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HIGHER REGULATORS AND VALUES OF L-FUNCTIONS

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In the work conjectures are formulated regarding the value of L-functions of motives and some computations are presented corroborating them.

INTRODUCTION

Let X be a complex algebraic manifold, and let $K_j(X)$, $H^j_{\mathcal{B}}(X, \mathbf{Q})$ be its algebraic K-groups and singulary cohomology, respectively. We consider the Chern character ch: $K_j(X)\otimes\mathbf{Q}\to\oplus H^{2i-j}_{\mathcal{B}}(X, \mathbf{Q})$. It is easy to see that there are the Hodge conditions on the image of ch: we have ch $(K_j(X))\subset\oplus(W_{2i}H^{2i-j}_{\mathcal{B}}(X, \mathbf{Q}))\cap(F^iH^{2i-j}_{\mathcal{B}}(X, \mathbf{C}))$, where $W_.$, $F^{\cdot}$ are the filtration giving the mixed Hodge structure on $H^{\cdot}_{\mathcal{B}}(X)$. For example, if X is compact, then ch $(K_j(X)) = 0$ for $j > 0$. It turns out that the Hodge conditions can be used, and, untangling them, it is possible to obtain finer analytic invariants of the elements of $K_.(X)$ than the usual cohomology classes. For the case of Chow groups they are well known: they are the Abel—Jacobi—Griffiths periods of an algebraic cycle. Apparently, these invariants are closely related to the values of L-functions; we formulate conjectures and some computations corroborating them.

In Sec. 1 our main tool appears: the groups $H^j_{\mathcal{D}}(x, \mathbf{Z}(i))$ of "topological cycles lying in the i-th term of the Hodge filtration." These groups are written in a long exact sequence

$$\ldots\to H^{j-1}_{\mathcal{B}}(X, \mathbf{C})\to H^j_{\mathcal{D}}(X, \mathbf{Z}(i))\xrightarrow{\varepsilon_{\mathcal{B}}\oplus\varepsilon_{\mathcal{F}}} H^j_{\mathcal{B}}(X, \mathbf{Z})\oplus F^iH^j_{\mathcal{B}}(X, C)\to\ldots.$$

On $H^{\cdot}_{\mathcal{D}}$ we construct a $\cup$ -product such that $\varepsilon_{\mathcal{B}}$ becomes a ring morphism, and we show that $H^{\cdot}_{\mathcal{D}}$ form a cohomology theory satisfying Poincaré duality. Therefore, it is possible to apply the machinery of characteristic classes to $H^{\cdot}_{\mathcal{D}}$ [22] and obtain a morphism $\mathrm{ch}_{\mathcal{D}}: K_j(X)\otimes\mathbf{Q}\to\oplus H^{2i-j}_{\mathcal{D}}(X, \mathbf{Q}(i))$. The corresponding constructions are recalled in Sec. 2. Let $H^{2i-j}_{\mathcal{A}}(X, \mathbf{Q}(i))\subset K_j(X)\otimes\mathbf{Q}$ be the eigenspace of weight i relative to the Adams operator [2]; then $\mathrm{ch}_{\mathcal{D}}$ defines a regulator — a morphism $r_{\mathcal{D}}: H^j_{\mathcal{A}}(X, \mathbf{Q}(i))\to H^j_{\mathcal{D}}(X, \mathbf{Q}(i))$. [It is thought that for any schemes there exists a universal cohomology theory $H^j_{\mathcal{A}}(X, \mathbf{Z}(i))$, satisfying Poincaré duality and related to Quillen's K-theory in the same way as in topology the singular cohomology is related to K-theory; $H^{\cdot}_{\mathcal{A}}$ must be closely connected with the Milnor ring.] In the appendix we study the connection between deformations of $\mathrm{ch}_{\mathcal{D}}$ and Lie algebra cohomologies; as a consequence we see that if X is a point, then our regulators coincide with Borel regulators. There we present a formulation of a remarkable theory of Tsygan—Feigin regarding stable cohomologies of algebras of flows. Finally, Sec. 3 contains formulations of the basic conjectures connecting regulators with the values of L-functions at integral points distinct from the middle of the critical strip; the arithmetic intersection index defined in part 2.5 is responsible for the behavior in the middle of the critical strip. From these conjectures (more precisely, from the part of them that can be applied to any complex manifold) there follow rather unexpected assertions regarding the connection of Hodge structures with algebraic cycles. The remainder of the work contains computations corroborating the conjectures in Sec. 3. Thus, in Sec. 7 we prove these conjectures for the case of Dirichlet series;

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Sec. 5 contains a result giving a partial proof of the conjecture for values at two of L-functions of curves uniformized by modular functions; Sec. 6 contains an analogous computation for the product of curves of this type.

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NOTATION

We shall use the standard language of cohomological algebra. If A is an Abelian category, then $\mathbf{D}(A)$ is the derived category of A; DF(A) is the filtered derived category; C(A) is the category of complexes; s is a functor assigning to a bicomplex the corresponding simple complex; $N:A^{\Delta} \to C(A)$ is the normalization functor (A^{Δ} are the cosimplicial objects of A). If $X^{\cdot}$ is a complex, then the complex $X^{\geqslant i}$ coincides with $X^{\cdot}$ in degrees $\geqslant i$ and is equal to zero in degrees $<i$ (the i-th term of the filtration group on X). We denote by $[Y^0 \to Y^1 \to Y^2 \to \ldots]$ the complex equal to zero in negative degrees and coinciding with $Y^{\cdot}$ in positive degrees. If T is the topology in $\mathcal{F}(T)$ — the category of sheaves of Abelian groups on T, then $C(T):=C(\mathcal{F}(T))$.

"An analytic space" is an analytic space over $\mathbf{R}$; we denote by $\mathcal{A}n$ the category of analytic spaces equipped with the usual topology.

Let $V \in \mathcal{A}n$. Then a sheaf $\mathcal{F}$ on V is a sheaf $\mathcal{F}_{\mathbf{C}}$ on $V(\mathbf{C})$ equipped with the action of an involution σ of complex conjugation on $V(\mathbf{C})$; the spectral sequence with second term $H^p(\mathbf{Z}/2, H^q(V(\mathbf{C}), \mathcal{F}_{\mathbf{C}}))$ converges to the cohomologies $H^{\cdot}(V, \mathcal{F})$ (the Leray sequence of the structural morphism $V \to \operatorname{Spec} \mathbf{R}$); in particular, for a $\mathbf{Q}$-sheaf we have $H^{\cdot}(V, \mathcal{F}) = H^{\cdot}(V(\mathbf{C}), \mathcal{F}_{\mathbf{C}})^{\sigma}$.

We denote by $\mathbf{C}_V$ or simply $\mathbf{C}$ the local system on V corresponding to a constant sheaf with stalk $\mathbf{C}$ on $V(\mathbf{C})$ with the action σ by means of complex conjugation. Identifying $\mathbf{C}_V$ with the subsheaf of constant functions in the structural sheaf $\mathcal{O}_V$, we obtain, if V is smooth, the isomorphism $H^{\cdot}(V, \mathbf{C}) = H^{\cdot}_{\mathcal{D}\mathcal{R}}(V)$. If $K \subset \mathbf{C}$ is a subgroup closed relative to conjugation, then let K_V (or simply K) be the subsheaf of $\mathbf{C}_V$ with stalk K. We need the following subgroups of this kind: for a subring $A \subset \mathbf{R}$ and $n \in \mathbf{Z}$ we set $A(n) := (2\pi i)^n A = A \cdot \mathbf{Z}(n) \subset \mathbf{C}$; then $A(i) \cdot A(j) = A(i + j)$. If $\mathcal{F}$ is a sheaf on V, then let $\mathcal{F}(n) := \mathcal{F} \otimes \mathbf{Z}(n)$; it is clear that for a sheaf of $\mathbf{C}_V$-modules $\mathcal{F}$ the sheaf $\mathcal{F}(n)$ can be canonically identified with $\mathcal{F}$.

If V is smooth, then let $\mathcal{O}_{V\infty} \supset \mathcal{O}_V$ be the sheaf of functions of class C^{∞} on V, let $\Omega^{\cdot}_{V\infty} \supset \Omega^{\cdot}_V$ be the corresponding de Rham complex, and let $\Omega^{n}_{V\infty} = \bigoplus_{p+q=n} \Omega^{p,q}_{V\infty}$; let $\mathbf{S}^{\cdot}_V$ be the subsheaf of $\mathbf{R}$-valued forms, $\Omega^{\cdot}_{V\infty} = \mathbf{S}^{\cdot}_V \otimes_{\mathbf{R}} \mathbf{C}_V$. If X is an algebraic manifold over $\mathbf{R}$, then we set $H^{\cdot}_{\mathcal{B}}(X, A(n)) = H^{\cdot}(X_{\text{an}}, A(n))$. For any cohomology theory $H^{\cdot}_{?}$ we denote by $H^{!\cdot}_{?}$ the corresponding cohomology groups (see 2.3).

CHAPTER 1. MAIN CONSTRUCTIONS AND CONJECTURES

1. $\mathcal{D}$-Cohomologies

1.1. $\mathcal{D}$-Cohomologies of Analytic Spaces.

We fix a subring $A \subset \mathbf{R}$.

Definition 1.1.1.[1] For $i \in \mathbf{Z}$ we define a complex $A(i)_{\mathcal{D}}$ of sheaves on $\mathcal{A}n$ by the formula

$$A(i)_{\mathcal{D}} := \operatorname{Cone}(F^i \oplus A(i) \to \Omega^{\cdot})[-1].$$

Here $\Omega^{\cdot}$ is the de Rham complex of holomorphic forms equipped with the filtration group $F^j := \Omega^{\geqslant j}$; the arrow $F^i \oplus A(i) \to \Omega^{\cdot}$ is the difference of the obvious imbeddings $F^i \hookrightarrow \Omega^{\cdot}$ and $A(i) \hookrightarrow \mathbf{C} \hookrightarrow \Omega^{\cdot}$. ■

Let ε_F, ε_A be the natural morphisms of $A(i)_{\mathcal{D}}$ into F(i), A(i), respectively. We have the exact triangle

$$\ldots \to \Omega^{\cdot}[-1]^{\alpha} \to A(i)_{\mathcal{D}} \xrightarrow{\varepsilon_F + \varepsilon_A} F(i) \oplus A(i) \to \ldots \qquad (*)$$

---

[1] Apparently, these complexes were first considered by Deligne (see [8]).

It is clear that for $i \leqslant 0$ ε_A is a quasiisomorphism. For i > 0, factoring $A(i)_{\mathcal{D}}$ by the cones id:$F^i \to F^i$, A(i) → A(i), we obtain the quasiisomorphisms

$$A(i)_{\mathcal{D}} \to [A(i) \to \mathcal{O} \to \ldots \to \Omega^{i-1}] \to [0 \to \mathcal{O}/A(i) \to \Omega^1 \to \ldots \to \Omega^{i-1}].$$

In particular, since $\mathcal{O}/\mathbf{Z}(1) \overset{\exp}{\simeq} \mathcal{O}^*$, the complex $\mathbf{Z}(1)_{\mathcal{D}}$ coincides in $\mathbf{D}^+(\mathcal{A}n)$ with $\mathcal{O}^*[-1]$. If V is a smooth manifold, then the morphism $\beta_V: \mathbf{C}/A(i)[-1] \to A(i)_{\mathcal{D}}$ in $\mathbf{D}^+(V)$ connected with the imbedding $\mathbf{C}/A(i) \hookrightarrow \mathcal{O}/A(i)$, is a quasiisomorphism for i > dim V.

The complex $A(i)_{\mathcal{D}}$ determines a corresponding contravariant cohomological functor $R\Gamma(\cdot, A(i)_{\mathcal{D}})$ on $\mathcal{A}n$ with values in the derived category $\mathbf{D}^+$(A-mod). Applying to (*) the functor RΓ, we obtain the exact triangle of functors

$$\ldots \to \mathbf{H}_{\mathcal{B}}(\cdot)[-1]^{\alpha} \to R\Gamma(\cdot, A(i)_{\mathcal{D}}) \xrightarrow{\varepsilon_F + \varepsilon_A} R\Gamma(\cdot, F^i) \oplus R\Gamma(\cdot, A(i)) \to \ldots \tag{*}$$

Generally, for any diagram $G: I \to \mathcal{A}n$ of analytic spaces its cohomologies $R\Gamma(G, A(i)_{\mathcal{D}})' \in \mathbf{D}^+(\text{A-mod}^I)$ are defined, and

$$R\Gamma(G, A(i)_{\mathcal{D}}) := R \varprojlim R\Gamma(G, A(i)_{\mathcal{D}})' \in \mathbf{D}^+(A\text{-mod}).$$

We require cohomologies of simplicial spaces ($I = \Delta^0$) and relative cohomologies

$$(I = \cdot \to \cdot,\ \mathcal{A}n^I = \mathrm{Mor}\,\mathcal{A}n,\ \mathbf{D}(A\text{-mod}^I) \subset \mathbf{DF}(A\text{-mod})).$$

If $V_{\bullet}$ is a simplicial space, then to

$$H^{\cdot}(V_{\cdot}, A(i)_{\mathcal{D}})\ (:= H^{\cdot} R\Gamma(V_{\cdot}, A(i)_{\mathcal{D}}))$$

there converges the spectral sequence with first term $E_1^{p,q} = H^q(V_p, A(i)_{\mathcal{D}})$. If f:U → V is a morphism, then we have the exact triangle

$$\ldots \to R\Gamma(f, A(i)_{\mathcal{D}}) \to R\Gamma(V, A(i)_{\mathcal{D}}) \to R\Gamma(U, A(i)_{\mathcal{D}}) \to \ldots$$

Remark 1.1.2. Let $C^{\cdot}(V, ?)$ be the complex of singular cochains of class C^{∞} with coefficients in ? on V(**C**). This is a complex of presheaves on $\mathcal{A}n$; let $\tilde{C}^{\cdot}(?)$ be the complex of sheaves corresponding to it. We set

$$C^{\cdot}_{\mathcal{D}}(\cdot, A(i)) := \mathrm{Cone}(\tilde{C}^{>i}(\mathbf{C}) \oplus A(i) \to \tilde{C}^{\cdot}(\mathbf{C}))[-1],$$

where the arrow is the difference of the obvious imbeddings $C^{\geqslant i} \to C^{\cdot}$ and $A(i) \to C^{\cdot}$. Integration over chains gives the morphism $\int: A(i)_{\mathcal{D}} \to C^{\cdot}_{\mathcal{D}}(A(i))$. Its composition with the natural projection

$$C^{\cdot}_{\mathcal{D}}(A(i)) \to [\tilde{C}^{\cdot}(\mathbf{C}/A(i))/\tilde{C}^{>i}(\mathbf{C}/A(i))][-1]$$

defines the morphisms

$$\int^j: H^j(V, A(i)_{\mathcal{D}}) \to H^{j-1}(V, \mathbf{C}/A(i)) \text{ при } j < i$$

and

$$H^i(V, A(i)_{\mathcal{D}}) \to \Gamma(V, \tilde{C}^{i-1}(\mathbf{C}/A(i))/d\Gamma(V, \tilde{C}^{i-2}).$$

If V is smooth and j < i, then $\int^j$ are isomorphisms inverse to β_V.

1.2. Multiplication on $\mathcal{D}$ -Cohomologies.

Definition 1.2.1. Let $\alpha \in \mathbf{R}$. We define the mapping

$$\cup_{\alpha}: A(i)_{\mathcal{D}} \otimes A(j)_{\mathcal{D}} \to A(i+j)_{\mathcal{D}}$$

by the formula

$$f_i \cup_{\alpha} f_j = f_i \Lambda f_j \in F^{i+j},\quad a_i \cup_{\alpha} a_j = a_i \cdot a_j \in A(i+j),$$
$$a_i \cup_{\alpha} f_j = f_i \cup_{\alpha} a_j = \omega_i \cup_{\alpha} \omega_j = 0,\quad f_i \cup_{\alpha} \omega_j = (-1)^{\deg f_i} \alpha \cdot f_i \Lambda \omega_j \in \Omega^{\cdot},$$
$$\omega_i \cup_{\alpha} f_j = (1-\alpha)\, \omega_i \Lambda f_j \in \Omega^{\cdot},\quad a_i \cup_{\alpha} \omega_j = (1-\alpha)\, a_i \cdot \omega_j \in \Omega^{\cdot},$$
$$\omega_i \cup_{\alpha} a_j = \alpha \omega_i \cdot a_j \in \Omega^{\cdot},$$

here $f_i \in F^i$, $\omega_i \in \Omega^{\cdot}$, $a_i \in A(i), \ldots$

LEMMA 1.2.2. a) $\cup_\alpha$ is a morphism of complexes.

b) All $\cup_\alpha$ are homotopic to one another; we denote by $\cup$ their common homotopy class.

c) The multiplication $\cup$ is associative and commutative up to homotopy.

d) The cycle $(1, 1)\in H^0(A(0)_{\mathcal{D}})$ is a two-sided identity for $\cup$.

e) We define the morphisms $A(i)_{\mathcal{D}}\otimes A(j)\to A(i+j)$, $A(i)_{\mathcal{D}}\otimes F^j\to F^{i+j}$, and $A(i)_{\mathcal{D}}\otimes\Omega^\cdot\to\Omega^\cdot$ as compositions of the natural multiplications on $A(\cdot)$, $F^\cdot$, and $\Omega^\cdot$ with the morphisms ε_A, ε_F, and $\varepsilon_\Omega = \varepsilon_A \sim \varepsilon_F$, respectively. Then these multiplications together with $\cup$ give a morphism of triangles $A(i)_{\mathcal{D}}\otimes(*)\to(*)$ in the homotopy category.

Proof. b) The homotopy between $\cup_\alpha$ and $\cup_\beta$ is given by the formula $\omega_i\otimes\omega_j\to(-1)^{\deg\omega_i}(\alpha-\beta)\,\omega_i\Lambda\omega_j$; the other components of it are equal to zero.

c) $\cup_0$ and $\cup_1$ are associative; if $S: A(i)_{\mathcal{D}}\otimes A(j)_{\mathcal{D}}\to A(j)_{\mathcal{D}}\otimes A(i)_{\mathcal{D}}$ is the permutation of factors, then $\cup_\alpha S=\cup_{1-\alpha}$. ■

We have thus obtained a commutative and associative multiplication $\cup: A(i)_{\mathcal{D}}\overset{L}{\otimes}A(j)_{\mathcal{D}}\to A(i+j)_{\mathcal{D}}$ in the derived category; it defines a multiplication on $\mathcal{D}$-cohomologies. It is clear that ε_A, ε_F are morphisms of graded rings.

Remark 1.2.3. Let $(M_\Omega, F^\cdot M_\Omega)$ be a differential, filtered $(\Omega^\cdot, F^\cdot)$-module, let M_A be a complex of A-modules, and let $M_A \to M_\Omega$ be a morphism of complexes of A-modules. We set $M(i)_{\mathcal{D}}:=\mathrm{Cone}(F^iM_\Omega\otimes M_A(i)\to M_\Omega)[-1]$. Formulas 1.2.1 give a natural pairing $A(i)_{\mathcal{D}}\otimes M(j)_{\mathcal{D}}\to M(i+j)_{\mathcal{D}}$, converting $M(\cdot)_{\mathcal{D}}$ into a $A(\cdot)_{\mathcal{D}}$-module.

1.2.4. We shall present a more convenient construction of the multiplication on $\mathcal{D}$-cohomologies with coefficients in **R**. We assume that our manifolds are smooth.

The projection $\pi_i:\mathbf{C}=\mathbf{R}(i)\oplus\mathbf{R}(i+1)\to\mathbf{R}(i)$ gives a morphism $\Omega^\cdot\to S^\cdot\otimes\mathbf{C}\to S\otimes\mathbf{R}(i)$, which makes it possible to define the complex $\tilde{\mathbf{R}}(i)_{\mathcal{D}}:=\mathrm{Cone}(\pi_{i-1}:F^i\to S^\cdot\otimes\mathbf{R}(i-1))[-1]$ together with the morphism $p_i:\mathrm{R}(i)_{\mathcal{D}}\to\tilde{\mathbf{R}}(i)_{\mathcal{D}}$; $p_i|_{F^i}=\mathrm{id}$, $p_i|_{\mathbf{R}(i)}=0$, $p_i|_{\Omega^\cdot}=\pi_{i-1}$. We define the multiplication $\tilde{\cup}:\tilde{\mathbf{R}}(i)_{\mathcal{D}}\otimes\tilde{\mathbf{R}}(j)_{\mathcal{D}}\to\tilde{\mathbf{R}}(i+j)_{\mathcal{D}}$ by the formula $f_i\tilde{\cup}f_j=f_i\Lambda f_j$, $s_i\tilde{\cup}s_j=0$, $f_i\tilde{\cup}s_j=(-1)^{\deg f_i}\pi_i f_i\Lambda s_j$, $s_i\tilde{\cup}f_j=s_i\Lambda\pi_j f_j$, where $f_i\in F^i$, $s_i\in S^\cdot\otimes\mathbf{R}(i-1)\subset S^\cdot\otimes\mathbf{C},\ldots$

LEMMA 1.2.5. a) p_i is a homotopy equivalence.

b) $\tilde{\cup}$ is a morphism of complexes, and $p_{i+j}\cup$ is homotopic to $\tilde{\cup}(p_i\otimes p_j)$.

Proof. b) The homotopy between $p_{i+j}\cup_\alpha$ and $\tilde{\cup}(p_i\otimes p_j)$ is given by the formula

$$\omega_i\otimes\omega_j\to(-1)^{\deg\omega_i}(\alpha\cdot\pi_{i-1}\omega_i\Lambda\pi_j\omega_j+(1-\alpha)\,\pi_i\omega_i\Lambda\pi_{j-1}\omega_j),$$

and the other components are 0. ■

Thus, real $\mathcal{D}$-cohomologies can be computed in terms of $\tilde{\mathbf{R}}(\cdot)_{\mathcal{D}}$-complexes, and the $\cup$-product on them can be computed in terms of $\tilde{\cup}$-multiplication. For example, for $i > 0$ we have $H^i(V,\mathbf{R}(i)_{\mathcal{D}})=\{f\in\Gamma(V, S^{i-1}\otimes\mathbf{R}(i-1)):df=\omega\pm\bar{\omega},\ \omega\in\Omega^i(V)/d\Gamma(V, S^{i-2}\otimes\mathbf{R}(i-1))$; in particular, $H^1(V,\mathbf{R}(1)_{\mathcal{D}})$ is the space of **R**-valued functions f of class C^∞ on V such that d_zf is a holomorphic differential; further, for such f, g we have $f\cup g=f\cdot\pi_1 d_z g-g\pi_1 d_z f\in H^2(V,\mathbf{R}(2)_{\mathcal{D}})$.

Remark 1.2.6. If we have any diagram of differential graded rings, then there is the Alexander–Whitney multiplication on its inverse homotopy limit. It is easy to see that the $\cup$-product on $A(*)_{\mathcal{D}}$ is the Alexander–Whitney multiplication corresponding to the diagram $A(*)\to\Omega^\cdot\leftarrow F^*$.

1.3. In order to become accustomed to $\mathcal{D}$-cohomologies, it is worthwhile to consider in detail the multiplication $\cup:H^1(V,\mathbf{Z}(1)_{\mathcal{D}})\otimes H^1(V,\mathbf{Z}(1)_{\mathcal{D}})\to H^2(V,\mathbf{Z}(2)_{\mathcal{D}})$. We have $H^1(V,\mathbf{Z}(1)_{\mathcal{D}})=\mathcal{O}^*(V)$, $H^2(V,\mathbf{Z}(2)_{\mathcal{D}})=H^1(V,\mathcal{O}^*\xrightarrow{d\log}\Omega^1)(1)$, so that $\cup:\mathcal{O}^*(V)\otimes\mathcal{O}^*(V)\to H^1(V,\mathcal{O}^*\to\Omega^1)(1)$. This pairing was constructed independently by Deligne and the author [19, 3, 11] and served as the point of departure for the present work.

As Deligne noted, $H^1(V,\mathcal{O}^*\to\Omega^1)$ is a group of isomorphism classes of invertible sheaves with a connection, so that $\cup$ assigns to a pair of invertible functions a sheaf with a connection; for $\dim V = 1$ all connections are integrable, and $H^1(V,\mathcal{O}^*\to\Omega^1)=H^1(V,\mathbf{C}^*)$. The next lemma is left as an exercise for the reader.

LEMMA 1.3.1. Let $f, g \in \mathcal{O}^*(V)$. Then

a. The curvature of $f \cup g$ is $d\,\mathrm{lcg}\, f \Lambda d \log g$.

b. The monodromy logarithm of $f \cup g$ over a loop γ is computed by the formula $\left(\int \log f d \times \log g - g(\alpha) \int d \log g\right) \in \mathbf{C}/\mathbf{Z}(2)$; here α is a point of γ, log f, log g are branches of the logarithm that are continuous off α, and the integrals are taken over γ beginning at α.

c. Let f, g be functions on the punctured disk which are meromorphic at the exercised point 0. Then the monodromy of $f \cup g$ over a loop about 0 coincides with the manual symbol $\{f, g\}_0$ at 0. ■

COROLLARY 1.3.2. (the Steinberg identity). If t, $1 - t \in \mathcal{O}^*(V)$, then $t \cup (1 - t) = 0$.

Indeed, by functoriality it suffices to verify the identity for t a parameter on V = $\mathbf{P}^1 \setminus \{0, 1, \infty\}$. Then $H^2(V, \mathbf{Z}(2)_{\mathcal{D}}) = H^1(V, \mathbf{C}^*(1))$, the group $H_1(V^\cdot)$ is generated by loops about 0 and 1, and everything follows from c. and the Steinberg identity for the manual symbol. ■

COROLLARY 1.3.2. On the category of algebraic curves over **R** there is a unique morphism of functors $K_2(X) \to H^1(X_{an}, \mathbf{C}^*(1))$ taking $\{f, g\}$ into $f \cup g$, where $f \cup g$ is given by formula 3.1.b.

Indeed, the Matsumoto theorem plus the preceding corollary define a morphism at a general point η over the curve X. The morphism extends to all of X by means of a commutative diagram with rows which are exact sequences of localization (the commutativity of the right square is given by 3.1.1c)

$$\begin{tikzcd}
 & K_2(X) \arrow[r] \arrow[d] & K_2(\eta) \arrow[r] \arrow[d] & \underset{x \in X(\mathbf{C})}{\oplus} \mathbf{C}^* \arrow[d, equal] \\
0 \arrow[r] & H^1(X_{\mathrm{an}}, \mathbf{C}^*(1)) \arrow[r] & H^1(\eta_{\mathrm{an}}, \mathbf{C}^*(1)) \arrow[r] & \underset{x \in X(\mathbf{C})}{\oplus} \mathbf{C}^*
\end{tikzcd}.$$

The cohomologies constructed possess the following shortcoming: for open V they are often infinite-dimensional [for example, $H^1(V, \mathbf{Z}(1)\mathcal{D}) = \mathcal{O}^*(V)$]. In order to obtain a more convenient theory for algebraic manifolds, we impose growth conditions at infinity. Before doing this, however, we must present some definitions from general topology.

1.4. Relative Cohomologies. Let $j: T \to \overline{T}$ be a topology morphism. With j there is connected a new topology $(\overline{T}, T)$: sheaves on $(\overline{T}, T)$ are triples $\mathcal{F} = (\mathcal{F}_{\overline{T}}, \mathcal{F}_T, \varphi_{\mathcal{F}})$, $\mathcal{F}_{\overline{T}}$, $\mathcal{F}_T$ are sheaves on $\overline{T}$, T, $\varphi_{\mathcal{F}}$, and j is a morphism $\mathcal{F}_{\overline{T}} \to \mathcal{F}_T$; a morphism $\alpha: \mathcal{F} \to \mathcal{G}$ is a pair of morphisms $\alpha_{\overline{T}}: \mathcal{F}_{\overline{T}} \to \mathcal{G}_{\overline{T}}$, $\alpha_T: \mathcal{F}_T \to \mathcal{G}_T$ such that $\alpha_T \varphi_{\mathcal{F}} = \varphi_{\mathcal{G}} \alpha_{\overline{T}}$. We define the left-exact functor $\Gamma(\overline{T}, T, \cdot)$: $C(\overline{T}, T) \to C(\mathrm{Ab})$ by the formula $\Gamma(\overline{T}, T, \mathcal{F}) := \mathrm{Cone}\,(\Gamma(\overline{T}, \mathcal{F}_{\overline{T}}) \to \Gamma(T, \mathcal{F}_T))[-1]$; let $R\Gamma(\overline{T}, T, \cdot): \mathbf{D}^+(\overline{T}, T) \to \mathbf{D}^+(\mathrm{Ab})$ be the right derived functor. We have an exact triangle of functors on $\mathbf{D}^+(\overline{T}, T)$:

$$R\Gamma(\overline{T}, T, \mathcal{F}) \to R\Gamma(\overline{T}, \mathcal{F}_{\overline{T}}) \to R\Gamma(T, \mathcal{F}_T) \to \ldots.$$

It is clear that $R\Gamma(\overline{T}, T, \cdot)$ is also a right derived function of the functor $\Gamma^0(\overline{X}, X, \cdot)$ taking the sheaf $\mathcal{F}$ into $\mathrm{Ker}\,(\Gamma(\overline{T}, \mathcal{F}_{\overline{T}}) \to \Gamma(T, \mathcal{F}_T))$.

We define a bifunctor $\underset{j}{\otimes}: C(\overline{T}, T) \times C(\overline{T}, T) \to C(\overline{T}, T)$ by the following formula. Let $\mathcal{F}^\cdot$, $\mathcal{G}^\cdot \in C(\overline{T}, T)$. We set $(\mathcal{F}^\cdot \underset{j}{\otimes} \mathcal{G}^\cdot)_{\overline{T}} := \mathcal{F}^\cdot_{\overline{T}} \otimes \mathcal{G}^\cdot_{\overline{T}}$, $(\mathcal{F}^\cdot \underset{j}{\otimes} \mathcal{G}^\cdot)_T$ is the shift by -1 of the cone of the morphism $\varphi_{\mathcal{F}} \otimes \mathrm{id} - \mathrm{id} \otimes \varphi_{\mathcal{G}}: (j^* \mathcal{F}^\cdot_{\overline{T}} \otimes \mathcal{G}^\cdot_T) \oplus (\mathcal{F}^\cdot_T \otimes j^* \mathcal{G}^\cdot_{\overline{T}}) \to \mathcal{F}^\cdot_T \otimes \mathcal{G}^\cdot_T$, while the connecting morphism $\varphi_{\mathcal{F} \underset{j}{\otimes} \mathcal{G}}$ is $j^* \otimes \varphi_{\mathcal{G}} + \varphi_{\mathcal{F}} \otimes j^*: \mathcal{F}^\cdot_{\overline{T}} \otimes \mathcal{G}^\cdot_{\overline{T}} \to (j^* \mathcal{F}^\cdot_{\overline{T}} \otimes \mathcal{G}^\cdot_T) \oplus (\mathcal{F}^\cdot_T \otimes j^* \mathcal{G}^\cdot_{\overline{T}})$. Let $\overset{L}{\underset{j}{\otimes}}: \mathbf{D}^+(\overline{T}, T) \times \mathbf{D}^+(\overline{T}, T) \to \mathbf{D}^+(\overline{T}, T)$ be the corresponding left derived functor. We have the obvious morphism $R\Gamma(\overline{T}, T, \mathcal{F}) \overset{L}{\otimes} R\Gamma(\overline{T}, T, \mathcal{G}) \to R\Gamma(\overline{T}, T, \mathcal{F} \overset{L}{\underset{j}{\otimes}} \mathcal{G})$, so that the pairing $\mathcal{F}^\cdot \overset{L}{\underset{j}{\otimes}} \mathcal{G}^\cdot \to \mathcal{H}^\cdot$ induces a multiplication on the relative cohomologies.

Remark. If T = $\overline{T}$, j = id, then $(\overline{T}, T)$ is the category of sheaf morphisms on T. We define an exact functor $I: C(\overline{T}, T) \to C(T)$ by the formula $I(\mathcal{F}^\cdot) = (\mathrm{Cone}\, \varphi_{\mathcal{F}})[-1]$; then $R\Gamma(\overline{T}, T, \mathcal{F}^\cdot) = R\Gamma(T, I(\mathcal{F}^\cdot))$, $I(\mathcal{F}^\cdot \underset{j}{\otimes} \mathcal{G}^\cdot) = I(\mathcal{F}^\cdot) \otimes I(\mathcal{G}^\cdot)$.

We now return to our main subject.

1.5. Complexes $A(i)_{\mathcal{D}}$ with Logarithmic Singularities. We consider the category Π of pairs $(\overline{V}, V)$, where $\overline{V}$ is a smooth analytic space, $j: V \hookrightarrow \overline{V}$ is an open subspace such that $\overline{V} \setminus V$ is a divisor with normal intersections on $\overline{V}$. For a pair $(\overline{V}, V)$ in Π let $\Omega^{\cdot}_{(V,\overline{V})}$ be the de Rham complex of holomorphic forms on $\overline{V}$ with logarithmic singularities along $\overline{V} \setminus V$. We filter the complex $\Omega^{\cdot}_{(V,\overline{V})}$ with the foolish filtration $F^{i}_{(V,\overline{V})} := \Omega^{\geqslant i}_{(V,\overline{V})}$.

We define a complex $A(i)_{\mathcal{D}}$ of sheaves on $(\overline{V}, V)$ by the formula

$$[A(i)_{\mathcal{D}}]_{\overline{V}} := F^{i}_{(V,\overline{V})}, \quad [A(i)_{\mathcal{D}}]_{V} := \mathrm{Cone}\,(A(i)_{V} \to \Omega^{\cdot}_{V}),$$

the connecting the morphism $[A(i)_{\mathcal{D}}]_{\overline{V}}|_{V} \to [A(i)]_{V}$ is the imbedding $F^{i}|_{V} \hookrightarrow \Omega^{\cdot}_{V}$

We have thus constructed complexes $A(i)_{\mathcal{D}}$ with logarithmic singularities along $\overline{V} \setminus V$. It is clear that if $\overline{V} = V$, then $I(A(i)_{\mathcal{D}})$ is the complex $A(i)_{\mathcal{D}}$ of part 1 (see the end of the preceding subsection). For $A = \mathbf{R}$ we can thus define complexes $\tilde{\mathbf{R}}(i)_{\mathcal{D}}$, quasiisomorphic to $\mathbf{R}(i)_{\mathcal{D}}$ (see 1.2.2).

The complexes $A(i)_{\mathcal{D}}$ depend functorially on $(\overline{V}, V) \in \Pi$. As in part 1, we can define a contravariant functor $R\Gamma(\overline{V}, V, A(i)_{\mathcal{D}})$ on Π or Π_{Δ}, define the relative cohomology connected with the morphism, etc. We have the exact triangle

$$\ldots \to \mathbf{H}_{\mathcal{D}\mathcal{R}}(V)[-1] \to R\Gamma(\overline{V}, V, A(i)_{\mathcal{D}}) \to R\Gamma(\overline{V}, F^{i}) \oplus R\Gamma(V, A(i)) \ldots . \qquad (*)$$

LEMMA 1.5.1. The formulas of Definition 1.2.1 give a morphism $A(i)_{\mathcal{D}} \underset{j}{\otimes} A(j)_{\mathcal{D}} \to A(i+j)_{\mathcal{D}}$; all other results of part 1.2 also obtain without change in the logarithmic situation. ■

We thus have a multiplication on $\mathcal{D}$ -cohomologies with logarithmic singularities. First example: we note that the complex $A(i)_{\mathcal{D}}$ is quasiisomorphic to the complex $F^{i}(V, \overline{V}) \to [\mathcal{O}_V / A(i) \to \Omega^{1}_{V} \to \ldots]$. From this we obtain the following result.

LEMMA 1.5.2. $H^{1}(\overline{V}, V, A(1)) = \{f \in \Gamma(V, \mathcal{O}_V / A(1)) : df \in \Gamma(V, \Omega^{1}_{(V,\overline{V})})\}$, $H^{i}(\overline{V}, V, A(1)_{\mathcal{D}}) = 0$ for $i \leqslant 0$. ■

1.6. $\mathcal{D}$ -Cohomologies of Algebraic Manifolds. We denote by $\tilde{\Pi} \subset \Pi$ the complete subcategory consisting of those pairs $(\overline{V}, V)$ for which $\overline{V}$ is a (smooth) projective algebraic variety. Let $\mathcal{V}_{\mathbf{R}}$ or simply $\mathcal{V}$ be the category of smooth quasiprojective schemes over $\mathbf{R}$. According to GAGA, we have the functor $\sigma: \tilde{\Pi} \to \mathcal{V}$, taking $(\overline{V}, V)$ into V.

LEMMA 1.6.1. Let $f: (\overline{U}, U) \to (\overline{V}, V)$ be a morphism in $\tilde{\Pi}$ such that $\sigma(f)$ is an isomorphism. Then f^* defines an isomorphism of the triangles (*) in part 1.5.

Proof. It follows from [15] that f^* gives an isomorphism of the two extreme terms of the triangles. Since the triangles are exact, $f^*: R\Gamma(\overline{V}, V, A(i)_{\mathcal{D}}) \to R\Gamma(\overline{U}, U, A(i)_{\mathcal{D}})$ is also an isomorphism. ■

For $X \in \mathcal{V}$ we shall find, according to Hironaka, $(\overline{X}, X) \in \tilde{\Pi}$. We set

$$\mathbf{H}_{\mathcal{D}}(X, A(i)) := R\Gamma(\overline{X}_{\mathrm{an}}, X_{\mathrm{an}}, A(i)_{\mathcal{D}}),$$

$$\mathbf{H}(X, F(i)) := R\Gamma(\overline{X}_{\mathrm{an}}, F^{i}_{(X,\overline{X})_{\mathrm{an}}}), \quad \mathbf{H}_{\mathcal{B}}(X, A(i)) := R\Gamma(X_{\mathrm{an}}, A(i)).$$

According to 1.6.1, these complexes do not depend [in D^{b}(A-mod)] on the choice of compactification and define an exact triangle of functors on $\mathcal{V}$:

$$\ldots \to \mathbf{H}_{\mathcal{D}\mathcal{R}}(X)[-1] \xrightarrow{\alpha} \mathbf{H}_{\mathcal{D}}(X, A(i)) \xrightarrow{\varepsilon_F + \varepsilon_A} \mathbf{H}(X, F(i)) \oplus \mathbf{H}_{\mathcal{B}}(X, A(i)) \to \ldots \qquad (*)$$

We have thus defined $\mathcal{D}$ -cohomologies $\mathbf{H}_{\mathcal{D}}(X, A(i))$ of smooth algebraic manifolds; $\mathcal{D}$ -cohomologies of smooth simplicial schemes and the relative cohomologies corresponding to a morphism of schemes are constructed in exactly the same way; the corresponding spectral sequences and exact triangles hold (see part 1).

We emphasize that for $X \in \mathcal{V}$ the natural morphism $\mathbf{H}_{\mathcal{D}}(X, A(i)) \to R\Gamma(X_{\mathrm{an}}, A(i)_{\mathcal{D}})$, generally speaking, is not an isomorphism; however, it is an isomorphism if X is compact or i > dim X.

The next result follows from the triangle (*) and [16].

LEMMA 1.6.2. Let $f: X. \to Y.$ be a morphism of simplicial schemes such that $f^*: H^{\cdot}(Y.(\mathbf{C}), \mathbf{Z}) \to H^{\cdot}(X.(\mathbf{C}), \mathbf{Z})$ is an isomorphism. Then f^* gives an isomorphism of the triangles (*)(X) and (*)(Y). In particular, $f^*:\mathbf{H}_{\mathscr{D}}(Y_{\cdot}, A(i)) \to \mathbf{H}_{\mathscr{D}}(X_{\cdot}, A(i))$ is an isomorphism. ■

Exercise 1.6.3. Determine the exact triangles

$$\cdot\cdot \to R\Gamma(X_{\mathrm{an}}, \mathbf{C}/A(i))[-1] \to \mathbf{H}_{\mathscr{D}}(X, A(i)) \to \mathbf{H}(X, F(i)) \to \cdot\cdot \qquad (**)$$

$$\cdot \to \left(\mathbf{H}_{\mathscr{DR}/F^i\mathbf{H}_{\mathscr{DR}}}\right)(X)[-1] \to \mathbf{H}_{\mathscr{D}}(X, A(i)) \to \mathbf{H}_{\mathscr{B}}(X, A(i)) \to \ldots \qquad (***)$$

Here $F^i\mathbf{H}_{\mathscr{DR}}(X)$ is the i-th term of the Hodge—Deligne filtration on $\mathbf{H}_{\mathscr{DR}}(X)$. ■

1.6.4. According to 1.2 and 1.5, we obtain a $\cup$ -product on $\mathscr{D}$ -cohomologies of schemes, simplicial schemes, and relative cohomologies. The morphisms ε_A and ε_F commute with multiplications. Further, this multiplication is consistent with the triangle (*): we have a morphism of triangles $\mathbf{H}_{\mathscr{D}}(\cdot, A(i)) \overset{L}{\otimes} (*) \to (*)$, in which the middle arrow is the $\cup$ -product and the two extreme arrows arise from the obvious structures of $\mathbf{H}_{\mathscr{B}}$ -modules on $\mathbf{H}_{\mathscr{DR}}$, $\mathbf{H}_{\mathscr{B}}$, and H_F by replacement of the rings $\varepsilon_A:\mathbf{H}_{\mathscr{D}} \to \mathbf{H}_{\mathscr{B}}$ (see 1.2.1.e).

1.6.5. In this subsection we show that $H_{\mathscr{D}}$ are cohomologies of certain complexes of sheaves in the Zariski topology. This enables us to appeal to [22] for the definition of the Chern character (see Sec. 2). Actually for our purposes the $H_{\mathscr{D}}$ already present on the category of simplicial schemes is sufficient, since, generally speaking, part 1.6.5 is not needed below.

We return to part 1.5. We note that the obvious functors of the direct and inverse image corresponding to morphisms in Π convert sheaves on the objects of Π into a Π-topology; the corresponding topology of sections we denote by Sh(Π) (see SGA4$\overline{\mathrm{V}}$bis). Its objects we call sheaves on Π. Thus, a sheaf $\mathscr{F}$ on Π is a collection of sheaves $\mathscr{F}_{(\overline{V},V)}$ on $(\overline{V}, V) \in \Pi$ and morphisms $f^*:\mathscr{F}_{(\overline{V},V)} \to f_*\mathscr{F}_{(\overline{U},U)}$ for $f:(\overline{U}, U) \to (\overline{V}, V)$ in Π satisfying the conditions $(f\circ g)^* = g^*\circ f^*$, $\mathrm{id}^* = \mathrm{id}$. For example, $A(i)_{\mathscr{D}}$ are complexes of sheaves on Π.

We equip $\mathscr{V}$ with the Zariski topology. We note that for $X \in \mathscr{V}$ the collection $\sigma^{-1}(X)$ of all regular compactifications of X forms a directed family. To the sheaf $\mathscr{F}$ on Π we assign the presheaf $X \to \varinjlim_{(\overline{X},X)\in\sigma^{-1}(X)} \Gamma^0(\overline{X}_{\mathrm{an}}, X_{\mathrm{an}}, \mathscr{F})$ on $\mathscr{V}$; we denote by $\sigma_*(\mathscr{F})$ the corresponding sheaf on $\mathscr{V}$. Thus, we have obtained a left-exact functor $\sigma_*:\mathrm{Sh}(\Pi) \to \mathrm{Sh}(\mathscr{V})$. Let $R\sigma_*$ be its right reduced functor; we set $A(i)_{\mathscr{D}\mathrm{Zar}} := R\sigma_* A(i)_{\mathscr{D}} \in \mathbf{D}^+(\mathscr{V})$. From 1.6.2 applied to an open hypercovering in the Zariski topology it follows that the canonical isomorphism $R\Gamma(X, A(i)_{\mathscr{D}\mathrm{Zar}}) = \mathbf{H}_{\mathscr{D}}(X, A(i))$ holds. Further, all other constructions also carry over to this language: it follows from 1.4 that there is the canonical commutative and associative multiplication $A(i)_{\mathscr{D}\mathrm{Zar}} \overset{L}{\otimes} A(j)_{\mathscr{D}\mathrm{Zar}} \to A(i+j)_{\mathscr{D}\mathrm{Zar}}$; the morphism $c_0:A \to A(0)_{\mathscr{D}\mathrm{Zar}}$, $c_1:\mathcal{O}^*[-1] \to A(1)_{\mathscr{D}\mathrm{Zar}}$ (see 1.5.2 and 1.7), etc. Translate the results of 1.6 and 1.7 to this language.

1.6.6. We present still another pair of properties of $\mathbf{H}_{\mathscr{D}}$. Let $A \supset \mathbf{Q}$. From (*), [16, (8.2.4)] and the Künneth formulas we obtain the following result.

LEMMA 1.6.6.1. Let X be a scheme. We assume that either $i > \min(l, \dim X)$ or $X = X' \times X''$, X' is compact, X'' is affine, and $l < 2i - \dim X''$. Then $\varepsilon_A H^l_{\mathscr{D}}(X, A(i)) = 0$. ■

Suppose now that X, Y are schemes whereby Y is connected; $y_1, y_2 \in Y(\mathbf{R})$; $\alpha \in H^l_{\mathscr{D}}(X \times Y, A(i))$; $\alpha_1, \alpha_2 \in H^l_{\mathscr{D}}(X, A(i))$ are the restrictions of α to the stalks $X \times y_1$, $X \times y_2$, respectively.

LEMMA 1.6.6.2 (on rigidity). We assume that X is compact and $l \leqslant 2i - 2$ or X is arbitrary and either $i > \dim X + 1$ or $i > l$. Then $\alpha_1 = \alpha_2$.

Proof. Replacing Y by a connected subscheme passing through y_1, y_2, it may be assumed that Y is a curve. The conditions of 1.6.6.1 are then satisfied and $\varepsilon_A(\alpha) = 0$. Therefore, from (***) it follows that α arose from a class $\tilde{\alpha} \in H^{l-1}_{\mathscr{DR}}(X \times Y)$. Since $\tilde{\alpha}_1 = \tilde{\alpha}_2$ from the rigidity of $H_{\mathscr{DR}}$, it follows that $\alpha_1 = \alpha_2$. ■

Remark 1.6.7. a. Let $k \subset \mathbf{R}$ be a subfield, let $\tilde{\Pi}_k$ be the category of pairs $(\overline{X}, X)$, where $\overline{X}$ is a smooth projective variety over k, and $X \subset \overline{X}$ is the augmentation to a divisor with normal intersections. For $(\overline{X}, X) \in \tilde{\Pi}_k$ we can define a complex $A(i)_{\mathscr{D}}$ on $(\overline{X}_{\mathrm{Zar}}, (X \underset{k}{\otimes} \mathbf{R}))_{\mathrm{an}}$, by replacing in 1.5 the holomorphic complex $F^i\Omega_{(X\otimes\mathbf{R}, X\otimes\mathbf{R})_{\mathrm{an}}}$ by the algebraic complex $F^i\Omega^{\cdot}_{(X,\overline{X})/k}$. Exactly as above, we obtain a cohomology theory $\mathbf{H}_{\mathscr{D}}(X/k)$ for smooth schemes over k included in the direct triangle $\ldots\to\mathbf{H}_{\mathcal{D}\mathcal{R}}(X/k)\underset{k}{\otimes}\mathbf{R}[-1]\to\mathbf{H}_{\mathcal{D}}(X/k, A(i))\to\mathbf{H}_{\mathcal{B}}(X\otimes\mathbf{R}, A(i))\oplus F^i\mathbf{H}_{\mathcal{D}\mathcal{R}}(X/k)\to\ldots$. If k = $\mathbf{R}$, then by GAGA we have $\mathbf{H}_{\mathcal{D}}(X/k)=\mathbf{H}_{\mathcal{D}}(X)$. All results of this and the next section carry over to this situation.

b. Using [16, 29], it is possible to define $\mathcal{D}$ -cohomologies on a category of schemes over $\mathbf{R}$ with any singularities by requiring that Lemma 1.6.2 be satisfied for them. However, for special schemes it would be nice to have a finer theory. For example, it is desirable that $H^j_{?}(X, \mathbf{Z}(1))=H^{j-1}(X, \mathcal{O}^*_X)$. In this work we shall not be interested in singularities with the exception of the appendix to Sec. 2.

1.7. Chern Classes of Vector Bundles. According to 1.5.2, for $X\in\mathcal{V}$ there is defined a canonical morphism $\mathcal{O}^*(X)\to H^1(X, A(1))$ (for A = $\mathbf{Z}$ this is an isomorphism). For a simplicial scheme X there is defined a morphism of cosimplicial groups $\mathcal{O}^*(X_.)\to H^1(\mathbf{H}_{\mathcal{D}}(X_., A(1))^{\Delta})$ (see part 1.1) whence (by 1.5.2) we obtain $\Gamma(X_., \mathcal{O}^*)[-1]\to\mathbf{H}_{\mathcal{D}}(X_., A(1))$. Since cohomologies can be computed by means of hypercoverings, from 1.6.2 it follows that this morphism extends in a unique manner to a functorial morphism $c_1{:}R\Gamma(X_., \mathcal{O}^*)[-1]\to\mathbf{H}_{\mathcal{D}}(X_., A(1))$. The morphism of exact triangles $R\Gamma(f, \mathcal{O}^*)[-1]\to H_{\mathcal{D}}(f, A(1))$ in the relative situation is defined analogously. All these morphisms arise from the morphism $c_1{:}\mathcal{O}^*[-1]\to A(1)_{\mathcal{D}\mathrm{Zar}}$ (see 1.6.5).

In particular, for any invertible sheaf $\mathcal{L}\in H^1(X_., \mathcal{O}^*)$ we obtain its Chern class $c_1(\mathcal{L})\in H^2_{\mathcal{D}}(X_., A(1))$.

Exercise 1.7.1. Show that $\varepsilon_A c_1(\mathcal{L})$ is the usual Chern class in Betti cohomology. Show that $c_1{:}H^1(X, \mathcal{O}^*)\to H^2_{\mathcal{D}}(X, \mathbf{Z}(1))$ is an imbedding, and if X is compact it is an isomorphism.

We shall show that the usual theorem on cohomologies of projective space holds for $\mathbf{H}_{\mathcal{D}}$. Namely, let E be an n-dimensional bundle over X, let π:P(E) → X be its projectivization, and let $\mathcal{J}(1)$ be the standard invertible sheaf on P(E).

Assertion 1.7.2. The mapping $\oplus c_1(\mathcal{O}(1))^j\cup\pi^*:\underset{0<j<n-1}{\oplus}\mathbf{H}_{\mathcal{D}}(X, A(i-j))[2j]\to\mathbf{H}_{\mathcal{D}}(\mathbf{P}(E), A(i))$ is an isomorphism.

Proof. It is necessary to use the triangle (*) and the consistency with it of the ∪ -product (6.2); according to 7.3 the morphism of 7.4 is an isomorphism on the second extreme terms of the triangle (by the usual theorem on cohomologies of projective space for $\mathbf{H}_{\mathcal{D}\mathcal{R}}$ and $\mathbf{H}_{\mathcal{B}}$ and by [15] for $\mathbf{H}_F$).

Proceeding from 1.7.2 a theory of Chern classes of vector bundles satisfying the usual formalism is constructed in the manner of Grothendieck. In particular, we construct the Chern character which is a morphism of rings ch: $k_0(X)-\oplus H^{2i}(X, A\otimes\mathbf{Q}(i))$. From 1.7.1 it follows that $\varepsilon_A c_i$ are the usual Chern classes in Betti cohomologies. We note that this fact uniquely determines c_i. Namely, we have the following result.

LEMMA 1.7.3. There exists a unique manner of assigning to each vector bundle E over $X_.\in\mathcal{V}_{\Delta}$ the class $c_i(E)\in H^{2i}_{\mathcal{D}}(X_., A(i))$ so that the following conditions are satisfied:

a. For any morphism f:Y. → X. and E over X. we have $f^*c_i(E) = c_if^*(E)$.

b. $\varepsilon_A c_i(E)\in H^{2i}_{\mathcal{B}}(X_., A(i))$ is the usual Chern class in Betti cohomologies.

Proof. Let G = GL_n. We consider the classifying space of G — the simplicial scheme B_G (see, for example, [16]): for any $X_.\in\mathcal{V}$ the morphisms from X into B_G are precisely the isomorphism classes of n-dimensional vector bundles on X. trivialized on X_0. Let E_{un} be the universal n-dimensional bundle on $B_{G.}$. The cohomologies of $B_{G.}$ and the Hodge structure on them are known: $H^{\cdot}(B_{G}.(\mathbf{C}), A)=\mathbf{Z}[c_1,\ldots,c_n]$, where $c_i = c_i(E_{un})$ have the pure weight (i, i). It is evident from (*) that $\varepsilon_A{:}H^{2i}_{\mathcal{D}}(B_{G}., A(i))\to H^{2i}_{\mathcal{B}}(B_{G}., A(i))$ is an isomorphism; therefore, condition b determines the Chern class of the universal bundle. From this it follows immediately that uniqueness holds: let E be any bundle over X.; we choose a hypercovering u: $\tilde{X}.$ → X. so that u*E is trivial on $\tilde{X}_0$; then u*E = f^*E_{un} for some f:$\tilde{X}.$ → $B_{G.}$. Since $u^*{:}\mathbf{H}_{\mathcal{D}}(X_.)\to\mathbf{H}_{\mathcal{D}}(\tilde{X}_.)$ is an isomorphism, c_iE are determined by condition a. The direct proof of existence and derivation of the product formula from the product formula for ordinary Chern classes are left to the reader. ■

1.8. Homologies. In this subsection we construct a homology theory corresponding to $\mathcal{D}$ -cohomology. For a smooth analytic space V let $\Omega^{p,q}_{V\infty}$ be the sheaf of (p, q)-forms of class C^{∞} on V, and let $\Omega'^{p,q}_{V\infty}$ be the sheaf of distributions over $\Omega^{-p,-q}_{V\infty}$. These sheaves form bicomplexes; let $\Omega^{\cdot}_{V\infty}$, $\Omega'^{\cdot}_{V\infty}$ be the corresponding simple complexes. We filter them with the foolish filtration according to $p: F^i\Omega^{(\cdot)}_{V\infty} = s\Omega^{(\cdot)>i}_{V\infty}$. The complex $\Omega^{\cdot}_{V\infty}$ is a filtered dg-algebra; the complex $\Omega^{'\cdot}_{V\infty}$ is a filtered module over $\Omega^{\cdot}_{V\infty}$. The natural imbedding $(\Omega^{\cdot}_V, F^i) \hookrightarrow (\Omega^{\cdot}_{V_\infty}, F^i) \hookrightarrow (\Omega'^{\cdot} \times [-2\dim V], F^{i+\dim V})$ is a filtered quasiisomorphism.

We define the filtered complex $(\Omega'^{\cdot}(V), F^i\Omega'(V)) := \Gamma_c(V, (\Omega'_{V\infty}, F^i\Omega'_{V\infty}))$ – the complex of sections of Ω' with compact support. We denote by $C'^i(V, A(j))$ the group of singular i-chains of class C^∞ on V with coefficients in the local system A(j). The groups $C'^{\cdot}(V, A(j))$ form a complex and integration over chains gives a morphism $C'^{\cdot}(V, A(j)) \to \Omega'^{\cdot}(V)$. The complexes $C'^{\cdot}(V, A(j))$ and $(\Omega'^{\cdot}(V), F^i\Omega'^{\cdot}(V))$ are covariant functors of V, and the morphism between them is a morphism of functors.

We now include logarithmic singularities. For $(\bar{V}, V) \in \Pi$ (see part 1.5) let $\Omega^{(\cdot)\cdot}_{(V,\bar{V})\infty} := \Omega^{\cdot}_{(V,\bar{V})} \otimes_{\Omega_{\bar{V}}} \Omega^{(\cdot)\cdot}_{\bar{V}\infty}$, $F^i\Omega^{(\cdot)}_{V\infty} := \Omega^{\cdot}_{(V,\bar{V})} \otimes_{\Omega_{\bar{V}}} F^i\Omega^{(\cdot)}_{\bar{V}\infty}$. We set $C'^{\cdot}(V, \bar{V}, A(i)) := C'^{\cdot}(\bar{V}, A(i)) / C'^{\cdot}(\bar{V} \setminus V, A(i))$ – the complex of relative singular chains, $\Omega'^{\cdot}(V, \bar{V}) := \Gamma_c(\bar{V}, \Omega^{\cdot}_{(V,\bar{V})\infty})$, $F^i\Omega'^{\cdot}(V, \bar{V}) := \Gamma_c(\bar{V}, F^i\Omega^{\cdot}_{(V,\bar{V})\infty})$. Let Π_* be a subcategory of Π, $\mathrm{Ob}\,\Pi_* = \mathrm{Ob}\,\Pi$, containing those morphisms $f: (\bar{V}, V) \to (\bar{U}, U)$ for which $f(\bar{V} \setminus V) \subset \bar{U} \setminus U$. It is easy to see that $C'^{\cdot}$ and $(\Omega'^{\cdot}, F^i\Omega'^{\cdot})$ are covariant functors on Π_*, and the arrow of the preceding paragraph defines a functorial morphism $C'^{\cdot}(V, \bar{V}, A(i)) \to \Omega'^{\cdot}(V, \bar{V})$.

We set $C'_{\mathscr{D}}(V, \bar{V}, A(i)) := \mathrm{Cone}(F^i\Omega'^{\cdot}(V, \bar{V}) \oplus C'^{\cdot}(V, \bar{V}, A(i)) \to \Omega'^{\cdot}(V, \bar{V}))$, where the arrow is the difference of the canonical imbeddings. This is a covariant functor on Π_*; we have the functorial exact triangle

$$(*)'(V, \bar{V}) \ldots \to \Omega'^{\cdot}(V, \bar{V})[-1] \to C'_{\mathscr{D}}(V, \bar{V}, A(i)) \to F^i\Omega'(V, \bar{V}) \oplus C'^{\cdot}(V, \bar{V}, A(i)) \to \ldots$$

As in part 1, we can define $C'_{\mathscr{D}}$-complexes of any diagram in Π_*; in particular, homologies of simplicial objects and relative homologies together with the corresponding spectral and exact sequences are defined.

We proceed to algebraic varieties. Let Sch_* be the category of schemes of finite type over **R** and of proper morphisms; let $\mathscr{V}_* \subset \mathrm{Sch}_*$ be the complete subcategory of smooth quasi-projective schemes; let $\tilde{\Pi}_* := \tilde{\Pi} \cap \Pi_*$ (see part 6), and let $\sigma_* := \sigma|_{\tilde{\Pi}}: \tilde{\Pi}_* \to \mathscr{V}_*$.

LEMMA 1.8.1. We assume that $f: (\bar{V}, V)_\cdot \to (\bar{U}, U)_\cdot$ in $\tilde{\Pi}_{*\Delta}$ induces an isomorphism between the Borel–Moore homologies $V_{\cdot an}$ and $U_{\cdot an}$. Then $f_*: (*)'(V, \bar{V}) \to (*)'(U, \bar{U})$ is a quasiisomorphism.

Proof. Indeed, the extreme terms of the triangle $(*)'(V, \bar{V})$ are the de Rham homologies of the scheme V, the i-th term of the Hodge–Deligne filtration on the de Rham homologies, and the singular Borel–Moore homologies of the scheme V. The lemma follows from the strict consistency of f_* with the Hodge filtration and the exactness of the triangle $(*)'$. ■

From [29] we obtain the following result.

LEMMA 1.8.2. Let C be a category. We call a functor $F: \mathrm{Sch}_{*\Delta} \to C$ $(G: \tilde{\Pi}_{*\Delta} \to C)$ topological if for any $f: X_\cdot \to Y_\cdot$ [respectively, $g: (\bar{X}, X)_\cdot \to (\bar{Y}, Y)_\cdot$] inducing an isomorphism between the Borel–Moore homologies $X_{\cdot an}$ and $Y_{\cdot an}$ the morphism F(f)(G(g)) is an isomorphism. Then the functor $F \rightsquigarrow F \cdot \sigma_*$ realizes an equivalence between the categories of topological functors on $\mathrm{Sch}_{*\Delta}$ and $\tilde{\Pi}_{*\Delta}$. ■

We apply 1.8.2 to the situation of 1.8.1. We obtain an exact triangle of functors on $\mathrm{Sch}_{*\Delta}$ with values in $\mathbf{D}^+$(A-mod):

$$\ldots \to \mathbf{H}'_{\mathscr{DR}}(X_\cdot) \to \mathbf{H}'_{\mathscr{D}}(X_\cdot, A(i)) \to F^i\mathbf{H}'_{\mathscr{DR}}(X_\cdot) \oplus \mathbf{H}'_{\mathscr{B}}(X_\cdot, A(i)) \to \ldots \qquad (*')$$

Here $\mathbf{H}'_{\mathscr{DR}}$ are the de Rham homologies, $F^i\mathbf{H}'_{\mathscr{DR}}$ is the i-th term of the Hodge–Deligne filtration on them, $\mathbf{H}'_{\mathscr{B}}(X_\cdot, A(i))$ are the Borel–Moore homologies with coefficients in A(i), and $\mathbf{H}'_{\mathscr{D}}(X_\cdot, A(i))$ are the desired $\mathscr{D}$-homologies. In particular, we have defined $\mathscr{D}$-homologies of any schemes (as constant simplicial schemes). The relative $\mathscr{D}$-homologies connected with the proper morphism $f: Y_\cdot \to X_\cdot$ are defined in exactly the same way: it is necessary to choose a morphism $\tilde{f}: (\bar{V}, V) \to (\bar{U}, U)$ in $\tilde{\Pi}_*$ together with a commutative diagram of proper morphisms

$$\begin{tikzcd} \sigma_*(\bar{V}, V)_\cdot \arrow[r] \arrow[d] & \sigma_*(\bar{U}, U)_\cdot \arrow[d] \\ Y_\cdot \arrow[r] & X_\cdot \end{tikzcd},$$

in which the vertical arrows induce an isomorphism in the homologies; then $\mathbf{H}'_{\mathcal{D}}(f, A(i)) :=$ Cone $(\tilde{f}_*: C'_{\mathcal{D}}(V, \bar{V}, A(i)) \to C'_{\mathcal{D}}(U, \bar{U}, A(i))$; we have the "vrai" exact triangle (in the sense of Deligne)

$$\ldots \to \mathbf{H}'_{\mathcal{D}}(V, A(i)) \to \mathbf{H}'_{\mathcal{D}}(U, A(i)) \to \mathbf{H}'_{\mathcal{D}}(f, A(i)) \to \ldots,$$

which does not depend on the arbitrariness in the choice of $\tilde{f}$ and is functorial in f.

Exercise 1.8.3. Define the spectral sequence corresponding to the simplicial scheme. [Here it is necessary to define a functor $(\mathbf{H}'_{\mathcal{D}\Delta}: \mathrm{Sch}_{*\Delta} \to \mathbf{D}^+(A\text{-mod}_\Delta)$; for this it is necessary to consider bisimplicial schemes.]

LEMMA 1.8.4. For any closed imbedding $f: Y \hookrightarrow X$ there is the canonical functorial morphism

$$\mathbf{H}'_{\mathcal{D}}(f, A(i)) = \mathbf{H}'_{\mathcal{D}}(Y \setminus X, A(i)).$$

Proof. We assume that X is smooth and Y is a divisor with normal intersections. We compactify X so that $(\bar{X} \setminus X) \cup Y$ is a divisor with normal intersections on $\bar{X}$; let $\bar{Y}$ be the closure of Y in $\bar{X}$. If we compute the homologies of Y by means of the usual "simplicial resolution" of Y (i.e., taking for $\bar{V}_. \supset V \to Y$ the coskeleton of the normalization of $\bar{Y}$), then $\tilde{f}_*: sNC'_{\mathcal{D}}(V, \bar{V}, A(i)) \to C'_{\mathcal{D}}(X, \bar{X}, A(i))$ is an imbedding, and the natural mapping of the factor into $C'_{\mathcal{D}}(X \setminus Y, \bar{X}, A(i))$ is a homotopy equivalence (more precisely, an isomorphism on Ω' and $F^i\Omega'$ and a homotopy equivalence on $C'_{\mathcal{B}}$). In the smooth case everything has been proved. If (X, Y) is arbitrary, then it is necessary to choose a smooth proper simplicial scheme $\bar{U}.$, an open subscheme $U_. \subset \bar{U}_.$ of it, and a proper morphism $\pi: U_. \to X$ so that $\pi^{-1}(Y) \cup (\bar{U}_. \setminus U_.)$ is a divisor with normal intersections on $\bar{U}.$ and the condition of cohomological descent is satisfied (surjectiveness of $U_n \to \mathrm{Cosk}_n\,\mathrm{sk}_{n-1} U_./X$). Thus, everything reduces to a simplicial version of the smooth situation. ■

LEMMA 1.8.5 (Poincaré duality). a) Let X be a smooth scheme, dim X = n. There is the canonical isomorphism $\mathbf{H}'_{\mathcal{D}}(X, A(i)) = \mathbf{H}_{\mathcal{D}}(X, A(i+n))[2n]$.

b) Let X be as in a), let $f: Y \hookrightarrow X$ be a closed imbedding, and let $j: U := X \setminus Y \hookrightarrow X$. There is a canonical isomorphism in **D**(A-mod) of exact triangles of the relative homologies and cohomologies

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbf{H}'_{\mathcal{D}}(Y, A(i)) \arrow[r] & \mathbf{H}'_{\mathcal{D}}(X, A(i)) \arrow[r] & {} \\
\cdots \arrow[r] & \mathbf{H}_{\mathcal{D}}(j, A(i+n))[2n] \arrow[r] \arrow[u, "\wr"] & \mathbf{H}_{\mathcal{D}}(X, A(i+n))[2n] \arrow[r] \arrow[u, "\wr"] & {} \\
{} \arrow[r] & \mathbf{H}'_{\mathcal{D}}(U, A(i)) \arrow[r, equal, "(1.8.4)"] & \mathbf{H}'_{\mathcal{D}}(f, A(i)) \arrow[r] & \cdots \\
{} \arrow[r] & \mathbf{H}_{\mathcal{D}}(U, A(i+n)[2n] \arrow[r] \arrow[u, "\wr"] & \cdots,
\end{tikzcd}$$

in which the two right vertical isomorphisms come from a). If $Y_1 \hookrightarrow Y_2 \hookrightarrow X$, then these isomorphisms are consistent with the morphisms of triangles corresponding to (X, Y_1) and (X, Y_2).

Proof. a) Let $(\bar{U}, U) \subset \tilde{\Pi}_*$, $\sigma_*(\bar{U}, U) = X$, $j: U \hookrightarrow \bar{U}_.$. We consider the complex of sheaves $\tilde{C}'_{(U,\bar{U})}(A(i))$ on $\bar{U}$ corresponding to the complex of presheaves $V \to C'(\bar{U}, A(i))/C'(\bar{U} \setminus (U \cap V), A(i))$. It is clear that $\tilde{C}'_{(U,\bar{U})}(A(i)) = j_* j^* \tilde{C}_{(U,\bar{U})}(A(i))$ and $j^*\tilde{C}'_{(U,\bar{U})}(A(i))$ is the flabby resolvent of the sheaf A(i + n)[2n] on U; moreover, the morphism $C'(U, \bar{U}, A(i)) \to \Gamma(\bar{U}, \tilde{C}'_{(U,\bar{U})}(A(i)))$ is a quasi-isomorphism. Further, the obvious imbedding $(\Omega_{(U,\bar{U})}, F^i) \hookrightarrow (\Omega''_{(U,\bar{U})\infty}[-2n], F^{i+n}\Omega'[-2n])$ is a filtered quasiisomorphism, and all $F^i\Omega'/F^{i+1}\Omega'$ are soft; therefore, $R\Gamma(\Omega_{(U,\bar{U})}, F^i) = \Gamma(\bar{U}, (\Omega'_{(U,\bar{U})\infty}, F^{i+n}\Omega')) \times$ [−2n]. We combine these two remarks into a single $\mathcal{D}$ -complex: we set $\tilde{C}'_{\mathcal{D}}(A(i))_{(U,\bar{U})} = \mathrm{Cone}(F^i\Omega' \oplus$ $C'(A(i)) \to \Omega')[-1]$; then

$$R\Gamma(\bar{U}, U, A(i)_{\mathcal{D}}) = \Gamma(\bar{U}, \tilde{C}'_{\mathcal{D}}(i-n))[-2n] \rightleftarrows C'_{\mathcal{D}}(U, \bar{U}, A(i-n))[-2n].$$

To prove a) it remains to verify that this quasiisomorphism does not depend on the choice of the compactification; this is an exercise for the reader.

b) We remark that it suffices to construct a morphism of triangles; it will be a quasi-isomorphism by a). If V is a divisor with normal intersections, then it is obtained by combining a) and the beginning of the proof of 1.8.4. If V is arbitrary, then it is necessary to proceed as at the end of 1.8.4. We obtain a simplicial scheme $\pi: (\bar{U}, U) \to (\bar{X}, X)$ and the desired morphism of triangles arises from the morphisms

$$A(i)_{\mathscr{D}(X,\overline{X})_{\mathrm{an}}}\to\pi_*\widetilde{C}'_{\mathscr{D}}(A(i))_{(U,\overline{U})_{\mathrm{an}}}$$

and

$$A(i)_{\mathscr{D}(U_0\setminus\pi_0^{-1}Y,\overline{U}_0)_{\mathrm{an}}}\to\widetilde{C}'_{\mathscr{D}}(A(i))\ (U_0\setminus\pi_0^{-1}Y,\overline{U}_0)_{\mathrm{an}}.$$

Exercise 1.8.6. a) Construct the exterior multiplication $\mathbf{H}'_{\mathscr{D}}(X)\overset{L}{\otimes}\mathbf{H}'_{\mathscr{D}}(Y)\to\mathbf{H}'_{\mathscr{D}}(X\times Y)$ and show that it is consistent with the morphisms of the direct image and the Poincaré duality.

b) We consider the category of schemes over a smooth scheme X and of proper morphisms. For any Y/X we define a $\cap$ -product $\mathbf{H}_{\mathscr{D}}(X,\ A(i))\overset{L}{\otimes}\mathbf{H}'_{\mathscr{D}}(Y,\ A(j))\to\mathbf{H}'_{\mathscr{D}}(Y,\ A(j+i))$, consistent with the morphisms of direct image (use 1.2.3). Show that under the isomorphism of the Poincaré duality the $\cap$ -product goes over into the $\cup$ -product on the relative cohomologies.

1.9. Cycles. Let Y be an irreducible scheme of dimension n. Then $\varepsilon_A:H'^{-2n}_{\mathscr{D}}(Y,\ A(-n))\to H'^{-2n}_{\mathscr{B}}(Y,\ A(-n))=A$ is an isomorphism; let $\mathrm{cl}_{\mathscr{D}}Y\in H'^{-2n}_{\mathscr{D}}(Y,\ A(-n))$ correspond to $1\in A$. From this for any scheme X there arises a mapping $\mathrm{cl}_{\mathscr{D}}:Z_n(X)\to H'^{-2n}_{\mathscr{D}}(X,\ A(-n))$, where $Z_n(X)$ is the group of cycles of dimension n on X; $\mathrm{cl}_{\mathscr{D}}([Y]):=i_*(\mathrm{cl}_{\mathscr{D}}Y)$ for an irreducible subscheme $i:Y\hookrightarrow X$. It is clear that $\varepsilon_A\mathrm{cl}_{\mathscr{D}}=\mathrm{cl}_{\mathscr{B}}$ is the usual mapping assigning to a cycle its class of Betti homologies. If X is smooth, then, passing to cohomologies by Poincaré duality, we obtain a mapping $\mathrm{cl}_{\mathscr{D}}:Z^n(X)\to H^{2n}_{\mathscr{D}}(X,A(n))$, where $Z^n(X)$ are cycles of codimension n.

We now suppose that X is smooth and compact and A = **Z**. From the triangle (*) we then have the exact sequence

$$0\to\mathscr{J}^n(X)\to H^{2n}_{\mathscr{D}}(X,\ \mathbf{Z}(n))\to\mathrm{Hdg}^n(X)\to 0,$$

where $\mathscr{J}^n(X)=H^{2n-1}(X_{\mathrm{an}},\ \mathbf{C})/H^{2n-1}(X_{\mathrm{an}},\ \mathbf{Z}(n))+F^nH^{2n-1}(X_{\mathrm{an}},\ \mathbf{C})$ is the n-th Jacobian of Griffiths, and $\mathrm{Hdg}^n(X)$ is the Hodge group of integral cycles of type (n, n).

LEMMA 1.9.1. Let $Y\in Z_n(X)$. If $\mathrm{cl}_{\mathscr{B}}(Y)\in H'^{-2n}_{\mathscr{B}}(X,\mathbf{Z}(-n))$ is equal to 0, then $\mathrm{cl}_{\mathscr{D}}(Y)$ coincides with the Abel–Jacobi–Griffiths periods of the cycle Y.

Proof. We consider the integral chain $i_*\mathrm{cl}_{\mathscr{B}}Y$. Integration over Y (or over $i_*\mathrm{cl}_{\mathscr{B}}Y$) gives a distribution $\mathrm{cl}(Y)\in F^{-n}\Omega'^{-2n}(X)$. Then $\mathrm{cl}_{\mathscr{D}}(Y)$ is the homology class of the cycle $(\mathrm{cl}_F(Y),\ i_*\mathrm{cl}_{\mathscr{B}}Y,\ 0)\in C'^{-2n}_{\mathscr{D}}(X,\ \mathbf{Z}(-n))$. If $\mathrm{cl}_{\mathscr{B}}(Y)$ is homologous to zero in X, then we choose $s\in C'^{-2n-1}(X,\ \mathbf{Z}(-n))$ so that $\mathrm{cl}\,s=i_*\mathrm{cl}_{\mathscr{B}}Y$. We subtract from $(\mathrm{cl}_F(Y),\ i_*\mathrm{cl}_{\mathscr{B}}Y,\ 0)$ the boundary (0, s, 0); we find that $\mathrm{cl}_{\mathscr{D}}(Y)$ is homologous to $(\mathrm{cl}_F(Y),\ 0,\ s)$. This is precisely the definition of the periods of the cycle Y: to compute them it is necessary to span Y by a film s and consider integrals of smooth forms over s.

1.9.2. Finally, we note that the mapping $\mathrm{cl}_{\mathscr{D}}$ commutes with the outer direct product; from 1.7 and 1.8.5b it follows that for a divisor Y on a smooth scheme X its class $\mathrm{cl}_{\mathscr{D}}(Y)$ coincides with $c_1(\mathcal{O}(Y))$.

1.10. The Hodge $\mathscr{D}$ -Conjecture. Let X be a smooth, compact scheme, and let a and b be two integers. The following conjecture is a special case of Conjecture 3.10.

Conjecture 1.10.1. If $a\leqslant 2\mathrm{b}$, then there is a closed subscheme $i:Y\hookrightarrow X$ such that dim × Y ⩽ b − a and the factor of $H'^{a}_{\mathscr{D}}(X,\ \mathbf{Z}(b))$ modulo the closure of the subgroup $i_*H'^{a}_{\mathscr{D}}(Y,\ \mathbf{Z}(b))$ is compact.

We clarify the meaning of the conjecture. We set

$$\mathscr{J}_{a,b}:=H'^{a-1}_{\mathscr{D}\mathscr{R}}(X)/H'^{a-1}_{\mathscr{B}}(X,\ \mathbf{Z}(b))+F^bH'^{a-1}_{\mathscr{D}\mathscr{R}}(X),$$
$$\mathrm{Hdg}_{ab}:=H'^{a}_{\mathscr{B}}(X,\mathbf{Z}(b))\cap F^bH'^{a}_{\mathscr{D}\mathscr{R}}(X).$$

We have the exact sequence

$$0\to\mathscr{J}_{\mathrm{ab}}\to H'^{a}_{\mathscr{D}}(X,\mathbf{Z}(b))\to\mathrm{Hdg}_{\mathrm{ab}}\to 0.$$

It follows from the condition $a\leqslant 2\mathrm{b}$ that $H'^{a-1}_{\mathscr{B}}(X,\ \mathbf{R}(b))\cap F^bH'^{a-1}_{\mathscr{D}\mathscr{R}}(X)=0$; therefore, $H'^{a}_{\mathscr{D}}(X,\ \mathbf{Z}(b))$ is a separable topological group with connected component of the identity $\mathscr{J}_{\mathrm{ab}}$.

If $a \neq 2b$, then $\mathcal{J}_{ab}=H'^{a}_{\mathcal{D}}(X,\mathbf{Z}(b))$; if $a = 2b$, then $\mathcal{J}_{ab}$ is compact and 1.10.1 coincides with the usual Hodge conjecture. If $b > 0$, then 1.10.1 follows from the easy Lefschetz theorem. We now suppose that $a < 2b \leqslant 0$. Then the factor of $\mathcal{J}_{ab}$ by the maximal compact subgroup — the image of $H'^{a-1}_{\mathcal{B}}(X,\mathbf{R}(b))$ — coincides with $H'^{a}_{\mathcal{D}}(X,\mathbf{R}(b))$; therefore, in 1.10.1 it is possible to replace $H'^{a}_{\mathcal{D}}(X,\mathbf{Z}(b))$ by $H'^{a}_{\mathcal{D}}(X,\mathbf{R}(b))$. We note that the natural mappings

$$H'^{a-1}_{\mathcal{B}}(X,\mathbf{R}(b-1))\cap F^{a-b}H'^{a-1}_{\mathcal{D}\mathcal{R}}(X)\to H'^{a-1}_{\mathcal{B}}(X,\mathbf{R}(b-1))/\pi_{a-1}F^{b}H'^{a-1}_{\mathcal{D}\mathcal{R}}(X)\to H^{a}_{\mathcal{D}}(X,\mathbf{R}(b))$$

are isomorphisms, so that 1.10.1 means that any (p, q)-cocycle with $p, q \geqslant j$ is a linear combination of cycles coming in the $\mathcal{D}$-sense from a subscheme of codimension j; cf. the usual Hodge conjecture. For example, we consider the case $a = 2b - 1$. Then the conjecture asserts that any cocycle in $H'^{a-1}_{\mathcal{B}}(X,\mathbf{R}(b-1))$ is homologous to a linear combination of distributions of the form l_φ; here $\varphi=\Sigma\varphi_\eta\in\oplus\mathcal{O}^*(\eta)$ is a collection of functions on general points of $(1 - b)$-dimensional irreducible subschemes of X such that the b-dimensional cycle $\Sigma\,\mathrm{div}\,\varphi_\eta$ is equal to 0; l_φ is a closed distribution on X such that $(1 - b, 1 - b)$-forms ω of class C^∞ on X we have $\int l_\varphi\cdot\omega=\sum_\eta\int_\eta\omega\cdot\log|\varphi_\eta|$. We remark that here the singularities of the support of φ are very crucial: if it is smooth, then all $\varphi_\eta=\mathrm{const}$, and we obtain an ordinary algebraic $(1 - b, 1 - b)$-cycle.

Finally, we note that the following assertion ensues from 1.10.1: for any integers i, $j \geqslant 0$ there is a closed subscheme $Y\subset X$, $\mathrm{codim}\,Y=i-j+1$ such that the image of $H^{i}_{\mathcal{D}\mathcal{R}}(X)$ in $H^{i}_{\mathcal{D}\mathcal{R}}(X\setminus Y)$ is contained in the sum of $F^{j}H_{\mathcal{D}\mathcal{R}}(X\setminus Y)$ and the complex conjugate subspace (use the exact sequence of relative $\mathcal{D}$-cohomologies with coefficients in **R**).

I do not know how to prove this even in the case of surfaces: a nontrivial example is the product of two modular curves treated in Sec. 6.

2. Regulators

2.1. Quillen's K-Theory. It is assumed that the reader is familiar with the basic concepts of K-theory; in this subsection we shall make only some general remarks.

2.1.1. Quillen's K-functor $\mathbf{K}$ is a contravariant functor from the category of schemes to the category of fibrant spectra. It can be extended in the usual way to a functor on any diagrams of schemes: for the diagram I we have $\mathbf{K}(I):=\underleftarrow{\mathrm{holim}}\,\mathbf{K}(X_i)$. In particular, K-functors of simplicial schemes are defined: $\mathbf{K}(X_.):=\underleftarrow{\mathrm{holim}}\,\mathbf{K}(X_i)$, and for a morphism $f:X. \to Y.$ of simplicial schemes there is defined the relative K-functor $\mathbf{K}(f) = \mathbf{K}(Y., X.)$ together with the exact triangle $\ldots\to\mathbf{K}(f)\to\mathbf{K}(Y_.)\xrightarrow{f^*}\mathbf{K}(X_.)\to\ldots$ in the homotopy category of spectra. K-groups are defined as the homotopy groups $\mathbf{K}:K_i(X) =\pi_i\mathbf{K}(X)$; if X is a scheme, then $K_i(X) = 0$ for $i < 0$; for a simplicial scheme X. we have the spectral sequence $E_r^{p,q}$ converging to $K_{-p-q}(X.)$ with $E_2^{p,q} = H^pK_{-q}(X_p)$. Further, there is a multiplication on the K-functor: there is a natural pairing of spectra $\{,\}:\mathbf{K}(X)\wedge\mathbf{K}(X) \to \mathbf{K}(X)$. It defines a multiplication $\{,\}$ on the K-groups of simplicial schemes and a multiplication $\{\ ,\ \}:K.(Y)\otimes K.(Y,\ X)\to K.(Y,\ X)$ in the relative situation; this multiplication is (gradedly) commutative.

2.1.2. I shall not recall the construction of the spectra $\mathbf{K}(X)$; below we shall need only one fact: if X is affine, then we have the canonical weak equivalence $K_0(X) \times \mathbf{Z}_\infty(B_G \times (X)) \to (\mathbf{K}(X))_0$. Here $K_0(X)$ is the (discrete) Grothendieck group of vector bundles on X, $B_G.$ is the standard simplicial classifying space of the group $G=\underrightarrow{\lim}\,GL_n$, $\mathbf{Z}_\infty$ is the Kan–Bousfield functor [14], and $(\mathbf{K}(X))_0$ is the null space of the spectrum $\mathbf{K}(X)$. From this for an affine simplicial scheme X we have $K_j(X_0)=\pi_j\,\mathrm{holim}\,\mathbf{Z}_\infty[B_{G}.(X_.)]$ for $j > 0$.

2.1.3. There is still another homological K-functor $\mathbf{K}'(X)$ from the category of schemes to the category of spectra which is contravariant relative to flat morphisms and covariant at least up to homotopy relative to proper morphisms.

For closed imbeddings $i:X\hookrightarrow Y$ the direct image i_* is a genuine morphism of spectra (and not only a homotopy class) and gives a canonical exact triangle of localization $\mathbf{K}'(X)\xrightarrow{i_*}\mathbf{K}'(Y)\xrightarrow{j^*}\mathbf{K}'(Y\setminus X)$ in the homotopy category of spectra (here $j:Y\setminus X\hookrightarrow Y$ is an open imbedding).

Remark. Actually the direct image relative to proper morphisms is also more or less a genuine morphism of spectra; in particular, K'-functors of diagrams of proper morphisms are defined (see [23]).

The homotopy groups of $\mathbf{K}'$ are the K'-groups of Quillen. Further K'(X) is equipped with a natural structure of a K(X)-module: there is a pairing $\mathbf{K}(X) \wedge \mathbf{K}'(X) \to \mathbf{K}'(X)$ which is a mapping of covariant functors relative to flat morphisms and satisfying the projection formula relative to proper morphisms.

There is the canonical morphism $\mathbf{K}(X) \to \mathbf{K}'(X)$. If X is regular, then this is an isomorphism (Poincaré duality); therefore, for regular X and a closed imbedding $Y \hookrightarrow X$ the exact triangle of localization gives an isomorphism $i_*: \mathbf{K}'(Y) \to \mathbf{K}(X, X \setminus Y)$ in the homotopy category. In particular, if Y is also regular there arises an exact triangle of localization $\mathbf{K} \times (Y) \xrightarrow{i_*} \mathbf{K}(X) \to \mathbf{K}(X \setminus Y)$ in K-theory. In Sec. 7 we require a version of it for relative K-functors.

LEMMA 2.1.4. Let $i: Y \hookrightarrow X$ be a closed imbedding, and let $S. \to X$ be a simplicial scheme over X. We set $j: U := X \setminus Y \hookrightarrow X$, $S_Y. := Y \underset{X}{\times} S.$, $S_U. := U \underset{X}{\times} S.$. We assume that all designated schemes are regular, and all morphisms $S_k \to X$ are Tor-independent of i. Then there is the natural exact triangle of localization

$$\mathbf{K}(Y, S_{Y}.) \xrightarrow{i_*} \mathbf{K}(X, S.) \xrightarrow{j^*} \mathbf{K}(U, S_{U}.).$$

Proof. From Quillen's theorem on the resolvent [29] it follows that $\mathbf{K}(X)$, $\mathbf{K}(S_k)$, $\mathbf{K}(X, S.)$ can be defined proceeding from the exact category of sheaves on X and S_k which are flat relative to any structural morphism $S_i \to X$, $S_i \to S_k$. For the K(X, S.) so defined there is the obvious morphism $i_*: \mathbf{K}(Y, S_Y.) \to \mathbf{K}(X, S.)$. The lemma now follows from the exact localization triangles for $S_{Y_k} \hookrightarrow S_k$ and $Y \hookrightarrow X$ and the fact that holim takes exact triangles into exact triangles. ■

2.2. Adams Operators, $\mathscr{A}$-Cohomologies. In [26, 31] Adams operators ψ^p, $p \in \mathbf{Z}^+$ on K-groups of quasiprojective schemes were defined. For affine X they are standard linear combinations of mappings $\mathbf{Z}_\infty[B_{G.}(X)] \to \mathbf{Z}_\infty[B_{G.}(X)]$ connected with the exterior degrees. For an affine simplicial scheme X. the exterior degrees define a mapping of the cosimplicial systems $\mathbf{Z}_\infty[B_{G.}(X.)]$; according to 3.1.2, we obtain Adams operators on K_j of affine simplicial schemes for j > 0. We now define the action of ψ^P on the K-groups of any regular scheme X: for this we replace X by an affine hypercovering; the K-functor does not change by this. Adams operators on the relative K-groups are defined similarly. Properties of ψ^P: they all commute with one another; if $K_.^{(i)}(X) \subset K_.(X) \otimes \mathbf{Q}$ is a subspace on which ψ^P acts by multiplication by $p^i (i \in \mathbf{Z}, i \geqslant 0)$, then $K_.^{(i)}(X)$ does not depend on p, and $K_.(X) \otimes \mathbf{Q} = \oplus K_.^{(i)}(X)$; the same holds for relative K-groups.

2.2.1. Notation. Let X be a regular scheme or an affine (simplicial) scheme. We set $H^j_{\mathscr{A}}(X, \mathbf{Q}(i)) := K^{(i)}_{2i-j}(X)$, $\mathrm{ch}_{\mathscr{A}}: K_l(X) \to \oplus H^{2i-l}_{\mathscr{A}}(X, \mathbf{Q}(i))$ – a sum of projections. We proceed similarly for relative cohomologies.

It is clear that on the $\mathscr{A}$-cohomologies there is a natural multiplication such that $\mathrm{ch}_{\mathscr{A}}$ is a ring isomorphism (we denote it by $\cup$ or {,}); there is a natural morphism $H^{\cdot}(X, \mathcal{O}_X^*) \to H^{\cdot+1}_{\mathscr{A}}(X, \mathbf{Q}(1))$, and the usual facts hold: the theorem on cohomologies of projective bundles, the exact sequence of relative cohomologies, etc.

2.2.2. It is clear that for a scheme X $H^j_{\mathscr{A}}(X, \mathbf{Q}(i)) = 0$ for j > 2i (since $K_n = 0$ for n < 0). If X is the spectrum of a field, then from [31] it follows that $H^j_{\mathscr{A}}(X, \mathbf{Q}(i)) = 0$ for j > i. Apparently, $H^j_{\mathscr{A}} = 0$ for j < 0, but I have no proof.

Remark. Recently V. V. Shekhtman defined the operation ψ^P at the level of spectra.

2.2.3. We now proceed to homologies. We fix a field k; let Sch_* be the category of quasiprojective schemes over k and proper morphisms. For $X \in \mathrm{Sch}_*$ on $K_.'(X) \otimes \mathbf{Q}$ it is possible to define the Adams operations ψ'^p as follows. We imbed X in a smooth scheme Y; then K'(X) = $K(Y, Y \setminus X)$. On $K(Y, Y \setminus X)$ there act the operators ψ^P. We define the operator ψ'^P on $K'(X) \otimes \mathbf{Q}$ by the formula $\psi'^p(l) = \psi^p(l) \cdot \theta_p^{-1}(\Omega^1_{Y/k})$; here $\vartheta_p(\Omega^1_{Y/k}) \in K_0(Y)$ is the cannibal class of the sheaf $\Omega^1_{Y/k}$. It is easily seen that the ψ'^P so defined does not depend on the imbedding $X \hookrightarrow Y$ and is a ψ^P-morphism of the K.(X)-module $K'.(X) \otimes \mathbf{Q}$. Further, all ψ'^P commute with one another and with morphisms of direct image, and also with morphisms of inverse image for etale $\tilde{X} \to X$.

As before, $K_{\cdot}'(X)\otimes\mathbf{Q}=\underset{i\in\mathbf{Z}}{\otimes}K_{\cdot}'^{(i)}(X)$, where $K'^{(i)}(X)\,(i\in\mathbf{Z})$ is the subspace corresponding to the eigenvalue p^i of the operator ψ^p; we set $H'^{j}_{\mathscr{A}}(X,\mathbf{Q}(i)):=K'^{(i)}_{2i-j}(X)$, $\tau_{\mathscr{A}}:K_l'(X)\otimes\mathbf{Q}\to\underset{i}{\oplus}H'^{2i-l}(X,\mathbf{Q}(i))$ – the tautological isomorphism. It is clear that on $H'_{\mathscr{A}}$ there is a natural structure of a $H'_{\mathscr{A}}$-module, and $\tau_{\mathscr{A}}$ is a $\mathrm{ch}_{\mathscr{A}}$-morphism of modules. Further, $H'_{\mathscr{A}}$ is a covariant functor relative to proper morphisms and a contravariant functor relative to etale morphisms; there is a long exact sequence of homologies for a closed imbedding.

2.2.4. The next fact follows from 2.2.2. Let X be a scheme, let a, $b\in\mathbf{Z}$, and let $\alpha\in H'^{a}_{\mathscr{A}}(X,\mathbf{Q}(b))$. Then there is a closed subscheme $i:Y\subset X$ such that $\dim Y\leqslant b-a$ and $\alpha\in i_*H'^{a}_{\mathscr{A}}(Y,\mathbf{Q}(b))$. Cf. Conjecture 1.10.1.

2.2.5. We assign to an n-dimensional, irreducible, reduced subscheme $Y\subset X$ the image $[\mathcal{O}_Y]\in K_0'(X)$ in $H'^{-2n}_{\mathscr{A}}(X,\mathbf{Q}(-n))$; the mapping $\mathrm{cl}_{\mathscr{A}}:Z_n(X)\to H'^{-2n}_{\mathscr{A}}(X,\mathbf{Q}(-n))$ obtained identifies $H'^{-2n}(X,\mathbf{Q}(-n))$ with the Chow group $\mathrm{CH}_n(X)\otimes\mathbf{Q}$ of n-dimensional cycles on X modulo rational equivalence. Further, if X is an m-dimensional smooth scheme, then the morphism of $H^{\cdot}_{\mathscr{A}}(X)$-modules $H^{\cdot}_{\mathscr{A}}(X,\mathbf{Q}(*))\to H'^{\cdot-2m}_{\mathscr{A}}(X,\mathbf{Q}(*-m))$, taking 1 into $\mathrm{cl}_{\mathscr{A}}[X]$, is an isomorphism (Poincaré duality).

2.3. Dealing with the "universal" case, we proceed to a sketch of the construction of the Chern character and the Riemann–Roch theorem for concrete cohomological functors, for example, for $\mathscr{D}$-cohomologies. For details of the proofs we refer to [22] or to the unpublished dissertation of V. V. Shekhtman.

Let $\mathscr{V}$ be a category of schemes over k containing all smooth quasiprojective schemes equipped with the Zariski topology. We fix a commutative ring $A\supset\mathbf{Q}$. We suppose that on $\mathscr{V}$ there is given a collection of complexes of sheaves of A-modules $\Gamma(i)\in\mathbf{D}^{\geqslant 0}(\mathscr{V},A)$, $i\in\mathbf{Z}$, and morphisms $\cup:\Gamma(i)\overset{L}{\underset{A}{\otimes}}\Gamma(j)\to\Gamma(i+j)$, $c_0:A\to\Gamma(0)$, $c_1:\mathcal{O}^*[-1]\to\Gamma(1)$ in $\mathbf{D}^+(\mathscr{V},A)$, satisfying the following axioms:

a) the $\cup$-product is commutative and associative; c_0 is the identity for $\cup$.

b) Let $n\in\mathbf{Z}$, $n>0$. We set $\xi=c_1(\mathcal{O}(1))\in H^2(\mathbf{P}_k^n,\Gamma(1))$. Then the natural morphism $\oplus\pi^*_{\mathbf{P}^n}(\xi)^i\cup\pi^*_X:\overset{n}{\underset{i=0}{\oplus}}\Gamma(j-i)[-2i]\to R\pi_{X*}\Gamma(j)_{\mathbf{P}^n\times X}$ is an isomorphism for any $X\in\mathscr{V}$ and $j\in\mathbf{Z}$ (here $\pi_{\mathbf{P}^n}$, π_X are the projections of $\mathbf{P}^n\times X$ onto the factors).

Following Grothendieck, for any bundle $\mathscr{E}$ over a scheme X (an ordinary or simplicial scheme) we can define its Chern class $c_{i\Gamma}(\mathscr{E})\in H^{2i}(X,\Gamma(i))$ and Chern character $\mathrm{ch}_\Gamma(\mathscr{E})\in\oplus H^{2i}(X,\Gamma(i))$ satisfying the usual identities.

We now define the Chern character in higher K-theory: the morphism ch_Γ: $K_j(X)\to\oplus H^{2i-j}\times(X,\Gamma(i))$ which coincides with the ordinary ch_Γ on $K_0(X)$. We note that for any pair X., Y. of simplicial schemes we have a morphism $A[\mathrm{Hom}(Y.,X.)]\to R\mathrm{Hom}(R\Gamma(X.,\Gamma(i))^\Delta,R\Gamma(Y.,\Gamma(i))^\Delta)$ in the derived category of $\Delta^0\times\Delta$ A-modules (here Hom(Y., X.) is considered as a $\Delta^0\times\Delta$-set; A[Hom(Y., X.)] is the corresponding free A-module; $R\Gamma(X.,\Gamma(i))^\Delta\in\mathbf{D}^+(A\text{-mod}^\Delta)$ are the sections of $\Gamma(i)$ considered as a cosimplicial group; $R\mathrm{Hom}:\mathbf{D}^+(A\text{-mod}^\Delta)^0\times\mathbf{D}^+(A\text{-mod}^\Delta)^0\to\mathbf{D}^+(A\text{-mod}^{\Delta^0\times\Delta})$ is the natural pairing). In particular, passing to normalizations, we obtain a morphism HJ(X., $\Gamma(i))\to\mathrm{Hom}(\mathrm{sN}A[\mathrm{Hom}(Y.,X.)],R\Gamma(Y.,\Gamma(i))[j])$. We apply this observation to the case X. = $B_G.$. To the Chern character of the universal bundle over X. there corresponds the morphism sN × A $[B_G.(Y.)]\to\oplus R\Gamma(Y.,\Gamma(i)[2i])$. If Y. is affine, then, combining this arrow with the canonical morphism $\mathrm{holim}\,\mathbf{Z}_\infty[B_G.(Y.)]\to\mathrm{holim}\,A[B_G.(Y.)]$, we obtain the desired morphism $K_j(Y_0)=\pi_j\mathrm{holim}\times\mathbf{Z}_\infty[B_G.(Y.)]\to\oplus H^{2i-j}(Y,\Gamma(i))$ for j > 0. In order to define ch_Γ for an arbitrary, not necessarily affine, scheme Y it is necessary to take an affine hypercovering $\tilde{Y}.\to Y$ and define ch_Γ as the composition $K_j(Y)\to K_j(\tilde{Y}.)\to\oplus H^{2i-j}(\tilde{Y}.,\Gamma(i))\rightleftarrows\oplus H^{2i-j}(Y,\Gamma(i))$.

<u>Assertion 2.3.1.</u> The Chern character ch_Γ is a ring morphism, $\mathrm{ch}(K_j^{(i)}(Y))\subset H^{2i-j}(Y,\Gamma(i))$. ■

From this there arises a canonical ring morphism $r_\Gamma:H^*_{\mathscr{A}}(Y,\mathbf{Q}(\cdot))\to H^*(Y,\Gamma(\cdot))$ such that $\mathrm{ch}_\Gamma=r_\Gamma\,\mathrm{ch}_{\mathscr{A}}$; it is clear that on $\mathcal{O}^*(Y)\otimes\mathbf{Q}=H^1_{\mathscr{A}}(Y,\mathbf{Q}(1))$ the morphism r_Γ coincides with c_1. Of course, r_Γ is the same as ch_Γ, but for us it will be more pleasant to deal with it.

As an exercise it is suggested that ch_Γ, $\mathbf{r}_\Gamma$ be defined for relative K-groups.

Remark 2.3.2. Recently V. V. Shekhtman defined ch_Γ at the level of spectra.

We proceed to homologies and the Riemann–Roch theorem. Let $\Gamma(\cdot)$ be a homology theory on $\mathscr{V}$, satisfying axioms a), b). We say that $\Gamma(\cdot)$ satisfies Poincaré duality if there exists a collection of functors $\mathbf{H}'(\cdot, \Gamma(i)): \mathrm{Sch}_* \to \mathbf{D}^+(A\text{-mod})$ together with the isomorphisms of Poincaré duality $\mathbf{H}'(Y, \Gamma(i)) = \mathbf{R}\Gamma_Y(X, \Gamma(i + \dim X)[2\dim X])$ for any pair $Y \subset X$, where X is smooth, Y is a closed subscheme in X, and the following axioms are satisfied:

c) If $Y_1 \overset{i}{\hookrightarrow} Y_2 \hookrightarrow X$ are closed imbeddings and X is smooth, then the morphism $\mathbf{H}'(Y_1, \Gamma(\cdot)) \to \mathbf{H}'(Y_2, \Gamma(\cdot))$ goes over under Poincaré duality into the canonical morphism $R\Gamma_{Y_1}(X, \Gamma(\cdot)) \to R\Gamma_{Y_2}(X, (\Gamma(\cdot))$.

d) Let $Y \subset X$ be smooth, $\dim X - \dim Y = 1$. Then the diagram

$$\begin{tikzcd}
\mathbf{Z} \arrow[r, equal] \arrow[d, "c_0"] & R\Gamma_Y(X, \mathcal{O}^*)[1] \arrow[d, "c_1"] \\
R\Gamma(Y, \Gamma(0)) \arrow[r] & R\Gamma_Y(X, \Gamma(1))[2]
\end{tikzcd}$$

is commutative (the upper isomorphism is canonical, while the lower arises from Poincaré duality).

e) Let X be a smooth scheme, let X_1, X_2 be smooth subschemes of X intersecting transversally, and let $Y \subset X_1$; let $N = \dim X - \dim X_1$. Then the diagram

$$\begin{tikzcd}
R\Gamma_Y(X_1, \Gamma(i)) \arrow[r, equal] \arrow[d] & R\Gamma_Y(X, \Gamma(i+N)[2N]) \arrow[d] \\
R\Gamma_{Y \cap X_2}(X_1 \cap X_2, \Gamma(i)) \arrow[r, equal] & R\Gamma_{Y \cap X_2}(X_2, \Gamma(i+N)[2N])
\end{tikzcd}$$

is commutative.

f) Let $f: X' \to X$ be a proper morphism of smooth schemes, let $Y \subset X$, $Y' := f^{-1}(Y)$, $N = \dim X - \dim X'$. Then the diagram

$$\begin{tikzcd}
& R\Gamma_Y(X, \Gamma(i)) \overset{L}{\otimes} R\Gamma(X', \Gamma(j)) \arrow[rr] \arrow[dl] & & R\Gamma_Y(X, \Gamma(i)) \overset{L}{\otimes} R\Gamma(X, \Gamma(j+N)[2N]) \arrow[dr] & \\
R\Gamma(Y', \Gamma(i)) \overset{L}{\otimes} R\Gamma(X', \Gamma(j) \arrow[drr] & & & & R\Gamma_Y(X, \Gamma(i+j+N)[2N]) \\
& & R\Gamma_{Y'}(X', \Gamma(i+j)) \arrow[urr] & &
\end{tikzcd}$$

is commutative. By the way, we note that H' and the Poincaré duality can be recovered uniquely on the basis of $\Gamma(i)$.

Thus, let $\Gamma(\cdot)$ be a homology theory satisfying Poincaré duality. We set $H'^i(X, \Gamma(j)) := H^i(\mathbf{H}'(X, \Gamma(j))$. Then C_0 defines for any irreducible scheme Y of dimension N a morphism $A \to H'^{-2N}(Y, \Gamma(-N))$, whence for any scheme X there arise the functorial morphisms $\mathrm{cl}_\Gamma: Z_n(X) \to H'^{-2n}(X, \Gamma(-n))$ (cf. 1.9).

Suppose now Y is an arbitrary scheme. We imbed Y in a smooth n-dimensional scheme X as a closed subscheme. Then $K_j'(Y) = K_j(X, X \setminus Y)$. Using these identifications, we define the morphism $\tau_\Gamma: K_j'(Y) \to \oplus H'^{2i-j}(Y, \Gamma(i))$ by the formula $\tau_\Gamma(\alpha) := \mathrm{ch}_\Gamma(\alpha)\mathrm{Td}(X)$, where $\mathrm{Td}(X) \in \oplus H^{2i}(X, \Gamma(i))$ is the Todd genus of the scheme X.

Assertion 2.3.3. τ_Γ does not depend on the choice of X and commutes with morphisms of direct image; we have $\tau_\Gamma(K_j'^{(i)}) \subset H'^{2i-j}(\Gamma(i))$. ■.

From this there arises a homology morphism $\mathbf{r}_\Gamma': H'^i_{\mathscr{A}}(Y, \mathbf{Q}(j)) \to H'^i(Y, \Gamma(j))$ such that $\mathbf{r}_\Gamma' \tau_{\mathscr{A}} = \tau_\Gamma$. If $Y \subset X$, and X is smooth, then this is a r_Γ-morphism of $H^\cdot(X)$-modules; moreover, $\mathbf{r}_\Gamma' \mathrm{cl}_{\mathscr{A}} = \mathrm{cl}_\Gamma$.

From the results of Sec. 1 it follows that $H_{\mathscr{D}}$ satisfies all the conditions of this subsection; thus, for schemes over $\mathbf{R}$ we obtain natural transformations $\mathbf{r}_{\mathscr{D}}: H^*_{\mathscr{A}}(X, \mathbf{Q}(\cdot)) \to H^*_{\mathscr{D}}(X, A(\cdot))$, $\mathbf{r}_{\mathscr{D}}': H'^*_{\mathscr{A}}(X, \mathbf{Q}(\cdot)) \to H'^*_{\mathscr{D}}(X, A(\cdot))$; we call them regulators. Here is their first nontrivial property.

Assertion 2.3.4. We assume that the smooth scheme X is compact and $l \leqslant 2i - 2$ or X is arbitrary and either $i > \dim X + 1$ or $l < i$. Then the groups $r_{\mathcal{D}} H^l_{\mathcal{A}}(X, \mathbf{Q}(i)) \subset H^l_{\mathcal{D}}(X, A(i))$ are no more than countable.

Proof. It may be assumed that X is a scheme over **C**. We choose a countable algebraically closed subfield $k \subset \mathbf{C}$, over which X is defined, i.e., $X = X_0 \otimes \mathbf{C}$ for some X_0 over k. We shall show that $r_{\mathcal{D}} H^l_{\mathcal{A}}(X, \mathbf{Q}(i)) = r_{\mathcal{D}} H^l_{\mathcal{A}}(X_0, \mathbf{Q}(i))$; since the last group is countable, this will imply 3.4.4. Indeed, let $\alpha \in H^l_{\mathcal{A}}(X, \mathbf{Q}(i))$. Then there is an algebra R of finite type over k, and element $\alpha_0 \in H^l_{\mathcal{A}}(X \underset{k}{\otimes} R, \mathbf{Q}(i))$, and $i \in \operatorname{Spec} R(\mathbf{C})$ such that $\alpha = i^*(\alpha_0)$ (see [28]). We choose a k-point i' in the same connected component of Spec R as i. According to 1.6.6.2, we have $r_{\mathcal{D}}(\alpha) = r_{\mathcal{D}}(i^*(\alpha_0)) = r_{\mathcal{D}}(i'^*(\alpha_0))$, as required. ■

From part 5.2 we obtain the following result.

COROLLARY 2.3.5. For $j > 1$ the images of Borel regulators $K_j(\mathbf{C}) \to \mathbf{R}$ are countable sets.

2.4. Cohomologies of Motifs. We shall show how to translate what has been said above to the language of Grothendieck motifs; we recall the basic constructions [6, 18].

2.4.1. We fix a number field E; for a quasiprojective scheme X we set $H^*_{\mathcal{A}}(X, E(\cdot)) := H^*_{\mathcal{A}}(X, \mathbf{Q}(\cdot)) \otimes E$. This is a bigraded E-algebra depending contravariantly on X. Let $\mathcal{V}_k$ be the category of smooth projective schemes over k; by Poincaré duality, $H^*_{\mathcal{A}}(X, E(\cdot))$ is also a covariant functor of $X \in \mathcal{V}_k$. We define the additive E-category of correspondences C(k, E) whose objects coincide with the objects of $\mathcal{V}$, while the morphisms are defined as follows: We denote by E[X] the object of C(k, E) corresponding to $X \in \mathcal{V}_k$; then

$$\operatorname{Hom}(E[X], E[Y]) := H^{2\dim Y}_{\mathcal{A}}(X \times Y, E(i)) = CH^{2\dim Y}(X \times Y) \otimes E;$$

composition of morphisms is defined as composition of correspondences: for $f \in \operatorname{Hom}(E[X_1], E[X_2])$, $g \in \operatorname{Hom}(E[X_2], E[X_3])$ we have $g \circ f = p_{13*}(p^*_{12}(f) \cup p^*_{23}(g))$, where $p_{ij}: X_1 \times X_2 \times X_3 \to X_i \times X_j$ are projections. We have the obvious functor $\mathcal{V}_k \to C(k, E)$, which is the identity on objects and assigns to a morphism the class of its graph. The functor $H_{\mathcal{A}} X, E(\cdot))$ on $\mathcal{V}_k$ extends to an additive E-functor $H^*_{\mathcal{A}}(?, \mathbf{Q}(\cdot))$ on $C(k, E): H^*(E[\mathbf{X}], \mathbf{Q}(\cdot)) := H^*_{\mathcal{A}}(X, E(\cdot))$; for $f \in \operatorname{Hom} \times (E[X_1], E[X_2])$, $\alpha \in H^*_{\mathcal{A}}(E[X_2])$ we have $f^*(\alpha) = \pi_{1*}(f \cup \pi_2^*(\alpha))$. The category of effective E-motifs $\mathcal{M}_{\text{eff}}(k, E)$ is defined as the pseudo-Abelian hull of C(k, E): to the objects of C(k, E) there are formally added the images of idempotent endomorphisms; $H^*_{\mathcal{A}}$ extends canonically to an additive E-functor on $\mathcal{M}_{\text{eff}}(k, E)$. On the category $\mathcal{M}_{\text{eff}}(k, E)$ there is a natural operation ⊗, induced by the direct product of manifolds over k; we note that $H^*_{\mathcal{A}}$ does not commute with the product: for the cohomologies $H^*_{\mathcal{A}}$ there is no Künneth formula. In the category $\mathcal{M}_{\text{eff}}(k, E)$ we have the natural decomposition $E[\mathbf{P}_k^1] = E[\operatorname{Spec} k] \oplus E(-1)_k$; $E(-1)_k$ is called the Tate motif. We define the category of E-motifs $\mathcal{M}(k, E) \supset \mathcal{M}_{\text{eff}}(k, E)$, by localizing $\mathcal{M}_{\text{eff}}(k, E)$ with respect to the functor $M \to M \otimes E(-1)$. Any motif has the form $M(k) := M \otimes E(k)$ for some $M \in \mathcal{M}_{\text{eff}}$, $k \in \mathbf{Z}$ [where $E(k) := E(-1)^{\otimes -k}$]; for $M_1, M_2 \subset \mathcal{M}_{\text{eff}}$ and $k \in \mathbf{Z}$ we have $\operatorname{Hom}_{\mathcal{M}_{\text{eff}}}(M_1, M_2) = \operatorname{Hom}_{\mathcal{M}}(M_1(k), M_2(k))$. Since for $k \leqslant 0$ and $M \in \mathcal{M}_{\text{eff}}$ we have $H^j_{\mathcal{A}}(M(-k), \mathbf{Q}(i)) = H^{j-2k}_{\mathcal{A}}(M, \mathbf{Q}(i-k))$ (the formula for the cohomologies of $\mathbf{P}^1 \times X$), we can extend $H^*_{\mathcal{A}}$ to a functor on $\mathcal{M}(k, E)$, by requiring that this equality hold for all $k \in \mathbf{Z}$.

We have thus defined a functor $H_{\mathcal{A}}$ on the category of E-motifs. Let Γ be a cohomology theory in the sense of 2.3 which satisfies Poincaré duality, and let A be the ring of coefficients of Γ. Then on $\mathcal{M}(k, E)$ there arises a functor $M \rightsquigarrow H^*_{\Gamma}(M, A(\cdot))$ together with the natural transformation $r_{\Gamma}: H_{\mathcal{A}} \to H_{\Gamma}$: we set $H^*_{\Gamma}(E[X], A(\cdot)) := H^*(X, \Gamma(\cdot)) \underset{\mathbf{Q}}{\otimes} E \ldots$.

Thus, on $\mathcal{M}(k, E)$ the usual cohomology functors are defined on $H^{\cdot}_{\mathcal{DR}}(M)$, $H^{\cdot}(M, \mathbf{Q}_l(*))$ and for $k = \mathbf{R}$, $H^{\cdot}_{\mathcal{B}}(M, \mathbf{Q}(*))$, $H^{\cdot}_{\mathcal{D}}(M, A(*))$.

We note that the $A \otimes E$ -modules $H^*_?(M, A(*))$ are always free (see [18], 2.5), so that it is possible to speak of their dimension; we shall need this in Sec. 3.

2.4.2. In the next section we require the $\mathcal{A}$ -cohomologies corresponding to the "integral" part of the motif over **Q**. We shall try to determine them. Let X be a smooth projective scheme over **Q**. We assume that X admits a regular model $X_{\mathbf{Z}}$ over **Z**. Then the image of $H^{*}_{\mathscr{A}}(X_{\mathbf{Z}}, \mathbf{Q}(\cdot))$ in $H^{*}_{\mathscr{A}}(X, \mathbf{Q}(\cdot))$ does not depend on the choice of $X_{\mathbf{Z}}$. Indeed, suppose $X'_{\mathbf{Z}}$ is another regular model. Let $X''_{\mathbf{Z}}$ be some third proper scheme over **Z** together with morphisms $X_{\mathbf{Z}} \xleftarrow{\pi} X''_{\mathbf{Z}} \xrightarrow{\pi'} X'_{\mathbf{Z}}$, which are isomorphic on each fiber; the assertion follows from the commutativity of the diagram

$$\begin{tikzcd}
K.(X_Z) \arrow[r, "\pi^*"] \arrow[drr] & K.(X''_Z) \arrow[r] \arrow[dr] & K.(X''_Z) \arrow[r, "\pi'_*"] \arrow[d] & K.'(X'_Z)=K.(X'_Z) \arrow[dl] \\
 & & K.(X) &
\end{tikzcd}$$

In Sec. 3 we must either restrict ourselves to schemes over **Q** admitting regular models over **Z** and their motifs or adopt the following conjecture.

Conjecture 2.4.2.1. Let X be a smooth projective scheme over **Q**, and let $X_{\mathbf{Z}}$ be a proper scheme over **Z** (possibly special) such that $X_{\mathbf{Z}} \otimes \mathbf{Q} = X$. Then the image of $K'(X_{\mathbf{Z}}) \otimes \mathbf{Q}$ in K' × (X) $\otimes \mathbf{Q} = K(X) \otimes \mathbf{Q}$ does not depend on the choice of $X_{\mathbf{Z}}$ and is invariant under the operations ψ and the morphisms of direct image. ■

If 2.4.2.1 is true, then we can define the groups $H^{*}_{\mathscr{A}}(M_{\mathbf{Z}}, \mathbf{Q}(\cdot))$ for any E-motif M over **Q** proceeding from the groups $\mathrm{ch}_{\mathscr{A}}(\mathrm{Im}(K'(X_{\mathbf{Z}}) \otimes \mathbf{Q} \to K(X) \otimes \mathbf{Q}) \subset H^{*}_{\mathscr{A}}(X, \mathbf{Q}(\cdot))$ for smooth projective schemes X over **Q**.

Conjecture 2.4.2.2. Let F be a field of finite characteristic, char F = p. Then $K_i(F) \otimes \mathbf{Q} = K^m_i(F) \otimes \mathbf{Q}$, where $K^m_{\cdot}(F)$ is the Milnor ring of the field F, and $K_i(F) \otimes \mathbf{Q} = 0$ for $i > \mathrm{degtr}\, F/\mathbf{F}_p$. ■

According to [30], 2.4.2.2 is equivalent to $H^a_{\mathscr{A}}(F, \mathbf{Q}(b)) \neq 0$ only for $0 \leqslant a = b \leqslant \mathrm{degtr}\, F/\mathbf{F}_p$. If this conjecture is true, then for any scheme $Y/\mathbf{F}_p$ the groups $H'^{a}_{\mathscr{A}}(Y, \mathbf{Q}(b)) \neq 0$ only for b ⩽ 0 and $-b \leqslant b - a \leqslant \dim Y$, and for any smooth projective $X/\mathbf{Q}$ we have $H^j_{\mathscr{A}}(X_{\mathbf{Z}}, \mathbf{Q}(i)) = H^j_{\mathscr{A}}(X, \mathbf{Q}(i))$ except possibly for those (i, j) for which $i \leqslant \dim X + 1$ and $i \leqslant j \leqslant 2i - 1$.

The following conjecture was once communicated to me by A. N. Parshin.

Conjecture 2.4.2.3. If Y is a smooth scheme over $\mathbf{F}_p$, then $K_i(Y) \otimes \mathbf{Q} = 0$ for i ≠ 0.

From this conjecture it follows that $H^j_{\mathscr{A}}(X, \mathbf{Q}(i))/H^i_{\mathscr{A}}(X_{\mathbf{Z}}, \mathbf{Q}(i))$ depends for $j \leqslant 2i - 2$ only on the degenerate fibers of $X_{\mathbf{Z}}$.

2.5. The Arithmetic Intersection Index.

In this subsection we present a multidimensional analogue of Arakelov's construction [1] of the Neron–Tate height of points on curves.

2.5.1. The Local Index at ∞.

Let $X_{\mathbf{R}} = X$ be a smooth proper scheme over **R** of dimension N, and let $z_i \in Z^{l_i}(X) \otimes A$, i = 0, 1 be cycles on X. We suppose that $l_0 + l_1 = N + 1$, $\mathrm{supp}\, z_0 \cap \mathrm{supp}\, z_1 = \emptyset$ and the classes $\mathrm{cl}_{\mathscr{D}}(Z_i) \in H^{2l_i}_{\mathscr{D}}(X, A(l_i))$ are equal to 0. We assign to them a class $(z_0 \cap z_1)_\infty \in H^{2N+1}_{\mathscr{D}}(X, A(N+1)) = H^{2N}_{\mathscr{D}\mathscr{R}}(X)/H^{2N}_{\mathscr{B}}(X, A(N+1))$ and a number $[z_0 \cap z_1] \in \mathbf{R}^{*+}$ as follows. Let $U_i = X \setminus \mathrm{supp}\, z_i : U_1 \cup U_2 = X$. Since $\mathrm{cl}_{\mathscr{D}} Z_i = 0$ in $H_{\mathscr{D}}(X)$, there are $\varphi_i \in H^{2l_i-1}_{\mathscr{D}}(U_i, A(l_i))$ such that $\partial_i \varphi_i = \mathrm{cl}_{\mathscr{D}}(z_i) \in H^{2l_i}_{\mathscr{D}}(X, U_i, A(l_i))$ in the exact sequence of pairs (X, U_i). Then $(z_0 \cap z_1)_\infty := \partial(\varphi_0 \cup \varphi_1)$, where $\partial : H^{\cdot}_{\mathscr{D}}(U_0 \cap U_1) \to H^{\cdot+1}(X)$ is the differential in Meyer–Vietoris sequence for {U_0, U_1}. We set $[z_0 \cap z_1]_\infty := \pi_*(z_0 \cap z_1)_\infty \in H^1_{\mathscr{D}}(\mathrm{Spec}\, \mathbf{R}, A(1)) = \mathbf{R}^{*+}$ (where $\pi : X \to \mathrm{Spec}\, \mathbf{R}$ is the structural morphism).

LEMMA 2.5.1. a) The class $(z_0 \cap z_1)_\infty$ (and the number $[z_0 \cap z_1]_\infty$) depends only on z_i and not on the choice of φ_i.

b) $(z_0 \cap z_1)_\infty = (z_1 \cap z_0)_\infty$.

c) Let $i_1 : \mathrm{Supp}\, z_1 \hookrightarrow X$ be an imbedding. Then $(z_0 \cap z_1)_{\mathscr{D}} = i_{1*}(i_1{}^*\varphi_0 \cap \mathrm{cl} z_1)$, $[z_0 \cap z_1]_\infty = \pi_*(i^*\varphi_0 \cap \mathrm{cl} z_1)$.

d) We assume that $\mathrm{cl}_{\mathscr{A}}(z_0) \in H^{2l_0}_{\mathscr{A}}(X_{\mathbf{R}}, \mathbf{Q}(l_0))$ is equal to 0. From the exact localization sequence there is a $\varphi_{0\mathscr{A}} \in H^{2l_0-1}_{\mathscr{A}}(U_0, \mathbf{Q}(l_0))$ such that $\partial_0 \varphi_0 = \mathrm{cl}_{\mathscr{A}}(z_0) \in H^{2l_0}_{\mathscr{A}}(X, U_0, \mathbf{Q}(l_6))$. Then $(z_0 \cap z_1)_\infty = r_{\mathscr{D}} i_{1*}(i_1{}^*\varphi_{0\mathscr{A}} \cap \mathrm{cl}_{\mathscr{A}} z_1)$, $[z_0 \cap z_1]_\infty = r_{\mathscr{D}} \pi_*(i_1{}^*\varphi_0 \cap \mathrm{cl}_{\mathscr{A}} z_1)$.

The proof of a) follows from the Meyer–Vietoris sequence for {U_0, U_1} and the sequences of pairs for (X, U_i); b) follows from the commutativity of ∪ ; for c) we consider the morphism and exact sequences

$$\begin{tikzcd}
H^{\cdot}(U_0)\oplus H^{\cdot}(U_1) \arrow[r] & H^{\cdot}(U_0\cap U_1) \arrow[r, "\partial"] & H^{\cdot+1}(X) \\
H^{\cdot}(U_0) \arrow[u] \arrow[r] & H^{\cdot}(U_0\cap U_1) \arrow[u, equal] \arrow[r, "\bar{\partial}_1"] & H^{\cdot+1}(U_0,U_0\cap U_1). \arrow[u, "e"']
\end{tikzcd}$$

We have $(z_0\cap z_1)_{\mathscr{D}}=\partial(\varphi_0\cup\varphi_1)=e\bar{\partial}_1(\varphi_0\cup\varphi_1)=e(\varphi_0\cup \mathrm{cl} z_1)=i_{1*}(i_1{}^*\varphi_0\cap \mathrm{cl} z_1)$, q.e.d. Finally, d) follows from c) and the fact that $r_{\mathscr{D}}$ is natural. ■

2.5.2. <u>The Local Index at Finite Points.</u> We consider a regular flat projective scheme $X_{\mathbf{Z}}$ over $\mathbf{Z}$ of dimension N + 1. Let $z_i \in Z^{l_i}(X_{\mathbf{Z}})$ be cycles on $X_{\mathbf{Z}}$ such that $l_0+l_1=N+1$ and $(\mathrm{supp}\, z_0\cap \mathrm{supp}\, z_1)_{\mathbf{Q}}\subset X_{\mathbf{Q}}$ is empty. We choose a finite set $\{p_i\}\subset \mathrm{Spec}\,\mathbf{Z}$ such that $S:=\mathrm{supp}\, z_0\cap \mathrm{supp}\, z_1\subset \bigcup X_{p_i}$ of fibers of X over p_i; let $U_i=X\setminus \mathrm{supp}\, z_i$. We set $z_0\cap z_1=\Sigma(z_0\cap z_1)_{p_i}=\mathrm{cl}\, z_0\cup \mathrm{cl}\, z_1 \in H^{2(+N1)}_{\mathscr{A}}(X_{\mathbf{Z}}, U_0\cup U_1, \mathbf{Q}(N+1))=H'^0(S,\mathbf{Q}(0))=H'^0(S_{p_i},\mathbf{Q}(0))$; here $\mathrm{cl}\, z_i \in H^{2l_i}_{\mathscr{A}}(X_{\mathbf{Z}}, U_i, \mathbf{Q}(l_i))$; finally, $[z_0\cap z_1]_{p_i}:=\pi_{p_{i*}}(z_0\cap z_1)_{p_i}(\pi_{p_i}:S_{p_i}\to \mathrm{Spec}\,\mathbf{Z}/p\mathbf{Z}$ are the structural morphisms). It is clear that the intersection index is commutative, and if $i_1:\mathrm{supp}\, z_1\hookrightarrow X$, we have $(z_0\cap z_1)=i_{1*}(i_1{}^*\mathrm{cl}\, z_0\cap z_1)$. We now assume that cl z_0 in $H^{2l_0}_{\mathscr{A}}(X_{\mathbf{Z}},\mathbf{Q}(l_0))$ is equal to 0. Then there is a $\varphi_{0\mathscr{A}}\in H^{2l_0-1}(U_0,\mathbf{Q}(l_0))$ such that $\mathrm{cl}\, z_0=\partial_0\varphi_{0\mathscr{A}}$. From the compatibility of the exact localization sequence with direct images it follows that $(z_0\cap z_1)=i_{1*}(\partial_0(i_1{}^*\varphi_0\cap z_1))$ and $[z_0\cap z_1]_{p_i}=\partial_{(p_i)}\pi_{(p_i)*}(i_1{}^*\varphi_0\cap z_1)$ [here (p_i) denotes localization outside $\Pi p_i=0$, $\partial_{(p_i)}:H^1(\mathrm{Spec}\,\mathbf{Z}/(p_i),\mathbf{Q}(1))\to H^0(\mathrm{Spec}\,\mathbf{Z}/p_i\mathbf{Z},\mathbf{Q})=\mathbf{Q}$, $\pi_{(p_i)}:?_{(p_i)}\to \mathrm{Spec}\,\mathbf{Z}_{(p_i)}$ is the structural morphism; we note that $H'(\mathrm{Spec}\,\mathbf{Z}_{(p)},\mathbf{Q}(1))=\mathbf{Z}^*_{(p_i)}\otimes\mathbf{Q}$ and $\partial(p_i)$ is $\mathrm{ord}(p_i)$].

2.5.3. <u>The Global Index</u>. We suppose that we are in the situation of 5.2 and z_i are homologically equivalent to 0 on a general fiber; this means that the image of z_i in $H^{2l_i}_{\mathscr{D}}(X_{\mathbf{R}},(\mathbf{R}(l_i))\overset{\varepsilon_{\mathscr{B}}}{\hookrightarrow} H^{2l_i}_{\mathscr{B}}(X_{\mathbf{R}},\mathbf{R}(l_i))$ is equal to 0. If z_i do not intersect on a general fiber, then according to 5.1 and 5.2 we have the definition of numbers $[z_0\cap z_1]_\infty\in \mathbf{R}^{*+}\xrightarrow{\ln}\mathbf{R}$ and $[z_0\cap z_1]_p\in\mathbf{Q}$ for $p\in \mathrm{Spec}\,\mathbf{Z}$, which are distinct from zero for a finite number p. We set $[z_0\cap z_1]_{\mathbf{Z}}:=[z_0\cap z_1]_\infty-\sum_p \ln p[z_0\cap c_1]_p$. If $\mathrm{cl}\, z_0\in H^{2l_0}_{\mathscr{A}}(X_{\mathbf{Z}},\mathbf{Q}(l_0))$ is equal to 0, then, choosing $\varphi_{0\mathscr{A}}$ as in part 5.2, we have, according to 5.2 and 5.1.2: $[z_0\cap z_1]_{\mathbf{Z}}=(r_{\mathscr{D}}-\Sigma\ln p\cdot \mathrm{ord}\, p_i)\pi_{(p_i)*}(i_1{}^*\varphi_0\cap z_1)=0$, since the morphism $r_{\mathscr{D}}-\Sigma\ln p\cdot \mathrm{ord}_{p_i}:H^1(\mathrm{Spec}\,\mathbf{Q},\mathbf{Q}(1))=\mathbf{Q}^*\otimes\mathbf{Q}\to\mathbf{R}$ is equal to 0 by the product formula. From this, since by the shift lemma any two classes in $H^{2l_i}_{\mathscr{A}}(X_{\mathbf{Z}},\mathbf{Q}(l_i))=\mathrm{CH}^{l_i}(X_{\mathbf{Z}})\otimes\mathbf{Q}$ can be represented by cycles not intersecting on a general fiber, we obtain a symmetric pairing $[\ ,\ ]_{\mathbf{Z}}$ between subgroups $H^{2l_i}_{\mathscr{A}}(X_{\mathbf{Z}},\mathbf{Q}(l_i))^0\subset H^{2l_i}_{\mathscr{A}}(X_{\mathbf{Z}},\mathbf{Q}(l_i))$, consisting of cycles homologous to 0 on a general fiber.

APPENDIX

DEFORMATIONS OF CHERN CLASSES

In this appendix we construct the tangential transformation to the Chern character — the Chern character in additive K-theory. A first consequence of this construction is that for Spec $\mathbf{R}$, Spec $\mathbf{C}$ our regulator coincides with the Borel regulator.

A1. <u>Small Algebras.</u> Let A be an Abelian tensor category. If $R^{\cdot}$ is a cosimplicial algebra, then the standard $\cup$ -product gives on the $N(R^{\cdot})$-normalization the $R^{\cdot}$-structure of a differential algebra. We note that if $R^{\cdot}$ is commutative $N(R^{\cdot})$ need not be. However there is the following result.

<u>Lemma—Definition A1</u>. We call a cosimplicial algebra $R^{\cdot}$ with identity small if it is commutative, generated by R^0 and R^1, and $(\mathrm{Ker}\, s_1)^2\subset R^1$ is equal to zero. We call a differential algebra $Q^{\cdot}$ with identity small if it is commutative and generated by Q^0, Q^1. Then N establishes an equivalence of the categories of small cosimplicial and small differential algebras. ■

Below we shall be interested in two Abelian tensor categories: the category Vect (k) of vector spaces over a field k of characteristic 0 and the category of complexes C(k): = $C^b \times$ (Vect k)); Vect $(k) \subset C(k)$ as a full subcategory of complexes equal to zero outside degree 0. We call cosimplicial algebras over Vect (k) (C(k)) c- (respectively, cd-) algebras; differential graded algebras over Vect (k) (C(k)) are called d- (respectively, dd-) algebras. Lemma A1 establishes an equivalence of the categories of small c- and d- (respectively, cd- and dd-) algebras.

For a small c-algebra $R^\cdot$ let $\bar{\Omega}^*(R^\cdot)$ be the universal small cd-algebra such that $\bar{\Omega}^0(R^\cdot) = R^\cdot$ (it is clear that $\bar{\Omega}^*(R^\cdot)$ is a factor of $\Omega^*(R^\cdot)$ by the cd-ideal generated by $[\mathrm{Ker}\, S_1{}^*: \Omega^*(R^1) \to \Omega^*(R^0)]^2$.). Similarly, for a small d-algebra Q we define the small dd-algebra $\bar{\Omega}^*(Q^\cdot)$; we have $\bar{\Omega}^*(NR^\cdot) = N\bar{\Omega}^*(R^\cdot)$. We note that the dd-algebra $\bar{\Omega}^*(Q^\cdot)$ is universal in the class of all commutative dd-algebras: if $A^{*\cdot}$ is a commutative dd-algebra, then any morphism of d-algebras $Q^\cdot \to A^{0\cdot}$ extends uniquely to a morphism $\Omega^*(Q^\cdot) \to A^{*\cdot}$.

Below we shall identify complexes over C(k) with bicomplexes; to the bicomplex $X^{*\cdot}$ there corresponds the complex $\to X^{*0} \to X^{*1} \to$ over C(k); correspondingly, normalization is carried out according to the gradation. For $X^{*\cdot}$ we denote by $H^i(X^{*\cdot})$ the cohomologies of the complex with convolute gradation; we denote by $X^{\geqslant i,\cdot}$ the i-th term of the foolish filtration with respect to *.

A2. The Weyl Complex. Let $\mathfrak{G}$ be a finite-dimensional Lie algebra over k. For a $\mathfrak{G}$-module V we denote by $C^\cdot(\mathfrak{G}, V)$ the standard complex of cochains of $\mathfrak{G}$ with coefficients in V.

We consider the complex $C^\cdot(\mathfrak{G}) := C^\cdot(\mathfrak{G}. k)$ — cochains of the trivial representation. $C^\cdot(\mathfrak{G})$ is a small d-algebra; we set $W^{*\cdot} := \bar{\Omega}^*(C^\cdot(\mathfrak{G}))$. It is clear that $W^{ij} = S^i\mathfrak{G}' \otimes \Lambda^{j-i}\mathfrak{G}'$ ($\mathfrak{G}'$ is the vector space dual to $\mathfrak{G}$), and the differential $d^* W^{ij} \to W^{i+1,j}$ is the Koshul' differential, while $d: W^{ij} \to W^{i,j+1}$ is the differential of the complex $C^\cdot(\mathfrak{G}, S^i(\mathfrak{G}'))$ of cochains of the i-th symmetric power of the coadjoint representation. In particular, for j > 0 the complex W^{*j} is d^*-acyclic. This implies

LEMMA A2.1. $H^j(W^{*\cdot}) = 0$ for $j \neq 0$; $H^0(W^{*\cdot}) = k$; $H^{2i}(W^{>i,\cdot}) = S^i(\mathfrak{G}')^{\mathfrak{G}}$. ■

We now suppose that $\mathfrak{G}$ is reductive; let $E_r^{p,q}$ be the spectral sequence of W relative to the filtration $W^{\geqslant p,\cdot}$.

LEMMA A2.2. There is the canonical isomorphism compatible with multiplication $E_1^{p,q} = H^{q-p}(\mathfrak{G}) \otimes S^p(\mathfrak{G}')^{\mathfrak{G}}$.

Proof. Indeed, $E_1^{p,q} =: H^{q-p}(\mathfrak{G}, S^p_c(\mathfrak{G}'))$. But if $\mathfrak{G}$ is reductive and V is a semisimple, finite-dimensional representation of $\mathfrak{G}$, then $H^\cdot(\mathfrak{G}, V) = H^\cdot(\mathfrak{G}) \otimes V^{\mathfrak{G}}$. ■

A3. de Rham Cohomologies of B_G. Let G be a reductive group over k, let $\mathfrak{G}$ be its Lie algebra, and let B_G be the classifying space of G (see 1.7.5). We need the following list of facts.

Assertion A3.1. a) There exists a canonical isomorphism $H^q_{\mathfrak{DR}}(G) \stackrel{\sim}{\to} H^q(\mathfrak{G})$, proceeding from the identification of the complex of cochains of $\mathfrak{G}$ with complex left-invariant forms on G.

b) There exists a unique ring morphism $H^{2\cdot}_{\mathfrak{DR}}(B_{G\cdot}) \to S^\cdot(\mathfrak{G}')^G$, which is functorial in G and such that for G = Gm it identifies $\mathfrak{G}' = S^1(\mathfrak{G}')^G$ with the invariant differentials on G = B_{G_1}. This morphism is an isomorphism. Further, $F^i H^{2i}_{\mathfrak{DR}}(B_{G\cdot}) = H^{2i}_{\mathfrak{DR}}(B_{G\cdot})$, $F^{i+1} H^{2i}_{\mathfrak{DR}}(B_{G\cdot}) = 0$ (F^i is the Hodge—Deligne filtration).

c) Let $E_r'^{p,q}$ be the Leray spectral sequence of de Rham cohomologies of the universal G-bundle $U_{G\cdot} \to B_{G\cdot}$. There is a canonical isomorphism between the spectral sequences E (A2.2) and DecE' compatible with multiplication and coinciding with a) on the terms $E_2'^{p,q}$ and with b) on the terms $E_2'^{p,p}$.

d) The algebra $H^\cdot_{\mathfrak{DR}}(G)$ is the exterior algebra spanning the primitive classes $\mathrm{Prim}^\cdot \times (G) \subset H^\cdot_{\mathfrak{DR}}(G)$. The algebra $H^\cdot_{\mathfrak{DR}}(B_{G\cdot})$ is a free commutative algebra; let $P^\cdot(B_{G\cdot}) = H^{>0}(B_{G\cdot})/(H^{>0}(B_{G\cdot}))^2$ be its generators. Transgression in the spectral sequence E' realizes an isomorphism

$$T: \mathrm{Prim}^{2i-1}(G) \to P^{2i}(B_{G\cdot}) \; ■$$

A4. Deformations of Chern Classes. In this subsection k = **R**. Let X. = Spec $R^{\cdot}$ be the spectrum of a small c-algebra, and let $A(i)_{\mathcal{D}} := \mathrm{Cone}(\bar{\Omega}^{\geqslant i}_{X\cdot\mathrm{an}} \oplus A(i) \to \Omega^{*}_{X\cdot\mathrm{an}})[-1]$ be a complex on $X_{\cdot\mathrm{an}}$. We assume that on X. there is given a G-torser which is trivial on X_0. We compute the images of its Chern classes in $H^{\cdot}(X_{\cdot\mathrm{an}}, \bar{A}(i)_{\mathcal{D}})$.

It suffices to consider the universal situation: $X_{\cdot} = B^{(1)}_{G\cdot} :=$ the largest small subscheme of B_G. Since $B^{(1)}_{G\cdot}$ is defined by the c-ideal $J \subset \mathcal{O}_{B_{G\cdot}}$, generated by I_1^2, where I_1 is the ideal distinguished point $s(e) \in B_{G1} = G$, we have $\left[N\mathcal{O}_{B^{(1)}_{G}}\right]^1 = \mathfrak{G}' = C^1(\mathfrak{G})$ and the following result.

LEMMA A4.1. $N\mathcal{O}_{B^{(1)}_{G\cdot}} = C^{\cdot}(\mathfrak{G})$. ■

Further, $H^j(B^{(1)}_{G\cdot\mathrm{an}}, A(i)) = 0$ for j > 0; from A4.1 and A2.1 we find that $H^j(B^{(1)}_{G\cdot\mathrm{an}}, \bar{\Omega}^{*}_{\mathrm{an}}) = H^j(B^{(1)}_{G\cdot}, \bar{\Omega}^{*}) = 0$ for j > 0. Therefore, $\varepsilon_F : H^{2i}(B^{(1)}_{G\cdot\mathrm{an}}, \bar{A}(i)_{\mathcal{D}}) \to H^{2i}(B^{(1)}_{G\cdot\mathrm{an}}, \bar{\Omega}^{>i})$ is an isomorphism for i > 0. We have the commutative diagram

$$\begin{tikzcd}
H^{2i}_{\mathcal{D}}(B_{G\cdot}, A(i)) \arrow[r, "\varepsilon_F"] \arrow[d] & H^{2i}(B_{G\cdot}, F(i)) \arrow[r, equal] \arrow[d] & H^{2i}_{\mathcal{DR}}(B_{G\cdot}) \arrow[d, "\varphi"] \\
H^{2i}(B^{(1)}_{G\cdot\mathrm{an}}, \bar{A}(i)_{\mathcal{D}}) \arrow[r, "\varepsilon_F", "\sim"'] & H^{2i}(B^{(1)}_{G\cdot}, \bar{\Omega}^{>i}) \arrow[r, equal] & H^{2i}(W^{>i,\cdot}).
\end{tikzcd}$$

We are interested in the left vertical arrow. To compute it it suffices to compute the right arrow φ.

THEOREM A4.2. The composition of φ with the isomorphism of A2.1 $H^{2i}(W^{>i,\cdot}) = S^i(\mathfrak{G}')^{\mathfrak{G}}$ coincides with the isomorphism of A3, b).

Proof. Indeed, this composition satisfies all the conditions of A3, b). ■

A5. Comparison with Borel Regulators. From A4.2 and A3, c) it follows that jets of Chern classes are canonical generators of $H^{\cdot}(\mathfrak{G})$. Namely, we have the following result.

COROLLARY A5.1. For i ⩾ 1 the composition $H^{2i}_{\mathcal{DR}}(B_{G\cdot}) \xrightarrow{\varphi} H^{2i}(W^{>i,\cdot}) \xrightarrow{\Pi 2,1} H^{2i-1}(W^{*\cdot}/W^{>i\cdot}) \to H^{2i-1}(W^{*\cdot}/W^{>1,\cdot}) = H^{2i-1}(\mathfrak{G}) \overset{\Pi 3.a}{=} H^{2i-1}_{\mathcal{DR}}(G)$ coincides with the composition $H^{2i}_{\mathcal{DR}}(B_{G\cdot}) \to P^{2i}(B_{G\cdot}) \xrightarrow{T^{-1}} \mathrm{Prim}^{2i-1}(G) \to H^{2i-1}_{\mathcal{DR}}(G)$. In particular, for G = GL_n the images of the Chern classes c_i are canonical generators $\mathbf{v}_i$ of the ring $H^{\cdot}(\mathfrak{gl}_n)$. ■

COROLLARY A5.2. Let $i : B_{GL_n(\mathbf{C})\cdot} \to B_{GL_n\mathbf{C}}$ be the morphism connected with the obvious mapping of the discrete group G(**C**) into the algebraic group G. Then $i^{*}(c_i) \in H^{2i}_{\mathcal{D}}(B_{GL_n(\mathbf{C})}, \mathbf{R}(i)) = H^{2i-1}_{\mathcal{B}} \times (B_{GL_n(\mathbf{C})\cdot}, \mathbf{R}(i)) = H^{2i-1}(GL_n(\mathbf{C}), \mathbf{R}(i))$ coincides with the class of [12, 13] constructed by means of continuous cohomologies. In particular, the Chern characters $K_{2i-1}(\mathbf{C}) \to \mathbf{R}(i-1)$, $K_{4i+1}(\mathbf{R}) \to \mathbf{R}(2i)$ coincide with Borel regulators.

Proof. We consider B.(**C**) as a manifold of class C^{∞}; by definition, we have $H_{\mathrm{cont}}(GL_n(\mathbf{C}), \mathbf{R}) := H^{\cdot}(B_{\cdot}(\mathbf{C}), S^0)$. According to the theorem of van Est $H^{\cdot}_{\mathrm{cont}}(GL_n(\mathbf{C}), \mathbf{R}) = H^{\cdot}(\mathfrak{gl}_n(\mathbf{C}), \mathfrak{u}_n, \mathbf{R})$ (the relative cohomologies of real Lie algebras; $\mathfrak{u}_n$ is the unitary algebra). We recall the construction of this isomorphism and of the Borel classes [12, 9, 21].

Let I_{1s} be the ideal of the distinguished point $e \in B_1$, and let $J_s \subset S^0$ be the cosimplicial ideal generated by I_{1s}^2. Then, as in A4.1, $H^{\cdot}(S^0/I_s) = H^{\cdot}(\mathfrak{gl}_n(\mathbf{C}), \mathbf{R})$; the arrow $S^0 \to S^0/J_S$ defines an imbedding $H^{\cdot}_{\mathrm{cont}}(GL_n(\mathbf{C}), \mathbf{R}) \to H^{\cdot}(\mathfrak{gl}_n(\mathbf{C}), \mathbf{R})$ with image $H^{\cdot}(\mathfrak{gl}_n(\mathbf{C}), \mathfrak{u}_n, \mathbf{R}) \subset H^{\cdot}(\mathfrak{gl}_n(\mathbf{C}), \mathbf{R})$. This is the desired isomorphism. By definition, the Borel class corresponds to the class $\pi_{i-1}(\mathrm{v}_i) \in H^{2i-1} \times (\mathfrak{gl}_n(\mathbf{C}), \mathfrak{u}_n, \mathbf{R}(i-1))$. Since the morphism i* decomposes into the composition $H^{2i}_{\mathcal{D}}(B_{\cdot}, \mathbf{R}(i)) \to H^{2i}(B_{\cdot}, \tilde{\mathbf{R}}(i)_{\mathcal{D}}) \to H^{2i-1}(B_{\cdot}(\mathbf{C}), S^0(i-1)) \to H^{2i-1}(GL_n(\mathbf{C}), \mathbf{R}(i-1))$, the corollary follows from A5.1 and the commutative diagram

$$\begin{tikzcd}
\mathbf{R}(i)_{\mathcal{D}} \arrow[r, "p_i"] & \tilde{\mathbf{R}}(i)_{\mathcal{D}} \arrow[r] & S^0(i-1)[-1] \arrow[dr] & \\
 & & & S^0(i-1)/I_s[-1]. \blacksquare \\
\Omega^{*}[-1] \arrow[uu] \arrow[r] & \mathcal{O}[-1] \arrow[r] & \mathcal{O}/J[-1] \arrow[ur, "\pi_{i-1}"'] &
\end{tikzcd}$$

A6. Cohomologies of Algebras of Flows — Additive K-Theory. We return to the situation at the beginning of part 4 (now k is any field of characteristic 0). According to A4.1 and A1 we have the bijections {G-torsers on X. trivialized on X_0} = {morphisms of d-algebras

$C^{\cdot}(\mathfrak{G})\to N(R^{\cdot})\}$ = {morphisms of dd-algebras $W^{*\cdot} \to N\bar{\Omega}^{*}(R^{\cdot})$}. The Chern classes of the torser are morphisms $S^{l}(\mathfrak{G}')^{G}=H^{2l-1}(\bar{W}^{*\cdot}/\bar{W}^{>l\cdot})\to H^{2l-1}N\bar{\Omega}^{*}(R^{\cdot})/\bar{\Omega}^{>l}(R^{\cdot})$ $(i\geqslant 1)$. We can define $\mathfrak{G}$ -torsers on any commutative dd-algebra $C^{*\cdot}$ as morphisms $C^{\cdot}(\mathfrak{G})\to C^{0\cdot}$ or, equivalently, as morphisms $W^{*\cdot} \to C^{*\cdot}$; to the $\mathfrak{G}$ -torsers there correspond their Chern classes... .

Here is an important example. Let R be a commutative algebra. We consider the commutative dd-algebra $C^{\cdot}(\mathfrak{G}\otimes R, \Omega^{*}(R/k))$ — the complex of cochains of $\mathfrak{G}\otimes R$ considered as a Lie algebra over k with coefficients in the trivial $\mathfrak{G}\otimes R$ -module $\Omega^{*}(R/k)$. On it there is a canonical $\mathfrak{G}$ -torser taking $l\in C^{l}(\mathfrak{G})=\Lambda^{l}(\mathfrak{G}')$ into $l_{R}\in C^{l}(\mathfrak{G}\otimes R, R)=C^{l}(\mathfrak{G}\otimes R, \Omega^{0}(R))$, $l_{R}(g_{1}\otimes r_{1},\ldots g_{l}\otimes r_{l})=l(g_{1},\ldots,g_{l})\otimes r_{1}\ldots r_{l}$. We assume that $\mathfrak{G}=\mathfrak{G}\mathfrak{l}_{n}$; we obtain the i-th Chern class of our torser: $c_{l}\in H^{2l-1}(\mathfrak{G}\otimes R, \Omega^{*}(R)/\Omega^{>l}(R))$. It defines mappings $c_{lj}:H_{j}(\mathfrak{G}\otimes R, k)\to H^{2l-1-j}(\Omega^{*}(R/k)/\Omega^{>l}(R/k))$.

We consider the stable situation $\mathfrak{G}=\varinjlim \mathfrak{G}\mathfrak{l}_{n}$. Then $H_{\cdot}(\mathfrak{G}\otimes R)$ is equipped in a natural way with the structure of a Hopf algebra; we denote by $K_{\cdot}^{\mathrm{add}}$ (R/k) its generators. This is an additive analogue of Quillen's K-functor. Restricting c_{ij} to K^{add}, we obtain the morphism

$$c_{lj}:K_{j}^{\mathrm{add}}(R/k)\to H^{2l-j-1}(\Omega^{*}(R/k)/\Omega^{>l}(R/k)).$$

Recently B. L. Zygan and B. L. Feigin proved the following remarkable theorem (for j = 2 this fact was established by Bloch [11] for any rings with $1/2\in R$; for R = k it is a corollary of A5.1).

<u>THEOREM.</u> If R is a smooth ring, then $\underset{l}{\oplus} c_{lj}:K_{j}^{\mathrm{add}}(R/k)\to\oplus H^{2l-j-1}(\Omega^{*}(R)/\Omega^{>l}(R))$ is an isomorphism. ■

In conclusion we note that it would be very interesting to compare deformations of $K_{\cdot}^{\mathrm{add}}$ and $K_{\cdot}$.

3. Values of L-Functions

We fix a number field E; let $\mathscr{M}=\mathscr{M}(\mathbf{Q}, E)$ be the category of Grothendieck E-motifs over Q (see 2.4). To a motif $M\in\mathscr{M}$ there correspond its L-functions $L^{(j)}(M, s):=L(H^{j}(M\otimes\bar{\mathbf{Q}}, \mathbf{Q}_{l}), s)$ — $E\otimes\mathbf{C}$ -valued analytic functions of complex argument s. We shall assume that the familiar conjectures regarding analytic continuation and the functional equation are satisfied for $L^{(j)}$. Thus, if $L_{\infty}^{(j)}(M, s)$ and $\varepsilon^{(j)}(M, s)$ are the ε-multiples corresponding to L, then $(L_{\infty}\cdot L)^{(j)}(M, s)=(\varepsilon_{\infty}\cdot\varepsilon)^{(j)}(M, s)\cdot(L_{\infty}\cdot L)^{(j)}(M^{0}, j+1-s)$ for some $M^{0}\in\mathscr{M}$; we have $E[X]^{0} = E[X]$ for $X\in\mathscr{V}_{\mathbf{Q}}$.

We recall that $\varepsilon^{(j)}$ have no zeros of poles, while $L_{\infty}^{(j)}$ has no zeros and its poles lie among integral points $\leqslant j/2$; from the explicit form of the L_{∞}-multiples we obtain the following result.

<u>LEMMA 3.1.</u> The order of the pole of $L_{\infty}(M^{0}, s)$ at the integral point s = n coincides with the order of the pole of $L_{\infty}(M^{0}, s)$ for s = n and is equal to $\dim_{E\otimes\mathbf{R}} H_{\mathscr{B}}^{j}(M\otimes\mathbf{R}, \mathbf{R}(j-n))/\pi_{j-n}F^{j+1-n}H_{\mathscr{D}\mathfrak{R}}^{j}(M\otimes\mathbf{R})$. ■

3.2. From the standard exact sequence

$$\ldots\to H_{\mathscr{B}}^{j}(M\otimes\mathbf{R}, \mathbf{R}(l-1))\to H_{\mathscr{D}}^{j+1}(M\otimes\mathbf{R}, \mathbf{R}(l))\to F^{l}H_{\mathscr{D}\mathfrak{R}}^{j+1}(M\otimes\mathbf{R})\to\ldots$$

we find for n ⩽ j/2 the short exact sequence

$$0\to F^{j+1-n}H_{\mathscr{D}\mathfrak{R}}^{j}(M\otimes\mathbf{R})\to H_{\mathscr{B}}^{j}(M\otimes\mathbf{R}, \mathbf{R}(j-n))\to H_{\mathscr{D}}^{j+1}(M\otimes\mathbf{R}, \mathbf{R}(j-n+1))\to 0.$$

We note that there is an E-structure on the first two terms of this sequence: they coincide with $[F^{j+1-n}H_{\mathscr{D}\mathfrak{R}}^{j}(M)]\otimes\mathbf{R}$ and $[H_{\mathscr{B}}^{j}(M\otimes\mathbf{R}, \mathbf{Q}(j-n))]\otimes\mathbf{R}$, respectively. Therefore, on

$$\det H_{\mathscr{D}}^{j+1}(M\otimes\mathbf{R}, \mathbf{R}(j-n+1))=\det H_{\mathscr{B}}^{j}\cdot\det^{-1}F^{j+1-n}H_{\mathscr{D}\mathfrak{R}}^{j}$$

there is the natural E-structure

$$\mathscr{L}(j, n):=\det H_{\mathscr{B}}^{j}(M\otimes\mathbf{R}, \mathbf{Q}(j-n))\cdot\det^{-1}(F^{j+1-n}H_{\mathscr{D}\mathfrak{R}}^{j}(M)).$$

<u>LEMMA 3.3.</u> The order of a zero of $L^{(j)}(M^{0}, s)$ at an integral point n < j/2 is equal to $d(j, n):=\dim H_{\mathscr{D}}^{j+1}(M\otimes\mathbf{R}, \mathbf{R}(j+1-n))$.

<u>Proof.</u> The Euler product for $L^{(j)}$ converges absolutely (and hence ≠ 0) for Re s > j/2 + 1 (the Weyl conjectures proved by Deligne + the conjectures regarding degenerate local

multiples). Therefore, from the functional equation it follows that if n < j/2, then the order of the zero of $L^{(j)}$ for s = n is equal to the order of the pole of $L_\infty^{(j)}$. The remainder consists of Lemma 3.1 and the exact sequence of 3.2. ■

We assume that n < j/2.

Conjecture 3.4. a) The morphism $r_{\mathscr{D}}\otimes\mathbf{R}: H_{\mathscr{A}}^{j+1}(M_{\mathbf{Z}}, \mathbf{Q}(j+1-n))\otimes\mathbf{R}\rightarrow H_{\mathscr{D}}^{j+1}(M\otimes\mathbf{R}, \mathbf{R}(j+1-n))$ is an isomorphism.

b) Let $c(j, n)\in E\otimes\mathbf{R}$ be a number such that $L^{(j)}(M^0, s)=c(j, n)(s-n)^{d(j,n)}+o(s-n)^{d(j,n)}$. Then $c(j, n)\cdot\mathscr{L}(j, n)=\det r_{\mathscr{D}}(H_{\mathscr{A}}^{j+1}(M_{\mathbf{Z}}, \mathbf{Q}(j+1-n)))$.■

3.5. We note that for almost all, n (for example, for n ≤ 0) $F^{j+1-n}H_{\mathscr{D}\mathfrak{R}}^{j}(M)=0$. For such n the conjecture contains the determinant of the matrix of periods of elements of $H_{\mathscr{A}}^{j+1}$ over topological cycles ("the intersection index of topological and algebraic cycles"). Is it possible, at least hypothetically, to determine these matrices themselves on the basis of the L-function (up to multiplication by rational numbers) and not only their determinants?

3.6. Using the functional equation, we can rewrite 3.4, b) as a conjecture regarding the values of L-functions at integral points lying in the region of absolute convergence. We use the following lemma.

LEMMA 3.6.1. Let m be an integer. Then

a) The product of any finite number of finite L-multiples $L_p^{(j)}(M, m)$ belongs to E* if it does not vanish. For m ⩾ (j + 1)/2 it is always finite.

b) Let χ be the E*-valued Dirichlet character with which Aut **C** acts on $[\det H^j(M\otimes\mathbf{C}, \mathbf{Q}_l)](dj/2)$ (where $d = \dim H^j$), and let $\varepsilon^{(j)}(M, n)\in(E\otimes\mathbf{C})^*$ be the constants in the functional equation. Then $\sigma\varepsilon^{(j)}(M, n) = \chi(\sigma)\varepsilon^{(j)}(M, n)$ for any $\sigma\in\mathrm{Aut}\,\mathbf{C}$ and $n\in\mathbf{Z}$.

c) Let $[L_\infty^{(j)}(M^0, s_0)]$ be the leading term of the asymptotics of $L_\infty^{(j)}(M^0, s)$ for $s = s_0$. Then for m ⩾ (j + 1)/2 we have $L_\infty^{(j)}(M, m)=c[L_\infty^{(j)}(M_0, j+1-m)]\in(E\otimes\mathbf{C})^*$, and $c\in(2\pi)^{d_0(M)+(j-2m)\cdot d(M)/2}\cdot\mathbf{Q}^*$. [here $d_l(M)=\dim H_{\mathscr{B}}^{l}(M\otimes\mathbf{R}, \mathbf{Q}(l+1))$, $d(M)=\dim H_{\mathscr{D}\mathfrak{R}}(M)$.]

Proof. a) The finiteness of the L-multiples follows for nondegenerate multiples from the Weyl conjectures; for degenerate multiples it is necessary to use the (unproved) conjectures regarding their weights. All the remaining is obvious. b) Use ([18], (5.4)). For more details see ([17], (5.5)). c) See ([18], (5.4)). ■

For $l_1, l_2\in(E\otimes\mathbf{C})^*$ we say that $l_1\sim l_2$, if $l_1\cdot l_2^{-1}\in E^*$ (see [18]). Thus 3.4, b) determines c(j, n) up to equivalence. We note that 3.6.1, b) determines the equivalence class of ε(M, n); in particular, it does not depend on $n\in\mathbf{Z}$; we denote it by ε(M). From the functional equation we obtain the following assertion.

COROLLARY 3.6.2. $c(j, j+1-m)\sim L^{(j)}(M, m)\cdot\varepsilon(M)^{-1}(2\pi i)^{d_m(M)+(j-2m)d(M)/2}$. ■

We shall verify 3.4, b) precisely in this form (see Chap. 2).

3.7. Suppose now that j is even and n = j/2. We have the isomorphisms

$$H_{\mathscr{D}}^{2n}(M\otimes\mathbf{R}, \mathbf{R}(n))\overset{\varepsilon_{\mathbf{R}}}{\rightarrow}F^n\cap H_{\mathscr{B}}^{2n}(M\otimes\mathbf{R}, \mathbf{R}(n))\overset{\alpha}{\rightarrow}H_{\mathscr{D}}^{2n+1}(M\otimes\mathbf{R}, \mathbf{R}(n+1)).$$

Let $\mathscr{L}_{\mathrm{he}}(M)=H_{\mathscr{A}}^{2n}(M, \mathbf{Q}(n))/H_{\mathscr{A}}^{2n}(M, \mathbf{Q}(n))^0$ be cycles modulo homological equivalence; $r_{\mathscr{D}}$ realizes the imbedding

$$\mathscr{L}_{\mathrm{he}}^{n}(M)\hookrightarrow H_{\mathscr{D}}^{2n}(M\otimes\mathbf{R}, \mathbf{R}(n))\hookrightarrow H_{B}^{2n}(M\otimes\mathbf{R}, \mathbf{R}(n)).$$

Conjecture 3.7.

a) The order of a zero of $L^{(j)}(M^0, s)$ at s = n = j/2 is equal to $d(j, n):=\dim H_{\mathscr{A}}^{2n+1}(M_{\mathbf{Z}}, \mathbf{Q}(n + 1)$.

b) The mapping $\alpha\varepsilon_{\mathbf{R}}r_{\mathscr{D}}+r_{\mathscr{D}}:\mathscr{L}_{\mathrm{he}}^{n}(M)\otimes\mathbf{R}\oplus H_{\mathscr{A}}^{2n+1}(M_{\mathbf{Z}}, \mathbf{Q}(n+1))\otimes\mathbf{R}\rightarrow H_{\mathscr{D}}^{2n+1}(M\otimes\mathbf{R}, \mathbf{R}(n+1))$ is an isomorphism.

c) Let c(j, n), as in 4.4, b), be the leading term of the asymptotics of $L^{(j)}(M^0, s)$ for s = n. Then $c(j, n)\cdot\mathscr{L}(j, n)=\det r_{\mathscr{D}}(\mathscr{L}_{\mathrm{he}}^{n}(M)\oplus H_{\mathscr{A}}^{2n+1}(M_{\mathbf{Z}}, \mathbf{Q}(n+1)))$. ■

Remark. This conjecture as well as the one following is closely connected with Tate's conjectures regarding algebraic cycles [32].

Conjectures 3.4 and 3.7 together with the functional equation determine, up to multiplication by an element of E*, the values of L-functions at any integral point with the exception of the middle of the critical strip. We now consider the middle, i.e., j = 2n − 1. Here a conjecture will be formulated not for L-functions of motifs over **Q** but for L-functions of schemes over **Z**.

Thus, let $X_{\mathbf{Z}}$ be a regular, flat, projective scheme over **Z**, and let $L^{(j)}(X_{\mathbf{Z}}, s)$ be the Euler product corresponding to the j-dimensional cohomologies of the reductions $X_{\mathbf{Z}}$. In part 2.5.3 we defined a pairing $[\ ,\ ]: H^{2n}_{\mathcal{A}}(X_{\mathbf{Z}}, \mathbf{Q}(n))^0 \otimes H^{2(\dim X_{\mathbf{Z}}-n)}_{\mathcal{A}}(X_{\mathbf{Z}}, \mathbf{Q}(\dim X_{\mathbf{Z}}-n))^0 \to \mathbf{R}$. The exact sequence of 4.2 for 2n = j + 1 becomes the isomorphism $F^n H^j_{\mathcal{DR}}(M) \otimes \mathbf{R} \xrightarrow{\sim} H^j_{\mathcal{B}}(M \otimes \mathbf{R}, \mathbf{Q}(j-n)) \otimes \mathbf{R}$; let $p(j, n) := \det F^n H^j_{\mathcal{DR}}(M) \cdot \det^{-1} H^j_{\mathcal{B}}(M \otimes \mathbf{R}, \quad \mathbf{Q}(j-n)) \in \mathbf{R}^*/\mathbf{Q}^*$.

Conjecture 3.8. a) The order of zero of $L^{(j)}(X_{\mathbf{Z}}, s)$ at s = n is equal to $\dim H^{2n}_{\mathcal{A}}(X_{\mathbf{Z}}, \mathbf{Q}(n))^0$. b) The pairing [,] is nondegenerate. c) The leading term of the asymptotics of $L^{(j)}(X_{\mathbf{Z}}, s)$ at s = n is equal to p(j, n) det [,] up to multiplication by elements of **Q***. ■

3.9. If our motif is the spectrum of a number field, then the conjectures are satisfied thanks to Borel's theorem and A5.2; in the case of Artin motifs the conjectures 3.4 and 3.7 coincide with the conjectures of Gross and Stark [24]. For the values at two of L-functions of elliptic curves over Q this is Bloch's conjecture [9, 10, 3].

Conjecture 3.8 for the value at one of L-functions of curves is consistent with the conjecture of Berch–Swinnerton-Dyer; this follows from the coincidence in the case of curves of the intersection index [,] and the Arakelov construction of the Neron–Tate height. Conjecture 3.8.2 is due to Swinnerton-Dyer.

For some integral points n ⩽ j + 1/2 the order of the pole of L_∞ at n is equal to zero; these are critical points in the sense of [18]. For these points $F^{j+1-n} H^j_{\mathcal{DR}}(M) \otimes \mathbf{R} \to H^j_{\mathcal{B}}(M \otimes \mathbf{R}, \mathbf{Q}(j-n)) \otimes \mathbf{R}$ is an isomorphism. The conjectures then assert that the corresponding groups $H_{\mathcal{A}}$ are equal to 0 and $L^{(j)}(M^0, n)$ coincides, up to multiplication by an element of E*, with the determinant of this isomorphism written in rational bases. Using Poincaré duality, we see that it coincides with the determinant of $H^j_{\mathcal{B}}(M^0 \otimes \mathbf{R}, \mathbf{Q}(n)) \otimes \mathbf{R} \to [H^j_{\mathcal{DR}}(M^0)/F^n H^j_{\mathcal{DR}}(M^0)] \otimes \mathbf{R}$. Thus, for critical n our conjecture reduces to the conjecture of Deligne [18].

Finally, we mention that part of the assertions of the conjectures can be formulated for varieties over any fields $k \subset \mathbf{R}$.

Conjecture 3.10. Let X be a smooth proper scheme over $k \subset \mathbf{R}$. Then $r_{\mathcal{D}}: H^j_{\mathcal{A}}(X, \mathbf{Q}(j-n)) \otimes \mathbf{R} \to H^j_{\mathcal{D}}(X \otimes_k \mathbf{R}, \mathbf{R}(j-n))$ for n ⩽ (j − 1)/2 is an isomorphism. ■

From 2.2.3 it follows that for k = **R** conjecture 3.10 implies conjecture 1.10.1 (for $a \neq 2b$).

CHAPTER 2. COMPUTATIONS

4. K_2 for Curves – Formulas for the Regulator

Let X be a smooth affine curve over **R**, let $\bar{X} \supset X$ be a smooth compactification of X, and let $P = \bar{X} \setminus X$. We recall that $H^1_{\mathcal{D}}(X, \mathbf{R}(1)) = \{\varphi \in \Gamma(X_{an}, \mathcal{O}_{X_{an}}/\mathbf{R}(1)) : d\varphi \in \Omega^1_{(X,\bar{X})}\} = \{f \in S^0(X_{an}) : f$ is summable, and if we consider f as a distribution on $\bar{X}_{an}$ we have $d_{\bar{z}} d_z f = \sum \alpha_i \delta_{x_i}$, where $\alpha_i \in \mathbf{R}$, δ_{x_i} is the δ-function at the point $x_i \in P\}$ (here $f = \mathrm{Re}\,\varphi$); $H^2_{\mathcal{D}}(X, \mathbf{R}(2)) = H^1(X_{an}, \mathbf{R}(1)) \subset H^1(X_{an}, \mathbf{C}) = H^1(\Omega^{\cdot}_{X\infty}(X_{an}))$; the ∪ -product $\Lambda^2 H^1_{\mathcal{D}} \to H^2_{\mathcal{D}}$ is given by the formula $f \cup g = f \cdot \pi_1(d_z g) - g \pi_1(d_z f)$ (see 1.2, 1.5). Let F^1 be the Hodge filtration on $H^1_{\mathcal{DR}}$; $\mathbf{R}[P] := H^0(P, \mathbf{R}) \supset \mathbf{R}[P]^0$ – cycles of degree 0 on P.

LEMMA 4.1. a) $H^1(X_{an}, \mathbf{C}) = H^1(\bar{X}_{an}, \mathbf{C}) \oplus (F^1(X) \cap \overline{F^1(X)})$, $H^1(X_{an}, \mathbf{R}(1)) = H^1(\bar{X}_{an}, \mathbf{R}(1)) \oplus (H^1(X_{an}, \mathbf{R}(1)) \cap F^1(X))$.

b) $\dim H^1_{\mathcal{D}}(X, \mathbf{R}(1)) = |P|$, and the residue morphism $\mathrm{div}: H^1_{\mathcal{D}}(X, \mathbf{R}(1)) \to \mathbf{R}[P]$, given by the formula $\mathrm{div}\, f = 2\Sigma(\mathrm{Res}_x d_z f)\, x$, identifies $H^1_{\mathcal{D}}(X, \mathbf{R}(1))/\mathbf{R}$ with $\mathbf{R}[P]^0$.

c) $H^1(X_{an}, \mathbf{R}(1)) \cap F^1(X) = \varepsilon H^1_{\mathcal{D}}(X, \mathbf{R}(1)) = \{d_z f,\ f \in H^1_{\mathcal{D}}(X, \mathbf{R}(1))\} \subset F^1(X) = \Omega^1(X, \bar{X})$. ■

Let $\Pi: H^1(X_{an}) \to H^1(\bar{X}_{an})$ be the projection connected with the decomposition of 1, a). Then 4.1, b) and c) define a pairing $[\ ,\ ]: \Lambda^2 \mathbf{R}[P]^0 \to H^1(\bar{X}_{an}, \mathbf{R}(1))$ such that $\Pi(f \cup g) = [\operatorname{div} f, \operatorname{div} g]$. We shall try to compute $\Pi(f \cup g)$. According to 4.1, b) and c), the image I of the morphism $\cup: \Lambda^2 H^1_{\mathscr{D}}(X, \mathbf{R}(1)) \to H^2_{\mathscr{D}}(X, \mathbf{R}(2)) = H^1(X_{an}, \mathbf{R}(1))$ decomposes into a direct sum $(I \cap \Omega^1_{\bar{X}^\infty}(\bar{X})^{cl}) \oplus (\mathbf{R} \cup H^1_{\mathscr{D}}(X, \mathbf{R}(1))$, whereby the restriction of Π to I coincides with the projection onto the first factor. We consider the pairing given by the formula $\langle \omega, \gamma \rangle = \frac{1}{2\pi i} \int_{X(\mathbf{C})} \omega \Lambda \gamma$, $\omega \in F^1(\bar{X})$, $\gamma \in \Omega^1(X^\infty)$ is summable on X. On closed forms γ of $\Omega^1(\bar{X}^\infty)$ this pairing coincides with the Poincaré duality; it identifies $H^1(\bar{X}_{an}, \mathbf{R}(1))$ with $\operatorname{Hom}(F^1(\bar{X}), \mathbf{R})$. We now note that all forms of I are measurable, and from Stokes' formula it follows easily that $\langle \omega, \mathbf{R} \cup H^1_{\mathscr{D}}(X, \mathbf{R}(1)) \rangle = 0$. Therefore, $\langle \omega, \Pi(f \cup g) \rangle = \frac{1}{2\pi i} \int \omega \Lambda (f \cup g)$. From this we obtain the following assertion.

Assertion 4.2. Let $f, g \in H^1_{\mathscr{D}}(X, \mathbf{R}(1))$, $\omega \in F^1(X) = \Omega^1(\bar{X})$. Then $\langle \omega, \Pi(f \cup g) \rangle = \langle \pi_0 \omega, \Pi(f \cup g) \rangle = \frac{1}{2\pi i} \int_{X(\mathbf{C})} f \cdot d_z g \Lambda \omega$.

Proof. Since f and g have singularities at the points P of the form $c \log |z|$, applying Stokes' formula, we find $\int \omega \Lambda d_{\bar{z}}(f \cdot g) = \int \omega \Lambda d(f \cdot g) = \int d(\omega \cdot f \cdot g) = 0$. Therefore

$$\int \omega \Lambda (f \cup g) = \int \omega \Lambda [f \cdot 1/2(d_z g - d_{\bar{z}} g) - g \cdot 1/2(d_z f - d_{\bar{z}} f)] = \int \omega \Lambda 1/2 [g d_{\bar{z}} f - f d_{\bar{z}} g] = \int f d_{\bar{z}} g \Lambda \omega. \blacksquare$$

Example 4.3. Let $\bar{X}$ be an elliptic curve, and let $\Gamma = H_1(\bar{X}_{an}, \mathbf{Z})$. The Poincaré duality gives an isomorphism $\Gamma = \operatorname{Hom}(\Gamma, \mathbf{Z}(1))$. Hence $\bar{X}(\mathbf{C}) = \Gamma \otimes \mathbf{R}/\Gamma = \operatorname{Hom}(\Gamma, \mathbf{R}(1)/\mathbf{Z}(1))$; we denote by $(\ ,\ )$: $X(\mathbf{C}) \otimes \Gamma \to \mathbf{R}(1)/\mathbf{Z}(1) \subset \dot{\mathbf{C}}^*$ the corresponding pairing. We fix a holomorphic differential ω on $\bar{X}$ such that $\int \omega \Lambda \bar{\omega} = 1$; it defines an imbedding $\Gamma \hookrightarrow \bar{\mathbf{C}}$ and an isomorphism $\bar{X}(\mathbf{C}) = \mathbf{C}/\Gamma$. Let $\alpha = \Sigma \alpha_i x_i \in \mathbf{R}[P]^0$. We define the function ε_α on $\bar{X}(\mathbf{C})$ as the Fourier transform of the function $\gamma \to \frac{\Sigma \alpha_i (x_i, \gamma)}{\gamma \bar{\gamma}}$ on Γ. It is easy to see that ε_α is a summable function of class C^∞ away from x_i and $d_{\bar{z}} d_z \varepsilon_\alpha = \Sigma \alpha_i \delta_{x_i}$. Therefore, $\varepsilon_\alpha \in H^1_{\mathscr{D}}(X, \mathbf{R}(1))$ and $\operatorname{div} \varepsilon_\alpha = \alpha$. It is clear that $d_{\bar{z}} \varepsilon_\alpha$ is the Fourier transform of the function $\gamma \to \sum \frac{\alpha_i (x_i, \gamma)}{\gamma}$. Finally, the integral of 4.2 can be evaluated by convolution of the Fourier transforms:

$$\langle \omega, [\alpha, \beta] \rangle = \sum_{i,j} \frac{\alpha_i \beta_j (x_i - x_j, \gamma)}{\gamma^2 \bar{\gamma}}. \tag{4.3.1}$$

4.4. The formula of 4.2 enables us to compute $\Pi r_{\mathscr{D}}\{\varphi, \psi\}$ for $\varphi, \psi \in \mathcal{O}^*(X)$ ($\{\varphi, \psi\} \in H^2_{\mathscr{A}}(X, \mathbf{Q}(2)) \subset K_2(X) \otimes \mathbf{Q}$): we have $r_{\mathscr{D}}\{\varphi, \psi\} = \ln|\varphi| \cup \ln|\psi|$. However, to verify the conjectures of Sec. 3 we would like to have elements of $H^2_{\mathscr{A}}(\bar{X})$, rather than of $H^2_{\mathscr{A}}(X)$. We shall use the following lemma of Bloch which provides an analogue of the decomposition 4.1, a) for $H_{\mathscr{A}}$.

LEMMA 4.4.1. Let X be a curve over a field k such that all points $P = \bar{X} \setminus X$ are defined over k, and any of their pairwise differences have finite order on the Jacobian of X. Then

a) $K_i(X) \otimes \mathbf{Q}$ is generated by the image of $K_i(\bar{X})$ and $\{K_{i-1}(k), \mathcal{O}^*(X)\}$.

b) $H^2_{\mathscr{A}}(X, \mathbf{Q}(2))$ decomposes into a sum of $H^2_{\mathscr{A}}(\bar{X}, \mathbf{Q}(2))$ and $\{\mathcal{O}^*(X), k^*\} \otimes \mathbf{Q}$. The intersection of these subspaces is $H^2_{\mathscr{A}}(\operatorname{Spec} k, \mathbf{Q}(2))$.

Proof. Since $\operatorname{div} \mathcal{O}^*(X) \otimes \mathbf{Q} = \mathbf{Q}[P]^0$, it follows that the image of the arrow $K_i(X) \otimes \mathbf{Q} \to K_{i-1}(P) \otimes \mathbf{Q} = K_{i-1}(k) \otimes \mathbf{Q}[P]$ of the exact localization sequence coincides with $K_{i-1}(k) \otimes \mathbf{Q}[P]^0$ (the image is always contained in $\mathbf{Q}[P]^0$). The lemma now follows from the exactness of the localization sequence. ■

Remark 4.4.2. Let X be any regular scheme, and let $U \subset X$ be an open subscheme. Then $H^2_{\mathscr{A}}(X, \mathbf{Q}(2)) \to H^2_{\mathscr{A}}(U, \mathbf{Q}(2))$ is an imbedding. Indeed, by induction on $\dim(X \setminus U)$ everything reduces to the case where $X \setminus U = \operatorname{Spec} K$, K a field. From the exact localization sequence it follows then that the kernel of our arrow is the image of $H^{2-2n}_{\mathcal{A}}(\operatorname{Spec} K, \mathbf{Q}(2-n))$, $n=\operatorname{codim}$ × (X − U). It is easy to see that all these groups are equal to 0 [for n > 1 this is obvious; if n = 1, then $H^0_{\mathcal{A}}(\operatorname{Spec} K, \mathbf{Q}(1)) \subset K_2(K) \otimes \mathbf{Q}$. But $K_2(K)$ is generated by symbols; hence $K_2(K) = H^2_{\mathcal{A}}(K, \mathbf{Q}(2))$ and $H^0_{\mathcal{A}}(K, \mathbf{Q}(1))=0$] ■

Therefore, in the situation of 4.4 there arises a canonical projection $\Pi: H^2_{\mathcal{A}}(X, \mathbf{Q}(2)) \to H^2_{\mathcal{A}}(\bar{X}, \mathbf{Q}(2))/H^2_{\mathcal{A}}(k, \mathbf{Q}(2))$. It is clear that $\Pi \circ \{\ ,\ \}: \mathcal{O}^*(X) \otimes \mathcal{O}^*(X) \otimes \mathbf{Q} \to H^2_{\mathcal{A}}(\bar{X}, \mathbf{Q}(2))/H^2_{\mathcal{A}}(k, \mathbf{Q}(2))$ passes through the arrow $\Lambda^2 \mathbf{Q}[P]^0 \to H^2_{\mathcal{A}}(\bar{X}, \mathbf{Q}(2))/H^2_{\mathcal{A}}(k, \mathbf{Q}(2))$. If k is a number field, then $H^2_{\mathcal{A}}(k, \mathbf{Q}(2)) \subset K_2(k) \otimes \mathbf{Q} = 0$, and we obtain the pairing

$$\{\ ,\ \}: \Lambda^2 \mathbf{Q}[P]^0 \to H^2_{\mathcal{A}}(\bar{X}, \mathbf{Q}(2)). \tag{4.4.3}$$

By definition, $\Pi\{f, g\} = \{\operatorname{div} f, \operatorname{div} g\}$ for $f, g \in \mathcal{O}^*(X)$.

If $k \subset \mathbf{R}$, then $\mathbf{R} \cdot r_{\mathcal{D}}\{\mathcal{O}^*(X), k^*\} = F^1(X \underset{k}{\otimes} \mathbf{R}) \cap \bar{F}^1(X \underset{k}{\otimes} \mathbf{R})$, so that $r_{\mathcal{D}}\Pi = \Pi r_{\mathcal{D}}$ and $r_{\mathcal{D}}\{\ ,\ \} = [\ ,\ ]$.

Therefore, 4.2 makes it possible to compute the regulators of elements of the form $\Pi\{\alpha, \beta\}$, $\alpha, \beta \in \mathbf{Q}[P]^0$. In the next section we carry out these computations for a modular curve, while now we shall say a few words regarding the case of an elliptic curve. Thus, let $\bar{X}$ be an elliptic curve over $\mathbf{Q}$, and let $\alpha, \beta \in \mathbf{Q}[\bar{X}(\mathbf{Q})_{\mathrm{tors}}]^0$. Then $r_{\mathcal{D}}\{\alpha, \beta\}$ can be computed by formula 4.3.1. If we knew that $0 \neq \{\alpha, \beta\} \in H^2_{\mathcal{A}}(\bar{X}_{\mathbf{Z}}, \mathbf{Q}(2)) \subset H^2_{\mathcal{A}}(\bar{X}, \mathbf{Q}(2))$, then, according to conjecture 3.4, $H^2_{\mathcal{A}}(\bar{X}_{\mathbf{Z}}, \mathbf{Q}(2))$ must be generated by (α, β} and $r_{\mathcal{D}}\{\alpha, \beta\}/L(\bar{X}, 2) \in \mathbf{Q}$, i.e., the value of the L-function of X at two coincides with the value of the Eisenstein–Kronecker series 4.3.1. If $\bar{X}$ admits complex multiplication, then this fact is well known (the L-function itself coincides with such a series), but for curves without complex multiplication this is surprising. Bloch and Grayson composed a program to compute $r_{\mathcal{D}}\{\alpha, \beta\}$ for Weyl curves. It was found that quite frequently $r_{\mathcal{D}}\{\alpha, \beta\}/L(\bar{X}, 2)$ to high accuracy does not belong to $\mathbf{Q}$. Fortunately, however, Bloch and Grayson showed that such {α, β} do not belong to $H^2_{\mathcal{A}}(\bar{X}_{\mathbf{Z}}, \mathbf{Q}(2))$. We note that the integral condition was omitted in the original formulation of the conjecture [9, 10, 3].

4.5. To verify the integral property of {α, β} on modular curves we use the following version of 4.4.

Let S be a Dedekind ring with field fractions k; let $\bar{M}_S$ be a projective curve over S; let $P_S \subset \bar{M}_S$ be a closed subscheme; let $M_S = \bar{M}_S \setminus P_S$; $\bar{M} = \bar{M}_S \otimes k$, ... We assume that $\bar{M}_S$ is regular, P_S/S is the disjoint union of several copies of Spec S, all pairwise difference of points in P have finite order on the Jacobian, and $(\operatorname{Pic} \operatorname{Spec} S) \otimes \mathbf{Q} = 0$.

LEMMA 4.5.1. If for any closed point $\alpha \in S$ and any $f \in \mathcal{O}^*(M)$ the order of a zero of f at general points of irreducible components of the fiber over α is constant (i.e., it depends only on α and f), then $\{\mathcal{O}^*(M), \quad \mathcal{O}^*(M)\} \otimes \mathbf{Q} \cap H^2_{\mathcal{A}}(\bar{M}, \mathbf{Q}(2)) \subset H^2_{\mathcal{A}}(\bar{M}_S, \mathbf{Q}(2)) + \{\mathcal{O}^*(M), k^*\} \cdot \mathbf{Q}$.

Proof. The condition of the lemma implies that $\mathcal{O}^*(M) \otimes \mathbf{Q} = k^* \cdot \mathcal{O}^*(M_S) \otimes \mathbf{Q}$. Therefore, any pairwise difference of divisors of P_S are divisors of elements of $\mathcal{O}^*(M_S) \otimes \mathbf{Q}$. The remainder follows from the exact localization sequence of the pair ($\bar{M}_S$, P_S). ■

COROLLARY 4.5.2. If S is the ring of integers in a field of algebraic numbers, then under the conditions of 4.5 we have $\{\mathbf{Q}[P]^0, \mathbf{Q}[P]^0\} \subset H^2_{\mathcal{A}}(\bar{M}_S, \mathbf{Q}(2))$. ■

5. Values at Two of L-Functions of Modular Curves

In this section for any curve over $\mathbf{Q}$ uniformized by modular functions we construct a subgroup in K_2 whose image in the cohomologies satisfies Conjecture 3.4.

Standard notation: G = GL_2, Z = Gm the center of G, $\mathbf{A} = \mathbf{R} \times \mathbf{A}^f$ adeles of the field $\mathbf{Q}$, $z = x + iy: H^{\pm} := \mathbf{P}^1_{\mathbf{R}} \setminus \mathbf{P}^1(\mathbf{R}) \hookrightarrow \mathbf{P}^1_{\mathbf{R}}$ the half plane with the usual right action of $G(\mathbf{R})$ ($H^{\pm}$ is an analytic space over $\mathbf{R}$).

5.1. Formulation of the Theorem. Let $M/\mathbf{Q}$ be a scheme of moduli of elliptic curves with structures of all levels, let $\bar{M}$ be the compactification of M, and let $P = \bar{M} \setminus M$. The group acts $G(\mathbf{A}^f)/Z(\mathbf{Q})$ acts from the left on these schemes. The scheme M is the projective limit of schemes of finite type $K \setminus M$, where K runs through open and compact subgroups in $G(\mathbf{A}^f)$; the same holds for $\bar{M}$ and P. We have the canonical isomorphism $(M \otimes \mathbf{R})_{\mathrm{an}} = H^{\pm} \times G(\mathbf{A}') / G(\mathbf{Q})$.

If H is a contravariant functor on schemes of finite type over $\mathbf{Q}$ [for example, $H^*_{\mathscr{D}}(\cdot\otimes\mathbf{R})$], then we set $H(M):=\varinjlim H(K\diagdown M)$; the same goes for $\overline{M}$ and P; then $G(\mathbf{A}^f)$ acts on H(M).

Let F be the union of cyclotomic fields, and let $S\subset F$ be the ring of integers. Our schemes are schemes over F in a natural way. They extend in a natural way to schemes $\overline{M}_S$, P_S, $M_S = \overline{M}_S \setminus P_S$ over S with an action of $G(\mathbf{A}^f)$ — projective limits of regular proper schemes of finite type over $\mathbf{Z}$ (see [20]).

According to 4.4.1, we have the imbeddings $H^2_{\mathscr{A}}(\overline{M}_S, \mathbf{Q}(2)) \subset H^2_{\mathscr{A}}(\overline{M}, \mathbf{Q}(2)) \subset H^2_{\mathscr{A}}(M, \mathbf{Q}(2))$. We set $\mathscr{P}:=(\{\mathcal{O}^*(M),\ \mathcal{O}^*(M)\}\mathbf{Q})\cap H^2_{\mathscr{A}}(\overline{M},\ \mathbf{Q}(2))=\{H^1_{\mathscr{A}}(M,\ \mathbf{Q}(1)),\ H^1_{\mathscr{A}}(M,\mathbf{Q}(1))\}\ \cap H^2_{\mathscr{A}}(\overline{M},\mathbf{Q}(2))$.

<u>THEOREM 5.1.1</u>. $\mathscr{P}\subset H^2_{\mathscr{A}}(\overline{M}_S, \mathbf{Q}(2))$.

For the proof see part 5.5.

We wish to compute the mapping of the regulator on $\mathscr{P}$. For this we decompose the motif $\overline{\mathbf{Q}}[\overline{M}]$ by the action $G(\mathbf{A}^f){:}\overline{\mathbf{Q}}[\overline{M}]=[\overline{M}]^0\oplus[\overline{M}]^1\oplus[\overline{M}]^2, [\overline{M}]^0=\overline{\mathbf{Q}}[\operatorname{Spec}F], [M]^2=[\overline{M}]^0(-1), [\overline{M}]^1=\Sigma M_V\otimes V$, where V runs through all irreducible parabolic $\overline{\mathbf{Q}}$-representations of $G(\mathbf{A}^f)$ of weight 2. Since $\mathscr{P}$ is a $G(\mathbf{A}^f)$-submodule of $H^2_{\mathscr{A}}(M,\mathbf{Q}(2))$ of weight 2, we have $\mathscr{P}\otimes\overline{\mathbf{Q}}=\Sigma\mathscr{P}_V\otimes V$, $\mathscr{P}_V\subset H^2_{\mathscr{A}}(M_V,\mathbf{Q}(2))$.

We recall that to each automorphic irreducible $\mathbf{C}$-representation of the π group $G(\mathbf{A}^f)$ there corresponds its L-function L(π, s), the ε-multiple ε(π, s), and the L_∞-multiple $L_\infty(\pi, s)$; we orthonormalize them so that the functional equation has the form $(L_\infty\cdot L)(\pi, s) = \varepsilon(\pi, s)(L_\infty\cdot L)(\pi^*, 2 - s)$. If V is an irreducible, automorphic $\mathbf{Q}$-representation of $G(\mathbf{A}^f)$, then V is defined over some field E which is finite over $\mathbf{Q}$, $V=V_E\underset{E}{\otimes}\overline{\mathbf{Q}}$. We define the L-function of V as an $E\otimes\mathbf{C}\subset\overline{\mathbf{Q}}\otimes\mathbf{C}$ -valued function whose components corresponding to the imbeddings $i{:}E\hookrightarrow\mathbf{C}$, are $L(V_F\underset{i}{\otimes}\mathbf{C}, s)$; ε(V, s) and $L_\infty(V, s)$ are defined similarly. For sufficiently large Re s the L-function is given by the Euler product $L(V, s) = \Pi L_p(V, s)$. We assume that V is parabolic of weight 2. Then $L(V, s) = \Pi L_p(V, s)$ for Re s > 1.5 and $L_\infty(V, s) = (2\pi)^{-s}\Gamma(s)$. Since the ε-multiple never vanishes, we have $L(V,0)=0,\ l_0(V):=\frac{d}{ds}L(V,s)|_{s=0}\neq 0$.

<u>THEOREM 5.1.2.</u> $r_{\mathscr{D}}(\mathscr{P}_V)=l_0(V)\cdot H^1_{\mathscr{B}}(M_V\otimes\mathbf{R},\mathbf{Q}(1))\subset H^1_{\mathscr{B}}(M_V\otimes\mathbf{R},\mathbf{R}(1))=H^2_{\mathscr{D}}(M_V\otimes\mathbf{R},\mathbf{R}(2))$.

For the proof see part 5.5.

We recall that $L(M_V, s) = L(V^*, s)$ (Eichler, Schmidt, Deligne,...) and that $\dim H^1_{\mathscr{B}}(M_V\otimes\mathbf{R},\mathbf{Q}(1)) = 1$ (the theorem on multiplicity one of the spectrum). From 5.1.1 and 5.1.2 it therefore follows that $H^2_{\mathscr{A}}(M_{V\mathbf{Z}},\mathbf{Q}(2))$ contains a subgroup satisfying Conjecture 3.4; of course, the same holds for any motifs decomposing into a sum of motifs of the form M_V.

5.2. In this subsection we reduce 5.1.2 to the computation of certain integrals. The scheme M is the projective limit of schemes of finite type; we shall explain how to integrate $M(\mathbf{C})$. Let φ be a 2-form of class C^∞ on $M(\mathbf{C})$, and let U be an open subset of $M(\mathbf{C})$. If $K\subset G(\mathbf{A}^f)$ is such that φ and U are K-invariant, then φ is a 2-form on $K\setminus M(\mathbf{C})$; we set $\int_U\varphi=-6^{-1}\chi(K\diagdown M(\mathbf{C}))^{-1}\int_{K\diagdown U}\varphi$. The normalization is chosen so that under the identification of $M(\mathbf{C})$ with $H^\pm(\mathbf{C}) \times G(\mathbf{A}^f)/G(Q)$ the preimage of this measure on $H^\pm(\mathbf{C}) \times G(\mathbf{A}^f)$ coincides with $\varphi\times d\mu^f$, where $d\mu^f$ is a Haar measure on $G(\mathbf{A}^f)$ such that $\int_{G(\hat{Z})}d\mu^f=1$. The canonical isomorphism $\int{:}H^2_{\mathscr{B}}(\overline{M}\otimes\mathbf{R},\mathbf{Q}(1))\rightrightarrows\mathbf{Q}$; is defined similarly; if $\alpha\in H^2_{\mathscr{B}}(K\diagdown\overline{M}\otimes\mathbf{R},\mathbf{Q}(1))$, then $\int\alpha:=-6^{-1}\chi(K\diagdown\overline{M})^{-1}\int\alpha$. It gives the Poincaré duality — the isomorphism of $G(\mathbf{A}^f)$-modules $H^1_{\mathscr{B}}(\overline{M}\otimes\mathbf{R},\mathbf{Q}(1))^*\simeq H^1_{\mathscr{B}}(\overline{M}\otimes\mathbf{R},\mathbf{Q})$.

We shall deal with $(\overline{\mathbf{Q}}\otimes\mathbf{C})^*$ -valued functions on the set of irreducible, parabolic $\overline{\mathbf{Q}}$-representations of weight 2 [i.e., on the set of irreducible components of $H^1_{\mathscr{B}}(\overline{M}\otimes\mathbf{R},\overline{\mathbf{Q}})$ or $\Omega^1(\overline{M})\otimes\overline{\mathbf{Q}}$], for example, with the values of L-functions or ε-functions. We say that two functions l_1 and l_2 are equivalent, $l_1 \sim l_2$, if $l_1\cdot l_2^{-1}$ takes values in $\overline{\mathbf{Q}}^*$. For example, for $z^f\in Z(\mathbf{A}^f)$ the value of the central character ω(V) on z^f, i.e., $\omega(V)(z^f)$ is equivalent to one.

We denote by l_1 the equivalence class of functions such that $\Omega^1(M_V)=l_1(V)\cdot H^1_{\mathscr{B}}(M_V,\mathbf{Q})$ under the period isomorphism $\Omega^1(M_V)\otimes\mathbf{R}\simeq H^1_{\mathscr{B}}(M_V,\mathbf{R})$ [we recall that $\Omega^1(M_V)$ and $H^1_{\mathscr{B}}(M_V,\mathbf{Q})$ are one-dimensional $\overline{\mathbf{Q}}$-spaces].

According to Manin—Drinfel'd, we find ourselves in the situation of 4.4. Using 4.2, we can reformulate 5.1.2 as follows.

THEOREM 5.1.2'. Let $V\subset\Omega^1(\bar{M})\otimes\bar{\mathbf{Q}}$ be an irreducible representation. Then for $f, g\in\mathcal{O}^*(M)$ and $v\subset V$ we have

$$(2\pi i)^{-1}\int_{M(\mathbf{C})}\log|f|\overline{d\log}\, g\Lambda v\in l_0(V^*)\cdot l_1(V)\cdot\bar{\mathbf{Q}}\subset\bar{\mathbf{Q}}\otimes\mathbf{C};$$

for some f, g, and v it belongs to $(\bar{\mathbf{Q}}\otimes\mathbf{C})^*$.

We rewrite 5.1.2 again. Let $\theta:\mathrm{Aut}\,\mathbf{C}\to\hat{\mathbf{Z}}^*$ be the character of the action on the roots of unity.

We recall (see 3.6) that for a $\bar{\mathbf{Q}}$-valued character χ of the group $\mathrm{Aut}\,F\xrightarrow{\theta}\hat{\mathbf{Z}}^*=\mathbf{A}^{f*}/\mathbf{Q}^{*+}$ $\varepsilon(\chi)\in(\bar{\mathbf{Q}}\otimes\mathbf{C})^*/\bar{\mathbf{Q}}^*$ denotes the equivalence class of $\varepsilon(\chi, n)$, $n\in\mathbf{Z}$, the ε-multiple corresponding to χ.

LEMMA 5.2.1. a) Let $n\in\mathbf{Z}$. Then $\varepsilon(V, n)\sim\varepsilon(\omega(V))$.
b) If $L(V, 1)\neq 0$, then $L(V, 1)\sim l_1(V)$; for any V there is an even χ such that $L(V\otimes\chi(\det), 1)\neq 0$.
c) For even χ we have $l_1(V\otimes\chi(\det))\sim\varepsilon(\chi)\cdot l_1(V)$.

The proof of a) follows, for example, from $\varepsilon(V, S) = \varepsilon(M_{V^*}, S)$ and $\omega(V) = \det M_{V^*}$ and 3.6; however, it is simpler to use the identity $\varepsilon_p(V, \psi(bx)) = \omega(V)(b)\varepsilon_p(V, \psi(X))$ for the local constants directly.

b) is well known (the connection of the values of L-functions at one with periods; see, for example [18]).

c) We decompose the motif $[\bar{M}]^0 = \bar{\mathbf{Q}}[F]$ by the action of $\mathrm{Aut}\,F=\mathbf{A}^{f*}/\mathbf{Q}^{*+}$: $[M]^0=\oplus[\chi]$; the sum goes over all $\bar{\mathbf{Q}}$-valued characters. From the definition it is evident that $G(\mathbf{A}^f)$ acts on $[\chi]$ by means of $\chi^{-1}(\det)$. Therefore, the canonical pairing $[\bar{M}]^0\times[\bar{M}]^1\to[\bar{M}]^1$ gives an isomorphism $[\chi^{-1}]\times M_V\xrightarrow{\sim}M_{V\otimes\chi(\det)}$. If χ is an even character, then $H^0_{\mathcal{DR}}([\chi])=F^0H^0_{\mathcal{DR}}([\chi])$, $H^0_{\mathcal{B}}\times([\chi]\otimes\mathbf{R}, \mathbf{Q})$ are one-dimensional $\bar{\mathbf{Q}}$-spaces. Hence, if for a critical motif N we denote by P(N) its period matrix, we always have $P(N[\chi])=P(N)\cdot P([\chi])$. Since $P([\chi])\sim\varepsilon(\chi)$ (see [18]), everything has been proved. ■

We return to 5.1.2. The functional equation and 5.2.1, a) give $l_0(V^*)\sim\pi^{-2}\varepsilon(\omega(V))^{-1}\cdot L(V, 2)$. We choose an even χ so that $L(V\otimes\chi(\det), 1)\in(\bar{\mathbf{Q}}\otimes\mathbf{C})^*$; then by 5.2.1, b), c) we have $l_1(V)\sim L(V\otimes\chi(\det), 1)\cdot\varepsilon(\chi)^{-1}$. Therefore, $l_0(V^*)\cdot l_1(V)\sim\pi^{-2}\cdot\varepsilon(\omega(V)\chi)^{-1}L(V\otimes\chi(\det), 1)L(V, 2)$, and we can rewrite 5.1.2 in the following form.

THEOREM 5.1.2". Let $V\subset\Omega^1(\bar{M})\otimes\bar{\mathbf{Q}}$ be an irreducible $\bar{\mathbf{Q}}$-representation. We choose some even χ so that $L(V\otimes\chi(\det), 1)\in(\mathbf{C}\otimes\bar{\mathbf{Q}})^*$. Then for $f, g\in\mathcal{O}^*(M)$ and $v\in V$ we have

$$(2\pi i)^{-1}\int\log|f|\overline{d\log g}\Lambda v\in\pi^{-2}\varepsilon(\omega(V)\cdot\chi)^{-1}\cdot L(V\otimes\chi(\det), 1)\, L(V, 2)\cdot\bar{\mathbf{Q}}\subset\mathbf{C}\otimes\bar{\mathbf{Q}},$$

and for some f, g, and v the integral belongs to $(\mathbf{C}\otimes\bar{\mathbf{Q}})^*$.

5.3. Eisenstein Series of $\mathcal{O}^*$ and (M). In this subsection we show that the factors under the integral sign in 5.1.2' coincide with the Eisenstein—Kronecker series. We need the following subgroups of G: $B=\left\{\begin{pmatrix}* & 0\\ * & *\end{pmatrix}\right\}$, $D=\left\{\begin{pmatrix}1 & 0\\ 0 & *\end{pmatrix}\right\}$, $U=\left\{\begin{pmatrix}1 & 0\\ * & 1\end{pmatrix}\right\}$, $B(\mathbf{Q})\supset B(\mathbf{Q}^+)=\{g\in B(\mathbf{Q}):\det g>0\}$.

LEMMA 5.3.1. We have $P(\mathbf{C})=G(\mathbf{A}^f)/B(\mathbf{Q}^+)U(\mathbf{A}^f)$ as $G(\mathbf{A}^f)$-sets. The action of $\sigma\in\mathrm{Aut}\,\mathbf{C}$ on P(C) in this notation is multiplication on the right by $\begin{pmatrix}1 & 0\\ 0 & \theta(\sigma)\end{pmatrix}\in D(\hat{\mathbf{Z}})$ ■

If X is a compact, topological $G(\mathbf{A}^f)$-space and V is a vector Q-space, then $G(\mathbf{A}^f)$ acts on V-valued measures and V-valued functions on X. We denote by $\mathcal{M}_V(X)$ and $\mathcal{F}_V(X)$, respectively, the algebraic parts of these representations; let $\mathcal{M}^0_V(X):=\mathrm{Ker}\left(\int_X:\mathcal{M}_V(X)\to V\right)$. Suppose for brevity that $\mathcal{M}_V:=\mathcal{M}_V(P(\mathbf{C}))$, $\mathcal{F}_V:=\mathcal{F}_V(P(\mathbf{C}))$, $\bar{\mathcal{M}}_V=\mathcal{M}_V(\mathrm{Aut}\,\mathbf{C}\setminus P(\mathbf{C}))=\mathcal{M}_V^{D(\mathbf{A}^f)}$, $\bar{\mathcal{F}}_V=\mathcal{F}_V^{D(\mathbf{A}^f)}$. It is clear that $\mathcal{M}_V=\mathcal{M}_{\mathbf{Q}}\otimes V$, $\mathcal{F}^*_V=\mathcal{M}_{V^*}\dots$.

We define the $G(\mathbf{A}^f)$-morphism $\mathrm{Res}:\Omega^1(M, \bar{M})\otimes\bar{\mathbf{Q}}\to\mathcal{M}^0_{\bar{\mathbf{Q}}}$ as follows (cf. the beginning of part 2). Let $v\in\Omega^1(M, \bar{M})\otimes\bar{\mathbf{Q}}$ and the sufficiently small open subgroup $K\subset G(\mathbf{A}^f)$ be such that v is

K-invariant, i.e., $v \in \Omega^1(K\setminus M)\otimes\bar{\mathbf{Q}}$. Then $\int\limits_{K\cdot x} \mathrm{Res}\, v := -\mathfrak{S}^{-1}\chi(K\setminus M)^{-1}\ \mathrm{Res}_{K\cdot x} v$ is the residue at the point K·x on $K\setminus\bar{M}$. This formula uniquely determines the measure Res v.

We have the exact sequence of representations $0\to\Omega^1(\bar{M})\otimes\bar{\mathbf{Q}}\to\Omega^1(M,\bar{M})\otimes\bar{\mathbf{Q}}\xrightarrow{\mathrm{Res}}\mathscr{M}^0_{\bar{\mathbf{Q}}}\to 0$, admitting a unique $G(\mathbf{A}^f)$-decomposition — the Eisenstein series. It coincides with $d\log\mathcal{O}^*(M\otimes\bar{\mathbf{Q}})\otimes\bar{\mathbf{Q}}$: by the Manin–Drinfel'd theorem $\mathrm{Res}\,d\log\mathcal{O}^*(M\otimes\bar{\mathbf{Q}})=\mathscr{M}^0_{\mathbf{Q}}$; hence, since Ker d log = Ker Res d log, we have $\Omega^1(M,\bar{M})\otimes\bar{\mathbf{Q}}=\Omega^1(\bar{M})\otimes\bar{\mathbf{Q}}\oplus d\log\mathcal{O}^*(M\otimes\bar{\mathbf{Q}})\otimes\bar{\mathbf{Q}}$.

LEMMA 5.3.2. a) $\mathrm{Res}\,d\log\mathcal{O}^*(M)\otimes\mathbf{Q}=\bar{\mathscr{M}}^0_{\mathbf{Q}}=\mathscr{M}^0_{\mathbf{Q}}(G(\mathbf{A}^f)/B(\mathbf{Q})\,(D\cdot U)(\mathbf{A}^f))$. b) $d\log\mathcal{O}^*(M)\otimes\mathbf{Q}=\oplus E_\chi\subset\Omega^1(M,\bar{M})\otimes\bar{\mathbf{Q}}$: the sum is over all even Dirichlet characters χ; here E_χ is the space of Eisenstein series with eigenvalues $1+\chi(p)\cdot p$ of the Hecke operators T_p.

Proof. Since

$$\mathrm{Res}\,d\log\mathcal{O}^*(M)=[\mathrm{Res}\,d\log\mathcal{O}^*(M\otimes\bar{\mathbf{Q}})]^{\mathrm{Aut}\,\bar{\mathbf{Q}}}$$

by Hilbert's theorem 90, part a) follows from 5.3.1. Part b) follows from a. ■

We have thus identified d log g in the integral 5.2.3 with an Eisenstein series. We shall show that log|f| is a (nonholomorphic) Eisenstein–Kronecker series.

Let h be a left $G(\hat{\mathbf{Z}})$-invariant function on $G(\mathbf{A}^f)$ such that $h\left(x\cdot\begin{pmatrix}a_1 & 0\\ * & a_2\end{pmatrix}\right)=h(x)\cdot|a_2/a_1|^f$, $(a_i\in\mathbf{A}^{f*})$, $h(1)=1$; $\mu^f_{G/U}$ is a left invariant measure on $G(\mathbf{A}^f)/U(\mathbf{A}^f)$: $\mu^f_{G/U}(G(\hat{\mathbf{Z}})/U(\hat{\mathbf{Z}}))=1$. If $\varphi\in\mathscr{M}_{\mathbf{R}}$, then we set $\hat{\varphi}^f=\varphi/\mu^f_{G/U}$. This is a function on $G(\mathbf{A}^f)/U(\mathbf{A}^f)$ such that $\hat{\varphi}^f/h$ is right $B(\mathbf{Q}^+)$-invariant. Let $\hat{\varphi}^\infty: H^\pm(\mathbf{C})\to\mathbf{R}$ coincide with $-2\pi y$ on H^+ and be equal to zero on $H^-(\mathbf{C})$. For $\sigma\in\mathbf{R}$ the function $\hat{\varphi}_\sigma(z,g^f):=y^{-\sigma}h(g^f)^{-\sigma}\cdot\hat{\varphi}^\infty(z)\cdot\hat{\varphi}(g^f)$ on $H^\pm(\mathbf{C})\times G(\mathbf{A}^f)$ is right $B(\mathbf{Q}^+)$-invariant. We set $\mathscr{E}_{\varphi,\sigma}:=\sum\limits_{\delta\in G(\mathbf{Q})/B(\mathbf{Q}^+)}\hat{\varphi}_\sigma((z,g^f)\delta)$. If $\sigma>0$, then the series converges absolutely and gives a function $\mathscr{E}_{\varphi,\sigma}: H^\pm(\mathbf{C})\times G(\mathbf{A}^f)/G(\mathbf{Q})\to\mathbf{R}$. If $\varphi\in\mathscr{M}^0_{\mathbf{R}}$, then the limit $\mathscr{E}_\varphi:=\lim\limits_{\sigma\to 0}\mathscr{E}_{\varphi,\sigma}$ exists; we have $\mathscr{E}_{g\varphi}=g\mathscr{E}_\varphi$ for $g\in G(\mathbf{A}^f)$ [4].

LEMMA 5.3.3. If $f\in\mathcal{O}^*(M\otimes\mathbf{C})$, then log|f| is equal to $\mathscr{E}_{\mathrm{Res}\,d\log f}$ up to a constant.

Proof. It is clear that $\mathrm{Res}\,d_z\mathscr{E}_\varphi=\varphi$, and hence everything follows from the fact that $\mathscr{E}_\varphi$ is real. ■

5.4. Rankin's Method. We recall (see [25]) that to each pair V_1, V_2 of irreducible automorphic $\bar{\mathbf{Q}}$-representations of $G(\mathbf{A}^f)$ there corresponds a $\bar{\mathbf{Q}}\otimes\mathbf{C}$ -valued L-function $L(V_1\times V_2, s)$. If V_2 is the space of Eisenstein series of weight (χ,χ'), then $L(V_1\times V_2,s)=L(V_1\otimes\chi(\det), s)\times L(V_1\otimes\chi'(\det), s)$; if both V_i are parabolic, then $L(V_1\times V_2,s)=L^{(2)}(M_{V_1^*}\times M_{V_2^*}, s)$.

THEOREM 5.4.1 (see [25]). Let $V_1\subset\Omega^1(\bar{M})\otimes\bar{\mathbf{Q}}$, $V_2\subset\Omega^1(M,\bar{M})\otimes\bar{\mathbf{Q}}$ be two irreducible $\bar{\mathbf{Q}}$-representations of $G(\mathbf{A}^f)$ with central characters ω_i, whereby $V_1\not\approx V_2{}^*$. Then $L(V_1\otimes V_2, 2)\neq\infty$, and for $v_i\in V_i$, $\varphi\in\bar{\mathscr{M}}^0_{\mathbf{Q}}$ a right $D(\mathbf{A}^f)$-invariant measure on $P(\mathbf{C})$ we have $(2\pi i)^{-1}\int\mathscr{E}_\varphi v_1\wedge\bar{v}_2\in\pi^{-2}\varepsilon\times(\omega_1\omega_2)^{-1}L(V_1\times V_2,2)\cdot\bar{\mathbf{Q}}\subset\bar{\mathbf{Q}}\otimes\mathbf{C}$. If $L(V_1\times V_2,2)\in(\bar{\mathbf{Q}}\otimes\mathbf{C})^*$, then for some φ, v_1, v_2 the integral belongs to $(\bar{\mathbf{Q}}\otimes\mathbf{C})^*$. ■

5.5. Proof of 5.1.2. According to 5.1.2", 5.3.2, and 5.3.3 it suffices for us to prove the following. Let $V\subset\Omega^1(\bar{M})\otimes\bar{\mathbf{Q}}$ be an irreducible $\bar{\mathbf{Q}}$-representation of $G(\mathbf{A}^f)$, and let E_χ be as in 5.3.2, b). Then for any $\varphi\in\bar{\mathscr{M}}^0_{\mathbf{Q}}$, $v_1\in V$, $v_2\in E_\chi$ we have $(2\pi i)^{-1}\int\mathscr{E}_\varphi v_1\wedge\bar{v}_2\in\pi^{-2}\varepsilon(\chi\cdot\omega(V))^{-1}\cdot L(\chi,1)\cdot L(V,2)\cdot\bar{\mathbf{Q}}\subset\bar{\mathbf{Q}}\otimes\mathbf{C}$, and if $L(\chi,1)\neq 0$, then for some φ, v_1, v_2 the integral is nonzero. This follows directly from 5.4.

Proof of 5.1.1. We shall show that we are in the situation of 4.5. It is evident from [20] that all conditions of the beginning of part 4.5 are satisfied. It remains to verify the condition of Lemma 4.5.

$G(\mathbf{A}^f)$ acts on $\bar{M}_S$, and the restriction of this action to $SL_2(\mathbf{A}^f)$ commutes with projection onto S. If $\alpha\in S$ has characteristic p, then $SL_2(\mathbf{A}^f)$ — the set of components of the fiber over α — coincides with $P^1(Q_p)$ equipped with the obvious action of $SL_2(\mathbf{A}^f)$ in terms of projection onto $SL_2(\mathbf{Q}_p)$.

We define the $SL_2(\mathbf{A}^f)$-morphism $\mathrm{ord}_\alpha:\mathcal{O}^*(M)\otimes\mathbf{Q}\to\mathcal{F}_{\mathbf{Q}}(\mathbf{P}^1(\mathbf{Q}_p))$ by the formula $\mathrm{ord}_\alpha(f)(x)=\mathrm{ord}_x f\cdot\mathrm{ord}_x^{-1}p$ [this means the following: take $K\subset G(\mathbf{A}^f)$ such that $f\in\mathcal{O}^*(K\setminus M)\otimes\mathbf{Q}$ and find the order of the zero $\mathrm{ord}_x f$ in the component of the fiber on $K\setminus M$ corresponding to x; the number $\mathrm{ord}_x f/\mathrm{ord}_x p$ does not depend on the choice of K]. We must show that the image of ord_α is $\mathbf{Q}\subset\mathcal{F}_{\mathbf{Q}}(\mathbf{P}^1(\mathbf{Q}_p))$ – the constant functions. If this were not so, there would arise a nonzero $SL_2(\mathbf{A}^f)$-morphism $\mathcal{O}^*(M)\otimes\mathbf{Q}\to\mathcal{F}_{\mathbf{Q}}(\mathbf{P}^1(\mathbf{Q}_p))/\mathbf{Q}$. But the representation on $\mathcal{F}_{\mathbf{Q}}(\mathbf{P}^1(\mathbf{Q}_p))/\mathbf{Q}$ is irreducible and nonautomorphic [since $SL_2(\mathbf{A}^f)$ acts by projection onto $SL_2(\mathbf{Q}_p)$], while $\mathcal{O}^*(M)\otimes\mathbf{Q}$ is automorphic. Contradiction.

6. Values at Two of the L-Functions of the Product of Two Modular Curves

For all notation see Sec. 5. Here we shall construct explicitly a large subgroup in $H^3_{\mathcal{A}}(\overline{M}\times\overline{M},\mathbf{Q}(2))\subset K_1(\overline{M}\times\overline{M})\otimes\mathbf{Q}$, and compute the mapping of the regulator on it by Rankin's method. The idea of the section is due to S. Bloch, who considered the case of the surface $X_0(37)\times X_0(37)$ (a letter to the author of March 19, 1982).

6.1. For $(\alpha,\beta)\in P\times P$, $\alpha\neq\beta$, let $C_{\alpha\beta}\subset\overline{M}\times\overline{M}$ be union of the diagonal, $\alpha\times\overline{M}$, and $\overline{M}\times\beta$. Then for each open compact $K\subset G(\mathbf{A}^f)$ the image of ${}_KC_{\alpha\beta}$ of the curve $C_{\alpha\beta}$ in $K\times K\setminus\overline{M}\times\overline{M}$ is a curve of finite type over $\mathbf{Q}$; we denote by $R\subset H^3_{\mathcal{A}}(\overline{\mathbf{Q}}[\overline{M}\times\overline{M}],\mathbf{Q}(2))$ the submodule generated over $\overline{\mathbf{Q}}[G(\mathbf{A}^f)^2]$ by the images of $H'^{-1}_{\mathcal{A}}({}_KG_{\alpha\beta},\mathbf{Q}(0))$ in

$$H'^{-1}_{\mathcal{A}}(K\times K\setminus\overline{M}\times\overline{M},\mathbf{Q}(0))\subset H^3_{\mathcal{A}}(\overline{M}\times\overline{M},\mathbf{Q}(2)),$$

where (α,β) runs through all pairs of distinct parabolic points.

We decompose the motif $\overline{\mathbf{Q}}[\overline{M}]$ by the action of $G(\mathbf{A}^f)$ (see 5.1) and consider the corresponding decomposition $\overline{\mathbf{Q}}[\overline{M}\times\overline{M}]$. The parts contributing to H^2 are $[\overline{M}]^0\times[\overline{M}]^2\oplus[\overline{M}]^2\times[\overline{M}]^0\oplus[\overline{M}]^1\times[\overline{M}]^1$. The first two terms are motifs of cyclotomic fields; regarding them, see Sec. 7. We consider $[\overline{M}]^1\times[\overline{M}]^1=\Sigma(M_{V_1}\times M_{V_2})\otimes V_1\otimes V_2=\Sigma M_{V_1\times V_2}\otimes V_1\otimes V_2$. If $V_1\simeq V_2\otimes\chi(\det)$ for some character χ, then $M_{V_1\times V_2}$ decomposes into a sum of motifs of the form $[\chi](1)$ and a critical motif.

We henceforth assume that $V_1\not\simeq V_2\otimes\chi(\det)$. Let $R_{V_1\times V_2}\subset H^3_{\mathcal{A}}(M_{V_1\times V_2},\mathbf{Q}(2))$ be the $V_1\otimes V_2$ - component of R; let

$$\Phi_{V_1\times V_2}:=F^2H^2_{\mathcal{DR}}(M_{V_1\times V_2})=F^1H^1_{\mathcal{DR}}(M_{V_1})\otimes F^1H^1_{\mathcal{DR}}(M_{V_2}),$$
$$\dim\Phi_{V_1\times V_2}=1.$$

The space $H^2_{\mathcal{B}}((M_{V_1\times V_2})\otimes\mathbf{R},\mathbf{R}(1))$ is two-dimensional; therefore, $H^3_{\mathcal{D}}((M_{V_1\times V_2})\otimes\mathbf{R},\mathbf{R}(2))=H^2_{\mathcal{B}}(\mathbf{R}(1))/\Phi\otimes\mathbf{R}$ is one-dimensional. On $H^3_{\mathcal{D}}$ there is a natural $\mathbf{Q}$-structure (see 3.1): if we set $\mathcal{L}_{V_1\times V_2}=\det H^2_{\mathcal{B}}(\mathbf{Q}(1))\cdot\Phi^*$, then $H^3_{\mathcal{D}}=\mathcal{L}\otimes\mathbf{R}$. We recall that $L(V_1\times V_2,s)$ is a holomorphic function of s, $L(V_1\times V_2,s)=L\left(M_{V_1^*\times V_2^*},s\right)$, $L(V_1\times V_2,1)=0$, but $l_1(V_1\times V_2):=\frac{d}{ds}L(V_1\times V_2,s)|_{s=1}\in(\overline{\mathbf{Q}}\otimes\mathbf{R})^*$.

THEOREM 6.1.1. We have $R_{V_1\times V_2}\subset H^3_{\mathcal{A}}((M_{V_1\times V_2})_{\mathbf{Z}},\mathbf{Q}(2))$ (see 2.4) and $\mathcal{D}(R_{V_1\times V_2})=l_1(V_1\times V_2)\cdot\mathcal{L}_{V_1\times V_2}\subset H^3_{\mathcal{D}}(M_{V_1\times V_2},\mathbf{R}(2))$.

Thus, $H^3_{\mathcal{A}}((M_{V_1\times V_2})_{\mathbf{Z}},\mathbf{Q}(2))$ contains a subspace for which in $H^3_{\mathcal{D}}$ Conjecture 3.7 is satisfied: $L(V_1\times V_2,s)=L\left(M^*_{V_1}\times M^*_{V_2},s\right)$.

We shall prove the theorem. We first show that $R_{V_1\times V_2}\subset H^3_{\mathcal{A}}((M_{V_1\times V_2})_{\mathbf{Z}},\mathbf{Q}(2))$.

Indeed, elements of $H'^{-1}_{\mathcal{A}}(K\setminus C_{\alpha\beta},\mathbf{Q}(0))$ are triples of functions $\varphi=(\varphi_\Delta,\varphi_\alpha,\varphi_\beta)$ on $K\setminus M$ such that $\varphi\in\mathcal{O}^*(K\setminus\overline{M}\setminus\{\alpha,\beta\})\otimes\mathbf{Q}$ and $\mathrm{ord}_\alpha\varphi_\Delta=-\mathrm{ord}_\alpha\varphi_\alpha$, $\mathrm{ord}_\beta\varphi_\Delta=-\mathrm{ord}_\beta\varphi_\beta$, $\mathrm{ord}_\beta\varphi_\alpha=-\mathrm{ord}_\alpha\varphi_\beta$. From this it is evident that $H'^{-1}_{\mathcal{A}}(K\setminus C_{\alpha\beta},\mathbf{Q}(0))$ decomposes into a sum of the subspace spanned by constant functions and a subspace generated by a triple $(\varphi,\varphi^{-1},\varphi)$, $\mathrm{ord}_\alpha\varphi=1$, $\mathrm{ord}_\beta\varphi=-1$. According to 5.5, we may assume that φ and $(\varphi,\varphi^{-1},\varphi)$ came from a standard proper model over $\mathbf{Z}$, therefore, such elements belong to $H^3_{\mathcal{A}}((\overline{M}\times\overline{M})_{\mathbf{Z}},\mathbf{Q}(2))$. The subspace spanned by constant functions goes over into zero under projection onto $H^3_{\mathcal{A}}(M_{V_1\times V_2},\mathbf{Q}(2))$, $V_1\neq V_2\otimes\chi(\det)$.

We now compute the regulators. Let $v_i\in\Omega^1(M_{V_i^*})$, $v_i\neq0$. Then $\pi_1(v_1\otimes\overline{v}_2)\in H^2_{\mathcal{B}}(M_{V_1^*\times V_2^*}\otimes\mathbf{R},\mathbf{R}(1))=H^2_{\mathcal{B}}(M_{V_1\times V_2}\otimes\mathbf{R},\mathbf{R}(1))^*$, is a generator of $(\mathcal{L}_{V_1\times V_2}\otimes\mathbf{R})^*=(H^2_{\mathcal{B}}(M_{V_1\times V_2}\otimes\mathbf{R},\mathbf{R}(1))/\Phi_{V_1\times V_2}\otimes\mathbf{R})^*$.

Theorem 6.1 now follows from the next lemma and 3.6.2 [since det $H^2(M_{V_1\times V_2}\bar{\otimes}\mathbf{Q}, \mathbf{Q}_l)$ is $\omega(V_1)^{-2}\cdot\omega(V_2)^{-2}$].

LEMMA 6.1.2.

a) $\langle \mathfrak{r}_{\mathcal{D}}R_{V_1\times V_2}, \pi_1(v_1\otimes\bar{v}_2)\rangle = \pi^{-2}\varepsilon(\omega(V_1)\omega(V_2))^{-1}L(V_1\times V_2, 2)\times\bar{\mathbf{Q}}\subset\bar{\mathbf{Q}}\otimes\mathbf{C}$,

b) $\langle \mathcal{L}_{V_1\times V_2}, \pi_1(v_1\otimes\bar{v}_2)\rangle = \varepsilon(\omega(V)\cdot\omega(V_2))\cdot\bar{\mathbf{Q}}\subset\bar{\mathbf{Q}}\otimes\mathbf{C}$.

Proof. a) Let $\varphi_{V_1\times V_2}$ be the projection onto $H^3_{\mathcal{A}}(M_{V_1\times V_2}, \mathbf{Q}(2))$ of the element corresponding to $\varphi=(\varphi_\Delta, \varphi_\alpha, \varphi_\beta)$. Then $\langle \mathfrak{r}_{\mathcal{D}}\varphi_{V_1\times V_2}, \pi_1(v_1\otimes\bar{v}_2)\rangle = (2\pi i)^{-1}\int\limits_{M(\mathbf{C})}\log|\varphi_\Delta|\cdot v_1\wedge\bar{v}_2\in\pi^{-2}\varepsilon(\omega(V_1)\omega(V_2))^{-1}\cdot L(V_1\times V_2, 2)\cdot\bar{\mathbf{Q}}$, according to 1.8 and 5.4. Since for distinct α, β, K functions of the form φ_Δ generate all of $\mathcal{O}^*(M)$ and $L(V_1\times V_2, 2)\in(\bar{\mathbf{Q}}\otimes\mathbf{C})^*$ everything follows from 5.4.

b) We carry out the computation. Let e_i^0 be a generator of $H^1_{\mathcal{B}}(M_{V_i}, \mathbf{Q})$, and let e_i^1 be a generator of $H^1_{\mathcal{B}}(M_{V_i}, \mathbf{Q}(1))$; let $e_i^{0'}\in H^1_{\mathcal{B}}\left(M_{V_i^*}, \mathbf{Q}\right)$, $e_i^{1'}\in\ldots$ be such that $\langle e_i^0, e_i^{1'}\rangle = \langle e_i^{0'}, e_i^1\rangle = 1$; let $u_i=\alpha_i^0e_i^0+\alpha_i^1e_i^1$ be a generator of $\Omega^1(M_{V_i})\subset H^1(M_{V_i}\otimes\mathbf{R}, \mathbf{C})(\alpha_i^j\in\mathbf{R})$. Then $\pi_1(U_1\otimes U_2)=(\alpha_1^0\alpha_2^1)\times(e_1^0\otimes e_2^1)+(\alpha_1^1\cdot\alpha_2^0)(e_1^1\otimes e_2^0)$ is a generator of $\Phi_{V_1\times V_2}$ and $l=(\alpha_1^1\cdot\alpha_2^0)^{-1}\cdot e_1^0\otimes e_2^1 \bmod \Phi_{V_1\times V_2}$ is a generator of $\mathcal{L}_{V_1\times V_2}$. We now recall that $V^*\simeq U\otimes\omega(V)^{-1}(\det)$, whence $M_{V^*}\simeq M_V\times[\omega(V)]$, therefore, $v_i = \varepsilon(\omega(V_i))\cdot(\alpha_i^0e_i^{0'}+\alpha_i^1e_i^{1'})$ is a generator of $\Omega^1\left(M_{V_i^*}\right)$ [see the proof of 5.2.1, c)]. Finally, we have $\langle l, \pi_1(v_1\otimes\bar{v}_2)\rangle = \varepsilon(\omega(V_1))\cdot\varepsilon(\omega(V_2)) = \varepsilon(\omega(V_1)\cdot\omega(V_2))$. ■

7. Cyclotomic Fields

In this section we construct an explicit basis in $H^1_{\mathcal{A}}(F, \mathbf{Q}(n+1)) = K_{2n+1}(F)\otimes\mathbf{Q}$ for cyclotomic F and compute the regulator mapping. We thus prove the conjecture of Sec. 3 for values of Dirichlet L-functions (this conjecture coincides with Gross's conjecture). The construction presented below is a generalization of Bloch's construction [9, 24] of a basis in K_3.

7.0. The symbols <...,...> (see [27]). We first define a "universal" symbol. The localization exact sequence gives isomorphisms $K_{i+1}(\mathbf{Z}[x_0, \ldots, x_n, (1-x_0\ldots x_n)^{-1}], (x_0)) \xrightarrow{\sim} K_i(\mathbf{Z}[x_0, \ldots, x_n]/(1-x_0, \ldots, x_n))$. We denote by $\langle x_0, \ldots x_n\rangle\in K_{n+1}(\mathbf{Z}[x_0, \ldots, x_n, (1-x_0\ldots x_n)^{-1}], (x_0))$ the preimage of the element $\{x_1, \ldots, x_n\}\in K_n(\mathbf{Z}[x_0, \ldots, x_n]/(1-x_0\ldots x_n))$; for n = 0 let $\langle x_0\rangle$ be the preimage of 1.

Suppose now that A is a commutative ring, $I\subset A$ is an ideal, $a_0, \ldots, a_n$ are elements of A with $a_0\in I$, and the element $1 - a_0\ldots a_n$ is invertible in A. There arises the morphism $\rho: \mathbf{Z}[x_0, \ldots, x_n, (1-x_0; x_n)^{-1}]\to A$, $\rho(x_i)=a_i$, $\rho((x_0))\subset I$; we denote by $\langle a_0, \ldots, a_n\rangle\in K_{n+1}(A, I)$ the image of $\langle x_0, \ldots, x_n$ under the morphism ρ. It is clear that $\psi^p\langle x_0, \ldots, x_n\rangle = p^{n+1}\langle x_0, \ldots, x_n\rangle$.

LEMMA 7.0.1. We assume that all a_i for i ⩾ 1 are invertible; we have $a_i\in K_1(A)$ for i ⩾ 1, $1-a_0\ldots a_n\in K_1(A, I)$. Then $\langle a_0, \ldots, a_n\rangle$ coincides with the symbol $\{1-a_0\ldots a_n, a_1, \ldots, a_n\}\in K_{n+1}(A, I)$.

Proof. It suffices to consider the universal case $A=\mathbf{Z}[x_0, x_1^{\pm1}, \ldots, x_n^{\pm1}, (1-x_0\ldots x_n)^{-1}]$, I = (x_0). Localization again gives an isomorphism $K_{n+1}(A, I)\xrightarrow{\sim}K_n(A/(1-x_0\ldots x_n)A)$, which takes both our symbols into the same element. ■

We can thus use the following notation. Let $f, a_1, \ldots, a_n\in A$ and the ideal $I\subset A$ be such that f is invertible in A, a_i are not zero divisors, and the element $a_0 := (1-f)a_1^{-1}\ldots a_n^{-1}$ belongs to $I\subset A$. We set

$$\{f, a_1, \ldots, a_n\} := \langle a_0, a_1, \ldots, a_n\rangle\in H^{n+1}_{\mathcal{A}}(A, I, \mathbf{Q}(n+1))\subset K_{n+1}(A, I)\otimes\mathbf{Q}.$$

We carry out all computations of regulators for $\mathcal{X}$ -cohomologies with rational coefficients.

LEMMA 7.0.2. Let A be a ring of functions on a smooth, affine variety over **R**, let $Y := \operatorname{Spec} A/I\subset X$, and let Z be a relative cycle on X(**R**) modulo Y(**R**). We assume that there is a branch log f of the logarithm f such that log f is single-valued on $\mathcal{Z}$ and is equal to 0 on $\partial\mathcal{Z}$. Then $\int\limits_{\mathcal{Z}}\mathfrak{r}_{\mathcal{D}}\{f, a_1, \ldots, a_n\} = \int\limits_{\mathcal{Z}}\log f\, d\log a_1\wedge\ldots\wedge d\log a_n$.

The proof is an exercise to 1.1, 1.2. ■

7.1. The Main Theorem. Let $A = \mathbf{Q}[t_1, \dots, t_n]$, $I = (\Pi t_i(1-t_i)) \subset A$. Thus, Spec A = $\mathbf{A}^n$ and Spec $A/I \subset \mathbf{A}^n$ is the union of the hyperplanes $t_i = 0$ and $t_i = 1$. We denote by $S./\mathbf{A}^n$ a relative simplicial scheme — "the resolution of singularities" of the scheme $S/\mathbf{A}^n$: this scheme $S./\mathbf{A}^n$ is the coskeleton $S./\mathbf{A}^n$ of the normalization of S. As usual, if F is a field, then $\mathbf{A}^n_F := \mathbf{A}^n \otimes F, \dots$; if f is a rational function on $\mathbf{A}^n_F$., then $\mathbf{A}^n_{F(f)} \subset \mathbf{A}^n_F$ is the complement to the divisor f.

The next result follows immediately from the spectral sequence connected with a simplicial scheme, the compatibility of the regulator with it, and Borel's theorem.

LEMMA 7.1.1. a) For any field F we have $K_l(\mathbf{A}^n_F, S_{\cdot F}) = K_{l+n}(F)$, $H^a_{\mathcal{A}}(\mathbf{A}^n_F, S_{\cdot F}, \mathbf{Q}(b)) = H^{a-n}_{\mathcal{A}}(F, \mathbf{Q}(b))$.

b) Let $\mathcal{X} = \{(t_1, \dots, t_n) \mid t_i \in \mathbf{R},\ 0 \leqslant t_i \leqslant 1\} \subset \mathbf{R}^n$ be a relative cycle on $\mathbf{R}^n$ modulo $S(\mathbf{R})$. If $\tilde{f} \in H^{n+1}_{\mathcal{A}}(\mathbf{A}^n_{\mathbf{C}}, S_{\cdot\mathbf{C}}, \mathbf{Q}(n+1))$ corresponds to $f \in K_{2n+1}(\mathbf{C}) \otimes \mathbf{Q} = H^1_{\mathcal{A}}(\mathbf{C}, \mathbf{Q}(n+1))$, then $r_{\mathcal{D}}(f) = \int_{\mathcal{X}} r_{\mathcal{D}}(\tilde{f}) \in \mathbf{C}/(2\pi i)^{n+1}\mathbf{Q}$. ■

c) If F is a number field, then the mapping $H^{n+1}_{\mathcal{A}}(\mathbf{A}^n_F, S_{\cdot F}, \mathbf{Q}(n+1)) \to \prod_i \mathbf{C}/(2\pi i)^{n+1}\mathbf{Q}$, taking γ into $\left(\int_{\mathcal{X}} r_{\mathcal{D}} i(\gamma)\right)$, where i runs through all C-points of the field F, is an imbedding; moreover, the composition of this mapping with the product of the projections $\mathbf{C}/(2\pi i)^{n+1}\mathbf{Q} \to \mathbf{R}$ of taking the real and imaginary parts is an imbedding. ■

The plan of what follows is to construct many elements of $H^{n+1}_{\mathcal{A}}(\mathbf{A}^n_F, S_{\cdot F}, \mathbf{Q}(n+1))$ with the help of symbols and compute their regulators. By the lemma we then will have elements of $H^1_{\mathcal{A}}(F, \mathbf{Q}(n+1))$ together with their regulators.

We fix a cyclotomic field F. Let $\omega \in F$,. $\omega \neq 1$ be a root of 1. If two collections (a_{ij}), (b_{ik}) of positive integers are given and the index i runs through the values 1,...,n, then we set $f_{a,b}(\omega) := \prod_j \left(1 - \omega \prod_i t_i^{a_{ij}}\right) \prod_k \left(1 - \omega \prod_i t_i^{b_{ik}}\right)^{-1}$ – a rational function on $\mathbf{A}^n_F$.

LEMMA 7.1.2. For any n, ω there are collections (a), (b) such that

$$1.\ (1 - f_{a,b}(\omega)) \prod t_i^{-1} \in I \cdot A_{F(f)};$$

$$2.\ C_{a,b} := \sum_j \prod_i a_{ij}^{-1} - \sum_k \prod_i b_{ik}^{-1} \neq 0.$$

For the proof see part 7.2.

We choose a, b as in the lemma and set $l_{ab}(\omega) := C_{ab}^{-1}\{f, t_1, \dots, t_n\} \in H^{n+1}_{\mathcal{A}}(\mathbf{A}^n_{F(f)}, S_{F(f)}, \mathbf{Q}(n+1))$; we denote by the same symbol the preimage of $l_{a,b}(\omega)$ under the canonical morphism $S_{\cdot F(f)}/\mathbf{A}^n_{F(f)} \to S_{F(f)}/\mathbf{A}^n_{F(f)}$.

LEMMA 7.1.3. The canonical morphism $H^{n+1}_{\mathcal{A}}(\mathbf{A}^n{}_F, S_{\cdot F}, \mathbf{Q}(n+1)) \xrightarrow{\mathrm{res}} H^{n+1}_{\mathcal{A}}(\mathbf{A}^n_{F(f)}, S_{\cdot F(f)}, \mathbf{Q}(n+1))$ is an imbedding.

Proof. It is easy to see that for any C-point i of the field F the cycle $\mathcal{X}$ lies in $\mathbf{A}^n_{\mathbf{C}}(if)$. Therefore, $\int_{\mathcal{X}} r_{\mathcal{D}} i(\gamma) = \int_{\mathcal{X}} r_{\mathcal{D}} i(\mathrm{res}\ \gamma)$ for $\gamma \in H^{n+1}_{\mathcal{A}}$, and the lemma follows from 7.1.1, c). ■

LEMMA 7.1.4. a) The element $l_{a,b}(\omega)$ lies in the image of the imbedding

$$H^{n+1}_{\mathcal{A}}(\mathbf{A}^n_F, S_{\cdot F}, \mathbf{Q}(n+1)) \hookrightarrow H^{n+1}_{\mathcal{A}}(\mathbf{A}^n_{F(f)}, S_{\cdot F(f)}, \mathbf{Q}(n+1)).$$

b) $l_{a,b}(\omega)$ depends only on ω (and does not depend on the choice of a and b in Lemma 7.1.2).

c) For any C-point i of the field F we have

$$\int_{\mathcal{X}} r_{\mathcal{D}} il(\omega) = \sum_{k \geqslant 1} \frac{i\omega^k}{k^{n+1}} \in \mathbf{C}/(2\pi i)^{n+1}\mathbf{Q}.$$

Proof. c) By Lemma 7.0.2 $\int\limits_{\mathfrak{X}} r_{\mathcal{D}} il_{a,b}(\omega) = C^{-1}\int\limits_{\mathfrak{X}} \log f d\log t_1\Lambda\ldots\Lambda d\log t_n = \int\limits_{\mathfrak{X}} \log(1-\omega t_1\ldots t_n)$ × $d\log t_1\Lambda\ldots\Lambda d\log t_n$ (here the branch of the logarithm is equal to 0 for $t_1 = \ldots = t_n = 0$). Making the change of variables $p_j = \prod\limits_{l=1}^{j} t_l$, we obtain the usual integral representation of the polylogarithm.

a) will be proved in part 7.2.

b) follows from a), c), and 7.1.1, c). ■

We have thus constructed a mapping l: (roots of 1 in $F)\setminus\{1\}\to H^{n+1}_{\mathcal{A}}(\mathbf{A}^n_F, S_{\cdot F}, \mathbf{Q}(n+1)) = H^1_{\mathcal{A}}(F, \mathbf{Q}(n+1))$.

THEOREM 7.1.5. The mapping l possesses the following properties:

a) For any $g\in\mathrm{Gal}\, F/\mathbf{Q}$ we have $l(g\omega) = gl(\omega)$.

b) $l(\omega^{-1}) = (-1)^n l(\omega)$.

c) For any **C**-point i of the field F we have

$$r_{\mathcal{D}} il(\omega) = \sum_{k>1} \frac{i\omega^k}{p^{n+1}} \in (2\pi i)^n \mathbf{R} \subset \mathbf{C}/(2\pi i)^{n+1}\mathbf{Q}.$$

Proof. a) follows from the naturality of all constructions; c) follows from 7.1.4, c). Part b) follows from c) and 7.1.1, c), since replacement of ω by ω^{-1} changes $r_{\mathcal{D}}$ into the complex conjugate quantity and hence multiplies $\mathrm{Re}(\mathrm{Im})r_{\mathcal{D}}$ by ±1; it remains to use injectivity.

By the way, from b) there follows the well known

COROLLARY. For any root of unity $\omega\in\mathbf{C}$ we have

$$\sum \frac{\omega^k}{k^{n+1}} \in (2\pi i)^{n+1}(\mathbf{Q}+i\mathbf{R}). \blacksquare$$

COROLLARY 7.1.6. Let F = Q(ω), ω ≠ 1, and let $c\in\mathrm{Gal}\, F/\mathbf{Q}$ be complex conjugation.

a) The mapping $\varphi_\omega : \mathbf{Q}[\mathrm{Gal}\, F/\mathbf{Q}]/(1-(-1)^n c)\cdot\mathbf{Q}[\mathrm{Gal}\, F/\mathbf{Q}] \to H^1_{\mathcal{A}}(F, \mathbf{Q}(n+1)) = K_{2n+1}(F)\otimes\mathbf{Q}$, $\varphi_\omega(g) = gl(\omega)$ is an isomorphism.

b) We decompose the motif $\overline{\mathbf{Q}}[F]$ by the action of $\mathrm{Gal}\, F/\mathbf{Q} : \overline{\mathbf{Q}}[F] = \oplus[\chi]$. Then $\dim H^1_{\mathcal{A}}([\chi], \mathbf{Q}(n+1))$ is equal to 1 if $\chi(c) = (-1)^n$ and to 0 if $\chi(c) = (-1)^{n+1}$. The element $\varphi_{\omega,\chi} := \Sigma\chi^{-1}(g)\varphi_\omega(g)$ is a generator of $H^1_{\mathcal{A}}([\chi], \mathbf{Q}(n+1))$. We have

$$r_{\mathcal{D}}\varphi_{\omega,\chi} \in \overline{\mathbf{Q}}^*\cdot(2\pi i)^n\cdot\frac{d}{ds}L(\chi, s)|_{s=-n} \subset \mathbf{C}\otimes\overline{\mathbf{Q}}/(2\pi i)^{n+1}\overline{\mathbf{Q}}.$$

In particular, for the motif [χ] Conjecture 3.4 is satisfied.

The proof follows immediately from 7.1.5: we first compute $r_{\mathcal{D}}\varphi_{\omega,\chi}$. If ω is the primitive root of degree f and χ is primitive, then

$$r_{\mathcal{D}}\varphi_{\omega}\chi = \sum_{a\in(\mathbf{Z}/f\mathbf{Z}),\, k>1} \chi^{-1}(a)\frac{\omega^{ak}}{k^{n+1}} = \sum \chi^{-1}(ak)\,\omega^{ak}\frac{\chi(k)}{k^{n+1}} = \varepsilon(\chi)\cdot L(\chi^{-1}, n+1)\in\overline{\mathbf{Q}}^*\cdot(2\pi i)^n\frac{d}{ds}L(\chi, s)|_{s=-n}.$$

If g|f and ψ is a primitive character modulo g, then

$$r_{\mathcal{D}}\varphi_{\omega,\chi} = (g/f)^{n+1}\prod_{p|f,\, p\nmid g}(1-\chi(p)\cdot p^{n+1})\, r_{\mathcal{D}}\varphi_{\omega^{f/g}\chi}.$$

From the obvious nonvanishing of the L-function at n + 1 it follows that $\varphi_{\omega,\chi}\neq 0$, if χ(C) = $(-1)^n$. Hence, φ_ω is a monomorphism and hence an isomorphism, since the dimensions are the same (Borel's theorem). ■

7.2. Proof of Lemmas 7.1.2 and 7.1.4, a). We begin with Lemma 7.1.2.

LEMMA 7.2.1. There is a pair of integral matrices (a_{ij}), (b_{ij}) of dimension n × 2^{n-1} such that

1) all a_{ij}, $b_{ij} \geqslant 2$;

2) for any i_0 there is a permutation $\sigma_{i_0} \in \Sigma_{2^{n-1}}$ such that $a_{ij} = b_{i\sigma(j)}$ for any $i \neq i_0$;

3) $\sum\limits_j \left(\prod\limits_i a_{ij}^{-1} - \prod\limits_i b_{ij}^{-1}\right) \neq 0.$

Derivation of 7.1.2 from 7.2.1. Condition 1) ensures that $(1-f)\Pi t_i^{-1} \in \Pi t_i \cdot A_f$, condition 2) that $(1-f) \in \Pi(1-t_i) A_f$, and hence from 1) and 2) we obtain condition 1.2.1. Condition 3) is 1.2.2. ■

We prove 7.2.1. Let $\alpha_1,\ldots,\alpha_n$, $\beta_1,\ldots,\beta_n$ be collections of integers with α_i, $\beta_i \geqslant 2$, $\alpha_i \neq \beta_i$ for any i. We construct the desired matrices $a_{ij} = a_{ij}(\alpha_1,\ldots,\alpha_n)$, $b_{ij} = b_{ij}(\beta_1,\ldots,\beta_n)$ by induction on n. If n = 1, we set $a_{11} = \alpha_1$, $b_{11} = \beta_1$. Suppose now that n > 1. We define the matrices a_{ij}, b_{ij} by the formula

$$a_{n_j} = \begin{cases} \alpha_n, & j \leqslant 2^{n-2} \\ \beta_n, & j > 2^{n-2}, \end{cases} \quad b_{nj} = \begin{cases} \beta_n, & j \leqslant 2^{n-2}, \\ \alpha_n, & j > 2^{n-2}; \end{cases}$$

for i < n

$$a_{nj} = b_{ij} = \begin{cases} a_{ij}(\alpha_1,\ldots,\alpha_{n-1}), & j \leqslant 2^{n-2}, \\ b_{i,j-2^{n-2}}(\beta_1,\ldots,\beta_{n-1}), & j > 2^{n-2}. \end{cases}$$

It is easy to see that 1) and 2) hold and the number of part 3) is $\prod\limits_i (\alpha_i^{-1} - \beta_i^{-1}) \neq 0$. ■

7.2.2. Proof of Lemma 7.1.4, a). Let n = 1. The localization exact sequence

$$H^2_{\mathscr{A}}(\mathbf{A}^1_F, S_F, \mathbf{Q}(2)) \to H^2_{\mathscr{A}}(\mathbf{A}^1_{F(f)}, S_{F(f)}\mathbf{Q}(2)) \to \bigoplus_v H^1_{\mathscr{A}}(F_v, \mathbf{Q}(1))$$

(in the sum v runs over all zeros and poles of f) shows that the image of the first arrow coincides with the intersection of the kernels of the manual symbols at the points v. Now all manual symbols $\{f, t\}_v$ are equal to zero by the Steinberg identity, as required.

Suppose now that $n \geqslant 2$.

Step A. We introduce notation. We set $A^n = \mathbf{A}^n_F$, $T^n = A^n(t_1 \ldots t_n)$ is the complement to the union of the coordinate planes $t_i = 0$; $S^n_T = S \cap T^n = \bigcup T_i^{n-1}$ (T_i^{n-1} is given by the equation $t_i = 1$); $T^{n-1} = \left\{(t_1,\ldots,t_n) \,\middle|\, \omega \prod\limits_i t_i = 1\right\}^{n} \subset T^n$; $U^n = T^n \setminus T^{n-1}$; $S^n_U = S^n \cap U^n$. Then $A^n = A_{(f)}{}^n \cup T^n$, and the corresponding Meyer–Vietoris exact sequence shows that

$$\begin{gathered} \operatorname{Im}\left(H^{n+1}_{\mathscr{A}}(A^n, S_., \mathbf{Q}(n+1)) \to H^{n+1}_{\mathscr{A}}(A^n_{(f)}, S_{\cdot(f)}, \mathbf{Q}(n+1)\right) \supset \\ \supset \operatorname{Ker}\left(H^{n+1}_{\mathscr{A}}(A^n_{(f)}, S_{\cdot(f)}, \mathbf{Q}(n+1)) \to H^{n+1}_{\mathscr{A}}(T^n_{(f)}, S^n_{T(f)} \cdot \mathbf{Q}(n+1))\right). \end{gathered}$$

Therefore, it suffices to prove that the restriction of $l_{a,b}(\omega)$ to $(T^n_{(f)}, S^n_{T(f).})$ is equal to zero.

Step B. We note that t_i are invertible on T^n and equal to 1 on T^{n-1}_i, i.e., $t_i \in H^1_{\mathscr{A}}(T^n, T_i^{n-1}, \mathbf{Q}(1))$. Therefore, $\{t_1,\ldots,t_n\} \in H^n_{\mathscr{A}}(T^n, \bigcup T_i^{n-1}, \mathbf{Q}(n)) = H^n_{\mathscr{A}}(T^n, S^n_T, \mathbf{Q}(n))$ is defined.

We use the following fact: if $\alpha, \beta \in H^\cdot(X, Y)$, $\bar{\alpha}$, $\bar{\beta}$ are the images of α, β in H(X), then $\{\alpha, \beta\} = \{\bar{\alpha}, \beta\} = \{\alpha, \bar{\beta}\} \in H^\cdot(X, Y)$. We apply it to $\alpha = f_{a,b}(\omega) \in H^1_{\mathscr{A}}(U^n, S^n_U, \mathbf{Q}(1))$, $\beta = \{t_1,\ldots,t_n\} \in H^n_{\mathscr{A}}(T^n, S^n_T, \mathbf{Q}(n))$. The restriction of $l(\omega)$ to $(T^n_{(f)}, S^n_{T(f)})$ is by 0.1 the symbol $\{\alpha, \bar{\beta}\} = \{\bar{\alpha}, \bar{\beta}\} = C^{-1} \times \left(\sum\limits_j \{1 - \omega\Pi t_i^{a_{ij}}, \{t_1,\ldots,t_n\}\} - \sum\limits_k \{1 - \omega\Pi t_i^{b_{ik}}, \{t_1,\ldots,t_n\}\}\right)$. Each term $\{1 - \omega\Pi t_i^{a_{ij}}, \{t_1,\ldots,t_n\}\} = \Pi a_{ij}^{-1}\{1 - \omega\Pi t_i^{a_{ij}'}, \{t_1^{a_{1j}},\ldots,t_n^{a_{nj}}\}\}$ is the preimage of the symbol $\{1 - \omega\Pi t_i, \{t_1,\ldots,t_n\}\} \in H^{n+1}_{\mathscr{A}}(U^n, S^n_U, \mathbf{Q}(n+1))$ under the mapping $t_i \to t_i^{a_{ij}}$.

Step C. It remains to show that the symbol is equal to zero. Let $\tilde{S}^n_T = S^n_T \cup T^{n-1}$, $\tilde{S}^n_{T\cdot} = \ldots$; we consider T^{n-1} as a torus with coordinates $t_1,\ldots,t_{n-1}$. Then $S^n \cap T^{n-1} = \tilde{S}^{n-1}_T$, and this intersection is transversal. According to 2.1.4, there is the exact localization sequence

$\ldots \xrightarrow{i_{n*}} H^{n+1}_{\mathscr{A}}(T^n, S^n_{T\cdot}, \mathbf{Q}(n+1)) \to H^{n+1}_{\mathscr{A}}(U^n, S^n_{U\cdot}, \mathbf{Q}(n+1)) \to H^n_{\mathscr{A}}(T^{n-1}, \tilde{S}^{n-1}_{T\cdot}, \mathbf{Q}(n)) \to \ldots$ From it and the monomorphicity of the restriction $H^{n+1}_{\mathscr{A}}(T^n, S^n_{T\cdot}, \mathbf{Q}(n+1)) \to H^{n+1}_{\mathscr{A}}(U^n, \mathbf{Q}(n+1))$ (which follows, as in 7.1.3, from the argument with regulators) it follows that the morphism $H^{n+1}_{\mathscr{A}}(U^n, S^n_{U\cdot}, \mathbf{Q}(n+1)) \to H^n_{\mathscr{A}}(T^{n-1}, \tilde{S}^{n-1}_{T\cdot}, \mathbf{Q}(\omega)) \oplus H^{n+1}_{\mathscr{A}}(U^n, \mathbf{Q}(n+1))$ is an imbedding. It is clear that the restriction of our symbol to U is equal to 0 (the Steinberg identity), and its image in $H^n(T^{n-1}, \tilde{S}^{n-1}_{T\cdot}, \mathbf{Q}(n))$ coincides with $i_n{}^*(\{t_1, \ldots, t_n\})$, where $i_n{}^*: H^{\cdot}_{\mathscr{A}}(T^n, S^n_{T\cdot}) \to H_{\mathscr{A}}(T^{n-1}, \tilde{S}^{n-1}_{T\cdot})$. Thus, it remains to show that $i^*_n = 0$. We use regulators: we shall first prove that the cohomological arrow is equal to 0 and then use injectivity (Borel's theorem).

Step D. The morphism i^*_n is included in the long exact sequence $\ldots \to H_{\mathscr{A}}(T^n, \tilde{S}^n_{T\cdot}) \to H_{\mathscr{A}}(T^n, S^n_{T\cdot}) \xrightarrow{i^*_n} H_{\mathscr{A}}(T^{n-1}, \tilde{S}^{n-1}_{T\cdot})$; there is an analogous exact sequence for the other cohomologies; these sequences are connected by the morphism r.

LEMMA 7.2.2.1. $H^{\cdot}_{\mathscr{A}}(T^n, S^n_{T\cdot})$ is a free $H^{\cdot}_{\mathscr{A}}(F)$ -module with generator $\{t_1, \ldots, t_n\}$. An analogous fact holds for the other cohomologies.

The next result follows from Borel's theorem.

COROLLARY 7.2.2.2. The morphism $\mathfrak{r}_{\mathscr{D}}: H^{\cdot}_{\mathscr{A}}(T^n, S^n_{T\cdot}) \to H^{\cdot}_{\mathscr{D}}(T^n \underset{\mathbf{Q}}{\otimes} \mathbf{R}, S^n_{T\cdot} \otimes \mathbf{R})$ is an imbedding.

LEMMA 7.2.2.3. For an imbedding $F \hookrightarrow \mathbf{C}$ we consider the manifolds $T^n_{\mathbf{C}} = T^n \otimes_F \mathbf{C}$, ... We have $H^n_{\mathscr{B}}(T^n_{\mathbf{C}}, \tilde{S}^n_{T\mathbf{C}}, \mathbf{Q}) \cong \mathbf{Q} \oplus \mathbf{Q}(-1) \oplus \ldots \odot \mathbf{Q}(-n)$ as mixed Hodge structures; the other $H^i = 0$. The morphism in Betti cohomologies is equal to 0.

Proof. Indeed, $H^n_{\mathscr{B}}(T^n_{\mathbf{C}}, S^n_{\mathbf{C}}, \mathbf{Q}) = \mathbf{Q}(-n) = \mathbf{Q} \cdot \mathfrak{r}_{\mathscr{B}}\{t_1, \ldots, t_n\}$ the other $H^i = 0$. From this and the exact sequence 2.2.4.0 for cohomologies we obtain by induction on n that $H^i_{\mathscr{B}}(T^n_{\mathbf{C}}, \tilde{S}^n_{T\cdot\mathbf{C}}, \mathbf{Q}) = 0$ for $i \neq n$, i^*_n is equal to 0 in the cohomologies, and the factors in the weighted filtration on $H^n_{\mathscr{B}}(T^n_{\mathbf{C}}, \tilde{S}^n_{T\mathbf{C}}, \mathbf{Q})$ are $\mathbf{Q}(-i)$, $0 \leqslant i \leqslant n$. It remains to prove that the weighted filtration on H^n is decomposable. By induction it suffices to verify that the term F^n in the Hodge filtration is defined over **Q**. A generator of F^n is the form $(2\pi i)^{-n} d \log t_1 \wedge \ldots \wedge d \log t_n$. We shall show that its periods over all cycles are rational. Let $\omega = \exp \frac{2\pi i}{k}$. We consider the unit cube in R^n and decompose it into n + 1 pieces by the hyperplanes $\sum_l x_l = j + \frac{1}{k}$, $j = 0, \ldots, n-1$. The images of these pieces under the mapping $t_i = \exp 2\pi i x_l$ form a basis in $H_n(T^n_{\mathbf{C}}, S^n_{T\mathbf{C}}, Q)$. The integrals of our form over them coincide with the volumes of the pieces. They are rational, as required. ■

The lemma can be reformulated as follows: in the exact sequence of mixed Hodge structures $\to H^{\cdot}_{\mathscr{B}}(T^n, \tilde{S}^n_{T\cdot}) \to H^{\cdot}_{\mathscr{B}}(T^n, S^n_{T\cdot}) \xrightarrow{i_n^*} H^{\cdot}_{\mathscr{B}}(T^{n-1}, \tilde{S}^{n-1}_{T\cdot}) \to \ldots$ the morphism i^*_n is equal to 0 and the remaining short exact sequence splits. From this we obtain

COROLLARY 7.2.2.4. The morphism i^*_n for $\mathscr{D}$ -cohomologies is equal to 0. ■

The last step: by induction on n we show that $\mathfrak{r}_{\mathscr{D}n}: H_{\mathscr{A}}(T^n, \tilde{S}^n_{T\cdot}) \to H_{\mathscr{D}}(T^n \otimes \mathbf{R}, \tilde{S}^n_{T\cdot} \otimes \mathbf{R})$ is an imbedding and the morphism i^*_n for $\mathscr{A}$ -theory is equal to 0. Suppose we know this for n − 1.

For the commutativity of $\mathfrak{r}_{\mathscr{D}}$ and i^*_n and 7.2.2.4 we have $\mathfrak{r}_{\mathscr{D}n-1} \circ i_n{}^* = 0$. But $\mathfrak{r}_{\mathscr{D}n-1}$ is injective, and hence $i^*_n = 0$. Injectiveness of $\mathfrak{r}_{\mathscr{D}n}$ now follows from the commutativity of the diagram

$$\begin{tikzcd}
0 \arrow[r] & H_{\mathscr{A}}(T^{n-1}, \tilde{S}^{n-1}) \arrow[r] \arrow[d, "\mathfrak{r}_{\mathscr{D}n-1}"] & H_{\mathscr{A}}(T^n, \tilde{S}^n) \arrow[r] \arrow[d, "\mathfrak{r}_{\mathscr{D}n}"] & H_{\mathscr{A}}(T^n, S^n) \arrow[r] \arrow[d, "\mathfrak{r}_{\mathscr{D}(n)}"] & 0 \\
0 \arrow[r] & H_{\mathscr{D}}(T^{n-1}, \tilde{S}^{n-1}) \arrow[r] & H_{\mathscr{D}}(T^n, \tilde{S}^n) \arrow[r] & H_{\mathscr{D}}(T^n, S^n) \arrow[r] & 0
\end{tikzcd}$$

and the injectivity of $\mathfrak{r}_{\mathscr{D}n-1}$ and $\mathfrak{r}_{\mathscr{D}(n)}$ (induction + 7.2.2.2). ■

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