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Appendix 2

Etale cohomology and duality in number fields

by Thomas Zink

In this Appendix we give a proof of Tate's global Duality Theorem using the étale cohomology. The proof is based on the Duality Theorem of Artin and Verdier given in [M].

Let X be the spectrum of the ring of integers of a number field K. The Duality Theorem given in [M] states that, for K totally imaginary and a constructible sheaf F on X, there is a perfect pairing

$$H^r(X, F) \times \mathrm{Ext}_X^{3-r}(F, G_{m,X}) \to H^3(X, G_{m,X}) \cong \mathbf{Q}/\mathbf{Z}.$$

In the case where X is not totally imaginary, we define the modified étale cohomology groups $\hat{H}^r(X, F)$, $r \in \mathbf{Z}$ (see 3.1.). The perfect pairing given above remains valid for any K if one replaces the usual étale cohomology by the modified one (see 3.4.). We show further how Tate's original theorem follows from the perfect pairing for the modified étale cohomology. The reader of the Appendix should be familiar with the étale cohomology and should know the contents of [M]. I want to thank H. J. Fitzner for useful conversations and for his help during the preparation of the manuscript.

1. Etale topology of algebraic number fields

Let K be a number field and D the ring of integers of K. In this section we initially list for convenience of the reader some definitions and properties of Abelian sheaves for the étale topology on $X = \mathrm{Spec}\, D$. Then we extend the category of Abelian sheaves by considering sheaves which formally have fibres at the real Archimedean points of K.

1.1. Recall that for an arbitrary scheme X, Et_X denotes the category whose objects are étale morphisms $p\colon U \to X$ and whose morphisms are X-morphisms. $\mathcal{S}_X$ denotes the category of Abelian sheaves on Et_X for the étale topology, i.e. the topology generated by surjective families of étale morphisms. Via the Yoneda embedding, Abelian group schemes over X define objects of $\mathcal{S}_X$.

If M is an Abelian group, the scheme $\bigsqcup_{m \in M} X$ is in a canonical way an Abelian group scheme over X. If no confusion is possible, we denote this scheme simply by M. An Abelian sheaf which is representable by a group scheme of that type is called to be *constant*.

A sheaf F on X is called to be *locally constant* if there exists a covering $p_i\colon U_i \to X$, $i \in I$, in Et_X, such that p_i^*F is constant.

Let L be a field and $G = \mathrm{Gal}(\bar{L}_{\mathrm{sep}}/L)$ the Galois group of its separable algebraic closure. For $X = \mathrm{Spec}\, L$, $\mathscr{S}_X$ is equivalent to the category of continuous G-modules. A sheaf is (locally) constant if (an open subgroup of) G acts trivially on the corresponding G-module.

1.2. Let us again assume $X = \mathrm{Spec}\, D$, D the ring of integers of K. Let $j_U\colon U \to X$ be an open subscheme of X and $i_S\colon S \to X$ its complement. Let F be an Abelian sheaf on X. We get a canonical morphism

1.2.1a. $\varphi_S\colon i_S^*F \to i_S^*j_{U*}j_U^*F$.

Associating to any sheaf F the triple $(i_S^*F, j_U^*F, \varphi_S)$, we obtain a functor from $\mathscr{S}_X$ to the category of triples (F_1, F_2, φ) where F_1 is an Abelian sheaf on S, F_2 an Abelian sheaf on U, and $\varphi\colon F_1 \to i_S^*j_{U*}F_2$ is a morphism. The decomposition lemma [GT], Chapter III, 2.5., states that this functor is an equivalence of categories. We wish to describe this more precisely in our special case.

Let $\gamma\colon \eta \to X$ be the general point of X, i.e., $\eta = \mathrm{Spec}\, K$, $\bar{\eta} = \mathrm{Spec}\, \bar{K}$ its algebraic closure and $G = \mathrm{Gal}(\bar{K}/K)$ the Galois group. Similarly, we define for a closed point $i_x\colon x \to X$, $G_x = \mathrm{Gal}\big(\overline{k(x)}/k(x)\big)$ as the Galois group of the residue class field $k(x)$. For each closed point $x \in X$, we fix also a decomposition group $D_x \subset G$. D_x is unique up to conjugation and isomorphic to the Galois group of the completion K_x of the field K at the point x. If $I_x \subset D_x$ denotes the inertial group of x, then $D_x/I_x \cong G_x$.

The general fibre $F_\eta = \gamma^*F$ of an Abelian sheaf F on X may be identified with a continuous G-module and the fibre $F_x = i_x^*F$ at a closed point x with a continuous G_x-module, see 1.1.

If we apply these considerations to $S = \{x\}$, the morphism 1.2.1a takes the following form

1.2.1b. $\varphi_x\colon F_x \to F_\eta^{I_x}$.

φ_x is called the *specialization map.*

We can now restate the decomposition lemma in the form needed.

1.2.2. Lemma. *Let $S = \{x_1, \ldots, x_n\}$ be a finite set of closed points of X and $U = X \setminus S$. The functor that associates the tuple*

$$(F_{x_1}, \ldots, F_{x_n}, F_U, \varphi_{x_1}\colon F_{x_1} \to F_\eta^{I_{x_1}}, \ldots, \varphi_{x_n}\colon F_{x_n} \to F_\eta^{I_{x_n}})$$

to an Abelian sheaf $F \in \mathscr{S}_X$ is an equivalence of the category $\mathscr{S}_X$ with the category of tuples $(M_1, \ldots, M_n, N, \varphi_{x_1}, \ldots, \varphi_{x_n})$ where M_i is a continuous G_{x_i}-module, N an Abelian sheaf on U, and $\varphi_i\colon M_i \to N_\eta^{I_{x_i}}$ is a G_{x_i}-homomorphism.

1.2.3. Definition. An Abelian sheaf F is called *constructible*, if its fibres F_x, $x \in X$, are finite G_x-modules and if it is locally constant on a non-empty open subscheme of X.

If F is locally constant in a neighbourhood of some point x, I_x acts trivially on F_η and $\varphi_x: F_x \to F_\eta^{I_x}$ is an isomorphism.

From 1.2.1a one can deduce:

The category of constructible Abelian sheaves is equivalent to the category of systems of Galois modules of the following type: For each point $x \in X$, there is a finite continuous G_x-module F_x. For each closed point $x \in X$, there is a morphism of G_x-modules $\varphi_x: F_x \to F_\eta^{I_x}$. φ_x is an isomorphism except for a finite number of points.

Note that, for each finite G-module M, I_x, $x \in X$, acts trivially on M except for a finite number of points.

1.3. Let $\bar{X}$ be the union of all closed points of X and the real Archimedean points of K supplied with the Zariski topology (i.e., a set is open iff it is either empty or the complement of a finite set). Denote the real Archimedean points of K by $v_1, \ldots, v_r$. These points are also called the *infinite points* of $\bar{X}$. Let $\bar{v}_1, \ldots, \bar{v}_r$ be fixed extensions to the algebraic closure $\bar{K}$ of K. We denote by I_{v_i} the decomposition group of $\bar{v}_i$. I_{v_i} is isomorphic to $\mathbf{Z}/2\mathbf{Z}$. We formally define $G_{v_i} = \{1\}$. If U is an open subscheme of X, we denote by $\bar{U}$ the union of the closed points of U and the infinite points.

Let $\tilde{U}$ be an arbitrary open subset of $\bar{X}$. Denote by $\tilde{v}_1, \ldots, \tilde{v}_s$ its infinite points and set $U = \tilde{U} \cap X$.

1.3.1. Definition. An *Abelian sheaf* $\tilde{F}$ on $\tilde{U}$ is a tuple

$$(\tilde{F}_{\tilde{v}_1}, \ldots, \tilde{F}_{\tilde{v}_s}, F, \varphi_{\tilde{v}_1}: F_{\tilde{v}_1} \to F_\eta^{I_{\tilde{v}_1}} \ldots, \varphi_{\tilde{v}_s}: F_{\tilde{v}_s} \to F_\eta^{I_{\tilde{v}_s}}),$$

where the $\tilde{F}_{\tilde{v}_i}$ are Abelian groups, F is an Abelian sheaf on U, and the φ_{v_i} are homomorphisms of groups.

We denote the Abelian category of Abelian sheaves on $\tilde{U}$ by $\mathcal{S}_{\tilde{U}}$. We call $\tilde{F}_{\tilde{v}_i}$ the *fibre at the point* $\tilde{v}_i$. The fibres F_x for $x \in U$ are also denoted by $\tilde{F}_x$.

The sheaf $\tilde{F}$ is called *constructible* if F is constructible and if all $\tilde{F}_{\tilde{v}_i}$, $i = 1, \ldots, s$, are finite Abelian groups.

1.3.2. For each $\tilde{v}_i$ we denote by $\mathcal{S}_{\tilde{v}_i}$ a copy of the category of Abelian groups. There are various functors relating the categories $\mathcal{S}_U$, $\mathcal{S}_{\tilde{U}}$ and $\mathcal{S}_{\tilde{v}_i}$:

$$\mathcal{S}_U \;\begin{array}{c} \xrightarrow{\;j_!\;} \\ \xleftarrow{\;j^*\;} \\ \xrightarrow{\;j_*\;} \end{array}\; \mathcal{S}_{\tilde{U}} \;\begin{array}{c} \xrightarrow{\;i^*_{\tilde{v}_k}\;} \\ \xleftarrow{\;i_{\tilde{v}_k *}\;} \\ \xrightarrow{\;i^!_{\tilde{v}_k}\;} \end{array}\; \mathcal{S}_{\tilde{v}_k}.$$

Recall the definitions [GT], Chapter III, 2.:

$$j_!F = (0, \ldots, 0, F, 0 \to F_\eta^{I_{v_1}}, \ldots, 0 \to F_\eta^{I_{v_s}}),$$
$$j^*\tilde{F} = F,$$
$$j_*F = \left(F_\eta^{I_{\tilde{v}_1}}, \ldots, F_\eta^{I_{\tilde{v}_s}}, F, \mathrm{id}_{F_\eta^{I_{\tilde{v}_1}}}, \ldots, \mathrm{id}_{F_\eta^{I_{\tilde{v}_s}}}\right),$$
$$i^*_{\tilde{v}_k}\tilde{F} = \tilde{F}_{\tilde{v}_k},$$
$$i_{\tilde{v}_k *}\tilde{F}_{\tilde{v}_k} = (0, \ldots, 0, \tilde{F}_{\tilde{v}_k}, \ldots, 0, 0 \to 0, \ldots, \tilde{F}_{\tilde{v}_k} \to 0, \ldots, 0 \to 0),$$
$$i^!_{\tilde{v}_k}F = \operatorname{Ker} \varphi_{\tilde{v}_k}.$$

Any functor is left adjoint to the one below it except for j_* and $i^*_{\tilde{v}_k}$. The functors $j_!$, j^*, i^*, i_* are exact and hence the functors j^*, j_*, i_*, $i^!$ preserve injective objects. Let $J \in \mathscr{S}_U$ and $J_{\tilde{v}_k} \in \mathscr{S}_{\tilde{v}_k}$, $k = 1, \ldots, s$, be injective objects. It follows that $j_* J \oplus \bigsqcup_{i=1} i_{\tilde{v}_k *} J_{\tilde{v}_k}$ is an injective object of $\mathscr{S}_{\tilde{U}}$. The reader easily verifies that any injective object in $\mathscr{S}_{\tilde{U}}$ is of this type. If we are dealing with several open sets of $\bar{X}$, we add to j and i the subscript U.

1.3.3. One can also interpret $\mathscr{S}_{\tilde{U}}$ as the category of Abelian sheaves over a suitable Grothendieck topology. We initially remark that one can interpret $\tilde{v}_1, \ldots, \tilde{v}_s$ as a subset of the real valued points $U(\mathbf{R}) = \mathrm{Hom}(\mathrm{Spec}\,\mathbf{R}, U)$ of U. Denote this subset by $\tilde{U}_\infty$. We are going to define the category $\mathrm{Et}_{\tilde{U}}$. For this we introduce a category $(\widetilde{\mathrm{Sch}})$ whose objects are pairs (Y, M) where Y runs over the category of schemes and M is a subset of $Y(\mathbf{R})$. A morphism $\tilde{\pi}$ of (Y_1, M_1) to (Y_2, M_2) in $(\widetilde{\mathrm{Sch}})$ is a morphism π of schemes of Y_1 to Y_2 which maps M_1 to M_2. We call a morphism *étale*, an *open immersion* or a *closed immersion* if the morphism of the underlying schemes has the corresponding property. We can identify $\tilde{U}$ with $(U, \{\tilde{v}_1, \ldots, \tilde{v}_s\})$. $(\widetilde{\mathrm{Sch}})$ has fibre products. The objects of $\mathrm{Et}_{\tilde{U}}$ are étale morphisms $\tilde{W} \xrightarrow{\tilde{\pi}} \tilde{U}$ in $(\widetilde{\mathrm{Sch}})$, and the morphisms in $\mathrm{Et}_{\tilde{U}}$ are the obvious ones. We also denote the object $\tilde{\pi}$ simply by $\tilde{W}$ if no confusion is possible. A family of morphisms $\tilde{\pi}_i \colon W_i \to U$, $i \in I$, is called a *covering* if the family $\pi_i \colon W_i \to U$, $i \in I$, is a covering in Et_U and if $\bigcup_i \pi_i(\mathbf{R})\,(\tilde{W}_{i\infty}) = \tilde{U}_\infty$.

Denote by $T_{\tilde{v}_k}$ a copy of the punctual topos, i.e., the topology which has the finite sets as the underlying category and surjective families of morphisms as the coverings. Let $j_{\tilde{U}} \colon \mathrm{Et}_{\tilde{U}} \to \mathrm{Et}_U$ be the functor which associates to $\tilde{W}$ the scheme W and $i_{\tilde{v}_k} \colon \mathrm{Et}_{\tilde{U}} \to T_{\tilde{v}_k}$, $k = 1, \ldots, s$, the functor which associates to $\tilde{W}$ the inverse image of $\tilde{v}_k$ under the map $\tilde{W}_\infty \to \tilde{U}_\infty$. If no confusion is possible, we denote $j_{\tilde{U}}$ simply by j.

Let $\mathscr{S}_{\tilde{U}}$ be the category of Abelian sheaves on $\mathrm{Et}_{\tilde{U}}$. We want to prove that $\mathscr{S}_{\tilde{U}}$ is equivalent with the category introduced in 1.3.1. Evidently the category $\mathscr{S}_{\tilde{v}_k}$ of Abelian sheaves on $T_{\tilde{v}_k}$ is equivalent with the category defined in 1.3.2. From the morphisms of topologies $j_{\tilde{U}}$, $i_{\tilde{v}_k}$ we get functors

$$\mathscr{S}_U \;\underset{j^*}{\overset{j_*}{\rightleftarrows}}\; \mathscr{S}_{\tilde{U}} \;\underset{i_{v_k *}}{\overset{i^*_{\tilde{v}_k}}{\rightleftarrows}}\; \mathscr{S}_{\tilde{v}_k}$$

From [GT], Chapter III, 2.4., we see that our assertion is equivalent to the following:

(i) $i_{v_k *}$, j^* are fully faithful.

(ii) An Abelian sheaf $F \in \mathscr{S}_{\tilde{U}}$ is of the form $\bigsqcup i_{\tilde{v}_k *} M_{\tilde{v}_k}$, where $M_{\tilde{v}_k} \in \mathscr{S}_{\tilde{v}_k}$ iff $j^*\tilde{F} = 0$.

(iii) The functor $i^*_{v_k} j_*$ is given by $F \mapsto F_\eta^{I_{\tilde{v}_k}}$.

We have immediately $j_* F(\tilde{W}) = F(W)$ and $i_{\tilde{v}_k *} M_{\tilde{v}_k}(\tilde{W}) = \mathrm{Hom}\big(i_{\tilde{v}_k}(\tilde{W}), M_{\tilde{v}_k}\big)$. Furthermore, if $\tilde{W} = \emptyset$, we have $j^*\tilde{F}(W) = \tilde{F}(\tilde{W})$. From this it is not hard to see (i).

For (ii) assume $j^*\tilde{F} = 0$. Let $\tilde{U}_i$ be the object $(U, \tilde{v}_i)$ and set $M_{\tilde{v}_i} = \tilde{F}(\tilde{U}_i)$. Let be $\big(W, w \in W(\mathbf{R}) \to (U, \tilde{v}_i)\big) \in \mathrm{Et}_{\tilde{U}}$ and $(V_j \to U)_{j \in J}$ a covering in Et_U. Then $(V_j, \emptyset) \to (U, \tilde{v}_i)$ together with $(W, w) \to (U, \tilde{v}_i)$ is a covering and from the axioms for sheaves we get that $\tilde{F}(W, w) \to \tilde{F}(U, \tilde{v}_i) = M_{\tilde{v}_i}$ is an isomorphism. By covering any $\widetilde{W}$ by objects of the form (W, w), we get that $\tilde{F}$ is of the type $\bigsqcup i_{\tilde{v}_k *} M_{\tilde{v}_k}$. Since the inverse is obvious, (ii) is proved.

(iii) A sheaf on $T_{\tilde{v}_k}$ is uniquely determined by its value at the set of one element $\{\tilde{v}_k\}$. By definition $i^*_{\tilde{v}_k} j_* F(\{\tilde{v}_k\}) = \varinjlim_{(W,w)} F(W)$, where $(W, w) \in \mathrm{Et}_{\tilde{U}}$ and $w \in W(\mathbf{R})$ is mapped to $\tilde{v}_k$.

The limit may be interpreted as the value of F at a maximal extension of K which is unramified at $\tilde{v}_k$. Since $\bar{K}^{I_{\tilde{v}_k}}$ is such an extension, we get $i^*_{\tilde{v}_k} j_* F(\{\tilde{v}_k\}) = F_\eta{}^{I_{\tilde{v}_k}}$.

1.3.4. Decomposition lemma. *Let $\tilde{V} \subset \tilde{U}$ be an open subset and $S = \{s_1, \ldots, s_n\}$ its complement. The category of Abelian sheaves on $\tilde{U}$ is equivalent to the category of tuples*

$$(E_{s_1}, \ldots, E_{s_n}, E, E_{s_1} \to E_\eta{}^{I_{s_1}}, \ldots, E_{s_n} \to E_\eta{}^{I_{s_n}}),$$

where E is an Abelian sheaf on $\tilde{U}$, E_{s_i} are G_{s_i}-modules and $E_{s_i} \to E_\eta{}^{I_{s_i}}$ are homomorphisms of G_{s_i}-modules.

This follows from 1.2.2. and 1.3.1.

2. Cohomology

The aim of this paragraph is to develop the necessary cohomology theory of Abelian sheaves over an open subset $\tilde{U}$ of $\bar{X}$. We do not apply the general cohomology theory of topos if we can cope without it.

We will use the same notations as in 1.3.

2.1. Definition. $H^0(\tilde{U}, \tilde{F})$ is the group $H^0(U, F) \times_{F_\eta} \tilde{F}_{\tilde{v}_1} \times_{F_\eta} \cdots \times_{F_\eta} \tilde{F}_{\tilde{v}_s}$ where the fibre product is taken with respect to the canonical morphism $H^0(U, F) \to F_\eta$ and with respect to the specialisation maps $\varphi_{\tilde{v}_i}: \tilde{F}_{\tilde{v}_i} \to F_\eta$. If we interpret $\tilde{F}$ as a sheaf on $\mathrm{Et}_{\tilde{U}}$, we have $H^0(\tilde{U}, \tilde{F}) = \tilde{F}(\tilde{U})$. Obviously, $H^0(\tilde{U}, -)$ is a left exact functor, and we denote by $H^p(\tilde{U}, -)$ $p \geqq 0$, its derived functors.

Exactly in the same way as in [GT], Chapter III, 2.10., we define local cohomology groups with support in an infinite point $\tilde{v}_k$.

2.2. $\quad H^p_{\tilde{v}_k}(\tilde{U}, \tilde{F}) = R^p i^!_{\tilde{v}_k} \tilde{F}$

where $R^p i^!_{\tilde{v}_k} F$ denote the derived functors of $i^!_{\tilde{v}_k}$.

We define the local cohomology groups with support in a finite point $x \in U$ by setting $H_x{}^p(\tilde{U}, \tilde{F}) = H_x{}^p(U, F)$.

Let $\tilde{V} \subset \tilde{U}$ be an open subset and S its complement. By means of the decomposition lemma 1.3.4 and [GT], Chapter III, 2.9., one derives the local cohomology sequence

2.2.1. $\cdots \to \coprod_{x\in S} H_s{}^i(\tilde U, \tilde F) \to H^i(\tilde U, \tilde F) \to H^i(\tilde V, \tilde F|_{\tilde V}) \to \cdots.$

If we set $\tilde V = \tilde U \setminus \tilde U_\infty$, we have $H^i(\tilde V, \tilde F|_{\tilde V}) = H^i(U, F)$. Hence the sequence 2.2.1. reduces the computation of $H^i(\tilde U, \tilde F)$ in the Grothendieck group to the computation of usual cohomology groups and local cohomology groups with support in infinite points.

The following lemma completes the considerations.

2.3. Lemma. *We have the following isomorphisms:*

$$H^0_{\tilde v_k}(\tilde U, \tilde F) = \operatorname{Ker} \varphi_{\tilde v_k},$$
$$H^1_{\tilde v_k}(\tilde U, \tilde F) = \operatorname{Coker} \varphi_{\tilde v_k},$$
$$H^i_{\tilde v_k}(\tilde U, \tilde F) = H^{i-1}(I_{\tilde v_k}, \tilde F_\eta), \qquad i \geqq 2.$$

Here the right-hand side of the last isomorphism is the group cohomology.

Proof. We use the axiomatical characterization of the derived functor and have to show that the right-hand sides of the isomorphisms of 2.3. form a δ-functor,

$$H^0_{\tilde v_k}(\tilde U, \tilde F) = \operatorname{Ker} \varphi_{\tilde v_k}$$

and

$$\operatorname{Coker} \varphi_{\tilde v_k} = H^{i-1}(I_{\tilde v_k}, \tilde F_\eta) = 0, \qquad i \geqq 2,$$

for an injective sheaf $\tilde F$. Let

$$0 \to \tilde F' \to \tilde F \to \tilde F'' \to 0$$

be an exact sequence of Abelian sheaves on U. Using the exact sequence for group cohomology, we get a commutative diagram with exact lines

$$\begin{array}{ccccccccc} 0 \to & \tilde F'_{\tilde v_k} & \longrightarrow & \tilde F_{\tilde v_k} & \longrightarrow & \tilde F''_{\tilde v_k} & \longrightarrow & 0 & \\ & \downarrow \varphi'_{v_k} & & \downarrow \varphi_{v_k} & & \downarrow \varphi''_{v_k} & & & \\ 0 \to & H^0(I_{\tilde v_k}, \tilde F_\eta{}') & \to & H^0(I_{\tilde v_k}, \tilde F_\eta) & \to & H^0(I_{\tilde v_k}, \tilde F''_\eta) & \to & H^1(I_{\tilde v_k}, \tilde F_\eta{}') & \to \cdots \end{array}$$

If we take the Ker-Coker sequence, we get that we are dealing with a δ-functor.

$H^0_{\tilde v_k}(\tilde U, \tilde F) = \operatorname{Ker} \varphi_{\tilde v_k}$ is obvious from the definition. Again from the definitions one checks that $\operatorname{Coker} \varphi_{\tilde v_k} = 0$ for an injective sheaf. Since $(i^*_{\tilde v_k} M)_\eta = 0$ for $M \in \mathscr{S}_{\tilde v_k}$, it remains to show that, for an injective sheaf F on U, it holds $H^p(I_{\tilde v_k}, F_\eta) = 0$. Consider the category $\mathscr{C}$ of objects $V \xrightarrow{p_V} U \in \mathrm{Et}_U$, such that V is connected, its function field is contained in $\bar K$ and fixed under the action of $I_{\tilde v_k}$. We have $\varprojlim_{V\in\mathscr{C}} V = \operatorname{Spec} \bar K^{I_{\tilde v_k}}$. Since the étale cohomology commutes with direct limits ([GT], Chapter III, 3.9.) and coincides with the Tate cohomology over the spectrum of a field, we get

$$\varinjlim_{V\in\mathscr{C}} H^p(V, p_V{}^*F) = H^p(I_{\tilde v_k}, F_\eta).$$

Since $p_V{}^*$ preserves flasque sheaves, the limit on the left-hand side is zero for $p \geqq 1$. That proves our assertion.

2.4. Corollary. *If the order of F_η is prime to 2, we have*

$$H^i(\tilde{U}, \tilde{F}) = H^i(U, F), \qquad i \geqq 2.$$

Since $H^p(I_{\bar{v}_k}, F_\eta) = 0$ for $p \geqq 1$, this follows from the local cohomology sequence.

2.5. As an example we compute the cohomology groups of the multiplicative group over $\bar{U}$. Our computation is based on that given in [M]. From now on we consider open subsets $\bar{U}$ of $\bar{X}$ which contain all infinite points of X.

Note that the fixed valuation $\bar{v}_i$ defines an embedding $\bar{v}_i\colon \bar{K} \to \mathbf{C}$, which is unique up to conjugation. Let us denote by $\bar{K}_+{}^{I_{v_i}}$ the elements of $\bar{K}$ which are mapped into $\mathbf{R}_+$ under $\bar{v}_i$. Obviously $\bar{K}_+{}^{I_{v_i}} \subset \bar{K}^{I_{v_i}}$.

2.5.1. Definition. The multiplicative group $G_{m,\bar{U}}$ is the tuple

$$(\bar{K}_+{}^{*I_{v_1}}, \ldots, \bar{K}_+{}^{*I_{v_r}}, G_m, U, \bar{K}_+{}^{*I_{v_1}} \subset \bar{K}^{*I_{v_1}}, \ldots, \bar{K}_+{}^{*I_{v_r}} \subset \bar{K}^{*I_{v_r}}).$$

Recall that we have functors $\gamma_{U*}\colon \mathcal{S}_\eta \to \mathcal{S}_U$ and $j_{\bar{U}*}\colon \mathcal{S}_U \to \mathcal{S}_{\bar{U}}$. Let $\gamma_{\bar{U}*} = j_{\bar{U}*}\gamma_{U*}$ be the composite functor. From the definitions one gets readily

$$\gamma_{\bar{U}*}G_{m,\eta} = (\bar{K}^{*I_{v_1}}, \ldots, \bar{K}^{*I_{v_r}}, \gamma_{U*}G_{m,\eta}, \mathrm{id}_{\bar{K}^*I_{v_1}}, \ldots, \mathrm{id}_{\bar{K}^*I_{v_r}}).$$

From [GT], Chapter IV, 1.4., or [M], one deduces an exact sequence

2.5.2. $$0 \to G_{m,\bar{U}} \to \gamma_{\bar{U}*}G_{m,\eta} \to \bigsqcup_{x\in U} i_{x*}\mathbf{Z} \oplus \bigsqcup_{i=1}^{r} i_{v_i*}\, \mathbf{Z}/2\mathbf{Z} \to 0.$$

We first compute the cohomology of $\gamma_{\bar{U}*}G_{m,\eta}$. By use of the local class field theory, we have $R^q\gamma_{U*}G_{m,\eta} = 0$, $q \geqq 1$, [M], 2.1. In complete analogy to the proof of Lemma 2.3, one checks

$$R^p j_{\bar{U}*}G = \bigsqcup_{j=1}^{r} i_{v_j*}H^p(I_{v_j}, G_{m,n}), \quad p > 0.$$

Using the spectral sequence for the composite functor $\gamma_{\bar{U}*}$, we get

$$R^p\gamma_{\bar{U}*}G_{m,\eta} \cong \bigsqcup_{j=1}^{r} i_{v_j*}H^p(I_{v_j}, \bar{K}^*), \quad p \geqq 1.$$

We consider now the Cartan-Leray spectral sequence, for the functor $\gamma_{\bar{U}*}$

$$E_2{}^{p,q} = H^p(\bar{U}, R^q\gamma_{\bar{U}*}G_{m,\eta}) \Rightarrow H^{p+q}(\eta, G_{m,\eta}).$$

Since for $q \geqq 1$ $R^q\gamma_{\bar{U}*}G_{m,\eta}$ is concentrated in the real Archimedean primes, we get $E_2{}^{p,q} = 0$, for $p > 0$ and $q > 0$. Hence, we have an exact sequence

2.5.3. $$0 \to H^1(\bar{U}, \gamma_{\bar{U}*}G_{m,\eta}) \to H^1(\eta, G_{m,\eta}) \to \bigoplus_{j=1}^{r} H^1(I_{v_j}, \bar{K}^*)$$

$$\to H^2(\bar{U}, \gamma_{\bar{U}*}G_{m,\eta}) \to H^2(\eta, G_{m,\eta}) \to \bigoplus_{j=1}^{r} H^2(I_{v_j}, \bar{K}^*)$$

$$\to H^3(\bar{U}, \gamma_{\bar{U}*}G_{m,\eta}) \to H^3(\eta, G_{m,\eta}).$$

Recall that by HILBERT's Theorem 90 $H^1(\eta, G_{m,\eta}) = 0$, $H^1(I_{v_j}, \bar{K}^*) = 0$. It follows from the class field theory that we obtain an exact sequence

2.5.4. $$0 \to H^2(\eta, G_{m,\eta}) \to \bigoplus_v Br(K_v) \xrightarrow{\text{inv}} \mathbf{Q}/\mathbf{Z} \to 0$$

where the direct sum is taken over all points v of the number field K and $Br(K_v)$ denotes the *Brauer group* of the completion of K with respect to v (cf. [CF], Chapter VII). The last arrow in the second line of 2.5.3. may be identified with the map

2.5.5. $$H^2(\eta, G_{m,\eta}) \to \bigoplus_{i=1}^{r} Br(K_{v_i}).$$

From 2.5.4. we get an exact sequence

2.5.6. $$0 \to H^2(\bar{U}, \gamma_{\bar{U}*}, G_{m,\eta}) \to \bigoplus_{x \in X} \mathrm{Br}(K_x) \xrightarrow{\text{inv}} \mathbf{Q}/\mathbf{Z} \to 0.$$

Finally we remark that 2.5.5. is surjective, and that by class field theory, $H^3(\eta, G_{m,\eta}) = 0$ ([CF], Chapter VII, 11.4.). We summarize our results.

2.5.7. Lemma. $H^0(\bar{U}, \gamma_{\bar{U}*}G_{m,\eta}) = K^*$,

$$H^1(\bar{U}, \gamma_{\bar{U}*}G_{m,\eta}) = 0,$$

$$0 \to H^2(\bar{U}, \gamma_{\bar{U}*}G_{m,\eta}) \to \bigoplus_{x \in X} \mathbf{Q}/\mathbf{Z} \to \mathbf{Q}/\mathbf{Z} \to 0 \textit{ is exact,}$$

$$H^3(\bar{U}, \gamma_{\bar{U}*}G_{m,\eta}) = 0.$$

As in [M], 2.3. we have

$$H^p\Big(\bar{U}, \coprod_{x \in U} i_{x*}\mathbf{Z} \oplus \coprod_{j=1} i_{v_j*}\mathbf{Z}/2\mathbf{Z}\Big) = \begin{cases} 0, & p \neq 0, 2, \\ \bigoplus_{x \in U} \mathbf{Q}/\mathbf{Z}, & p = 2. \end{cases}$$

We now apply $H^p(\bar{U}, -)$ to the exact sequence 2.5.2.

2.5.8. $$0 \to H^0(\bar{U}, G_{m,\bar{U}}) \to K^* \to \coprod_{x \in U} \mathbf{Z} \oplus \coprod_{i=1} \mathbf{Z}/2\mathbf{Z} \to H^1(\bar{U}, G_{m,\bar{U}}) \to 0.$$

$H^0(\bar{U}, G_{m,\bar{U}})$ are the elements of K which are units at all points of U and which are positive at the infinite points $v_1, \ldots, v_r$. We denote this group by D^*_{U+}.

Let $\alpha \in K^*$. We denote by α_{v_i} the real number which is the image of α under the embedding defined by v_i. The map $K^* \to \coprod_{i=1}^{r} \mathbf{Z}/2\mathbf{Z}$ is given by

$$\alpha \to (\operatorname{sgn} \alpha_{v_1}, \ldots, \operatorname{sgn} \alpha_{v_r}).$$

By the approximation lemma ([CF], Chapter II, 6.1.) this map is surjective. Denote its kernel by K_+^*. The map

$$K \to \coprod_{x \in U} \mathbf{Z} \oplus \coprod_{i=1}^{r} \mathbf{Z}/2\mathbf{Z}$$

from 2.5.8. induces a map $K_+{}^* \xrightarrow{b} \coprod_{x \in U} \mathbf{Z}$. Then

$$H^1(\bar{U}, G_{m,\bar{U}}) = \coprod_{x \in U} \mathbf{Z}/b(K_+{}^*).$$

We call this group the *restricted Picard group* of U and denote it by $\mathrm{Pic}^+ U$.

Since now the computation of $H^i(\bar{U}, G_{m,\bar{U}})$ for $i = 2, 3$ is a trivial diagram chasing, we give only the result.

2.5.9. Proposition. *For $U \neq X$, we have*

$$H^0(\bar{U}, G_{m,\bar{U}}) = D^*_{U,+} \quad H^1(\bar{U}, G_{m,\bar{U}}) = \mathrm{Pic}^+ U,$$

$$0 \to H^2(\bar{U}, G_{m,\bar{U}}) \to \bigoplus_{x \in X \setminus U} \mathbf{Q}/\mathbf{Z} \to \mathbf{Q}/\mathbf{Z} \xrightarrow{\sim} 0,$$

$$H^3(\bar{U}, G_{m,\bar{U}}) = 0.$$

For $U = X$, we have

$$H^0(\bar{X}, G_{m,\bar{X}}) = D_+{}^*, \quad H^1(\bar{X}, G_{m,\bar{X}}) = \mathrm{Pic}^+ X,$$

$$H^2(\bar{X}, G_{m,\bar{X}}) = 0, \quad H^3(\bar{X}, G_{m,\bar{X}}) \to \mathbf{Q}/\mathbf{Z}.$$

2.6. Our aim is to prove the following theorem.

Theorem. *Let U be an open subscheme of X. Let $\bar{F}$ be a constructible sheaf on $\bar{U}$, then $H^p(\bar{U}, \bar{F})$ are finite Abelian groups and $H^p(\bar{U}, \bar{F}) = 0$, $p \geqq 4$.*

Of course, in the case where K is totally imaginary or if the order of F_η is not divisible by 2 this theorem follows from [SGA 4], Chapter X, 6.1.

We begin with the proof of two technical lemmas.

2.6.1. Let $\bar{V} \xrightarrow{\bar{\pi}} \bar{U}$ be an object of $\mathrm{Et}_{\bar{U}}$. Assume that a finite group G acts on this object. We say that $\bar{\pi}$ is an *unramified Galois covering* with the Galois group G if the following conditions are satisfied:

(i) the underlying map of schemes $\pi\colon V \to U$ with the canonical G-action is an unramified Galois covering,

(ii) G acts on $\bar{V}_\infty$ without fixed points,

(iii) the space of orbits $\bar{V}_\infty/G$ is mapped bijectively to $\bar{U}_\infty$.

This is the same as an principal homogeneous G-space over $\bar{U}$ in the topology of $\mathrm{Et}_{\bar{U}}$.

Let M be a finite G-set. G acts on the scheme $\coprod_{m \in M} V$ by acting on V and M. We denote the quotient of this action by $V \times_G M$. The set $\bar{V}_\infty \times_G M$ may be identified with a subset of $(V \times_G M)$ $(\mathbf{R})$. Let $\bar{V} \times_G M$ be the resulting object of $\mathrm{Et}_{\bar{U}}$. The functor from the category of finite G-sets to $\mathrm{Et}_{\bar{U}}$ which associates $\bar{V} \times_G M$ to M gives a Hochschild-Serre sequence ([GT], Chapter III, 4.7.).

Lemma. *Let $\bar{F}$ be an Abelian sheaf on $\bar{U}$. Then there exists a spectral sequence*

$$H^p\big(G, H^q(\bar{V}, \bar{\pi}^* \bar{F})\big) \Rightarrow H^{p+q}(\bar{U}, \bar{F}).$$

2.6.2. Lemma (effaceability). *Let $\bar{F}$ be a constructible sheaf on $\bar{U}$ and $c \in H^p(\bar{U}, \bar{F})$, $p \geqq 1$, a cohomology class. There exists an injection $\bar{F} \to \bar{F}'$ of $\bar{F}$ into a constructible sheaf $\bar{F}'$, such that $H^p(\bar{U}, \bar{F}) \to H^p(\bar{U}, \bar{F}')$ kills c.*

Proof. Since H^p commutes with direct limits, we have to show that the injective hull $\bar{I}$ of $\bar{F}$ is the direct limit of its constructible subsheaves. We know that $\bar{I}$ is a direct sum of sheaves of the kind $j_{U*}\, I$ and $i_{v_k *} I_k$. Hence, we conclude by [SGA 4], Chapter IX, 2.7.2.

Proof of the theorem. Recall that two sheaves are called *punctually equivalent* if its difference in the Grothendieck group of all sheaves lies in the subgroup generated by punctual sheaves, i.e., sheaves concentrated to a finite number of points. Clearly, we may replace in the proof $\bar{F}$ by a punctually equivalent one. To prove that the cohomology is finite, it suffices by 2.2.1. and 2.2. to show that $H^p(U, F)$ is finite, which is obvious from [M]. For the second statement of the theorem, we need still another lemma.

2.6.3. Lemma. *Let $\bar{F}$ be a sheaf on $\bar{U}$. Assume that the residue class characteristics of U are prime to the order n of F_η, and that F_η is unramified as a G_η-module at all points of U, i.e., I_x acts trivially on F_η, for all $x \in U$. Then $H^3(\bar{U}, \bar{F}) = 0$.*

Proof. First of all, by the usual argument, we can assume $n = p$ to be a prime and $p \cdot F_\eta = 0$. Since we can replace $\bar{F}$ by a punctual equivalent one, we may assume $p \cdot \bar{F} = 0$ and $\bar{F}$ to be isomorphic to the direct image of F_η. Then $\bar{F}$ is locally constant since F_η is unramified. The proof will be given by induction on $\dim_{\mathbf{Z}/p\mathbf{Z}} F_\eta$. Let $G_p \subset G_\eta$ be a p-Sylow group. Since F_η is a $\mathbf{Z}/p\mathbf{Z}$-vector space, there is an element $x \in F_\eta$ fixed under the action of G_p. Let $H \subset G_\eta$ be the stabilizer of x. Since the action of G_η on F_η is unramified at the points of U, there exists an unramified Galois covering $V \xrightarrow{\pi} U$ with Galois group $G = G_\eta/H$ of order prime to p.

Let us first treat the case $p = 2$. In this case $\bar{V} \xrightarrow{\bar{\pi}} \bar{U}$ is also unramified since the I_{v_k} have order 2 and hence can not be contained in G. Therefore we can apply the Hochschild-Serre sequence

$$H^p\big(G, H^q(\bar{V}, \pi^*\bar{F})\big) \Rightarrow H^n(\bar{U}, \bar{F}).$$

This sequence degenerates because the order of G is prime to p. Hence $H^q(\bar{V}, \pi^*\bar{F})^G = H^q(\bar{U}, \bar{F})$, and we are reduced to prove our assertion for $\pi^*\bar{F}$. Since x is fixed by H, and F is locally constant, we find an exact sequence of sheaves on V

$$0 \to \mathbf{Z}/2\mathbf{Z} \to \pi^*F \to F' \to 0.$$

By induction we reduce to the case $F = \mathbf{Z}/2\mathbf{Z}$. Since the residue characteristics of V are prime to p, we may apply the Kummer sequence

$$0 \to j_{\bar{V}!}\mathbf{Z}/2\mathbf{Z} \to G_{m,\bar{V}} \to G_{m,\bar{V}} \to 0$$

where $j_{\bar{V}!}$ is the extension by zero (see 1.3.2.). Now the statement follows from 2.5.9).

If $p \neq 2$ the proof is completely analog, but one has to start from the Hochschild-Serre sequence for $\pi\colon V \to U$ taking into account 2.4.

Let us return to the proof of the theorem. Let $V \subset U$ be an open subscheme. We have the local cohomology sequence

$$\coprod_{x \in U \setminus V} H_x^i(\overline{U}, \overline{F}) \to H^i(\overline{U}, \overline{F}) \to H^i(\overline{V}, \overline{F}) \to \coprod_{x \in U \setminus V} H_x^{i+1}(\overline{U}, \overline{F}).$$

It follows from [M] that $H_x^i(\overline{U}, \overline{F}) = 0$, $i \geqq 4$. Therefore, we may replace U by any open subscheme V. Let $p \geqq 4$. From the effaceability and finiteness of the cohomology, we find an injection $\overline{F} \xrightarrow{\alpha} \overline{I}$ into a constructible sheaf $\overline{I}$, such that the map $H^p(\overline{U}, \overline{F}) \to H^p(\overline{U}, \overline{I})$ is zero. Let $\overline{F}'$ be the cokernel of α. From the cohomology sequence we get a surjection $H^{p-1}(\overline{U}, \overline{F}') \to H^p(\overline{U}, \overline{F})$. Applying induction on p, we may assume $p = 4$. By shrinking U we may arrange that $\overline{F}'$ satisfies the hypotheses of Lemma 2.6.3. Hence the proof of the theorem is complete.

As a corollary we obtain part (c) of TATE's theorem (see these notes).

Corollary. *Let S be a not necessarily finite set of points of X which contains the infinite points. Denote by K_S the maximal algebraic extension of K which is unramified outside S and by G_S its Galois group. Let M be a G_S-module whose order is prime to the residue characteristics of points not lying in S. There is a canonical isomorphism*

$$H^p(G_S, M) \to \coprod_{i=1}^{r} H^p(I_{v_i}, M) \quad \textit{for} \quad p \geqq 3.$$

Proof. Let $\mathscr{C}$ be the category of all étale maps $V \to X$, such that V is connected and its function field lies in K_S. We evidently have $\varprojlim_{V \in \mathscr{C}} V = \operatorname{Spec} K_S$. Let $\overline{F} = \gamma_{\overline{X}*} M$ where $\gamma_{\overline{X}*}$ is the functor $\mathscr{S}_\eta \to \mathscr{S}_{\overline{X}}$. For each sufficiently small $V \in \mathscr{C}$, the restriction $\overline{F}|_{\overline{V}}$ satisfies the hypotheses of Lemma 2.6.3 and consequently $H^p(\overline{V}, \overline{F}|_{\overline{V}}) = 0$, $p \geqq 3$. From the local cohomology sequence for $\overline{V}$ and the infinite points, we get isomorphisms

$$H^p(V, F|_V) \to \coprod_{i=1}^{r} H^p(I_{v_i}, M), \quad p \geqq 3.$$

Finally we conclude by passing to the direct limit over all V.

3. Artin-Verdier duality

3.1. Modified étale cohomology

We consider an open subscheme U of X and an Abelian sheaf F on U.

3.1.1. Definition. The modified sheaf $\hat{F}$ is the sheaf on $\overline{U}$ given by the tuple

$$\big(H_0(I_{v_1}, F_\eta), \ldots, H_0(I_{v_r}, F_\eta), F,$$
$$N: H_0(I_{v_1}, F_\eta) \to H^0(I_{v_1}, F_\eta), \ldots, N: H_0(I_{v_r}, F_\eta) \to H^0(I_{v_r}, F_\eta)\big).$$

Here $H_0(I_{v_i}, -)$ denotes the covariants under the action of I_{v_i} and N is the norm map.

3.1.2 Lemma. *$F \mapsto \hat{F}$ is a right exact functor which preserves injective objects.*

Proof. The right exactness follows from that of H_0. Assume that F is an injective sheaf. Then its direct image j_*F on $\overline{U}$ is injective. Hence $H^p_{v_i}(\overline{U}, j_*F) = 0$, $p \geqq 1$, and by Lemma 2.3 $H^p(I_{v_i}, F_\eta) = 0$, $p \geqq 1$. Since the cohomology of I_{v_i} is cyclic, it follows that all modified cohomology groups $\hat{H}^p(I_{v_i}, F_\eta)$ are zero ([CF], Chapter IV). Hence, the norm map is an isomorphism, and we conclude that $j_*F = \hat{F}$, which proves the lemma.

3.1.3. Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of Abelian sheaves on U. Using the exact sequence for homology of I_{v_k}, we get an exact sequence

3.1.3.1.
$$\to \coprod_{k=1}^{r} i_{v_k *} H^{-2}(I_{v_k}, F_\eta) \to \coprod_{k=1}^{r} i_{v_k *} H^{-2}(I_{v_k}, F_\eta'') \to \hat{F}' \to \hat{F} \to \hat{F}'' \to 0.$$

Consider the exact sequences

$$0 \to \overline{K} \to \hat{F} \to \hat{F}'' \to 0,$$
$$0 \to \overline{L} \to \hat{F}' \to \overline{K} \to 0.$$

The sheaf $\overline{L}$ is concentrated in the infinite points. Consequently, $H^p(\overline{U}, \overline{L}) = 0$ for $p \geqq 1$. From the cohomology sequences we get exact sequences

$$0 \to H^0(\overline{U}, \overline{L}) \to H^0(\overline{U}, \hat{F}') \to H^0(\overline{U}, \overline{K}) \to 0,$$
$$H^p(\overline{U}, \hat{F}') \cong H^p(\overline{U}, \overline{K}), \quad p \geqq 1,$$
$$0 \to H^0(\overline{U}, \overline{K}) \to H^0(\overline{U}, \hat{F}) \to H^0(\overline{U}, \hat{F}'') \to H^1(\overline{U}, \hat{F}') \to \cdots.$$

And finally from 3.1.3.1.

$$\cdots \to \bigoplus_{k=1}^{r} \hat{H}^{-2}(I_{v_k}, F_\eta) \to \bigoplus_{k=1}^{r} \hat{H}^{-2}(I_{v_k}, F_\eta'') \to H^0(\overline{U}, \overline{L}) \to 0.$$

3.1.4. Definition. For $p \in \mathbf{Z}$ we set

$$\hat{H}^p(U, F) = \begin{cases} H^p(\overline{U}, \hat{F}), & p \geqq 0, \\ \bigoplus_{k=1}^{r} \hat{H}^{p-1}(I_{v_k}, F_\eta), & p < 0. \end{cases}$$

The $\hat{H}^p(U, F)$ are called the *modified étale cohomology groups.*

Fitting together the above exact sequences, we get a long exact sequence

3.1.4.1. $\cdots \to \hat{H}^p(U, F') \to \hat{H}^p(U, F) \to \hat{H}^p(U, F'') \to \hat{H}^{p+1}(U, F') \to \cdots, \quad p \in \mathbf{Z}.$

3.1.5. With help of Lemma 2.3, one checks easily that for $v \in \overline{U}_\infty$

$$H_v^p(\overline{U}, \hat{F}) = \hat{H}^{p-1}(I_v, F_\eta), \quad p \geqq 0.$$

We define therefore the local cohomology groups as follows.

Definition.

$$H_v{}^p(U, F) = \begin{cases} H_v{}^p(\bar{U}, \hat{F}), & p \geqq 0,\ v \in \bar{U}_\infty, \\ \hat{H}^{p-1}(I_v, F_\eta), & p < 0. \end{cases}$$

Let S be a finite subset of U, and let $\bar{S}$ be the union of S with all infinite points. Denote by $V = U \setminus S$ the complement, then we have an exact sequence (see 2.2.1.)

3.1.5.1. $$\cdots \to \coprod_{s \in \bar{S}} H_s{}^p(U, F) \to \hat{H}^p(U, F) \to H^p(V, F|_V)$$

$$\to \coprod_{s \in \bar{S}} H_s{}^{p+1}(U, F) \to \cdots.$$

With the convention that $H^p(V, F|_V) = H_s{}^p(U, F) = 0$, for $p < 0$ and $s \in S$, the sequence makes sense also for $p < 0$.

3.1.6. Let L be a finite extension of K, and V the integral closure of U in L. We have a finite morphism $f\colon V \to U$. For any Abelian sheaf F on V, we have canonical isomorphisms

3.1.6.1. $\hat{H}^p(V, F) = \hat{H}^p(U, f_*F), \quad p \in \mathbf{Z}.$

Proof. Let $\theta = \operatorname{Spec} L$ be the general point of V. Then G_θ is a subgroup of G_η. We form the induced module $\operatorname{Ind}_{G_\eta}^{G_\theta} F_\theta$. Since the functor $\operatorname{Ind}_{G_\eta}^{G_\theta}$ is adjoint to the restriction functor which regards a G_η-module as a G_θ-module, we have a canonical isomorphism

$$\operatorname{Ind}_{G_\eta}^{G_\theta} F_\theta \cong (f_*F)_\eta.$$

We study more closely the structure of $\operatorname{Ind}_{G_\eta}^{G_\theta} F_\theta$. Let w be a real Archimedean point of K. Denote by $v_1, \ldots, v_t$ all Archimedean points of L which lie over w. Assume that $v_1, \ldots, v_r$ are the real ones. We choose fixed extensions $\bar{v}_1, \ldots, \bar{v}_t$ and take $\bar{w} = \bar{v}_1$. Furthermore, we choose $\sigma_i \in G_\eta$, such that $\bar{v}_1\sigma_i = \bar{v}_i$. Let $\tau_1, \ldots, \tau_t$ be the generators of the decomposition groups $I_{\bar{v}_1}, \ldots, I_{\bar{v}_t}$. Evidently, we have $\sigma_i^{-1}\, \tau_1 \sigma_i = \tau_i$ and

$$G_\eta = \left(\bigcup_{i=1}^{t} G_\theta \sigma_i^{-1}\right) \cup \left(\bigcup_{i=r+1}^{t} G_\theta \sigma_i^{-1} \tau_1^{-1}\right).$$

Consequently, the induced module may be written as

$$\operatorname{Ind}_{G_\eta}^{G_\theta} F_\theta = \coprod_{i=1}^{r} \sigma_i F_\theta \oplus \coprod_{j=r+1}^{t} (\sigma_j F_\theta \oplus \tau_1 \sigma_j F_\theta).$$

The action of τ_1 on $\operatorname{Ind}_{G_\eta}^{G_\theta} F_\theta$ may be described as follows. For $j \geqq r+1$, it acts by interchanging $\sigma_j F_\theta$ and $\tau_1 \sigma_j F_\theta$, and, for $j \leqq r$, it acts on $\sigma_j F_\theta$ by the defining equation $\tau_1 \sigma_j y = \sigma_j \tau_j y$, for $y \in F_\theta$.

Since we have chosen $I_{\bar{v}_1} = I_{\bar{w}}$, one gets readily

$$H^0(I_{\bar{w}}, \operatorname{Ind}_{G_\eta}^{G_\theta} F_\theta) = \bigsqcup_{i=1}^{r} H^0(I_{\bar{v}_i}, F_\theta) \oplus \bigsqcup_{i=r+1}^{t} F_\theta,$$

$$H_0(I_{\bar{w}}, \operatorname{Ind}_{G_\eta}^{G_\theta} F_\theta) = \bigsqcup_{i=1}^{r} H_0(I_{\bar{v}_i}, F_\theta) \oplus \bigsqcup_{i=r+1}^{t} F_\theta,$$

where the norm map induces the identity on $\bigsqcup_{i=r+1}^{t} F_\theta$.

Since we know that $H^0(V, F) = H^0(U, f_*F)$ ([GT], Chapter III, 4.11.), one checks readily from 2. and 3.1.4. that

$$\hat{H}^0(V, F) = \hat{H}^0(U, f_*F).$$

Because f_* is exact in our case and preserves injective modules and because of 3.1.2. it follows that, for $p \geqq 0$, exact effaceable δ-functors stand on both sides of 3.1.6.1. This proves our assertion for $p \geqq 0$. For $p < 0$, we are dealing only with a group theoretic question and we can argue by the lemma of Shapiro.

3.2. We are now ready to proof the central result of this Appendix. From Definition 2.5.1 the reader checks without difficulty $G_{m,\bar{U}} = \hat{G}_{m,U}$. Because of 2.5.9. we get further a canonical isomorphism $\hat{H}^3(X, G_{m,X}) \cong \mathbf{Q}/\mathbf{Z}$. To simplify the notation, we make the convention that $\operatorname{Ext}_-^r(-, -)$ is zero, for negative r.

Theorem (Artin-Verdierduality). *Let F be a constructible sheaf on X. The Yoneda pairing*

3.2.1. $\hat{H}^r(X, F) \times \operatorname{Ext}_X^{3-r}(F, G_{m,X}) \to \hat{H}^3(X, G_{m,X}) \cong \mathbf{Q}/\mathbf{Z}, \quad r \in \mathbf{Z},$

is perfect.

If K is totally imaginary, the modified cohomology coincides with the usual étale cohomology. In this case the theorem is just 2.4. of [M].

3.2.2. Lemma. *For each constructible sheaf F, the canonical morphism*

3.2.2.1. $H^p(X, \underline{\operatorname{Hom}}_X(F, G_m)) \to \operatorname{Ext}_x^p(F, G_m)$

is an isomorphism, for $p \geqq 4$.

Proof. We treat first the case where F is concentrated in a closed point $x \in X$. That means, F may be written in the form $i_{x*}N$, where $i_x\colon x \to X$ is the inclusion. Since i_{x*} is exact and left adjoint to $i_x^!$, we get a spectral sequence

$$\operatorname{Ext}_x^p(N, R^q i_x^! G_m) \Rightarrow \operatorname{Ext}_X^p(F, G_m).$$

Since further by [M], 1.2., $R^1 i_x^! G_m = \mathbf{Z}$ and $R^q i_x^! G_m = 0$, $q \neq 1$, this sequence reduces to isomorphisms

$$\operatorname{Ext}_X^{p-1}(N, \mathbf{Z}) = \operatorname{Ext}_X^p(F, G_m).$$

By the local duality theorem (see [M], 1., Remark d), $\mathrm{Ext}_x^{p-1}(N, \mathbf{Z})$ is dual to $H^{3-p}(G_x, N)$ and hence zero, for $p \geqq 4$. Since $\underline{\mathrm{Hom}}(F, G_m)$ is zero in our case, we get the assertion for punctual sheaves.

Let again F be any constructible sheaf. Choose an open set U in X, such that the restriction $F|_U$ is locally constant and has the order prime to the residue class characteristics of U. Let $j\colon U \to X$ be the inclusion. Since the cokernel of the injection $j_!F|_U \to F$ is concentrated to a finite number of points, it suffices to prove our assertion for $j_!F|_U$. Since $j_!$ and j^* are adjoint and exact, we get

$$\mathrm{Ext}_X{}^p(j_!F|_U, G_{m,X}) \cong \mathrm{Ext}_U{}^p(F|_U, G_{m,U}), \quad p \in \mathbf{Z}.$$

Since the local Ext groups $\underline{\mathrm{Ext}}_U{}^p(F|_U, G_{m,U})$ vanish for $p > 0$ (see [M]) we obtain

$$\mathrm{Ext}_U{}^p(F|_U, G_{m,U}) \cong H^p\big(U, \underline{\mathrm{Hom}}(F|_U, G_{m,U})\big).$$

Since finally local cohomology is zero in dimension greater than 3, we deduce

$$H^p\big(U, \underline{\mathrm{Hom}}(F|_U, G_{m,U})\big) \cong H^p\big(X, \underline{\mathrm{Hom}}(j_!F|_U, G_{m,U})\big), \quad p \geqq 4.$$

This proves the lemma.

3.2.3. Corollary. *There is a canonical isomorphism*

$$\mathrm{Ext}_X{}^p(F, G_{m,X}) \cong \coprod_{i=1}^{r} H^p(I_{v_i}, F_\eta{}^*), \quad p \geqq 4,$$

where $F_\eta{}^* = \mathrm{Hom}(F_\eta, G_{m,\eta})$ *is the dual Galois module.*

Proof. This follows readily from 3.2.2., 2.6., the local cohomology sequence and 2.3.

Remark. Let $\mathscr{S}'_{v_i}$ be the category of I_{v_i}-modules. The functor $F \mapsto F_\eta$ from $\mathscr{S}_X$ to $\mathscr{S}'_{v_i}$ is exact and preserves injective modules and gives therefore rise to a map

$$\mathrm{Ext}_X{}^p(F, G_m) \to \mathrm{Ext}^p_{\mathscr{S}'_{v_i}}(F_\eta, G_{m,\eta}) = H^p(I_{v_i}, F_\eta{}^*).$$

This provides the canonical map of 3.2.3.

Proof of the theorem. We consider first the case $r < 0$. By the Corollary 3.2.3 and Definition 3.1.4, it suffices to prove that the cup-product induces a perfect pairing

$$\hat{H}^{2-p}(I_{v_i}, F_\eta) \times \hat{H}^p(I_{v_i}, F_\eta{}^*) \to H^2(I_{v_i}, G_{m,\eta}) \hookrightarrow \hat{H}^3(X, G_{m,X}).$$

If the order of F_η is not divisible by 2, the groups are zero and the assertion is obvious. Hence by descent we may assume $F = \mathbf{Z}/2\mathbf{Z}$. Let $\delta \varepsilon\, H^2(I_{v_i}, \mathbf{Z})$ be a generator. We know that the cup-product with δ induces, for any I_{v_i}-module M, an isomorphism

$$\hat{H}^p(I_{v_i}, M) \to \hat{H}^{p+2}(I_{v_i}, M).$$

It follows that our assertion is equivalent to the assertion that

$$\hat{H}^p(I_{v_i}, F_\eta) \times \hat{\hat{H}}{}^p(I_{v_i}, F_\eta{}^*) \to \hat{H}^0(I_{v_i}, G_{m,\eta})$$

is a perfect pairing, for $p = 0, 1$ and $F_\eta = \mathbf{Z}/2\mathbf{Z}$. This is a trivial verification. Our proof in the case $r < 0$ is therefore complete.

By replacing the usual étale cohomology by the modified one in the general machinery developed in [M], 3., we could finish our proof. We go a slightly different way by proving the following lemma. Since our theorem is proved in [M] for K totally imaginary, this will complete the proof.

Lemma. *Assume that our theorem is true for K totally imaginary. Assume furthermore that the pairing* 3.2.1. *is perfect for any K, for any constructible F and $r < r_0$. Then the pairing is perfect for $r = r_0$.*

Proof. A consideration similar to that in the proof of 3.2.2. shows that our theorem is true for sheaves with support in a finite number of closed points. Choose a finite extension L of K which is totally imaginary. Let Y be the spectrum of the ring of integers of L and $\pi\colon Y \to X$ the projection. Consider the map

$$F \to \pi_* \pi^* F.$$

We get two exact sequences

3.2.4. $$0 \to H \to F \to E \to 0,$$
$$0 \to E \to \pi_*\pi^*F \to G \to 0.$$

Since H has finite support, the theorem is true for H. Hence it is sufficient to show that 3.2.1. is perfect, for $F = E$ and $r = r_0$. For an Abelian group M, let $\tilde{M} = \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$ be its Pontryagin dual. The pairing 3.2.1. gives a map

$$m^r(F)\colon \hat{H}^r(X, F) \to \mathrm{Ext}_X{}^{3-r}(F, G_{m,X})^{\sim}.$$

Hence from 3.2.4. we get a commutative diagram with exact lines (compare with [M], 3.)

3.2.5.
$$\begin{array}{ccc}
\hat{H}^{r_0-1}(X, \pi_*\pi^*F) & \to & \hat{H}^{r_0-1}(X, G) \to \\
\downarrow & & \downarrow \\
\mathrm{Ext}_X{}^{4-r_0}(\pi_*\pi^*F, G_{m,X}) & \to & \mathrm{Ext}_X{}^{4-r_0}(G, G_{m,X}) \to
\end{array}$$
$$\begin{array}{ccccc}
\to \hat{H}^{r_0}(X, E) & \to & \hat{H}^{r_0}(X, \pi_*\pi^*F) & \to & \hat{H}^{r_0}(X, G) \\
\downarrow{\scriptstyle m^{r_0}(E)} & & \downarrow{\scriptstyle m^{r_0}(F)} & & \downarrow{\scriptstyle m^{r_0}(G)} \\
\to \mathrm{Ext}_X{}^{3-r_0}(E, G_{m,X}) & \to & \mathrm{Ext}_X{}^{r_0}(\pi_*\pi^*F, G_{m,X}) & \to & \mathrm{Ext}_X{}^{3-r_0}(G, G_{m,X})
\end{array}$$

By 3.1.6., we have $\hat{H}^r(X, \pi_*\pi^*F) = \hat{H}^r(Y, \pi^*F)$ and, by the norm theorem [M], 2.7.,

$$\mathrm{Ext}_X{}^{3-r}(\pi_*\pi^*F, G_{m,X}) = \mathrm{Ext}_Y{}^{3-r}(\pi^*F, G_{m,Y}).$$

Since we assume that our theorem is true for Y, we get that the first and the fourth vertical arrow in 3.2.5. are isomorphisms. The second arrow is an isomorphism by induction. Hence $m^{r_0}(E)$ and consequently $m^{r_0}(F)$ are injective. Applying this result to G, we see that $m^{r_0}(G)$ is injective. An easy diagram chasing yields now that $m^{r_0}(E)$ is an isomorphism. This proves the lemma and completes the proof of the theorem.

3.3. The purpose of this section is to show that Theorem 3.2 is essentially equivalent to TATE's global Duality Theorem (see [T], Theorem 3.1, or these notes Theorem 1).

Let U be an open set of X and let $S = X \setminus U$ be its complement. We denote by G_S the fundamental group $\pi_1(U)$ of U. This is the Galois group of the maximal extension of K which is unramified outside S.

3.3.1. Proposition. *Let F be a locally constant sheaf on U. Assume that the order of F is prime to the residue class characteristics of U. There is a canonical isomorphism*

$$H^i(G_S, F_\eta) = H^i(U, F).$$

Proof. The Hochschild-Serre sequence ([GT], Chapter III, 4.7.) reads

$$H^p\big(G_S, \varinjlim_{W|U} H^q(W, F|_W)\big) \Rightarrow H^n(U, F).$$

Here W runs over all unramified connected Galois coverings W of U. Evidently it suffices to prove

$$\varinjlim_{W|U} H^q(W, F|_W) = \begin{cases} 0, & q > 0, \\ F_\eta, & q = 0. \end{cases}$$

Note that this is trivial, for $q = 0$.

Considering only sufficiently great coverings W, we may assume that $F|_W = \mathbf{Z}/n\mathbf{Z}$, and that the function field L of W contains the n-th roots of unity and is totally maginary. Hence we can use the Kummer sequence

$$0 \to \mathbf{Z}/n\mathbf{Z} \to G_{m,W} \xrightarrow{n} G_{m,W} \to 0.$$

Using 2.5.9. we may write the cohomology sequence as follows

$$0 \to H^0(W, \mathbf{Z}/n\mathbf{Z}) \to D_W{}^* \xrightarrow{n} D_W{}^* \to H^1(W, \mathbf{Z}/n\mathbf{Z}) \to \operatorname{Pic} W \xrightarrow{n} \operatorname{Pic} W$$
$$\to H^2(W, \mathbf{Z}/n\mathbf{Z}) \to H^2(W, G_{m,W}) \xrightarrow{n} H^2(W, G_{m,W}) \to H^3(W, \mathbf{Z}/n\mathbf{Z}) \to 0$$

where $D_W{}^*$ denotes the elements of L, which are units at all points of W.

From 2.6.3. we know that $H^p(W, \mathbf{Z}/n\mathbf{Z}) = 0$, for $p \geqq 3$. It remains to show that $\varinjlim_W H^p(W, \mathbf{Z}/n\mathbf{Z}) = 0$, for $p = 1, 2$.

Let L' be the Hilbert class field of L and let W' be the integral closure of W in L'. By the principal ideal theorem of the class field theory, the map $\operatorname{Pic} W \to \operatorname{Pic} W'$ is trivial. Hence, we get $\varinjlim_W \operatorname{Pic} W = 0$.

Furthermore, we get $\varinjlim_W D_W{}^*/D_W{}^{*n} = 0$. Indeed, if a is an element of $D_W{}^*$, then the equation $x^n - a = 0$ defines a Galois extension L'' of L which is unramified along W and in which a is an n-th power.

We denote, for an Abelian group M, by ${}_nM$ the kernel of multiplication by n. The only thing that remains to be shown is

$$\varinjlim_W {}_nH^2(W, G_{m,W}) = 0.$$

Again from 2.5.9., we get an exact sequence

$$0 \to {}_nH^2(W, G_{m,W}) \to \bigoplus_{q \in S_W} {}_n(\mathbf{Q}/\mathbf{Z}) \to {}_n(\mathbf{Q}/\mathbf{Z})\,.$$

Here S_W denotes the set of finite points of L not lying in W. Recall that the first arrow is the composition of the following maps

$${}_nH^2(W, G_{m,W}) \to {}_nH^2(\theta, G_{m,\theta}) \to \bigoplus_{q \in S_W} {}_nH^2(\bar{L}_q \mid L_q, L_q) \to \bigoplus_{q \in S_W} {}_n(\mathbf{Q}/\mathbf{Z})\,.$$

Here θ denotes the general point of W, L_q the completion of L at q and $H^2(\bar{L}_q \mid L_q, \bar{L}_q{}^*)$ the Galois cohomology group

$$H^2\big(\mathrm{Gal}(\bar{L}_q \mid L_q), L_q{}^*\big) = Br(L_q)\,.$$

Obviously, it suffices to show that

$$\varinjlim_{W} \bigoplus_{q \in S_W} {}_nH^2(\bar{L}_q \mid L_q, \bar{L}_q{}^*) = 0\,.$$

Let $E \mid L$ be an extension which is unramified along W. Let further q' be a point over q. By use of the local class field theory ([CF], Chapter VI, 1.2.), there is a commutative diagram

$$\begin{array}{ccc} {}_nH^2(\bar{L}_q \mid L_q, \bar{L}_q{}^*) & \xrightarrow{\text{inv}} & \frac{1}{n}\,\mathbf{Z}/\mathbf{Z} \\ \downarrow & & \downarrow {\scriptstyle [E_{q'} : L_q]} \\ {}_nH^2(\bar{E}_{q'} \mid E_{q'}, \bar{E}_{q'}^*) & \xrightarrow{\text{inv}} & \frac{1}{n}\,\mathbf{Z}/\mathbf{Z} \end{array}$$

Our assertion follows if we can construct an extension $E \mid L$, which is unramified along W and the local degrees $[E_{q'} : L_q]$ of which, for $q \in S_W$, $q' \mid q$, are all divisible by n. Let again $L' \mid L$ be the Hilbert class field. In L' all ideals $q \in S_W$ become principal, say generated by f_q. Evidently, it suffices to take $E = L'\big(\sqrt[n]{f_q}\big)_{q \in S_W}$.

We are now ready to prove that Tate's long exact sequence for finite ramification (see these notes Theorem 1) follows from the Duality Theorem 3.4.

Let F_η be a G_η-module. For a point s of X, we denote by $H^i(K_s, F_\eta)$ the group $H^i(D_s, F_\eta)$ if s is a finite point and the group $\hat{H}^i(I_s, F_\eta)$ if s is an infinite point (see 1.2. and 1.3.).

3.3.2. Proposition. *Let M be a finite G_S-module. Assume that the order of M is prime to the residue class characteristics of U. Then there exists an exact sequence*

3.3.2.1. $$0 \to H^0(G_S, M) \to \coprod_{s \in \bar{S}} H^0(K_S, M) \to H^2(G_S, M^*)^{\sim} \to H^1(G_S, M)$$
$$\to \coprod_{s \in \bar{S}} H^1(K_s, M) \to H^1(G_S, M^*)^{\sim} \to H^2(G_S, M) \to \coprod_{s \in \bar{S}} H^2(K_s, M)$$
$$\to H^0(G_S, M^*)^{\sim} \to 0\,.$$

Recall that we denote by $M^* = \mathrm{Hom}(M, G_{m,\eta})$ the *dual Galois module* and by $\sim$ the *Pontryagin dual.*

Proof. Since G_S is the fundamental group of U, M defines a locally constant sheaf F on U. Denote by $j\colon U \to X$ the inclusion. As in the proof of 3.2.2., we have isomorphisms

$$\mathrm{Ext}_X{}^p(j_!F, G_{m,X}) \cong \mathrm{Ext}_U{}^p(F, G_{m,U}) \cong H^p\big(U, \underline{\mathrm{Hom}}\,(F, G_{m,U})\big).$$

We denote the sheaf $\underline{\mathrm{Hom}}(F, G_{m,U})$ by F^*. This sheaf is locally constant since the residue class characteristics of U are prime to the order of F. The general fibre of F^* is M^*. Applying 3.3.1. to F, we get an isomorphism

$$\mathrm{Ext}_X{}^p(j_!F, G_{m,X}) \cong H^p(G_S, M^*).$$

The local cohomology sequence 3.1.3.1. in our situation reads

3.3.2.3. $$\to \bigsqcup_{s\in\bar{S}} H_s{}^p(X, j_!F) \to \hat{H}^p(X, j_!F) \to H^p(U, F) \to \bigsqcup_{s\in\bar{S}} H_s{}^{p+1}(X, j_!F).$$

We want to show that this sequence may be identified with 3.3.2.1. By 3.3.1. we have $H^p(U, F) = H^p(G_S, M)$. Further, by the Duality Theorem, we have

$$\hat{H}^p(X, j_!F) \cong \mathrm{Ext}_X{}^{3-p}(j_!F, G_{m,X})^\sim \cong H^{3-p}(G_S, M^*)^\sim.$$

Hence, it remains to interpret the local cohomology groups $H_s{}^p(X, j_!F)$. If s is a point at infinity it is obvious from 3.1.5. that

$$H_s{}^p(X, j_!F) = \hat{H}^{p-1}(I_s, M) = H^{p-1}(K_s, M).$$

Let $s \in S$ be a finite point, and denote by $\tilde{X}_s \xrightarrow{\varphi} X$ the henselization of X in s. By excision [M], 1., we have $H_s{}^p(X, j_!F) = H_s{}^p(\tilde{X}_s, \varphi^* j_!F)$. The last cohomology group can be computed from the local cohomology sequence and [GT], Chapter III, 4.9. (for that see [M], 1., Remark e). We get

$$H_s{}^p(X, j_!F) \cong H^{p-1}(D_s, M) \cong H^{p-1}(K_s, M).$$

It remains only to check the zeros in 3.3.2.1. The last zero follows from the fact that by the corollary to 2.6. the map

$$H^3(G_S, M) \to \bigsqcup_{s\in\bar{S}\setminus S} H^3(K_s, M)$$

is an isomorphism. The first zero follows by Pontryagin duality. This completes the proof of 3.3.2.

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