c is defined by an exact sequence of Coleman [*Co*]*

$$0\to\mathbf{Z}_p(1)\to U'\xrightarrow{\varphi}\mathcal{O}_K[[G]]\to\mathbf{Z}_p(1)\to 0\qquad(4.7)$$

*where U' is the pro-p part of U ($U/U'\cong\mathbf{F}_q^\times$), so $\chi(c)=1$. Here $\varphi(u)\in\mathcal{O}_K[[G]]$ for $u\in U'$ is characterized by $(1-p^{-1}f)\log(g_u)=\varphi(u)z$. (In [*Co*], Coleman considers the case $K=\mathbf{Q}_p$, but the exact sequence (4.7) holds for any unramified finite extension K of $\mathbf{Q}_p$ with the same proof.) Consequently, the number (4.6.1) equals*

$$\chi(c\circ b\circ a\circ\exp:\mathcal{O}_K\dashrightarrow Hom_{\mathbf{Z}_p}(\mathcal{O}_K,\mathbf{Z}_p(r))).$$

CLAIM 4.8: The map $c \circ b \circ a \circ \exp$ coincides with the map

$$K \to \mathrm{Hom}_{\mathbf{Q}_p}(K, \mathbf{Q}_p(r))$$
$$x \mapsto \{y \mapsto (r-1)!^{-1} \operatorname{Tr}_{K/\mathbf{Q}_p}((1-p^{-r}f)(x) \cdot (1-p^{r-1}f)^{-1}(y)) \otimes \nu^{\otimes r}\}$$

where ν is the fixed generator of $\mathbf{Z}_p(1)$. Note this will confirm that

$$\chi(c \circ b \circ a \circ \exp) = |1-q^{-r}|_p^{-1} \cdot |(r-1)!|_K$$

and complete the proof of (4.2). Q.E.D.

To justify the claim, we use the main result (2.1) from §2, that the boundary map $\partial : K \to H^1(K, \mathbf{Q}_p(r))$ from the Fontaine-Messing sequence

$$0 \to \mathbf{Q}_p(r) \to J_{\mathbf{Q}}^{[r]} \to B_{\mathrm{crys}}^{+} \to 0$$

is given by

(4.8.1) $$\partial(x) = \operatorname{Tr}_{K/\mathbf{Q}_p}(x\phi_{CW}^r)/(r-1)!$$

It follows from the description of exp given in (4.5)(b) that

(4.8.2) $$\exp = \partial \circ (1-p^{-r}f).$$

Also, writing $\phi_{CW}^r(u) = \phi^r(u) \otimes \nu^{\otimes r}$ we have by definition $\phi^r(u) = (zd/dz)^r \log(g_u)|_{z=1}$ with g_u as in (2.2). Using

$$(zd/dz) \circ f = pf \circ (zd/dz)$$

and properties of g_u, we find

$$\begin{aligned}(1-p^{r-1}f)\phi^r(u) &= ((zd/dz)^r(1-p^{-1}f)\log(g_u))\big|_{z=1} \\ &= ((zd/dz)^r \varphi(u)z)\big|_{z=1} = \kappa_r(\varphi(u)).\end{aligned}$$

Here $\kappa_r : \mathcal{O}_K[[G]] \to \mathcal{O}_K$ is the $\mathcal{O}_K$-algebra homomorphism extending the character $G \to \mathbf{Z}_p^\times$ given by the action of G on $\mathbf{Z}_p(r)$. Thus

(4.8.3) $$\phi_{CW}^r(u) = (1-p^{r-1}f)^{-1}\kappa_r(\varphi(u)) \otimes \nu^{\otimes r}.$$

The claim follows by combining (4.8.1), (4.8.2), and (4.8.3).

This completes the proof of (4.2).

5. Global conjectures

In this section we formulate our conjecture (5.15) for special values of Hasse L-functions. After some discussion of H^1 of global Galois representations, we consider Tamagawa measures, Tamagawa numbers, and the conjectural Tamagawa number formula. Compatibility of the conjecture with isogeny is proven in (5.14), and a geometric analog is demonstrated in (5.21).

In what follows, $\mathbb{A}_f = \hat{\mathbf{Z}} \otimes \mathbf{Q}$ denotes the ring of finite adeles of $\mathbf{Q}$.

We first discuss global versions of the finite and geometric parts of H^1. K denotes a finite extension of $\mathbf{Q}$, and $\mathcal{O}_K$ denotes the ring of integers of K. For a place v of K, K_v will be the completion of K at v.

Definition 5.1. Let $\Lambda = \mathbf{Z}_\ell, \mathbf{Q}_\ell, \hat{\mathbf{Z}}$, or $\mathbb{A}_f$, and let T be a free Λ-module of finite rank endowed with a continuous Λ-linear action of $\mathrm{Gal}(\overline{K}/K)$. For a non-empty open set $U \subset \mathrm{Spec}(\mathcal{O}_K)$, we define $H^1_{f,U}(K,T) \subset H^1(K,T)$ to be the set of cohomolmogy classes whose images in $H^1(K_v,T)$ belong to $H^1_f(K_v,T)$ (resp. to $H^1_g(K_v,T)$) for any finite place $v \in U$ (resp. $v \notin U$). We define

$$H^1_g(K,T) = \varinjlim_U H^1_{f,U}(K,T).$$

If $\Lambda = \mathbf{Z}_\ell$, and if T is unramified on U and $\ell \notin U$, we regard $H^1_{f,U}(K,T)$ as a sub-$\mathbf{Z}_\ell$-module of $H^1(U,T) = \varprojlim H^1(U_{et}, T/\ell^n T)$.

To give some feeling for these groups, we formulate a conjecture relating $H^1_{f,U}$, H^1_g, and K-theory. Let X be a smooth proper scheme over K. Fix $m, r \in \mathbf{Z}$, and let

$$\Psi = \begin{cases} gr^r(K_{2r-m-1}(X) \otimes \mathbf{Q}) & \text{if } m \neq 2r-1 \\ (CH^r(X) \otimes \mathbf{Q})_{\text{homologically}\sim 0} & \text{if } m = 2r-1 \end{cases}$$

where gr is taken with respect to the γ-filtration [Be1], and "hom $\sim$ 0" denotes the subgroup of cycles homologically equivalent to 0.

Beilinson [op. cit.] has conjectured that if $m \neq 2r-1,\ 2r-2$, and if X has proper regular model $\mathcal{X}$ over $\mathbf{Z}$ and $\Phi \subset \Psi$ denotes the image $gr^r(K_{2r-m-1}(\mathcal{X}) \otimes \mathbf{Q})$ in Ψ, then his regulator map induces an isomorphism

$$\Phi \otimes \mathbf{R} \cong H^m_{DR}(X_{\mathbf{R}}/\mathbf{R})/(\mathrm{Fil}^r\, H^m_{DR}(X_{\mathbf{R}}/\mathbf{R}) + H^m(X(\mathbf{C}), \mathbf{R}(2\pi i)^r))^+.$$

Here $X_{\mathbf{R}} = X \times_{\mathbf{Q}} \mathbf{R}$, Fil is the Hodge filtration on the de Rham cohomology, $X(\mathbf{C})$ is the set of $\mathbf{C}$-points of X as a $\mathbf{Q}$-scheme, endowed with the natural topology, and $(\ )^+$ is the $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$-fixed part, with $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ acting on both $X(\mathbf{C})$ and $\mathbf{R}(2\pi i)^r$.

We want to formulate a conjecture which will be analogous to Beilinson's conjectures and also to Tate's conjecture

$$CH^r(X)/CH^r(X)_{\mathrm{hom}\sim 0} \otimes \mathbb{A}_f \cong H^0(K, H^{2r}(X_{\overline{K}}, \mathbb{A}_f(r))).$$

Note there is a canonical homomorphism

$$\Psi \to H^1(K, H^m(X_{\overline{K}}, \mathbb{A}_f(r))). \tag{5.2}$$

Indeed, Soulé [So] has constructed chern class maps

$$K_{2r-m-1}(X) \to H^{m+1}(X, \mathbf{Z}/n\mathbf{Z}(r))$$

for the étale topology. A weight argument using the Weil conjectures shows $K_{2m-m-1}(X) \to H^{m+1}(X_{\overline{K}}, \mathbb{A}_f(r))$ is zero if $m \neq 2r-1$, and the construction of (5.2) follows using the spectral sequence

$$E_2^{i,j} = H^i(K, H^j(X_{\overline{K}}, \mathbf{Z}/n\mathbf{Z}(r))) \Rightarrow H^{i+j}(X, \mathbf{Z}/n\mathbf{Z}(r)).$$

The following conjecture was independently formulated in a slightly different form by Jannsen.

Conjecture 5.3. (i) *The above homomorphism induces an isomorphism*

$$\Psi \otimes \mathbb{A}_f \cong H^1_g(K, H^m(X_{\overline{K}}, \mathbb{A}_f(r))).$$

(ii) *For each non-empty open set U of Spec(O_K), there exists a sub-$\mathbf{Q}$-vector space $\Phi_U \subset \Psi$ characterized by the property*

$$\Phi_U \otimes \mathbb{A}_f \cong H^1_{f,U}(K, H^m(X_{\overline{K}}, \mathbb{A}_f(r)))$$

via the isomorphism (i).
(iii) *If $m \neq 2r-1$ and X has a proper regular model $\mathcal{X}$ over U, then Φ_U coincides with the image of $gr^r(K_{2r-m-1}(\mathcal{X}) \otimes \mathbf{Q})$ in Ψ. If $m = 2r-1$, then $\Phi_U = \Psi$.*

This conjecture is true if $X = \mathrm{Spec}(K)$ and $m = 0$, $r \geq 0$ by Soulé [So2]. In the case $m = r = 1$, the conjecture is equivalent to (5.4.1) or (5.4.2) below, applied to the Picard variety A of X.

Proposition 5.4. *Let A be an abelian variety or an algebraic torus over K, and let T be its Tate module. The Kummer sequences induce an injection $A(K) \otimes \hat{\mathbf{Z}} \hookrightarrow H^1_g(K, T)$. This is an isomorphism when A*

is a torus. If A is an abelian variety, the following two conditions are equivalent:

(5.4.1) $$A(K) \otimes \hat{\mathbf{Z}} \cong H^1_g(K, T);$$

(5.4.2) *The ℓ-primary part* Ш$(A)\{\ell\}$ *of*

$$(A) = Ker(H^1(K, A) \to \oplus H^1(K_v, A))$$

is finite for all ℓ.

The proof is omitted.

We next discuss a Galois representation with a de Rham structure a *motivic pair* which provides a good axiomatic foundation for our Tamagawa measures. From now on it will be convenient to take our base field to be $\mathbf{Q}$. This simplifies definitions, and as usual one areduces to this case by Weil restriction of scalars, anyway. We write p for either a prime number or the real place, ∞, of $\mathbf{Q}$. For each p, we fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. We write $V_p = V \otimes \mathbf{Q}_p$.

Definition 5.5. A motivic pair (V, D) is a pair of finite dimensional $\mathbf{Q}$-vector spaces with the following extra structure (i)–(iii) satisfying axioms (P1)–(P4).

(i) $V \otimes \mathbb{A}_f$ has a continuous $\mathbb{A}_f$-linear galois action such that $V \subset V \otimes \mathbb{A}_f$ is stable under $\mathrm{Gal}(\mathbf{C}/\mathbf{R}) \subset \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$.

(ii) D has a decreasing filtration $(D^i)_{i \in \mathbf{Z}}$ by $\mathbf{Q}$-subspaces such that $D^i = (0)$ for $i \gg 0$ and $D^i = D$ for $i \ll 0$.

(iii) For $p < \infty$ we are given an isomorphism of $\mathbf{Q}_p$-vector spaces

$$\theta_p : D_p \cong DR(V_p)$$

preserving filtrations. For $p = \infty$, we are given an isomorphism of $\mathbf{R}$-vector spaces

$$\theta_\infty : D_\infty \cong (V_\infty \otimes_{\mathbf{R}} \mathbf{C})^+$$

Here $D_p = D \otimes \mathbf{Q}_p$, and $DR(\ )$ is defined with respect to the action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_q)$. We denote the $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ fixed part by $(\ )^+$, where the action of $\sigma \in \mathrm{Gal}(\mathbf{C}/\mathbf{R})$ on $V_\infty \otimes \mathbf{C}$ is $\sigma \otimes \sigma$. We shall regard the maps θ_p as being identifications.

These data are subject to the following axioms:

(P1) There exists a non-empty open set U of $\mathrm{Spec}(\mathbf{Z})$ such tht for any $p \in U$, V_ℓ is unramified at p for $\ell \neq p$ and V_p is crystalline.

(P2) Let M be a $\mathbf{Z}$-lattice in V and let L be a $\mathbf{Z}$-lattice in D. Then there exists a finite set S of primes of $\mathbf{Q}$ ("bad primes") with $\infty \in S$ and such that for all $p \neq S$, V_p is a crystalline representaton of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, condition (*) in §4 is satisfied by the filtration on $DR(V)$, $L \otimes \mathbf{Z}_p$ is a strongly divisible lattice in $D_p = \mathrm{Crys}(V_p)$,and $M \otimes \mathbf{Z}_p$ coincides with the $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$-stable lattice in V_p corresponding to $L \otimes \mathbf{Z}_p$ via (5.5)(iii).

(P3) Let $p < \infty$, and let $P_p(V_\ell, u)$ be the polynomial $P(V_\ell, u)$ of §4, defined for the $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-module V_ℓ. Then $P_p(V_\ell, u) \in \mathbf{Q}[u]$ for all ℓ and these polynomials are independent of ℓ.

(P4) If $p < \infty$, there exists a $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$-stable $\mathbf{Z}$-lattice $T \subset V \otimes \mathbb{A}_f$ such that $H^0(\mathbf{Q}_{p,nr}, T \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ is divisible for almost all ℓ. (This condition is easily seen to be independent of the choice of T.)

Definition 5.5.1. A motivic pair (V, D) has *weights* $\leq w$ if for each $p < \infty$, the polynomial $P_p(V, u)$ has the form $\prod(1 - \alpha_i u)$ with $|\alpha_i| \leq p^{w/2}$ in $\mathbf{C}[u]$, and if $D_\infty^i \cap V_\infty^+ = (0)$ for $i > w/2$.

Suppose (V, D) has weights $\leq w$, and S is a finite set of places of $\mathbf{Q}$ containing ∞. The L-function $L_S(V, s)$ is defined by

$$L_S(V, s) = \prod_{p \notin S} P_p(V, p^{-s})^{-1}.$$

This product converges absolutely for $\mathrm{Re}(s) > w/2 + 1$.

Let (V, D) be a motivic pair with weights ≤ -1. Fix a $\mathbf{Z}$-lattice M in V such that $M \otimes \hat{\mathbf{Z}}$ is $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-stable in $V \otimes \mathbb{A}_f$. (Note $\mathbf{Z}$-lattices in V are in one-to-one correspondence with $\hat{\mathbf{Z}}$-lattices in $V \otimes \mathbb{A}_f$, so there are many such M.) We define groups $A(\mathbf{Q}_p)$ for $p \leq \infty$ associated to M, which are analogous to groups of $\mathbf{Q}_p$-rational points of a commutative algebraic group. Here the letter A is simply a notation. When (V, D) has weights ≤ -3, we define the Tamagawa measure on $\prod_{p \leq \infty} A(\mathbf{Q}_p)$.

First, let

$$(5.6) \qquad A(\mathbf{Q}_p) = \begin{cases} H_f^1(\mathbf{Q}_p, M \otimes \hat{\mathbf{Z}}) & \text{if } p < \infty \\ ((D_\infty \otimes_{\mathbf{R}} \mathbf{C})/((D_\infty^0 \otimes_{\mathbf{R}} \mathbf{C}) + M))^+ & \text{if } p = \infty \end{cases}$$

(The inclusion $M \hookrightarrow D_\infty \otimes_{\mathbf{R}} \mathbf{C}$ is given by the identification $D_\infty \otimes_{\mathbf{R}} \mathbf{C} = V_\infty \otimes \mathbf{C}$.) We regard $A(\mathbf{Q}_p)$ for $p < \infty$ as a compact group with the natural topology, and $A(\mathbf{R})$ as a locally compact group.

For $p \leq \infty$, we have the exponential homomorphism

$$(5.7) \qquad \exp : D_p/D_p^0 \dashrightarrow A(\mathbf{Q}_p)$$

which is a local isomorphism defined on a neighborhood of zero in D_p/D_p^0. Indeed, for $p < \infty$, our hypothesis $w \leq -1$ implies $P_p(V,1) \neq 0$, so we know from (4.1)(ii) that

$$\exp : D_p/D_p^0 \cong H_f^1(\mathbf{Q}_p, V_p)$$

From (P3) and (P4), we easily see that

$$A(\mathbf{Q}_p)/H_f^1(\mathbf{Q}_p, M \otimes \mathbf{Z}_p) \cong \prod_{\ell \neq p} H^0(\mathbf{Q}_p, M \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)$$

is finite, so (5.7) is a local isomorphism as claimed. For $p = \infty$, we define the exponential homomorphism to be the evident canonical map, which is defined on the total space D_∞/D_∞^0.

Now assume the weights are ≤ -3. We define the Tamagawa measure on $\prod A(\mathbf{Q}_p)$ as follows. Fix an isomorphism

$$\omega : \det_{\mathbf{Q}}(D/D^0) \cong \mathbf{Q}.$$

For each $p \leq \infty$ this gives

$$\det_{\mathbf{Q}_p}(D_p/D_p^0) \cong \mathbf{Q}_p$$

This trivialization of the determinant gives a Haar measure on the p-adic space D_p/D_p^0 and hence a Haar measure $\mu_{p,\omega}$ on $A(\mathbf{Q}_p)$ via the exponential map. By (4.1), for a sufficiently large finite set S of places of $\mathbf{Q}$ containing ∞, we have for $p \neq S$

$$\mu_{p,\omega}(A(\mathbf{Q}_p)) = P_p(V,1). \tag{5.8}$$

Since the weights are ≤ -3, the product

$$L_S(V,0)^{-1} = \prod_{p \notin S} \mu_{p,\omega}(A(\mathbf{Q}_p))$$

converges, so the product measure $\mu = \prod_{p \leq \infty} \mu_{p,\omega}$ on $\prod_{p \leq \infty} A(\mathbf{Q}_p)$ is defined. Since for $a \in \mathbf{Q}^\times$, $\mu_{p,a\omega} = |a|_p \mu_{p,\omega}$, the product formula implies that μ is independent of the choice of ω.

Definition 5.9. μ is the Tamagawa measure for the motivic pair (D,V).

If we only assume the weights of (D,V) are ≤ -1, we can define the Tamagawa measure μ on $\prod A(\mathbf{Q}_p)$ if we assume $L_S(V,s)$ can be analytically continued to $\mathbf{Re}(s) > -\varepsilon$ for some $\varepsilon > 0$. Indeed, in this case, let $r = \mathrm{ord}_{s=0} L_S(V, s)$, and define

$$\mu = |\lim_{s\to 0} s^r L_S(V, s)^{-1}| \cdot \prod_{p\in S} \mu_{p,\omega} \cdot \prod_{s\notin S}(P_p(V,1)^{-1}\mu_{p,\omega}). \tag{5.9.1}$$

We hope that a motif over $\mathbf{Q}$ gives a motivic pair. Consider, for example, the pure motif $H^m(X)(r)$ where X is a smooth, proper, (not necessarily geometrically connected) scheme over $\mathbf{Q}$. Define

$$V = H^m(X(\mathbf{C}), \mathbf{Q}((2\pi i)^r); \quad D = H^m_{DR}(X/\mathbf{Q}).$$

By Artin's theorem, $V \otimes \mathbb{A}_f \cong H^m_{et}(X_{\overline{\mathbf{Q}}}, \mathbb{A}_f)(r)$ whence the galois acton. The filtration on D is deduced fromt he Hodge filtration on H_{DR} by

$$D^i \stackrel{\text{def}}{=} \mathrm{Fil}^{r+i} H^m_{DR}(X/\mathbf{Q}).$$

The isomorphism

$$\theta_\infty : D_\infty \cong (\mathbf{C} \otimes_{\mathbf{R}} V_\infty)^+$$

is standard. The de Rham conjecture of Fontaine [**Fo**] for the scheme $X_{\mathbf{Q}_p}$ for $p < \infty$ proved recently by Faltings [**Fa**] says that there exists a canonical isomorphism

$$\theta_p : D_p \cong DR(V_p).$$

The work of Fontaine and Messing shows that (V, D) has the properties (P1), (P2). Moreover, (P3) holds for almost all p ([**Del**], [**FM**]), and (P3) and (P4) hold if $m = 1$. Thus (V, D) is a motivic pair if $m = 1$. Although (P1)–(P4) involve a number of unproven properties of motives, we will see in (5.15.2) below that it is possible to formulate a conjecture about the ℓ-part of the Tamagawa number under much weaker assumptions.

We now introduce the global points $A(\mathbf{Q})$. Let (V, D) be a motivic pair of weights ≤ -3, and assume that we are given a finite dimensional $\mathbf{Q}$-vector space Φ endowed with an isomorphism of $\mathbf{R}$-vector spaces

$$R_\infty : \Phi \otimes \mathbf{R} \cong D_\infty/(D^0_\infty + V^+_\infty)$$

and an isomorphism of $\mathbb{A}_f$-modules

$$R_{\mathrm{Gal}} : \Phi \otimes \mathbb{A}_f \cong H^1_{f,\mathrm{Spec}(\mathbf{Z})}(\mathbf{Q}, V \otimes \mathbb{A}_f).$$

Fix a $\mathbf{Z}$-lattice M in V such that $M \otimes \hat{\mathbf{Z}}$ is galois stable in $V \otimes \mathbb{A}_f$. Define $A(\mathbf{Q}) \subset H^1_{f,\mathrm{Spec}(\mathbf{Z})}(\mathbf{Q}, M \otimes \hat{\mathbf{Z}})$ to be the inverse image of $R_{\mathrm{gal}}(\Phi)$. Note that $A(\mathbf{Q})$ is a finitely generated abelilan group such that

$$A(\mathbf{Q}) \otimes \hat{\mathbf{Z}} = H^1_{f,\mathrm{Spec}(\mathbf{Z})}(\mathbf{Q}, M \otimes \hat{\mathbf{Z}}) \text{ and } A(\mathbf{Q}) \otimes \mathbf{Q} = \Phi.$$

Lemma 5.10. (i) $A(\mathbf{Q})_{\text{tor}} \cong H^0(\mathbf{Q}, M \otimes \mathbf{Q}/\mathbf{Z})$.
(ii) $A(\mathbf{Q}_p)_{\text{tor}} \cong H^0(\mathbf{Q}_p, M \otimes \mathbf{Q}/\mathbf{Z})$ for $p \leq \infty$.

Proof. Exercise

Of course there are natural homomorphisms $A(\mathbf{Q}) \to A(\mathbf{Q}_p)$ for $p < \infty$, as well as $A(\mathbf{Q}) \to A(\mathbf{R})/A(\mathbf{R})_{\text{cpt}} = D_\infty/(D_\infty^0 + V_\infty^+)$. Here $A(\mathbf{R})_{\text{cpt}}$ denotes the maximal compact subgroup of $A(\mathbf{R})$. (5.10)(ii) above for $p = \infty$ implies that $A(\mathbf{Q})_{\text{tor}} \subset A(\mathbf{R})_{\text{tor}}$, so we can choose $h : A(\mathbf{Q}) \to A(\mathbf{R})$ lifting the above map. (We expect, of course, that if (V, D, Φ) comes from a motif, a canonical such lifting is given; but this is not necessary to define the Tamagawa number.) We define

$$\text{Tam}(M) = \mu((\prod A(\mathbf{Q}_p))/A(\mathbf{Q})). \tag{5.11}$$

Our hypotheses imply that the image of $A(\mathbf{Q})$ in $A(\mathbf{R})/A(\mathbf{R})_{\text{cpt}}$ is discrete and co-compact, so $\text{Tam}(M)$ is defined.

When (V, D) is associated to the motive $H^m(X)(r)$, the conjectures of Beilinson suggest that

$$\Phi = \text{Image}(gr^r(K_{2r-m-1}(\mathcal{X}) \otimes \mathbf{Q}) \to gr^r(K_{2dr-m-1}(X) \otimes \mathbf{Q})$$

with R_{gal} given by the chern class map and R_∞ by the Beilinson regulator, has the required properties. If so, we say that the triple (V, D, Φ) comes from the motive $H^m(X)(r)$.

Let (V, D, Φ, M) be as above, and let $A(\mathbf{Q})$ and $A(\mathbf{Q}_p)$ be the corresponding groups. In order to formulate our Tamagawa number conjecture, we consider the map

$$\alpha_M : \frac{H^1(\mathbf{Q}, M \otimes \mathbf{Q}/\mathbf{Z})}{A(\mathbf{Q}) \otimes \mathbf{Q}/\mathbf{Z}} \longrightarrow \bigoplus_{p \leq \infty} \frac{H^1(\mathbf{Q}_p, M \otimes \mathbf{Q}/\mathbf{Z})}{A(\mathbf{Q}_p) \otimes \mathbf{Q}/\mathbf{Z}} \tag{5.12}$$

Define

$$Ш(M) = \text{Ker}(\alpha_M). \tag{5.13}$$

Proposition 5.14 (i) *For any prime number ℓ, the ℓ-primary part $Ш(M)\{\ell\}$ is finite.*
(ii) $\text{Coker}(\alpha_M)$ *is finite and is isomorphic to the Pontryagin dual of the finite group $H^0(\mathbf{Q}, M^* \otimes \mathbf{Q}/\mathbf{Z}(1))$, with $M^* = Hom(M, \mathbf{Z})$.*
(iii) *Assume $Ш(M)$ is finite, and define $\chi(M) \in \mathbf{R}^\times$ by*

$$\chi(M) = \text{Tam}(M) \cdot \#(Ш(M)) \cdot \#(H^0(\mathbf{Q}, M^* \otimes \mathbf{Q}/\mathbf{Z}(1)))^{-1}.$$

Then for any $\mathbf{Z}$*-lattice* $M' \subset V$ *such that* $M' \otimes \hat{\mathbf{Z}}$ *is galois stable in* $V \otimes \mathbb{A}_f$*, we have* $Ш(M')$ *also finite and* $\chi(M) = \chi(M')$.

Conjecture 5.15.(Tamagawa number conjecture) *Assume the triple* (V, D, Φ) *comes from a motif. Let* M *be a* $\mathbf{Z}$*-lattice in* V *such that* $M \otimes \hat{\mathbf{Z}}$ *is galois stable in* $V \otimes \mathbb{A}_f$*. Then* $Ш(M)$ *is finite, and*

$$\mathrm{Tam}(M) = \frac{\#(H^0(\mathbf{Q}, M^* \otimes \mathbf{Q}/\mathbf{Z}(1)))}{\#(Ш(M))}.$$

By (5.14), the validity of this conjecture is independent of the choice of M. If (V, D, Φ) comes from $H^m(X)(r)$, we can take

$$M = H^m(X(\mathbf{C}), \mathbf{Z}(2\pi i)^r)/\,\mathrm{tors}.$$

However, when the motive is only the image of a projector, there is frequently no canonical choice for M so it is nice to have a conjecture which is "isogeny invariant".

The conjecture can be rewritten to emphasize the role of the L-function (5.15.1)

$$L_S(V, 0) = \frac{\#(Ш(M))}{\#(H^0(\mathbf{Q}, M^* \otimes \mathbf{Q}/\mathbf{Z}(1)))}\mu_{\infty,\omega}(A(\mathbf{R})/A(\mathbf{Q}))\cdot \prod_{p \in S-\infty} \mu_{p,\omega}(A(\mathbf{Q}_p))$$

where S is sufficiently large (depending on ω as well as (V, D, Φ, M)).

Remark 5.15.2. Suppose (V, D, Φ, M) corresponds to $H^m(X)(r)$ with $m - 2r \le -3$ and X smooth and proper over $\mathbf{Q}$. The problem of even defining the two sides of (5.15) involves difficult unsolved questions relating to global behavior at all primes. Such questions as finiteness of $Ш(M)$ and axiom (P4) can be avoided if we are content to work modulo $\mathbf{Z}_{(\ell)}^\times = \{a/b | a, b \in \mathbf{Z}; \ell \nmid ab\}$. Indeed, we can forget $V_{\ell'}$ for $\ell' \neq \ell$, and we only need assume $P_\ell(V_\ell, 1) \neq 0$ and find $\Phi \subset gr^r(K_{2r-m-1}(X) \otimes \mathbf{Q})$ such that

$$\Phi \otimes \mathbf{R} \cong D_\infty/(D_\infty^0 + V_\infty^+);$$
$$\Phi \otimes \mathbf{Q}_\ell \cong \mathrm{Ker}(H^1(\mathbf{Q}, V_\ell) \to H^1(\mathbf{Q}_\ell, V_\ell)/H^1_f(\mathbf{Q}_\ell, V_\ell) \oplus \prod_{p \neq \ell} H^1(\mathbf{Q}_p, V_\ell)).$$

(Note $P_\ell(V_\ell, 1) \neq 0$ holds at least if X is projective and has good reduction at ℓ by [FM], [Fa] and the Weil conjectures.) To define the ℓ-Tamagawa number

$$\mathrm{Tam}^{(\ell)}(M) \in \mathbf{R}^\times/\mathbf{Z}_{(\ell)}^\times$$

we use the groups

$$A^{(\ell)}(\mathbf{Q}_p) \stackrel{\text{def}}{=} \begin{cases} H^1_f(\mathbf{Q}_\ell, M\otimes \mathbf{Z}_\ell) & \text{if } p=\ell \\ H^1(\mathbf{Q}_\ell, M\otimes \mathbf{Z}_\ell)_{\text{tor}} & \text{if } p\neq \ell, p<\infty \end{cases}$$

One *can* show finiteness for

$$Ш^{(\ell)}(M) \stackrel{\text{def}}{=}$$
$$\operatorname{Ker}\Big(\frac{H^1(\mathbf{Q}, M\otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)}{\text{image}(\Phi)} \to \frac{H^1(\mathbf{Q}_\ell, M\otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)}{H^1_f(\mathbf{Q}_\ell, M\otimes \mathbf{Z}_\ell)\otimes \mathbf{Q}/\mathbf{Z}} \bigoplus$$
$$\bigoplus_{p\neq \ell} H^1(\mathbf{Q}_p, M\otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)\Big)$$

and we can actually ask if

$$\operatorname{Tam}^{(\ell)}(M)\cdot \#Ш^{(\ell)}(M)\cdot \#H^0(\mathbf{Q}, M^*\otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell(1)) = 1 \text{ in } R^\times/\mathbf{Z}^\times_{(\ell)}.$$

We turn now to the proof of (5.14). The finitenes of $Ш\{\ell\}$ and $\operatorname{Coker}(\alpha_M)\{\ell\}$ follows from

Lemma 5.16. *Let ℓ be a prime number, U a non-empty open set of Spec($\mathbf{Z}$) not containing ℓ, T a free $\mathbf{Z}_\ell$-module of finite rank with a continuous action of Gal($\overline{\mathbf{Q}}/\mathbf{Q}$), and $V = T\otimes \mathbf{Q}_\ell$. Assume conditions (a)–(d) below hold:*

(a) *V is unramified on U.*
(b) *V is a de Rham representation of Gal($\overline{\mathbf{Q}}_\ell/\mathbf{Q}_\ell$).*
(c) *$P_p(V,1)\neq 0$ for any $p\notin U$, $p\neq\infty$.*
(d) *$P_p(V(-1),1)\neq 0$ for any $p\in U$.*

Let

$$\alpha_T : \frac{H^1(\mathbf{Q}, T\otimes \mathbf{Q}/\mathbf{Z})}{H^1_{f,\text{Spec}(\mathbf{Z})}(\mathbf{Q}, T)\otimes \mathbf{Q}/\mathbf{Z}} \to \bigoplus_{p\leq\infty} \frac{H^1(\mathbf{Q}_p, T\otimes \mathbf{Q}/\mathbf{Z})}{H^1_f(\mathbf{Q}_p, T)\otimes \mathbf{Q}/\mathbf{Z}}$$
$$(5.16.1) \qquad \beta_T : H^2(\mathbf{Q}, T\otimes \mathbf{Q}/\mathbf{Z}) \to \bigoplus_{p\leq\infty} H^2(\mathbf{Q}_p, T\otimes \mathbf{Q}/\mathbf{Z}).$$

(For $p=\infty$, $H^1_f(\mathbf{Q}_p, T)\otimes \mathbf{Q}/\mathbf{Z}$ is understood to be 0.) Then $\ker(\alpha_T)$ *is finite, β_T is surjectrive,* $\operatorname{Coker}(\alpha_T)$ *and Ker(β_T) are of co-finite type (i.e., $\cong (\mathbf{Q}_\ell/\mathbf{Z}_\ell)^r \oplus$ (finite) for some $r<\infty$), and*

$$H^1_{f,\text{Spec}(\mathbf{Z})}(\mathbf{Q}, V) = \dim(V) - \dim(DR(V)^0) - \dim(V^+)$$
$$+ \operatorname{corank}(\operatorname{Coker}(\alpha_T)) + \operatorname{corank}(\operatorname{Ker}(\beta_T)).$$

In particular, if $H^1_{f,\mathrm{Spec}(\mathbf{Z})}(\mathbf{Q}, V) = \dim V - \dim(DR(V)^0) - \dim V^+$ then $\mathrm{Coker}(\alpha_T)$ and $\ker(\beta_T)$ are finite.

Remark 5.16.2. The bijectivity of β_T is proven in ([Ja],Th. 3) in a general setting.

Proof of 5.16. Let

$$\alpha_{T,U} : \frac{H^1(U, T\otimes \mathbf{Q}/\mathbf{Z})}{H^1_{f,\mathrm{Spec}(\mathbf{Z})}(\mathbf{Q},T)\otimes \mathbf{Q}/\mathbf{Z}} \to \bigoplus_{p\notin U} \frac{H^1(\mathbf{Q}_p, T\otimes \mathbf{Q}/\mathbf{Z})}{H^1_f(\mathbf{Q}_p,T)\otimes \mathbf{Q}/\mathbf{Z}}$$

$$\beta_{T,U} : H^2(U, T\otimes \mathbf{Q}/\mathbf{Z}) \to \bigoplus_{p\notin U} H^2(\mathbf{Q}_p, T\otimes \mathbf{Q}/\mathbf{Z}).$$

$\beta_{T,U}$ is surjective by Tate duality, and localization for étale cohomology yields

$$\mathrm{Ker}(\alpha_{T,U}) \cong \mathrm{Ker}(\alpha_T)$$
$$0 \to \mathrm{Coker}(\alpha_{T,U}) \to \mathrm{Coker}(\alpha_T) \to \mathrm{Ker}(\beta_{T,U}) \to \mathrm{Ker}(\beta_T) \to 0.$$

By the definition of $H^1_{f,\mathrm{Spec}(\mathbf{Z})}$ and the finite generation of $H^*(U,T)$, it follows that $\mathrm{Ker}(\alpha_{T,U})$, and hence $\mathrm{Ker}(\alpha_T)$, is finite. Also $\mathrm{Coker}(\alpha_T)$ and $\mathrm{Ker}(\beta_T)$ are of co-finite type, and

$$\begin{aligned}
&\mathrm{corank}(\mathrm{Coker}(\alpha_T)) + \mathrm{corank}(\mathrm{Ker}(\beta_T)) = \mathrm{corank}(\mathrm{Coker}(\alpha_{T,U})) \\
&\quad + \mathrm{corank}(\mathrm{Ker}(\beta_{T,U})) = \\
&\sum_{i=0}^{2}(-1)^i \,\mathrm{corank}\, H^i(U, T\otimes \mathbf{Q}/\mathbf{Z}) - \sum_{\substack{p\notin U\\ p\neq\infty}} \sum_{i=0}^{2}(-1)^i\, \mathrm{corank}(H^i(\mathbf{Q}_p, T\otimes \mathbf{Q}/\mathbf{Z}) \\
&\quad + \dim H^1_{f,\mathrm{Spec}(\mathbf{Z})}(\mathbf{Q}, V) - \dim H^1_f(\mathbf{Q}_\ell, V).
\end{aligned}$$

We have

$$\sum(-1)^i \,\mathrm{corank}\, H^i(U, T\otimes \mathbf{Q}/\mathbf{Z}) = \dim(V^+) - \dim(V) \qquad \text{[Ta1]}$$

$$\sum(-1)\,\mathrm{corank}\, H^i(\mathbf{Q}_p, T\otimes \mathbf{Q}/\mathbf{Z}) = \begin{cases} -\dim(V) & \text{if } p = \ell \\ & \qquad \text{(cf [Se])} \\ 0 & \text{if } p\neq \ell, p\neq\infty \end{cases}$$

$$\dim H^1_f(\mathbf{Q}_\ell, V) = \dim(D/D^0) \qquad (4.1).$$

These prove the formula in (5.14).

Proof of the remainder of (5.14) (ii). Let ℓ be a given prime number. By (3.8), we have

$$\frac{H^1(\mathbf{Q}_p, M \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)}{A(\mathbf{Q}_p) \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell} \cong \mathrm{Hom}(H^1_f(\mathbf{Q}_p, M^* \otimes \mathbf{Z}_\ell(1)), \mathbf{Q}_\ell/\mathbf{Z}_\ell)$$

Let $U \subset \mathrm{Spec}(\mathbf{Z})$ be non-empty open such that $M \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell$ is unramified over U, and consider the diagram

$$\begin{array}{ccccccc} H^1(U, M \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell) & \rightarrow & \bigoplus_{p \notin U} H^1(\mathbf{Q}_p, M \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell) & \rightarrow & H^1(U, M^* \otimes \mathbf{Z}_\ell(1))^* & \rightarrow & 0 \\ \downarrow & & \downarrow & & & & \\ \frac{H^1(U, M \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)}{A(\mathbf{Q}) \otimes \mathbf{Q}/\mathbf{Z}} & \xrightarrow{\alpha_{U,\ell}} & \bigoplus_{p \notin U} \frac{H^1(\mathbf{Q}_p, M \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)}{A(\mathbf{Q}_p) \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell} & & & & \end{array}$$

(The top row is exact by Tate duality.) Since $\mathrm{Coker}(\alpha_{U,\ell})$ is finite, we see that

$$\begin{aligned} \mathrm{Coker}(\alpha_{U,\ell}) &\cong H^1(U, M^* \otimes \mathbf{Z}_\ell(1))^*/(\mathrm{div}) \\ &\cong \mathrm{Hom}(H^1(U, M^* \otimes \mathbf{Z}_\ell(1))_{\mathrm{tor}}, \mathbf{Q}_\ell/\mathbf{Z}_\ell) \\ &\cong \mathrm{Hom}(H^0(U, M^* \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell(1)), \mathbf{Q}_\ell/\mathbf{Z}_\ell) \\ &\cong \mathrm{Hom}(H^0(\mathbf{Q}, M^* \otimes Q_\ell/\mathbf{Z}_\ell(1)), \mathbf{Q}_\ell/\mathbf{Z}_\ell). \end{aligned}$$

This last expression is independent of U, so

$$\mathrm{Coker}(\alpha_{U,\ell}) \cong \mathrm{Coker}(\alpha_M)\{\ell\}.$$

Further, our weight hypothesis implies $\mathrm{wt}(M^*(1)) \geq 1$, so

$$\mathrm{Coker}(\alpha_M) \cong \mathrm{Hom}(H^0(\mathbf{Q}, M^* \otimes \mathbf{Q}/\mathbf{Z}(1)), \mathbf{Q}/\mathbf{Z})$$

is finite. This proves (5.14)(ii).

Proof of 5.14(iii). We may assume $M' \subset M$ and $\#(M/M')$ is a power of ℓ. Let $A'(\mathbf{Q})$ and $A'(\mathbf{Q}_p)$ be groups corresponding to M'. For U a sufficiently small open set in $\mathrm{Spec}(\mathbf{Z})$, we have by (5.8)

$$\frac{\mathrm{Tam}(M)}{\mathrm{Tam}(M')} = \frac{\prod_{p \notin U} \chi(A'(\mathbf{Q}_p) \rightarrow A(\mathbf{Q}_p))}{\chi(A'(\mathbf{Q}) \rightarrow A(\mathbf{Q}))}. \tag{5.17}$$

(Here χ is as in (4.6).) As in the proof of (5.16), we have

$$\frac{\chi(\alpha_M)}{\chi(\alpha_{M'})} = \frac{\prod_{p\notin U}\chi\left[\frac{H^1(\mathbf{Q}_p,M'\otimes\mathbf{Q}_\ell/\mathbf{Z}_\ell)}{A'(\mathbf{Q}_p)\otimes\mathbf{Q}_\ell/\mathbf{Z}_\ell} \to \frac{H^1(\mathbf{Q}_p,M\otimes\mathbf{Q}_\ell/\mathbf{Z}_\ell)}{A(\mathbf{Q}_p)\otimes\mathbf{Q}_\ell/\mathbf{Z}_\ell}\right]}{\chi\left[\frac{H^1(U,M'\otimes\mathbf{Q}_\ell/\mathbf{Z}_\ell)}{A'(\mathbf{Q})\otimes\mathbf{Q}_\ell/\mathbf{Z}_\ell} \to \frac{H^1(U,M\otimes\mathbf{Q}_\ell/\mathbf{Z}_\ell)}{A(\mathbf{Q})\otimes\mathbf{Q}_\ell/\mathbf{Z}_\ell}\right]}. \tag{5.18}$$

A theorem of Tate ([**Ta3**]) gives

$$\begin{aligned} H^q(U,M/M') &\cong \bigoplus H^q(\mathbf{Q}_p,M/M') \text{ for } q\ge 3;\\ \prod_{0\le q\le 2} \#H^q(U,M/M')^{(-1)^q} &= \prod_{p\notin U}\prod_{0\le q\le 2} H^q(\mathbf{Q}_p,M/M')^{(-1)^q} \end{aligned} \tag{5.19}$$

Lemma 5.20. (i) *For $K=\mathbf{Q}$ or $K=\mathbf{Q}_p$, there is an exact sequence*
(5.20.1)

$$\begin{aligned} &0\to H^0(K,M/M')\to A'(K)\to A(K)\to\\ &H^1(K,M/M')\to \frac{H^1(K,M'\otimes\mathbf{Q}/\mathbf{Z})}{A'(K)\otimes\mathbf{Q}/\mathbf{Z}} \to \frac{H^1(K,M\otimes\mathbf{Q}/\mathbf{Z})}{A(K)\otimes\mathbf{Q}/\mathbf{Z}}\\ &\to H^2(K,M/M')\to H^2(K,M'\otimes\mathbf{Q}/\mathbf{Z})\to H^2(K,M\otimes\mathbf{Q}/\mathbf{Z}). \end{aligned}$$

(ii) *For U as above, there is an exact sequence*

$$\begin{aligned} &0\to H^0(K,M/M')\to A'(\mathbf{Q})\to A(\mathbf{Q})\to\\ &H^2(U,M/M')\to \frac{H^1(U,M'\otimes\mathbf{Q}_\ell/\mathbf{Z}_\ell)}{A'(K)\otimes\mathbf{Q}_\ell/\mathbf{Z}_\ell} \to \frac{H^1(U,M\otimes\mathbf{Q}_\ell/\mathbf{Z}_\ell)}{A(K)\otimes\mathbf{Q}_\ell/\mathbf{Z}_\ell}\\ &\to H^2(U,M/M')\to H^2(U,M'\otimes\mathbf{Q}_\ell/\mathbf{Z}_\ell)\\ &\to H^2(U,M\otimes\mathbf{Q}_\ell/\mathbf{Z}_\ell). \end{aligned} \tag{5.20.2}$$

The proof of the lemma is easy, and we omit it. It is now straightforward if somewhat tedious to combine formulas (5.17)–(5.19) with the above exact sequences and deduce $\chi(M)=\chi(M')$. This finishes the proof of (5.14).

We now show that an analogoue of (5.15) for function fields in one variable over finite fields is true for the non p part (p = characteristic) with no reference to a motif. We do not know how to treat the p-primary part.

Let C be a smooth proper connected curve over a finite field k of characteristic p, with function field K. Let $\ell\neq p$ be a prime number and let T be a free $\mathbf{Z}_\ell$-module of finite rank endowed with a continuous action of $\mathrm{Gal}(K_{\mathrm{sep}}/k)$. Assume that T is unramified on some non-empty open set of C.

For a place v of K, let K_v be the completion of K at v, and define $H^1_f(K_v,T) \subset H^1(K_v,T)$ and $P_v(T,u) \in \mathbf{Q}_\ell[u]$ just as in the number field case. For a non-empty open set U of C, define $H^1_{f,U}(K,T) \subset H^1(K,T)$ again as in the number field case, and let

$$Z(U,T,u) = \prod P_v(T, u^{\deg(v)}) \in \mathbf{Q}_\ell[[u]].$$

Then by Grothendieck, $Z(U,T,u) \in \mathbf{Q}_\ell(u)$, and we have also

$$H^1_{f,U}(K,T) = H^1(U, j_*T); \quad j : U' \hookrightarrow U,\ T \text{ unramified on } U'.$$

Proposition 5.21. *With notations as above, assume that for some isomorphism* $\iota : \overline{\mathbf{Q}}_\ell \cong \mathbf{C}$, *the sheaf* T *is* ι*-mixed* [De3] *on some open set of* C.

(i) *Assume* T *has* ι*-weights* ≤ -1. *Then for any place* v *of* K, $H^1_f(K_v,T)$ *is finite and is isomorphic to* $H^0(K_v, T\otimes \mathbf{Q}/\mathbf{Z})$. *If* T *is unramified at* v, *the order of this group is equal to* $|P_v(T,1)|_\ell$,

(ii) *Assume* T *has* ι*-weight* ≤ -2. *Then* $H^1_{f,C}(K,T)$ *is finite and is isomorphic to* $H^0(K, T\otimes\mathbf{Q}/\mathbf{Z})$.

(iii) *Assume* T *has* ι*-weights* ≤ -3. *Then the kernel and cokernel of* $\alpha : H^1(K, T\otimes \mathbf{Q}/\mathbf{Z}) \to \bigoplus_{\text{all } v} H^1(K_v, T\otimes\mathbf{Q}/\mathbf{Z})$ *are finite.*

(iv) *Assume* T *has* ι*-weights* ≤ -3. *If* U *is a non-empty open set of* C *on which* T *is unramified, we have*

$$\frac{\prod \#H^0(K_v, T\otimes\mathbf{Q}/\mathbf{Z})}{\#H^0(K, T\otimes\mathbf{Q}/\mathbf{Z})}\cdot |Z(U,T,1)|_\ell = \frac{\#\operatorname{Coker}(\alpha)}{\#\,Ker(\alpha)}$$

(v) *Assume* T *has* ι*-weights* ≤ -3. *Then* Coker(α) *is isomorphic to the Pontryagin dual of the finite group* $H^0(K, T^*\otimes\mathbf{Q}/\mathbf{Z}(1))$.

Proof. The proof used Deligne's theory of weights. We prove here only (iii) and (iv), leaving the rest for the reader. Assume T unramified over U. By [Gr],

$$Z(U,T,u) = \det(1 - fu : R\Gamma_c(U_{\bar{k}}, T\otimes\mathbf{Q}))^{-1}. \tag{5.22}$$

The eigenvalues of f on $H^q_c(U_{\bar{k}}, T\otimes\mathbf{Q})$ are not 1 by [De3], and hence the kernel and cokernel of $1-f$ on $H^q_c(U_{\bar{k}}, T\otimes\mathbf{Q}/\mathbf{Z})$ are finite. This yields finiteness for $H^q_c(U, T\otimes\mathbf{Q}/\mathbf{Z})$, and hence by (5.22)

$$|Z(U,T,1)|_\ell = \chi(1-f : R\Gamma_c(U_{\bar{k}}, T\otimes\mathbf{Q}/\mathbf{Z})) = \chi(R\Gamma_c(U, T\otimes\mathbf{Q}/\mathbf{Z})).$$

On the other hand, by Jannsen's theorem [**Ja**]

$$H^2(K, T\otimes \mathbf{Q}/\mathbf{Z}) \cong \bigoplus_{\text{all } v} H^2(K_v, T\otimes \mathbf{Q}/\mathbf{Z}).$$

Tate duality yields, therefore

$$0 \to H^0_c(U) \to H^0(K) \to \oplus_{v\in C-U} H^0(K_v) \to H^1_c(U) \to H^1(K) \to \oplus_{\text{all } v} H^1(K_v) \to H^2_c(U) \to 0$$

$$H^q_c(U) = (0) \text{ for } q \geq 3, \text{ (cohomology with coefficients in } T\otimes \mathbf{Q}/\mathbf{Z}).$$

This proves (5.21)(iii) and (iv).

For a motif of weights ≤ -1 which is not necessarily of weights ≤ -3, we formulate the conjecture on the special value of L-functions at $s = 0$ as follows. We consider here two cases

(i) motives of pure weight -1

(ii) anisotropic motives of pure weight -2, where "anisotropic" means that $H^0(\mathbf{Q}, V^* \otimes \mathbf{A}_f(1)) = (0)$.

It is probable that (considered modulo torsion) a motif of weights ≤ -1 is a successive extension of motives of types (i) and (ii), together with

(iii) motives of the form (Artin motif)(1)

(iv) motives of weights ≤ -3.

Motives of type (iii) are treated by the classical Tamagawa number formula, and we have already treated motives of type (iv), so it suffices in some sense to consider cases (i) and (ii). (Taking such pure pieces makes the statement of the conjecture simpler.) In these cases the Tamagawa number conjecture again has the form

$$\text{Tam}(M) = \#H^0(\mathbf{Q}, M^*\otimes \mathbf{Q}/\mathbf{Z})/\#\text{Ш}(M).$$

The right hand side is defined as before. To define $\text{Tam}(M)$, we assume analytic continuation of the L-function and define the Tamagawa measure μ on $\prod A(\mathbf{Q}_p)$ by (5.9.1). In case (ii) we assume

$$\Phi\otimes\mathbf{R} \cong D_\infty/(D^0_\infty + V^+_\infty)$$

and hence $\text{Tam}(M) = \mu(\prod A(\mathbf{Q}_p)/A(\mathbf{Q}))$ is defined. In case (i), we define

$$\text{Tam}(M) = \mu(\prod A(\mathbf{Q}_p))\cdot H/\#A(\mathbf{Q})_{\text{tor}}$$

where H is the descriminant of the height pairing

$$A(\mathbf{Q}) \times A^*(\mathbf{Q}) \to \mathbf{R} \quad ([\mathbf{Be}], [\mathbf{B}])$$

Here $A^*(\mathbf{Q}) \subset H^1_{f,\mathrm{Spec}(\mathbf{Z})}(\mathbf{Q}, M^* \otimes \hat{\mathbf{Z}}(1))$ is defined to be the inverse image of Image$(\Phi^*) \subset H^1_{f,\mathrm{Spec}(\mathbf{Z})}(\mathbf{Q}, V^* \otimes \mathbf{A}_f(1))$ with Φ^* as follows. In the case of a motif $H^{2r-1}(X)(r)$ with X smooth and proper of dim n,

$$\Phi^* = (CH^{n+1-r}(X) \otimes \mathbf{Q})_{\mathrm{hom}\sim 0}.$$

In general, motives of type (i) should be direct summands of such defined by correspondence projectors, and one takes for Φ^* the image of the cycles under this correspondence.

Finally, the reader may be concerned that our groups $A(\mathbf{Q}_p)$ ar always compact for $p < \infty$, while $A(\mathbf{R})$ need not be compact. This break in the parallelism between $\mathbf{R}$ and $\mathbf{Q}_p$ arises because in defining A we chose to work with Φ rather than Ψ. It can be remedied as follows. Consider the groups V, D, and Ψ corresponding to $H^m(X)(r)$ with $m - 2r \leq -1$. Let $M \subset V$ be a lattice with $M \otimes \hat{\mathbf{Z}}$ galois stable. Define groups $B(\mathbf{Q}_p) \supset A(\mathbf{Q}_p)$ by taking $B(\mathbf{R}) = A(\mathbf{R})$ and $B(\mathbf{Q}_p)$ = inverse image in $H^1_g(\mathbf{Q}_p, M \otimes \hat{\mathbf{Z}})$ of

$$\mathrm{Im}(\Psi \to H^1_g(\mathbf{Q}_p, V \otimes \mathbf{A}_f)/H^1_f(\mathbf{Q}_p, V \otimes \mathbf{A}_f)).$$

(We assume (5.3).) The topology on $A(\mathbf{Q}_p)$ is extended to $B(\mathbf{Q}_p)$ by taking $B(\mathbf{Q}_p)/A(\mathbf{Q}_p)$ to be discrete. We define $B(\mathbf{Q}) \subset A(\mathbf{Q})$ using Ψ in place of Φ.

Assume $m - 2r \leq -3$. Then $B(\mathbf{Q}_p) = A(\mathbf{Q}_p)$ for almost all p. Define

$$\mathrm{Tam}(M)^\sim = \mu\Big(\prod B(\mathbf{Q}_p)/B(\mathbf{Q})\Big)$$

$$Ш(M)^\sim = \mathrm{Ker}\Big(\frac{H^1(\mathbf{Q}, M \otimes \mathbf{Q}/\mathbf{Z})}{B(\mathbf{Q}) \otimes \mathbf{Q}/\mathbf{Z}} \to \bigoplus_{p \leq \infty} \frac{H^1(\mathbf{Q}_p, M \otimes \mathbf{Q}/\mathbf{Z})}{B(\mathbf{Q}_p) \otimes \mathbf{Q}/\mathbf{Z}}\Big)$$

Then conjecture (5.15) is equivalent to

$$\mathrm{Tam}(M)^\sim = \frac{\# H^0(\mathbf{Q}, M^* \otimes \mathbf{Q}/\mathbf{Z}(1))}{\# Ш(M)^\sim}.$$

6. The Riemann zeta function

Theorem 6.1. *Let $r \geq 2$ be given.*

(i) *If r is even, the Tamagawa number conjecture is true modulo a power of 2 for the motif $\mathbf{Q}(r)$.*

(ii) *Let r be odd. Then the Tamagawa number conjecture is true modulo a power of 2 for the motif $\mathbf{Q}(r)$ if the conjecture (6.2) below is true in the case $\alpha = 1$.*

The unfortunate power of 2 ambiguity in the theory is due to technical problems the Fontaine-Messing theory has at primes dividing 2.

The compatibility conjecture necessary to to prove the theorem for odd twists of $\mathbf{Q}$ concerns cyclotomic elements

$$c_r(\alpha) \in gr^r_\gamma(K_{2r-1}(\mathbf{Q}(\alpha)) \otimes \mathbf{Q})$$

defined by Beilinson [Bel]. Here α is a root of 1, and $c_r(\alpha)$ is characterized by the property that for any $\iota : \mathbf{Q}(\alpha) \to \mathbf{C}$, the image of $c_r(\alpha)$ in $\mathbf{C}/\mathbf{Q}(2\pi i)^r$ under the regulator map associated to ι is equal to the class of $-(r-1)!\sum_{i=1}^{\infty} \iota(\alpha)^n/n^r$.

Deligne and Soulé defined cyclotomic elements in $H^1(\mathbf{Q}, \hat{\mathbf{Z}}(r))$. For an n-th root of 1, there exists an element

$$c_{r,n}(\alpha) \in H^1(\mathbf{Q}(\alpha), \hat{\mathbf{Z}}(r))$$

which is characterized moodulo torsion elements by the following property. For any $m \geq 1$ and any field F over $\mathbf{Q}(\alpha)$ containing all m-th roots of α, the image of $c_{r,n}(\alpha)$ in $H^1(F, \mathbf{Z}/m\mathbf{Z}(r)) \cong (F^\times/F^{\times m}) \otimes \mathbf{Z}/m\mathbf{Z}(r-1)$ is equal to

$$\sum_{\zeta}\{1-\zeta\} \otimes [\zeta^n]^{\otimes(r-1)}.$$

where if $\alpha \neq 1$ (resp. $\alpha = 1$) ζ ranges over all m-th roots of α (resp. all m-th roots of α except 1). (cf. [De4] [Ih] [So2] [So5].)

Conjecture 6.2. *The images of $c_r(\alpha)$ and $n^{1-r}c_{r,n}(\alpha)$ coincide up to sign in $H^1(\mathbf{Q}(\alpha), \mathbf{A}_f(r))$.*

This conjecture is proved in Soulé [So5] in the case $r = 2$.

In [De4], Deligne constructed an extension of motives in his sense

$$0 \to \mathbf{Q}(r) \to E \to \mathbf{Q} \to 0$$

whose class has the same image as $c_r(1)$ in $\mathbf{C}/\mathbf{Q}(2\pi i)^r$ and the same image as $c_{r,1}(1)$ in $H^1(\mathbf{Q}, \hat{\mathbf{Z}}(r))$. However, the authors do not know if Deligne's extension corresponds to an element of $gr^r_\chi(K_{2r-1}(\mathbf{Q}(\alpha)) \otimes \mathbf{Q})$.

The rest of this section is devoted to the proof of (6.1). We do not assume (6.2) until the end.

Lemma 6.3. *Assume $r \geq 2$.*

(i) $\mathrm{Tam}(\mathbf{Z}(r)) = \pm\frac{\#H^0(\mathbf{Q},\mathbf{Q}/\mathbf{Z}(1-r))}{\#H^0(\mathbf{Q},\mathbf{Q}/\mathbf{Z}(r))} \cdot \frac{2}{\zeta(1-r)}$.

(ii) *If r is odd,* $\mathrm{Tam}(\mathbf{Z}(r)) = \pm\frac{\#H^0(\mathbf{Q},\mathbf{Q}/\mathbf{Z}(1-r))}{\chi(A(\mathbf{Q}):\mathbf{Z}\cdot c_r(1))}$.

Here $\mathbf{Z} \cdot c_r(1) \subset A(\mathbf{Q}) \otimes \mathbf{Q}$ is the free abelian group generated by $c_r(1)$, and by definition

$$\chi(A(\mathbf{Q}) : \mathbf{Z} \cdot c_r(1)) = [A(\mathbf{Q}) : L]/[\mathbf{Z} \cdot c_r(1) : L]$$

for any free subgroup L of $A(\mathbf{Q})$ of finite index whose image in $A(\mathbf{Q}) \otimes \mathbf{Q}$ is contained in $\mathbf{Z} \cdot c_r(1)$.

Proof. Take the canonical base $1 \in \mathbf{Q} = H^0_{DR}(\mathrm{Spec}(\mathbf{Q}))$, and let μ_p be the corresponding measure on $A(\mathbf{Q}_p)$ for $p \leq \infty$. Then, if $p \neq \infty$, the table (3.9) gives $A(\mathbf{Q}_p) = H^1(\mathbf{Q}_p, \widehat{\mathbf{Z}}(r))$ and by (4.2) we have

$$\begin{aligned}\mu_p(A(\mathbf{Q}_p)) &= |(r-1)!|_p(1-p^{-r})\#H^0(\mathbf{Q}_p, \mathbf{Q}_p/\mathbf{Z}_p(1-r)) \\ &= |(r-1)!|_p(1-p^{-r})\#H^0(\mathbf{Q}, \mathbf{Q}_p/\mathbf{Z}_p(1-r)).\end{aligned}$$

Hence

$$\prod_{p<\infty} \mu_p(A(\mathbf{Q}_p)) = \frac{\#H^0(\mathbf{Q}, \mathbf{Q}/\mathbf{Z}(1-r))}{(r-1)! \cdot \zeta(r)}$$

If r is even, $A(\mathbf{R}) = \mathbf{R}/(2\pi)^r\mathbf{Z}$ with the Lebesgue measure μ_∞, and we have

(6.4)

$$\begin{aligned}\mathrm{Tam}(\mathbf{Z}(r)) &= (\#A(\mathbf{Q}))^{-1} \prod_{p\leq\infty} \mu_p(A(\mathbf{Q}_p)) \\ &= \frac{1}{\#H^0(\mathbf{Q}, \mathbf{Q}/\mathbf{Z}(r))} \cdot (2\pi)^r \cdot \frac{\#H^0(\mathbf{Q}, \mathbf{Q}/\mathbf{Z}(1-r))}{(r-1)!\zeta(r)} \\ &= \pm \frac{\#H^0(\mathbf{Q}, \mathbf{Q}/\mathbf{Z}(1-r))}{\#H^0(\mathbf{Q}, \mathbf{Q}/\mathbf{Z}(r))} \cdot \frac{2}{\zeta(1-r)}.\end{aligned}$$

If r is odd, $A(\mathbf{R}) = A(\mathbf{C})^+ = (\mathbf{C}/(2\pi i)^r\mathbf{Z})^+ \cong \mathbf{R} \oplus \mathbf{Z}/2\mathbf{Z}$, $A(\mathbf{Q}) \hookrightarrow A(\mathbf{R})$ and the calculation reads

$$\begin{aligned}\mathrm{Tam}(\mathbf{Z}(r)) &= \mu\Big(\prod_{p\leq\infty} A(\mathbf{Q}_p)/A(\mathbf{Q})\Big) \\ &= \mu_\infty(A(\mathbf{R})/A(\mathbf{Q}))\#H^0(\mathbf{Q}, \mathbf{Q}/\mathbf{Z}(1-r))((r-1)!\zeta(r))^{-1} \\ &= \#H^0(\mathbf{Q}, \mathbf{Q}/\mathbf{Z}(1-r))\chi(A(\mathbf{Q}) : \mathbf{Z} \cdot c_r(1))^{-1}.\end{aligned}$$

Note that here we did not use conjecture (6.2). This proves the lemma.

To finish the proof of Theorem 6.1, we must show

(6.5) $\zeta(1-r) = 2 \cdot \#(\text{Ш}) \cdot \#H^0(\mathbf{Q}, \mathbf{Q}/\mathbf{Z}(r))^{-1}$ for r even

(6.6) $\#(\text{Ш}) = \chi(A(\mathbf{Q}) : \mathbf{Z} \cdot c_r(1))$ for r odd.

Lemma 6.7 *If $p \neq 2$, the p-primary part of Ш is isomorphic to $H^2(\mathbf{Z}[1/p], \mathbf{Z}_p(r))$. If r is even, it is also isomorphic to $H^1(\mathbf{Z}[1/p], \mathbf{Q}_p/\mathbf{Z}_p(r))$.*

Proof. Consider the commutative diagram of exact sequences:

$$\begin{array}{ccccccc} 0 \longrightarrow & \frac{H^1(\mathbf{Z}[1/p],\mathbf{Q}_p/\mathbf{Z}_p(r))}{A(\mathbf{Q})\otimes\mathbf{Q}_p/\mathbf{Z}_p} & \longrightarrow & \frac{H^1(\mathbf{Q},\mathbf{Q}_p/\mathbf{Z}_p(r))}{A(\mathbf{Q})\otimes\mathbf{Q}_p/\mathbf{Z}_p} & \longrightarrow & \bigoplus_{\ell\neq p} H^0(\mathbf{F}_\ell,\mathbf{Q}_p/\mathbf{Z}_p(r-1)) & \longrightarrow 0 \\ & \downarrow s & & \downarrow t & & \| & \\ 0 \longrightarrow & \frac{H^1(\mathbf{Q}_p,\mathbf{Q}_p/\mathbf{Z}_p(r))}{A(\mathbf{Q}_p)\otimes\mathbf{Q}_p/\mathbf{Z}_p} & \longrightarrow & S & \longrightarrow & \bigoplus_{\ell\neq p} H^0(\mathbf{F}_\ell,\mathbf{Q}_p/\mathbf{Z}_p(r-1)) & \longrightarrow 0 \end{array}$$

where

$$S = \bigoplus_{\ell} \frac{H^1(\mathbf{Q}_\ell,\mathbf{Q}_p/\mathbf{Z}_p(r))}{A(\mathbf{Q}_\ell)\otimes\mathbf{Q}_p/\mathbf{Z}_p} = \frac{H^1(\mathbf{Q}_p,\mathbf{Q}_p/\mathbf{Z}_r(r))}{A(\mathbf{Q}_p)\otimes\mathbf{Q}_p/\mathbf{Z}_p} \oplus \bigoplus_{\ell\neq p} H^0(\mathbf{F}_\ell,\mathbf{Q}_p/\mathbf{Z}_p(r-1)).$$

We have

$$Ш\{p\} = \mathrm{Ker}(t) \cong \mathrm{Ker}(s) \cong \mathrm{Ker}(H^2(\mathbf{Z}[1/p],\mathbf{Z}_p(r)) \to H^2(\mathbf{Q}_p,\mathbf{Z}_p(r)).$$

The map on the right is zero, as follows from the Tate duality sequence

$$H^2(\mathbf{Z}[1/p],\mathbf{Z}_p(r)) \longrightarrow \quad H^2(\mathbf{Q}_p,\mathbf{Z}_p(r)) \quad \longrightarrow H^0(\mathbf{Z}[1/p],\mathbf{Q}_p/\mathbf{Z}_p(1-r))^* \longrightarrow 0$$

$$H^0(\mathbf{Q}_p,\mathbf{Q}_p/\mathbf{Z}_p(1-r))^*$$

This proves Lemma (6.7)

The proof of (6.5) now follows from the result of Mazur-Wiles [MW]

$$\zeta(1-r) = \pm \prod_{p<\infty} \frac{\#H^1(\mathbf{Z}[1/p],\mathbf{Q}_p/\mathbf{Z}_p(r))}{\#H^0(\mathbf{Z}[1/p],\mathbf{Q}_p/\mathbf{Z}_p(r))} \qquad (r \geq 2 \text{ even}).$$

We consider (6.6). Let

$G = \mathrm{Gal}(\mathbf{Q}_p(\zeta_{p^\infty})/\mathbf{Q}_p = \mathrm{Gal}(\mathbf{Q}(\zeta_{p^\infty})/\mathbf{Q})$;
$P = \mathrm{Gal}(\mathbf{Q}_p(\zeta_{p^\infty})^{ab}/\mathbf{Q}_p(\zeta_{p^\infty}))$; $\mathcal{X} = \pi_1(\mathrm{Spec}(\mathbf{Z}[1/p][\zeta^{p^\infty}])^{ab}$;
$C =$ the $\hat{\mathbf{Z}}[[G]]$ – submodule of P generated by $(1-\zeta_{p^n})_n \in \varprojlim \mathbf{Q}_p(\zeta_{p^n})^\times \subset P$.

We have a canonical G-homomorphism $P/C \to \mathcal{X}$. Consider the following commutative diagram of exact sequences

$$\begin{array}{ccc}
0 & & \\
\downarrow & & \\
H^2(\mathbf{Z}[1/p],\mathbf{Z}_p(r))^* & & 0 \\
\downarrow & & \downarrow \\
H^1(\mathbf{Z}[1/p],\mathbf{Q}_p/\mathbf{Z}_p(1-r)) & \xrightarrow{a} & H^0(G,(P/C)^*(1-r)) \\
\downarrow & & \downarrow \\
H^1(\mathbf{Q}_p,\mathbf{Q}_p/\mathbf{Z}_p(1-r)) & \xrightarrow{b} & H^0(G,P^*(1-r)) \\
\downarrow & & \downarrow \\
H^1(\mathbf{Z}[1/p],\mathbf{Z}_p(r))^* & \xrightarrow{c} & H^0(G,C^*(1-r)) \\
\downarrow & & \downarrow \\
H^2(\mathbf{Z}[1/p],\mathbf{Q}_p/\mathbf{Z}_p(1-r)) & \xrightarrow{d} & H^1(G,(P/C)^*(1-r)) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Here $*$ denotes $\mathrm{Hom}_{\mathrm{cont}}(\ ,\mathbf{Q}_p/\mathbf{Z}_p)$, so $P^* = H^1(\mathbf{Q}_p(\zeta_{p^\infty}),\mathbf{Q}_p/\mathbf{Z}_p)$. The map a factors as $a = a_1 \circ a_2$:

$$H^1(\mathbf{Z}[1/p],\mathbf{Q}_p/\mathbf{Z}_p(1-r)) \overset{a_2}{\cong} H^0(G,\mathcal{X}^*(1-r)) \xrightarrow{a_1} H^0(G,(P/C)^*(1-r))$$

with a_2 bijective. (Note $\mathcal{X}^* = H^1(\mathbf{Z}[1/p][\zeta_{p^\infty}],\mathbf{Q}_p/\mathbf{Z}_p)$.) Moreover, the map b is bijective, and c is the surjection induced by

$$\varprojlim \mathbf{Z}[1/p][\zeta_{p^n}]^\times \longrightarrow \varprojlim H^1(\mathbf{Z}[1/p][\zeta_{p^n}],\mathbf{Z}_p(r))(1-r)$$
$$\xrightarrow{\text{Norm}} H^1(\mathbf{Z}[1/p][\zeta_{p^n}],\mathbf{Z}_p(r))(1-r).$$

Finally, d factors as

$$H^2(\mathbf{Z}[1/p],\mathbf{Q}_p/\mathbf{Z}_p(1-r)) \overset{d_2}{\cong} H^1(G,\mathcal{X}^*(1-r)) \xrightarrow{d_1} H^1(G,(P/C)^*(1-r))$$

with d_2 bijective

By Mazur-Wiles [MW], the kernel and cokernel of

$$(P/C)^+(p) \to \mathcal{X}^+(p)$$

(where $(\ )^+$ denotes the Gal($\mathbf{C}/\mathbf{R}$)-invariants and (p) the pro-p-part) have the same images in the Grothendieck group of

$$\{\text{finitely generated torsion } \mathbf{Z}_p[[G]]\text{-modules}\}\Big/\{\text{finite } \mathbf{Z}_p[[G]]\text{-modules}\}.$$

This shows

$$\#\operatorname{Coker}(a_1)\cdot\#\operatorname{Ker}(a_1)^{-1} = \#\operatorname{Coker}(d_1)\cdot\#\operatorname{Ker}(d_1)^{-1}.$$

Consequently we have

$$(6.8) \qquad \#H^2(\mathbf{Z}[1/p], \mathbf{Z}_p(r)) = \#\operatorname{Ker}(c).$$

On the other hand, by the definition of c, we see that

$$(6.9) \qquad \#\operatorname{Ker}(c) = [H^1(\mathbf{Z}[1/p], \mathbf{Z}_p(r)) : \mathbf{Z}_p\cdot c_{r,1}(1)].$$

If we assume the case $\alpha = 1$ of (6.2), we have

$$\chi(A(\mathbf{Q}), \mathbf{Z}\cdot c_r(1)) = \prod[H^1(\mathbf{Z}[1/p], \mathbf{Z}_p(r)) : \mathbf{Z}_p\cdot c_{r,1}(1)],$$

and hence (6.8)and (6.9) prove (6.6). This completes the argument.

7. Complex multiplication

In this section, for an elliptic curve over $\mathbf{Q}$ with complex multiplication, we reduce the ℓ-primary part of our Tamagawa number conjecture for $H^1(E)(2)$ (i.e., for $L(H^1(E), 2)$, where E is such an elliptic curve) to the question $\dim gr^2(K_2(E)\otimes\mathbf{Q}) \stackrel{?}{=} 1$ and to a certain problem on the Galois cohomology of the twist of the Tate module $T_\ell E(1)$, for good primes ℓ (cf. (7.3)). The latter problem is solve in the case ℓ is regular for E (cf. (7.4)).

7.1. In this section, we review the "cyclotomic elements" in K_2 of elliptic curves (cf. [B14] [DW]). Generally let k be a field and E an elliptic curve over k. For non-zero integers a, n such that $(a, n) = 1$ and such tht all points in ${}_aE \cup {}_nE$ are k-rational (here ${}_bE = \operatorname{Ker}(b : E \to E)$ for $b \in \mathbf{Z}$), and for $\beta \in {}_nE - \{0\}$, we define

$$c_n^a(\beta) \in \Gamma(E_{\text{zar}}, K_2)/K_2(k)$$

as follows. Take functions g, s, t_γ ($\gamma \in {}_aE - \{0\}$) on E such that $\mathrm{div}(g) = a^2(0) - {}_aE$, $\mathrm{div}(s) = N(\beta - n(0)$, $\mathrm{div}(t_\gamma) = a(\gamma) - a(0)$. Then,

$$c_n^a(\beta) \stackrel{\text{def}}{=} a\{g(\beta)^{-1}g, s\} - \sum_{\substack{a\gamma=0 \\ \gamma\neq 0}} \{s(\gamma), t_\gamma\} \in K_2(k(E))/K_2(k).$$

It is easily checked that $c_n^a(\beta)$ belongs to the kernel of the tame symbol map

$$K_2(k(E))/K_2(k) \to \bigoplus_{\substack{x\in E \\ \text{closed}}} k(x)^\times,$$

which is $\Gamma(E_{\text{zar}}, K_2)/K_2(k)$, and that $c_n^a(\beta)$ is independent of the choices of g, s, t_γ. Note

$$\text{(7.1.2)} \qquad c^a(\beta) \stackrel{\text{def}}{=} (an)^{-1}c_n^a(\beta) \in \Gamma(E_{\text{zar}}, K_2) \otimes \mathbf{Q}$$

is independent of the chocie of $n \neq 0$ such that $n\beta = 0$

In the rest of this section and §§(7-2)–(7.4), let E be an elliptic curve over $\mathbf{Q}$ with complex multiplication by the ring of integers $\mathcal{O}_K$ of a quadratic imaginary field K. Since we will be interested in the motif $H^1(E)(2)$, we denote by D and V the corresponding $\mathbf{Q}$-vector spaces $H^1_{DR}(E/\mathbf{Q})$ and $H^1(E(\mathbf{C}), \mathbf{Q}(2\pi i)^2)$, respectively, and we identify V_ℓ with $T_\ell E(1) \otimes \mathbf{Q}$. Fix $K \xrightarrow{\subset} \mathbf{C}$, let ψ be the Hecke character of K of A_0-type of type $(1,0)$ associated to E, and let f be the conductor of ψ. We fix a generator δ of $f^{-1}H_1(E(\mathbf{C}), \mathbf{Z})$ for the $\mathcal{O}_K$-module structure by complex multiplication. By the identification

$${}_fE(\mathbf{C}) = f^{-1}H_1(E(\mathbf{C}), \mathbf{Z})/H_1(E(\mathbf{C}), \mathbf{Z}),$$

$\delta \bmod H_1(E(\mathbf{C}), \mathbf{Z})$ is a generator of $E(\mathbf{C})$ as an $\mathcal{O}_K/f$-module. Take $v \in \mathcal{O}_K - \{0\}$ such that $v\delta \in H_1(E(\mathbf{C}), \mathbf{Z})^+$ and for a non-zero integer a which is prime to f, let

$$\text{(7.1.3)} \qquad \begin{aligned} c_v^a(E) &= N_v N_{K(f)/K}(c^a(\delta)) \\ &\in (\Gamma((E \otimes K)_{\text{zar}}, K_2) \otimes \mathbf{Q})^{\mathrm{Gal}(K/\mathbf{Q})} \\ &= \Gamma(E_{\text{zar}}, K_2) \otimes \mathbf{Q} = gr^2(K_2(E) \otimes \mathbf{Q}) \end{aligned}$$

where $K(f)$ denotes the ray class field of modulus f over K, we regard $c^a(\delta)$ as an element of $\Gamma((E \otimes K(f))_{zar}, K_2) \otimes \mathbf{Q}$ (it is seen easily that $c_n^a(\delta)$ is invariant under $\mathrm{Gal}(\overline{Q}/K(f))$; note that K_2 of an algebraic number field is a torsion group) $N_{K(f)/K}$ is the norm map

$$\Gamma((E \otimes K(f))_{\text{zar}}, K_2) \otimes \mathbf{Q} \to \Gamma((E \otimes K)_{\text{zar}}, K_2) \otimes \mathbf{Q},$$

and N_v is the norm map

$$\Gamma((E \otimes K)_{\mathrm{zar}}, K_2) \otimes \mathbf{Q} \to \Gamma((E \otimes K)_{\mathrm{zar}}, K_2) \otimes \mathbf{Q}$$

associated to the finite flat morphism $v : E \to E$ (complex multiplication). Using the fact that multiplication by $m \neq 0$ on E induces multiplication by m on $\Gamma(E, K_2)/K_2(k)$ we can show that

$$c_v(E) \underset{dfn.}{=} (a^2 - \psi(a)^{-1})^{-1} c_v^a(E) \tag{7.1.4}$$

is independent of a and $c_{nv}^a(E) = nc_v^a(E)$ for $n \in \mathbf{Z} - \{0\}$. Note that $\psi(a) = \pm a$. It is conjectured that $\dim_{\mathbf{Q}}(gr^2(K_2(E) \otimes \mathbf{Q})) = 1$.

It is known ([B14] [DW]) that

the regulator map $gr^2(K_2(E) \otimes \mathbf{Q}) \to D_\infty / V_\infty^+$ sends $c_v(E)$ to

$$\pm 2^{-1} (\lim_{s \to 0} s^{-1} L(H^1(E), s)) \cdot v\delta \tag{7.1.5}$$

where the L-function $L(H^1(E), s)$ includes the Euler factors at all bad places, and we regard $v\delta$ as an element of $H^1(E(\mathbf{C}), \mathbf{Z} \cdot 2\pi i)^+ \subset D_\infty$ via the Poincaré duality $H_1(E(\mathbf{C}), \mathbf{Z}) \cong H^1(E(\mathbf{C}), \mathbf{Z}\ 2\pi i)$ which preserves the action of $\mathrm{Gal}(\mathbf{Z}/\mathbf{R})$.

In (7.2)and (7.5) below, we relate the above cyclotomic elements to the following cyclotomic elements in $H^1(\mathbf{Q}, T_\ell E(1))$ considered by Soulé [So6] (Soulé treats $H^1(\mathbf{Q}, T_\ell E(r))$ for all $f \geq 1$). This implies that an analogue of the conjecture (6.2) for K_2 of elliptic curves is true.

In the rest of this section, we fix a prime $\ell \neq 2, 3$ such that E has a good reduction at ℓ. Let the fixed generator δ of $f^{-1} H_1(E(\mathbf{C}), \mathbf{Z})$ and v, a be as above, and let

$$\begin{aligned} e_v^a(E) &\stackrel{\text{def}}{=} (N_{K_i/K}(\{g(\ell^{-i}\delta) g(\gamma_i)^{-1}\} \otimes [v\gamma_i]))_{i \geq 1} \\ &\in (\lim_i H^1(K, {}_{\ell^i}E(1))^{\mathrm{Gal}(K/Q)} = H^1(\mathbf{Q}, T_\ell E(1)) \end{aligned}$$

where g is a function on E such that $\mathrm{div}(g) = a^2(0) - {}_aE$, γ_i is the image of $\ell^{-i}\delta$ in ${}_{\ell^i}E$ under ${}_{\ell^i f}E(\mathbf{C}) = {}_{\ell^i}E(\mathbf{C}) \oplus {}_f E(\mathbf{C})$, K_i is the ray class field of modulus $\ell^i f$ over K, and $N_{K_i/K}$ is the norm map

$$K_i^\times / K_i^{\times \ell^i} \otimes {}_{\ell^i}E \cong H^1(K_i, {}_{\ell^i}E(1)) \to H^1(K, {}_{\ell^i}E(1)).$$

It is easily see that

$$e_v(E) \stackrel{\text{def}}{=} (a^2 - \psi(a)^{-1})^{-1} e_v^a(E) \tag{7.1.6}$$

is independent of a, and $e_v(E) \in H^1(\mathbf{Q}, T_\ell E(1))$ in $H^1(\mathbf{Q}, V_\ell)$ (the last fact is seen by choosing suitable a). Note the groups

$$H^1(\mathbf{Q}, T_\ell E(1)), \quad H^1_{f,\mathrm{Spec}(\mathbf{Z})}(\mathbf{Q}, T_\ell E(1)), \quad H^1(\mathbf{Z}[1/\ell], j_* T_\ell E(1))$$

($j : \mathrm{Spec}(\mathbf{Q}) \to \mathrm{Spec}(\mathbf{Z}[1/\ell])$) coincide, and have no torsion.

In the following (7.1)–(7.4), let E, ℓ, δ, a, v be as above.

Proposition 7.2. *The regulator map sends $c_v(E)$ to $\kappa e_v(E)$ where κ is the value at $s = 0$ of the Euler factor at ℓ of the L-function $L(H^1(E), s)$.*

Proposition 7.3. *Let Φ be the one-dimensional $\mathbf{Q}$-vector subspace of $\mathrm{gr}^2(K_2(E) \otimes \mathbf{Q})$ generated by $c_v(E)$ with v as above (so Φ is independent of the choice of v). Then the ℓ-Tamagawa number of $M = H^1(E(\mathbf{C}), \mathbf{Z}(2\pi i)^2)$ with respect to Φ (cf. (5.15.2)) is defined if $e_v(E)$ generates $H^1(\mathbf{Q}, V_\ell)$. If this is the case, $H^2(\mathbf{Z}[1/\ell], j_* E(1))$ is finite and the ℓ-Tamagawa number conjecture (5.15.3) is equivalent to*

$$\#H^1(\mathbf{Z}[1/\ell], j_* T_\ell E(1))/\mathbf{Z}_\ell e_v(E) = \#H^2(\mathbf{Z}[1/\ell], j_* T_\ell E(1)). \tag{7.3.1}$$

for v prime to ℓ.

Remark 7.3.2. Kolyvagin and Rubin have recently proved the Iwasawa main conjecture ([dS] III §1). Using this and an argument similar to (6.6), we can prove (7.3.1) in the case ℓ splits in K and $e_v(E)$ generates $H^1(\mathbf{Q}, V_\ell)$.

By using the method of Soulé in [So6], we can show

Proposition 7.4. *If ℓ is "regular" for E ([So6], 3.3.1), $H^1(\mathbf{Q}, V_\ell)$ is generated by $e_v(E)$ and the ℓ-Tamagawa number conjecture is true.*

Now we prove (7.2)–(7.4). We deduce (7.2) from

Proposition 7.5. *Let k be a field, E an elliptic curve over k, a and n non-zero integers, and ℓ a prime number. Assume $(a, n) = 1$, $\ell \nmid a$, $\ell \neq \mathrm{char}(k)$, and all points in ${}_{\ell^\infty}E \cup {}_aE \cup {}_{n^2}E$ are k-rational. Then, for $\beta \in E_n - \{0\}$, the image of $c_n^a(\beta)$ in $H^1(k, E_{\ell^\nu}(1)) \cong k^\times/(k^\times)^{\ell^\nu} \otimes E_{\ell^\nu}$ is equal to*

$$-\sum_{\ell^\nu \gamma = \beta} a\{g(\gamma)\} \otimes [n\gamma] + \sum_{\substack{\ell^\nu \gamma = 0 \\ \gamma \neq 0}} a\{g(\gamma)\} \otimes [n\gamma].$$

where g is a function on E such that $\mathrm{div}(g) = {}_aE - a^2(0)$.

In the situation of (7.2), let k be the minimal extension of $\mathbf{Q}$ for which all points of ${}_{\ell^\infty}E \cup {}_aE \cup {}_{N(f)^2}E$ are k-rational. Then,

$$H^1(\mathbf{Q}, T_\ell(E(1)) \to H^1(k, T_\ell E(1))$$

is injective. By comparing the image of $e_v^a(E)$ in $H^1(k, T_\ell E(1))$ and that of $c_v^a(E)$ described by (7.5), we obtain (7.2) (assuming (7.5)).

For the proof of (7.5), we need the following two lemmas. Let the assumptions and notations be as in (7.5).

Lemma 7.6. *For any integer b which is prime to a, $N_b(g)g^{-1}$ is a constant where N_b is the norm map associated to the finite flat morphism $b: E \to E$. If we denote this constant by λ_b, we have $c_n^a(\beta) \to -a\{\lambda_n g(\beta)\} \otimes [\beta]$ in $H^1(k, {}_nE(1))$.*

Lemma 7.7. *Let $m = \ell^i$ for some $i \geq 1$. Then, the image of $c_n^a(\beta)$ in $H^1(k, T_\ell E(1))$ is equal to that of*

$$\sum_{m\gamma=\beta} c_{mn}^a(\gamma) - n \sum_{\substack{m\gamma=0\\ \gamma\neq 0}} c_m^a(\gamma).$$

vskip 1pc

We prove (7.5) assuming (7.6), (7.7). Let $m = \ell^i$. If $m\gamma = \beta$, the image of $c_{mn}^a(\gamma)$ in $H^1(k, {}_{mn}E(1))$ is equal to $-a\{\lambda_{mn} g(\gamma)\} \otimes [\gamma]$ by (7.6) (which we apply by replacing n by mn) and hence its image in $H^1(k, {}_mE(1))$ is equal to $-a\{\lambda_{mn} g(\gamma)\} \otimes [n\gamma]$. On the other hand, if $m\gamma = 0$ and $\gamma \neq 0$, the image of $c_m^a(\gamma)$ in $H^1(k, {}_mE(1))$ is equal to $-a\{\lambda_m g(\lambda)\} \otimes [\gamma]$ by (7.6) (with n replaced by m). Hence by (7.7) and by using $\lambda_{mn} = \lambda_m \lambda_n^{m^2}$, we see that the image of $c_n^a(\gamma)$ in $H^1(k, {}_mE(1))$ is equal to (7.5.1).

Proof of (7.6). We prove first a weaker version of (7.6):

(7.8) let $U = E - ({}_aE \cup {}_nE)$. Then, $c_n^a(\beta)$ and $-\{\lambda_n g(\beta)\} \otimes \beta$ have the same image in

$$H^1(k, H^1(\overline{U}, \mathbf{Z}/n\mathbf{Z}(2)) \subset H^2(U, \mathbf{Z}/n\mathbf{Z}(2))/\operatorname{Image}(H^2(k, \mathbf{Z}/n\mathbf{Z}(2))).$$

Proof. Let s be a function on E such that $\operatorname{div}(s) = n(\beta) - n(0)$. Then

$$\operatorname{div}(s \circ [n]) = n\Big(\sum_{n\gamma=\beta} (\gamma) - \sum_{n\gamma=0} (\gamma)\Big),$$

where $[n]$ denotes the multiplication $n : E \to E$. Since $\sum_{nx=\beta}(\gamma) - \sum_{n\gamma=0}(\gamma)$ is a principal divisor, by replacing s by cs for some $c \in k^\times$ if necessary, we have

$$s \circ [n] = (s')^n \quad \text{for some } s' \in \mathcal{O}(U')^\times$$

where $U' = [n]^{-1}(U)$. By $N_n(g) = \lambda_n g$, we have in $H^2(U, \mathbf{Z}/n\mathbf{Z}(2))$,

$$\{\lambda_n g, s\} = \{N_n(g), s\} = N_n\{g, s \circ [n]\} = nN_n\{g, s'\} = 0$$

where the last two N_n are the norm map $H^2(U', \mathbf{Z}/n\mathbf{Z}) \to H^2(U, \mathbf{Z}/n\mathbf{Z}(2))$. Furtherfore $s(\gamma) \in (k^\times)^n$ for any $\gamma \in {}_aE - \{0\}$ by $s(\gamma) = (s \circ [n])(n^{-1}\gamma)$. These prove

$$c_n^a(\beta) \to -a\{\lambda_n g(\beta), s\} \quad \text{in } H^2(U, \mathbf{Z}/n\mathbf{Z}(2)).$$

Since $[\beta] \to s$ under

$$\begin{aligned} {}_nE \cong H^1(E, \mathbf{Z}/n\mathbf{Z}(1))/(H^1(k, \mathbf{Z}/n\mathbf{Z}(1)) \to \\ H^1(U, \mathbf{Z}/n\mathbf{Z}(1))/H^1(k, \mathbf{Z}/n\mathbf{Z}(1)) \cong \mathcal{O}(U)^\times/(k^\times \cdot \mathcal{O}(U)^{\times n}), \end{aligned}$$

this proves (7.8).

For the proof of (7.6), it remains to show thast

$$H^1(k, H^1(\overline{E}, \mathbf{Z}/n\mathbf{Z})(2))) \to H^1(k, H^1(\overline{U}, \mathbf{Z}/n\mathbf{Z}(2)))$$

is injective. Twisting by (-1) (note k contains all n-th roots of 1) and taking the Galois cohomology of the exact sequence

$$0 \to H^1(\overline{E}, \mathbf{Z}/n\mathbf{Z}(1)) \to H^1(\overline{U}, \mathbf{Z}/n\mathbf{Z}(1)) \to (\oplus_{x \in E-U}\mathbf{Z}/n\mathbf{Z})^0 \to 0,$$

where $(\ )^0$ denotes the kernel of the summation $\oplus\mathbf{Z}/n\mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$, we are reduced to proving the surjectivity of

$$H^0(k, H^1(\overline{U}, \mathbf{Z}/n\mathbf{Z}(1))) \to (\oplus_{x \in E-U}\mathbf{Z}/n\mathbf{Z})^0$$

Let $x \in E - U$. Since all points in ${}_{n^2}E$ are k-rational by the assumption, we have $x = ny$ for some $y \in E(k)$. Hence

$$(x) - (0) = n((y) - (0)) + \mathrm{div}(h)$$

for some $h \in k(E)^{\times}$. Then h defines an element of

$$H^0(k, H^1(\overline{U}, \mathbf{Z}/n\mathbf{Z}(1)) = \mathrm{Ker}(H^0(k, \overline{k}(E)^{\times}/\overline{k}(E)^{\times n}) \to \bigoplus_{\substack{x \in \overline{U} \\ \text{closed}}} \mathbf{Z}/n\mathbf{Z})$$

whose image in $(\bigoplus_{x \in E-U} \mathbf{Z}/n\mathbf{Z})^0$ equals $(x) - (0)$.

Proof of (7.7). Since $H^1(k, T_\ell E(1))$ is torsion free and N_m acts on $H^1(k, T_\ell E(1))$ as the multiplication by m, it is enough to prove

$$N_m\Big(\sum_{m\gamma=\beta} c^a_{mn}(\gamma) - n \sum_{\substack{m\gamma=0 \\ \gamma\neq 0}} c^a_m(\gamma)) = mc^a_n(\beta) \tag{7.9}$$

in $\Gamma(E_{\mathrm{zar}}, K_2)/K_2(k)$.

The proof of (7.9) is a straightforward computation on symbols and is omitted.

Proposition (7.3) follows from (7.2), (7.1.4) and the following lemma.

Lemma 7.10. *Let X be a smooth projective scheme over $\mathbf{Q}$ having potentially good reductions at all finite places of $\mathbf{Q}$. Consider the motif $H^m(X)(r)$ with $m, r \in \mathbf{Z}$ such that $r > \sup(m, 1)$, and let V and D be the associated $\mathbf{Q}$-vector spaces. Fix a prime number ℓ such that X has good reduction at ℓ and such that $\ell > r+1$. Assume that $P_p(H^m(\overline{X}, \mathbf{Q}_\ell), T) \in \mathbf{Q}[T]$ for all $p < \infty$ and $L(H^m(X), s) = \prod_{p<\infty} P_p(H^m(\overline{X}, \mathbf{Q}_\ell), p^{-s})^{-1}$ has a meromorphic analytic continuation to the whole s-plane satisfying the conjectural functional equation. Assume further that we find a $\mathbf{Q}$-vector subspace Φ in the image of*

$$gr^r_\gamma(K_{2r-m-1}(X) \otimes \mathbf{Q})$$

such that

$$\Phi \otimes R \xrightarrow[\cong]{} D_R/V_R^+, \quad \Phi \otimes Q_\ell \xrightarrow[\cong]{} H^1_{f,\mathrm{Spec}(\mathbf{Z})}(\mathbf{Q}, V_\ell).$$

Let S be a finite set of places of $\mathbf{Q}$ containing ∞, ℓ and all finite places at which X has bad reduction, and let $U = \mathrm{Spec}(\mathbf{Z}) - S$. Then for any $\mathbf{Z}$-lattice M in V such that $M \otimes \mathbf{Z}$ is $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-stable, $H^2(U, M \otimes \mathbf{Z}_\ell)$ is finite and we have an equation in $\mathbf{R}^{\times}/\mathbf{Z}^{\times}_{\ell)}$

$$\begin{aligned}\mathrm{Tam}^{(\ell))}(M)\#Ш^{(\ell)}(M)(\#H^0(\mathbf{Q}, M^* \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell(1))^{-1} \\ = \lim_{s\to m+1-r}(L_S(H^m(X), s)^{-1}(s-m-1+r)^n) \\ \cdot \mathrm{vol}((D_\infty/V_\infty^+)/\mathrm{Image}(A^{(\ell)}(\mathbf{Q})))\#(A^{(\ell)}(\mathbf{Q})_{\mathrm{tor}})^{-1}\#H^2(U, M \otimes \mathbf{Z}_\ell)\end{aligned}$$

where n is the order of $L_S(H^m(X),s)$ at $s=m+1-r$ and other notations are as in (5.15.2).

Proof. Let $L_p(s)$ be the local Euler factor of $L(H^m(X),s)$ for $p<\infty$ and let $L_\infty(s)$ be the gamma factor. We may assume that there exists a $\mathbf{Z}$-lattice L in D such that the Hodge filtration and the frobenius of D_ℓ induce on $L\otimes\mathbf{Z}_\ell$ a structure of a strongly divisible lattice and the $\mathrm{Gal}(\overline{\mathbf{Q}}_\ell/\mathbf{Q}_\ell)$-module $M\otimes\mathbf{Z}_\ell$ corresponds to $L\otimes\mathbf{Z}_\ell$. Take $\omega:\det_{\mathbf{Q}}(D)\xrightarrow{\cong}\mathbf{Q}$ which induces $\det_{\mathbf{Z}}(L)\xrightarrow{\cong}\mathbf{Z}$. By the functional equation

$$L_S(H^m(X),r)\prod_{p\in S}L_p(r)=\lim_{s\to m+1-r}L_S(H^m(X),s)\prod_{p\in S}L_p(s)$$

in $\mathbf{R}^x/\mathbf{Z}^x_{(\ell)}$, (7.10) is reduced to

Lemma 7.11. *We have $\mathbf{R}^x/\mathbf{Z}^x_{(\ell)}$:*

(1) *For $p<\infty$,*

$$L_p(r)^{-1}L_p(m+1-r)=\mu_{p,\omega}(A^{(\ell)}(Q_p))\#(H^1(\mathbf{Q}_p,M\otimes\mathbf{Q}_\ell/\mathbf{Z}_\ell)/(A^{(\ell)}(\mathbf{Q}_p)\otimes\mathbf{Q}_\ell/\mathbf{Z}_\ell))^{-1}$$

(2) *On the other hand we have*

$$L_\infty(r)^{-1}\lim_{s\to m+1-r}L_\infty(s)(s-m-1+r)^n$$
$$=\mu_{\infty,\omega}(A(\mathbf{R})/A^{(\ell)}(\mathbf{Q}))\,\mathrm{vol}((D_\infty/V_\infty^+)/\,\mathrm{Image}(A^{(\ell)}(\mathbf{Q})))^{-1}\#(A^{(\ell)}(\mathbf{Q})_{\mathrm{tor}}).$$

Proof. The case $p=\ell$ of (1) follows from (4.1) and the duality (3.8). The case $p\neq\ell$ of (1) is an easy exercise and left to the reader. Finally (2) is a problem of the ratio of two volumes on D_∞/V_∞^+; one is the measure in (7.10) and the other is induced via the isomorphism $A(\mathbf{R})/A(\mathbf{R})_{cpt}\cong D_\infty/V_\infty^+$ by $\mu_{\infty,\omega}$ on $A(\mathbf{R})$ and by the Haar measure on $A(\mathbf{R})_{cpt}$ with the total volume one. We are reduced to

7.12. *With the assumption and notations as above, the composite*

$$det_{\mathbf{Z}}(M)\otimes_{\mathbf{Z}}\mathbf{C}\cong det_{\mathbf{Q}}(D)\otimes_{\mathbf{Q}}\mathbf{C}\xrightarrow{\omega}\mathbf{C}$$

sends $\det_{\mathbf{Z}}(M)\otimes_{\mathbf{Z}}\mathbf{Z}_{(\ell)}$ *onto* $\mathbf{Z}_{(\ell)}(2\pi i)^{(r-\frac{1}{2}m)\dim(V)}$.

Proof. By Poincaré duality and by the hard Lefschetz, we have an isomorphism of motives (with $\mathbf{Q}$-coefficients)

$$\det(H^m(X)(r))\otimes\det(H^m(X)((r))\xrightarrow{\cong}\mathbf{Q}((2r-m)\dim(V)).$$

The de Rham, the ℓ-adic and the Hodge realization of this isomorphism satisfy in $\mathbf{R}^x/\mathbf{Z}^x_{(\ell)}$

$$\begin{aligned}&\chi(\det_{\mathbf{Z}}(L)\otimes\det_{\mathbf{Z}}(L)\dashrightarrow\mathbf{Z})\\&=\chi(\det_{\mathbf{Z}_\ell}(M\otimes\mathbf{Z}_\ell)\otimes\det_{\mathbf{Z}_\ell}(M\otimes\mathbf{Z}_\ell)\dashrightarrow\mathbf{Z}_\ell((2r-m)\dim(V)))\\&=\chi(\det_{\mathbf{Z}}(M)\otimes\det_{\mathbf{Z}}(M)\dashrightarrow\mathbf{Z}(2\pi i)^{(2r-m)\dim(V)})\end{aligned}$$

in $\mathbf{R}^x/\mathbf{Z}^x_{(\ell)}$. Hence the commutative diagram

$$\begin{array}{ccc}(\det_{\mathbf{Z}}(L)\otimes_{\mathbf{Z}}\mathbf{C})\otimes_{\mathbf{C}}(\det_{\mathbf{Z}}(L)\otimes_{\mathbf{Z}}C) & \xrightarrow{\cong} & \mathbf{Z}\otimes_{\mathbf{Z}}\mathbf{C}\\ \downarrow & & \downarrow\\ (\det_{\mathbf{Z}}(M)\otimes_{\mathbf{Z}}\mathbf{C})\otimes_{\mathbf{C}}(\det_{\mathbf{Z}}(M)\otimes_{\mathbf{Z}}\mathbf{C}) & \xrightarrow{\cong} & \mathbf{Z}(2\pi i)^{(2r-m)\dim(V)}\otimes_{\mathbf{Z}}\mathbf{C}\end{array}$$

shows that the left vertical arrow sends $\det(M)\otimes\det(M)\otimes\mathbf{Z}_{(\ell)}$ onto $\det_{\mathbf{Z}}(L)\otimes\det_{\mathbf{Z}}(L)\otimes\mathbf{Z}_{(\ell)}(2\pi i)^{(2r-m)\dim(V)}$. This proves (7.12)

7.13. Here we review the theory of Hurwitz numbers from the point of view of Fontaine-Messing theory. This will be used for the proof of (7.4).

For a non-zero integer n, let θ_a be the function on E characterized by the properties $\operatorname{div}(\theta_a)=12((a^2(0)-{}_{a^2}E)$ and $N_b(\theta_a)=\theta_a$ for any non-zero element $b\in\mathcal{O}_K$ which is prime to a (cf. [dS], II 2.3). Assume ℓ splits in K, let $\ell\mathcal{O}_K=\mathfrak{m}\overline{\mathfrak{m}}$ with $\mathfrak{m}$ a maximal ideal of $\mathcal{O}_K$, and put $\tau=\psi(\mathfrak{m})$ (so τ is a generator of $\mathfrak{m}$). Let T be the component of $T_\ell E$ on which $\mathcal{O}_K$ acts via $\mathcal{O}_K\to\mathcal{O}_K$. T is a free $\mathbf{Z}_\ell$-module of rank one. For $r\in\mathbf{Z}$, we define $e_{r,E}\in H^1(K,T^{\otimes r})$ as

$$e_{r,E}=12^{-1}(a^2-\psi(a)^{-r})^{-1}(N_{K_i'/K}(\{\theta_a(\tau^{-i}\delta)\}\otimes[\gamma_i']^{\otimes r})_i\in H^1(K,T^{\otimes r})$$

where γ' denotes the ${}_{\mathfrak{m}^i}E$-component of the image of $\ell^{-i}\delta$ in ${}_{\ell^if}E(\mathbf{C})$ and K_i' denotes the ray class field of modulus $\mathfrak{m}^if$ over K. Then, $e_{r,E}$ is independent of the choice of a. On the other hand, let L be the filtered Dieudonné module corresponding to the $\operatorname{Gal}(\overline{K}_{\mathfrak{m}}/K_{\mathfrak{m}})$- module T in the sense of Fontaine-Laffaille. Then $L^0=0$, $L^{-1}=L$ and $L\otimes\mathbf{Q}\xrightarrow{\cong}H^1_{DR}(E/\mathbf{Q})/\operatorname{Fil}^1H^1_{DR}(E/\mathbf{Q})\otimes K_{\mathfrak{m}}$. For $r\geq 2$, let

$$\partial^r:L^{\otimes r}\otimes\mathbf{Q}\xrightarrow{\cong}H^1(K_{\mathfrak{m}},T^{\otimes r})\otimes\mathbf{Q}$$

be the boundary map obtained from (1.13) $\otimes T^{\otimes r}(-r)$ and from the isomorphism $L^{\otimes r}\cong H^0(K_{\mathfrak{m}},B^+_{\mathrm{crys}}\otimes T^{\otimes r}(-r))$. Then, if $r<p-1$, ∂^r gives

an isomorphism without $\otimes\mathbf{Q}$. The ℓ-adic theory of Hurwitz numbers (cf. [dS] II§4) is interpreted as:

(7.13.1) Let $r \geq 3$. Then, if ω is a differential $H^1_{DR}(E/\mathbf{Q})/\operatorname{Fil}^1 \xrightarrow{\cong} \mathbf{Q}$ and (ω, δ) denotes the integration of ω on δ, $(\omega,\delta)^{-r}L(\psi^{-r},0)$ belongs to K and the cup product with $c_{-r,E}$

$$L^{\otimes r} \otimes \mathbf{Q} \underset{\cong}{\xrightarrow{\partial^r}} H^1(K_m, T^{\otimes r}) \otimes \mathbf{Q} \overset{\cup c_{-r,E}}{\longrightarrow} H^2(K_{\mathfrak{m}}, \mathbf{Q}_\ell(1)) = \mathbf{Q}_\ell$$

coincides with

$$\pm(1-\tau^r p^{-1})((\omega,\delta)^{-r}L(\psi^{-r},0)) \cdot \omega^{\otimes r}.$$

In fact, this theory is described usually by using the Coates-Wiles homomorphisms applied to the norm compatible system $(\theta_a(\tau^{-i}\delta))_i$ via an isomorphism of formal groups $\widehat{E} \cong \widehat{G}_m$ over $\widehat{K}_{\mathfrak{m},nr}$ (cf. [dS], II §4), but the relation between the Coates-Wiles homomorphism and Fontaine-Messing theory proved in §2 enables us to rewrite the theory as above.

As a relation with cyclotomic elements in (7.1), the formula

$$e_v(E) = (1-\overline{\tau})ve_{1,E} \quad \text{in } H^1(K, T(1)) \tag{7.13.2}$$

is proved easily. By (7.2), if $\widehat{e}_v(E) \in H^1(\mathbf{Z}[1/\ell], T_\ell E(1))$ denotes the image of $c_v(E)$, we have

$$ve_{1,E} = (1-\tau)\widehat{e}_v(E) \quad \text{in } H^1(K, T(1)). \tag{7.13.3}$$

(Note $\kappa^{-1} = (1-\tau)(1-\overline{\tau})$ with κ as in (7.2).)

(7.14). Now we prove (7.4). In [So6], Soulé proved that $e_v(E)$ generates $H^1(\mathbf{Q}, V_\ell)$ and $H^2(\mathbf{Z}[1/\ell], j_*V_\ell) = 0$ in the case ℓ is regular for E. The following argument is inspired by [So6]. If ℓ is regular for E, the following conditions are satisfied ([Y]).

(i) ℓ splits in K.

(ii) $(\omega,\delta)^{2-\ell}L(\psi^{2-\ell},0)$ is an ℓ-unit if the differential ω is a generator of $\Gamma(\mathfrak{F}, \Omega_{\mathfrak{F}/\mathbf{Z}_{(\ell)}})$ for the proper smooth model $\mathfrak{F}$ of E over $\mathbf{Z}_{(\ell)}$.

(iii) Write $\ell O_K = \mathfrak{m}\overline{\mathfrak{m}}$ as in (7.13). Then, if F denotes the field $K({}_{\mathfrak{m}}E)$, $\operatorname{Pic}(O_F)/\ell\operatorname{Pic}(O_F)$ has no nontrivial element on which the action of $\operatorname{Gal}(F/K)$ is the same as that on ${}_{\mathfrak{m}}E(1)$.

Under the conditions (i)–(iii) we prove

$$\#(H^1(\mathbf{Z}[1/\ell], j_*T_\ell E(1))/\mathbf{Z}_\ell c_v(E)) = \#H^2(\mathbf{Z}[1/\ell], j_*T_\ell E(1)) = \kappa^{-1}$$

in $\mathbf{Q}^\times/\mathbf{Z}_{(\ell)}^\times$. Here and in the following, v is taken to be prime to ℓ.

By (ii) and (7.13.1), $e_{2-\ell,E}$ generates $H^1(K_\mathfrak{m}, T^{\otimes(2-\ell)}(1))$. By using $(T/\ell)^{\otimes(2-\ell)} \cong T/\ell$, we see that $e_{1,E}$ generates $H^1(K_\mathfrak{m}, T(1))$. Hence by (7.13.3), $\widehat{e}_v(E)$ generates $H^1(\mathbf{Z}[1/\ell], j_* T_\ell E(1))$ and we obtain

$$\#(H^1(\mathbf{Z}[1/\ell], j_* T_\ell E(1))/\mathbf{Z}_\ell e_v(E)) = \kappa^{-1}$$

in $\mathbf{Q}^\times/\mathbf{Z}_{(\ell)}^\times$.

Next we prove $\#H^2(\mathbf{Z}[1/\ell], j_* T_\ell E(1)) = \kappa^{-1}$ in $\mathbf{Q}^\times/\mathbf{Z}_{(\ell)}^\times$. It is seen easily that $\#H^2(\mathbf{Q}_\ell, T_\ell E(1)) = \kappa^{-1}$ in $\mathbf{Q}^\times/\mathbf{Z}_{(\ell)}^\times$ and hence it suffices to prove that the canonical map $H^2(\mathbf{Z}[1/\ell], j_* T_\ell E(1)) \to H^2(\mathbf{Q}_\ell, T_\ell E(1))$ is bijective. By Tate duality, the kernel (resp. cokernel) of this map is isomorphic to the Pontrjagin dual of

$$\ker(H^1(\mathbf{Z}[1/\ell], j_* H) \xrightarrow{i} H^1(\mathbf{Q}_\ell, H)) \quad (\text{resp. } H^0(\mathbf{Q}, H))$$

where $H = H^1(E \otimes \overline{\mathbf{Q}}, \mathbf{Q}_\ell/\mathbf{Z}_\ell)$. The injectivity of i is proved by using the conditions (iii) and the fact $H^0(\mathbf{Q}, H) = 0$ is seen easily.

Remark 7.15. (7.3) suggests that an Iwasawa main conjecture should exist even in the case ℓ does not split in K.

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