TWO-VARIABLE p-ADIC L-FUNCTIONS

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1. Introduction

This is a write-up of my talk in the Stanford reading group on the work of Bertolini-Darmon. The objective of my talk is to present a construction a two-variable p-adic L-function attached to a Hida family, and show that, in some sense, it p-adically interpolates special values of L-functions of the specializations of the Hida family.

Let me be very imprecise in this passage and give a quick introduction to what lies ahead. Let f be an ordinary eigenform of weight $k \geq 2$ living in a Hida family f_∞. To f, one can associate a concrete object called a modular symbol, $\tilde{I}_f$, and one can even divide by an appropriate period, and obtain an algebraically-defined modular symbol, I_f. Let a, b be elements in $\mathbb{P}^1(\mathbb{Q})$. The modular symbol is defined such that to any path from a to b in the completed complex upper half-plane, and any homogenous polynomial of degree $k-2$ in two variables X, Y, it associates an element in our coefficient field, essentially, by integrating f against P on the path. In particular, given $a = 0, b = \infty$, and $P(X,Y) = X^{s_0-1}Y^{k-1-s_0}$ (for integers $1 \leq s_0 \leq k-1$), we obtain an element in our coefficient field denoted

$$L(I_f, k, s_0) = (-1)^{s_0-1} I_f\{0 \to \infty\}(X^{s_0-1}Y^{k-1-s_0}). \tag{1.0.1}$$

Let us call these the special values of the L-function of the modular symbol I_f. Results of Birch and Manin show that $L(I_f, k, s_0)$ coincides with the classical special value $L(f, s_0)$ up to a certain factor. Moreover, there is a way to twist I_f by a Dirichlet character, so that this procedure produces the special value of the L-function of f twisted by the Dirichlet character. It is natural, then, to try to p-adically interpolate the values $L(I_f, k, s_0)$ in both variables k, s. The starting idea is to do so, through p-adically interpolating the modular symbols $\{I_{f_\kappa}\}_\kappa$, corresponding to various specializations of f_∞, to a *new kind* of modular symbol I_{f_∞} associated to f_∞, and applying the same formalism as in (1.0.1).

The interpolation of the modular symbols is done through replacing homogenous polynomials of degree $k-2$, which are really functions in one variable, with certain functions in two variable: more precisely, continuous functions on (the primitive vectors of) $\mathbb{Z}_p^2$. The modular symbols obtained this way are called Λ-adic modular symbols; they specialize in various weights to produce classical modular symbols as above. A Λ-adic modular symbol Φ associates to any $a, b \in \mathbb{P}^1(\mathbb{Q})$, and any continuous function on (the primitive part) of $\mathbb{Z}_p^2$, an element in $\mathbb{C}_p$. In particular, given $a = 0, b = \infty$, and the two-variable function[1] $f(x,y) = x^{s-1}y^{k-1-s}\ \mathbf{1}_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times}$, we obtain an element

[1]The inclusion of $\mathbf{1}_{\mathbb{Z}_\mathbf{p}^\times \times \mathbb{Z}_\mathbf{p}^\times}$ is for removing the Euler factor at p.

$$L(\Phi, k, s) = (-1)^{s-1}\Phi\{0 \to \infty\}(X^{s-1}Y^{k-1-s}\mathbf{1}_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times})$$

Thinking of s, k as p-adic variables[2], this *is* the two-variable p-adic L-function attached to Φ. And it is rather formal to prove that, in some sense, it does p-adically interpolate the special values of L-functions of the specializations.

Given the Birch and Manin's result mentioned above, it is enough to show that a Λ-adic modular symbol can be constructed which p-adically interpolates the modular symbols attached to the specializations of the Hida family f_∞. This is what Greenberg and Stevens do. In the following, I will make the above ideas precise.

2. Notation and background

The conventions are, for the most part, in accordance with [BD]. However, the presentation closely follows the article [GS], where the conventions are different from those of [BD]. To keep with the notation of the rest of the talks in the seminar, I will rephrase all the content from [GS] in the conventions of [BD], and hint, at some points, at how the notation relates to that of [BD]. So keep that in mind, when referring to [GS] for more details.

Let us quickly recall some notations and results from the previous lectures. Let p be a prime number, and M an integer prime to p. Let $N = Mp$. Let $k \geq 2$ be an integer, and $f \in S_k(\Gamma_0(N))$ be a U_p-eigenform, $U_p(f) = a_p f$, where a_p is a p-adic unit.

For any field K, Let $P_k(K)$ denote the K-vector space of all homogeneous polynomials of degree $k-2$ in two variables X, Y. It is equipped with a right action of $\mathrm{GL}_2(K)$, where $\gamma \in \mathrm{GL}_2(K)$ acts on $P \in P_k(K)$ via the formula

$$P|_\gamma(X, Y) = P(\gamma(X, Y)^T).$$

Let $V_k(K)$ be the K-linear dual of $P_k(K)$. The above action, induces a left action of $\mathrm{GL}_2(K)$ on $V_k(K)$. For $\eta \in V_k(K)$ and $P \in P_k(K)$, we have

$$(\gamma\eta)(P) = \eta(P|_\gamma).$$

We let $P_k = P_k(\mathbb{C}_p)$, and $V_k = V_k(\mathbb{C}_p)$. In particular, both have left actions of $\mathrm{GL}_2(\mathbb{Q}_p)$. A modular symbol with values in an abelian group G is a function

$$\phi : \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) \to G,$$

sending (a, b) to $\{a, b\}$, satisfying $\phi\{a \to b\} + \phi\{b \to c\} = \phi\{a \to c\}$, for all $a, b, c \in \mathbb{P}^1(\mathbb{Q})$.

Assume G is equipped with a fixed left action of $\mathrm{GL}_2(\mathbb{Q})$. If $\gamma \in \mathrm{GL}_2(\mathbb{Q})$, and ϕ is a modular symbol with values in G, we set

$$(\gamma\phi)\{r \to s\} = \gamma(\phi\{\gamma^{-1}r \to \gamma^{-1}s\}).$$

For any subgroup $\Gamma \subset \mathrm{GL}_2(\mathbb{Q})$, we let $\mathrm{Symb}_\Gamma(G)$ denote the group of modular symbols which are fixed under the above action of Γ.

[2] k, s should be thought of elements in $\mathcal{X} := \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$.

Let f be a modular form of weight $k \geq 2$ and level $\Gamma_0(N)$. To f, one can associate a modular symbol $\tilde{I}_f \in \mathrm{Symb}_{\Gamma_0(N)}(V_k(\mathbb{C}))$, defined as follows

$$\tilde{I}_f\{a \to b\}(P) = 2\pi i \int_a^b f(z)P(z,1)dz.$$

Let $\tilde{I}_f^{\pm}$ denote the plus/minus eigencomponent of $\mathrm{Symb}_{\Gamma_0(N)}(V_k(\mathbb{C}_p))$ under the (involution) action of the matrix $c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. A result of Shimura states that there are two (non-canonical) complex periods $\Omega_f^{\pm}$, such that both

$$I_f^{\pm} := \tilde{I}_f^{\pm}/\Omega^{\pm}$$

belong to $\mathrm{Symb}_{\Gamma_0(N)}(V_k(K_f))$, where K_f is the number field generated by the coefficients of the q-expansion of f. Our convention is to fix a sign, $w_\infty = \pm 1$, and let I_f (resp., Ω_f) denote the corresponding choice from the two $I_f^{\pm}$ (resp., $\Omega_f^{\pm}$). From now on, we will consider I_f as an element of $\mathrm{Symb}_{\Gamma_0(N)}(V_k)$.

There is an action of the Hecke algebra on $\mathrm{Symb}_{\Gamma_0(N)}(V_k)$, induced by the Hecke correspondences on the modular curve of level $\Gamma_0(N)$. For any modular form f, and Hecke operator T, we have $I_{Tf} = TI_f$. The action of U_p on $\phi \in \mathrm{Symb}_{\Gamma_0(N)}(V_k)$ can be explicitly described as

$$U_p(\phi) = \sum_{a=0}^{p-1} \begin{pmatrix} p & -a \\ 0 & 1 \end{pmatrix} \phi.$$

Definition 2.1. For any $\phi \in \mathrm{Symb}_{\Gamma_0(N)}(V_k)$. We define

$$L(\phi) = \phi\{0 \to \infty\}.$$

Let χ be a Dirichlet character of conductor m. We define the twist of ϕ by χ to be

$$\phi \otimes \chi = \sum_{a=0}^{m-1} \chi(a)(\begin{pmatrix} m & -a \\ 0 & 1 \end{pmatrix} \phi).$$

For any $1 \leq s \leq k-1$, define

$$L(\phi, \chi, s) = (-1)^{s-1} L(\phi \otimes \chi)(X^{s-1}Y^{k-1-s}).$$

More explicity, we have

$$L(\phi, \chi, s) = (-1)^{s-1} \sum_{a=0}^{m} \chi(a)\phi\{a/m \to \infty\}((mX - aY)^{s-1}Y^{k-1-s}).$$

If χ is trivial, we simply write $L(\phi, s)$ for $L(\phi, \chi, s)$.

Remark 2.2. Our $L(I_f, \chi, s)$ is the same as Bertolini-Darmon's $L^*(f, \chi, s)$ up to a factor of $(-1)^s m^{s-1}$ (See [BD, Prop 1.3]).

We recall the following theorem of Birch and Manin which relates the above values obtained from the modular symbol attached to a modular form to the spacial values of the L-function of that modular form. See [MTT, §8.6], for example.

Theorem 2.3. *Let $k \geq 2$, and $1 \leq s \leq k-1$ be an integer. Let χ be of conductor p^m, and set $w_\infty = (-1)^{s-1}\chi(-1)$. Assume $f \in S_k(\Gamma_0(N))$ is an eigenform, and let $\phi = I_f$ be the modular symbol associated to it. Then, we have*

$$L(I_f, \chi, s) = -\frac{m^{s-1}(s-1)!\tau(\chi)}{(2\pi i)^{s-1}\Omega_f} L(f, \chi, s),$$

where $\tau(\chi)$ is the Gauss sum, and Ω_f is the complex period introduced above.

3. The p-adic L-function of an ordinary modular form

This chapter is meant as a motivation for Greenberg-Stevens's construction of two-variable p-adic L-functions. The construction applies the formalism which emerges in this chapter to Λ-adic modular symbols attached to Hida families.

We recall the construction of p-adic L-functions attached to modular forms. We continue to consider p to be a prime number, and M an integer prime to p, and $N = Mp$. Let $k \geq 2$ be an integer, and $f \in S_k(\Gamma_0(N))$ be a U_p -eigenform, $U_p(f) = a_p f$, where a_p is a p-adic unit.

We recall the definition of the weight space $\mathcal{W}$. It is a rigid analytic space over $\mathbb{Q}_p$, whose $\mathbb{C}_p$ points are given by

$$\mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{cont}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$$

A modular form of weight $k \geq 2$, level $\Gamma_1(Mp^m)$, with p-nebentypus δ, will be assigned a weight character $\kappa \in \mathcal{W}(\mathbb{C}_p)$ given by

$$\kappa(x) = x^{k-2}\delta(x).$$

In particular, the weight character of $f \in S_k(\Gamma_0(N))$, will be $\kappa(x) = x^{k-2}$. Let $\Lambda = \mathbb{Z}_p[|1 + p\mathbb{Z}_p|]$, and $\tilde{\Lambda} = \mathbb{Z}_p[|\mathbb{Z}_p^\times|]$. Then, we have the following equivalences

$$\{\text{bounded analytic functions on } \mathcal{W}\} \longleftrightarrow \tilde{\Lambda} \longleftrightarrow \{p-\text{adic measures on } \mathbb{Z}_p^\times\}$$

For a measure μ on $\mathbb{Z}_p^\times$, let L_μ denote the corresponding analytic function on $\mathcal{W}$. One way to construct the p-adic L-function of a modular form f is to first construct a p-adic measure μ_f, and then use the analytic function L_{μ_f} to construct the $L_p(f, s)$.

We recall the construction of μ_f [MS], [MTT].

Definition 3.1. Fix $w_\infty = \pm 1$. The p-adic measure μ_f attached to f is defined via

$$\mu_f(\mathbf{1}_{a+p^n\mathbb{Z}_p}) = a_p^{-n} I_f\{a/p^n \to \infty\}(Y^{k-2}),$$

for all $a \in \mathbb{Z}$.

In the rest of this section we sketch an alternate formalism to obtain these p-adic measures, which can naturally be interpolated. We first make some definitions.

Definition 3.2. Let $\mathbb{B}_1$ be the p-adic open unit disc. For any integer $k \geq 0$, we define

$$\mathcal{A}_k = \{\text{rigid analytic functions defined on a neighborhood of } \mathbb{Z}_p \text{ in } \mathbb{B}_1\},$$

$$\mathcal{D}_k = \mathrm{Hom}_{cont}(\mathcal{A}_k, \mathbb{C}_p).$$

For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p)$, and $f \in \mathcal{A}_k$, we define $\gamma f \in \mathcal{A}_k$, defined as

$$\gamma f(z) = f(\frac{az+b}{cz+d})(cz+d)^k.$$

This gives a left action of $\mathrm{GL}_2(\mathbb{Z}_p)$ on $\mathcal{A}_k$, and induces a left action of $\mathrm{GL}_2(\mathbb{Z}_p)$ on $\mathcal{D}_k$, so that for any $\delta \in \mathcal{D}_k$, we have $\gamma\delta(\gamma f) = \delta(f)$.

Remark 3.3. Since every step function on $\mathbb{Z}_p^\times$ is an element of $\mathcal{A}_k$, we obtain a natural inclusion $\mathcal{D}_k \to \mathrm{Meas}(\mathbb{Z}_\mathrm{p}^\times)$.

There is a $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariant inclusion $P_k \to \mathcal{A}_k$ given by $P(X,Y) \mapsto f(z) = P(z,1)$. This induces a $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariant map $\mathcal{D}_k \to V_k$, and in turn, a map

$$\mathrm{Symb}_{\Gamma_0(N)}(\mathcal{D}_k) \to \mathrm{Symb}_{\Gamma_0(N)}(V_k).$$

Theorem 3.4. *Let* $\mathrm{Symb}_{\Gamma_0(N)}(-)^{ord}$ *denote the* U_p*-ordinary part of* $\mathrm{Symb}_{\Gamma_0(N)}(-)$*. The map*

$$\mathrm{Symb}_{\Gamma_0(N)}(\mathcal{D}_k)^{ord} \to \mathrm{Symb}_{\Gamma_0(N)}(V_k)^{ord}$$

is an isomorphism.

Let φ_f denote the lifting of I_f under the above isomorphism. We claim that the p-adic L-measure $\mathfrak{m}_f$ can be obtained as a simple evaluation of φ_f.

Proposition 3.5. *For* f *as above, we have*

$$\mu_f = \varphi_f\{0 \to \infty\}.$$

Proof. For any $a \in \mathbb{Z}$, we write

$$\begin{aligned}
\varphi_f\{0\to\infty\}(\mathbf{1}_{a+p^n\mathbb{Z}_p}) &= a_p^{-n} U_p^n \varphi_f\{0\to\infty\}(\mathbf{1}_{a+p^n\mathbb{Z}_p}) \\
&= a_p^{-n} \sum_{j=0}^{p^n-1} \begin{pmatrix} p^n & -j \\ 0 & 1 \end{pmatrix} \varphi_f\{0\to\infty\}(\mathbf{1}_{a+p^n\mathbb{Z}_p}) \\
&= a_p^{-n} \sum_{j=0}^{p^n-1} \varphi_f\{j/p^n\to\infty\}(\mathbf{1}_{a+p^n\mathbb{Z}_p}(p^n z - j)) \\
&= a_p^{-n} \varphi_f\{(p^n-a)/p^n \to\infty\}(\mathbf{1}_{a+p^n\mathbb{Z}_p}) \\
&= a_p^{-n} I_f\{a/p^n\to\infty\}(Y^{k-2}).
\end{aligned}$$

□

4. Λ-adic modular symbols and their p-adic L-function

4.1. **Λ-adic Modular Symbols.** As before, p prime, $(p, M) = 1$, and $N = pM$. Let $(\mathbb{Z}_p^2)'$ be the subset of primitive vectors in $\mathbb{Z}_p^2$. First, we define spaces of measures. Let $\mathrm{Cont}(\mathbb{Z}_p^2)$ be the space of continuous $\mathbb{C}_p$-valued functions on $\mathbb{Z}_p^2$, and $\mathrm{Step}(\mathbb{Z}_p^2)$ be the subspace of locally constant functions. We define

$$\tilde{\mathbb{D}}_* = \mathrm{Hom}_{\mathbb{C}_p}(\mathrm{Step}(\mathbb{Z}_p^2), \mathbb{C}_p)$$

to be the space of $\mathbb{C}_p$-valued measures on $\mathbb{Z}_p^2$. Every such measure has a unique extension to an element of $\mathrm{Hom}_{cont}(\mathrm{Cont}(\mathbb{Z}_p^2), \mathbb{C}_p)$, where the continuous functions on $\mathbb{Z}_p^2$ are considered with supremum norm. If $\mu \in \tilde{\mathbb{D}}_*$, $h \in \mathrm{Cont}(\mathbb{Z}_p^2)$, and $K \subset \mathbb{Z}_p^2$ is a compact open subset, we define

$$\int_K h\ d\mu = \int h\mathbf{1}_K\ d\mu = \mu(h\mathbf{1}_K).$$

As usual, we set $\mu(K) = \int \mathbf{1}_K\ d\mu$.

Let $\mathbb{D}_*$ be the subspace of $\tilde{\mathbb{D}}_*$ consisting of measures which are supported on $(\mathbb{Z}_p^2)'$. There is a projection $\tilde{\mathbb{D}}_* \to \mathbb{D}_*$ defined via the restriction of measures from $\mathbb{Z}_p^2$ to $(\mathbb{Z}_p^2)'$, making $\mathbb{D}_*$ a direct summand of $\tilde{\mathbb{D}}_*$.

There is a left action of $\mathrm{M}_2(\mathbb{Z}_p)$ on $\tilde{\mathbb{D}}_*$: for $\gamma \in \mathrm{M}_2(\mathbb{Z}_p)$, and $h \in \mathrm{Cont}(\mathbb{Z}_p^2)$, we have

$$\int_{\mathbb{Z}_p^2} h\ d(\gamma\mu) = \int_{\mathbb{Z}_p^2} h|_\gamma\ d\mu,$$

where $h|_\gamma(v) = h(\gamma v^T)$. The kernel of the projection $\tilde{\mathbb{D}}_* \to \mathbb{D}_*$ consists of those measures supported on $(p\mathbb{Z}_p)^2$, and, hence, is preserved by the above action. Therefore, the action of $\mathrm{M}_2(\mathbb{Z}_p)$ descends to $\mathbb{D}_*$. Explicitly, If $\gamma \in \mathrm{M}_2(\mathbb{Z}_p)$, $h : (\mathbb{Z}_p^2)' \to \mathbb{C}_p$ a continuous function, and $\mu \in \mathbb{D}_*$, then this action can be described as

$$\int_{(\mathbb{Z}_p^2)'} h\ d(\gamma\mu) = \int_{(\mathbb{Z}_p^2)'} \tilde{h}|_\gamma\ d\mu, \tag{4.1.1}$$

where $\tilde{h}$ is the extension by zero of h to $\mathbb{Z}_p^2$, and, hence,

$$\tilde{h}|_\gamma(x, y) = \begin{cases} h(\gamma(x, y)^T), & \text{if } \gamma(x, y)^T \in (\mathbb{Z}_p^2)' \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we have the continuous action of $\mathbb{Z}_p^\times$ on $\tilde{\mathbb{D}}_*$ and $\mathbb{D}_*$ via diagonal matrices. This shows that both $\tilde{\mathbb{D}}_*$ an $\mathbb{D}_*$ are continuous $\mathbb{Z}_p[[\mathbb{Z}_P^\times]] \otimes \mathbb{C}_p$-modules.

We define two spaces of measure valued modular symbols

$$\tilde{\mathbb{W}} = \mathrm{Symb}_{\Gamma_0(M)}(\tilde{\mathbb{D}}_*),$$

$$\mathbb{W} = \mathrm{Symb}_{\Gamma_0(M)}(\mathbb{D}_*).$$

These spaces carry an action of the Hecke algebra. The natural projection $\tilde{\mathbb{D}}_* \to \mathbb{D}_*$ induces a Hecke-equivariant surjective map $\tilde{\mathbb{W}} \to \mathbb{W}$.

4.2. **Two-variable p-adic L-functions.** Let $\Phi \in \mathbb{W}$. We define

$$\mu_\Phi = \Phi\{0 \to \infty\} \in \mathbb{D}_*.$$

Using μ_Φ, we can define an analytic function on $\mathcal{W} \times \mathcal{W}$ as follows. Let $\kappa, \sigma \in \mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{cont}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$. We define

$$L_p(\Phi, \kappa, \sigma) = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} \kappa(y)\sigma(-x/y)\ d\mu_\Phi(x, y).$$

If σ is *arithmetic*, i.e., $\sigma(x) = x^r\epsilon(x)$ for some integer $r \geq 0$, and some Dirichlet character $\epsilon : (\mathbb{Z}/p^m)^\times \to \mathbb{C}_p^\times$, we extend σ to $\mathbb{Z}_p$, by setting $\sigma(p) = p^r$ if ϵ is trivial, and $\sigma(p) = 0$, otherwise. With this definition at hand, we can define the *improved* two-variable p-adic L-function attached to Φ via

$$L_p^*(\Phi, \kappa, \sigma) = \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} \kappa(y)\sigma(-x/y)\ d\mu_\Phi(x, y),$$

where $\kappa, \sigma \in \mathcal{W}(\mathbb{C}_p)$, and σ is arithmetic.

Finally, we define a different form of the two-variable p-adic L-function attached to Φ which is presented in two variables in $\mathcal{W}(\mathbb{Q}_p)$. We set

$$\mathcal{W}(\mathbb{Q}_p) = \mathrm{Hom}_{cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times) \cong \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z} =: \mathcal{X},$$

considered additively. If $k = (k_1, k_2) \in \mathcal{X}$, and $x \in \mathbb{Z}_p^\times$, we set

$$x^k = <x>^{k_1} (x/<x>)^{k_2},$$

where $<x>$ denotes the projection of x onto $1 + p\mathbb{Z}_p$.

Definition 4.3. Let $\Phi \in \mathbb{W}$. Let $k, s \in \mathcal{X}$. Let $\kappa, \sigma \in \mathcal{W}(\mathbb{Q}_p)$ be defined as $\kappa(x) = x^{k-2}$, and $\sigma(x) = x^{s-1}$. We set

$$L_p(\Phi, k, s) = L_p(\Phi, \kappa, \sigma).$$

This is an analytic function on $\mathcal{X}$.

4.4. **The specialization maps.** Let $\kappa \in \mathcal{W}(\mathbb{C}_p)$ be an arithmetic weight, $\kappa(x) = x^{k-2}\delta(x)$, for some integer $k \geq 2$. We define the specialization map (in weight k, and character δ) to be

$$\rho_\kappa : \mathbb{D}_* \to V_k$$

via the formula

$$\rho_\kappa(\mu)(P) = \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} P(x, y)\delta(y)\ d\mu(x, y).$$

The map ρ_κ induces a Hecke-equivariant map

$$\rho_\kappa : \mathbb{W} = \mathrm{Symb}_{\Gamma_0(M)}(\mathbb{D}_*) \to \mathrm{Symb}_{\Gamma_0(N)}(V_k).$$

For $\Phi \in \mathbb{W}$, we often denote $\rho_\kappa(\mu)$ with Φ_κ. If $\kappa(x) = x^{k-2}$ for an integer $k \geq 2$, we write Φ_k for Φ_κ.

4.5. **The U_p operator.** The $\mathbb{Z}_p[[\mathbb{Z}_p^*]]$-module $\mathbb{W}$ is equipped with an action of the Hecke algebra. We recall the explicit definition of U_p. Recall that $\gamma \in \mathrm{GL}_2(\mathbb{Q})$ acts on $\Phi \in \mathbb{W}$ as follows:

$$(\gamma\Phi)\{r \to s\} = \gamma(\Phi\{\gamma^{-1}r \to \gamma^{-1}s\}).$$

We define

$$U_p\Phi = \sum_{a=0}^{p-1} \begin{pmatrix} p & -a \\ 0 & 1 \end{pmatrix} \Phi.$$

Using the explicit description of the action of $\mathrm{M}_2(\mathbb{Z}_p)$ on $\mathbb{D}_*$ (4.1.1), we have

$$\text{(4.5.1)} \qquad \int_{(\mathbb{Z}_p^2)'} f(x,y) d\mu_{U_p\Phi}(x,y) = \sum_{a=0}^{p-1} \int_{\mathbb{Z}_p \times \mathbb{Z}_P^\times} f(px - ay, y) d\mu_a(x,y),$$

where $\mu_a = \Phi\{a/p \to \infty\}$.

4.6. **The interpolation property.** We can now prove the p-adic interpolation property of $L_p(\Phi, -, -)$. Recall the notions of $L(\phi, s)$ and $L(\phi, \chi, s)$ for a classical modular symbol ϕ, given in Definition 2.1.

Proposition 4.7. *Let $\Phi \in \mathbb{W}$ satisfy $U_p\Phi = a_p\Phi$ with $a_p \neq 0$. Let $s, k \in \mathbb{Z} \subset \mathcal{X}$ satisfy $1 \leq s) \leq k-1$. Let $\kappa(x) = x^{k-2}$. Then*

$$L_p(\Phi, k, s) = (1 - a_p(k)^{-1}p^{s-1})L(\Phi_k, s).$$

This follows immediately from the following result.

Proposition 4.8. *Let $\Phi \in \mathbb{W}$. Let $\kappa, \sigma \in \mathcal{W}(\mathbb{C}_p)$ such that σ is arithmetic: $\sigma(x) = x^{s-1}\epsilon(x)$ for an integer $s \geq 1$, and some Dirichlet character ϵ of conductor p^m. Recall that $\sigma(p) = p^{s-1}$ if ϵ is trivial, and $\sigma(p) = 0$ otherwise. We have*

1. *$L_p(U_p\Phi, \kappa, \sigma) = L_p^*(U_p\Phi, \kappa, \sigma) - \sigma(p)L_p^*(\Phi, \kappa, \sigma)$.*
2. *Assume, further, that κ is arithmetic: $\kappa(x) = x^{k-2}\delta(x)$, for an integer $k \geq 2$, and some Dirichlet character δ . Then,*

$$L_p^*(U_p^m\Phi, \kappa, \sigma) = L(\Phi_\kappa, \epsilon, s).$$

Proof. To prove the first statement, using Equation 4.5.1, we can write

$$\begin{aligned} L_p(U_p\Phi, \kappa, \sigma) &= \int_{(\mathbb{Z}_p^2)'} \kappa(y)\sigma(-x/y)\mathbf{1}_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times}\, d\mu_{U_p\Phi} \\ &= \sum_{a=0}^{p-1} \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} \kappa(y)\sigma(\frac{-px+ay}{y})\mathbf{1}_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times}\, d\mu_a(x,y). \end{aligned}$$

Similarly, we write

$$\begin{aligned} L_p^*(U_p\Phi, \kappa, \sigma) &= \int_{(\mathbb{Z}_p^2)'} \kappa(y)\sigma(-x/y)\mathbf{1}_{\mathbb{Z}_p \times \mathbb{Z}_p^\times}\, d\mu_{U_p\Phi} \\ &= \sum_{a=0}^{p-1} \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} \kappa(y)\sigma(\frac{-px+ay}{y})\mathbf{1}_{\mathbb{Z}_p \times \mathbb{Z}_p^\times}\, d\mu_a(x,y). \end{aligned}$$

It is easy to check that all the corresponding terms are equal, except for $a = 0$. For $a = 0$, the term in $L_p(U_p\Phi, \kappa, \sigma)$ is zero. It follows that

$$\begin{aligned} L_p^*(\Phi,\kappa,\sigma) - L_p(\Phi,\kappa,\sigma) &= \int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} \kappa(y)\sigma(\frac{-px}{y})\ d\mu_0(x,y) \\ &= \sigma(p)\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} \kappa(y)\sigma(\frac{-x}{y})\ d\mu_\Phi(x,y) \\ &= \sigma(p)L_p(\Phi,\kappa,\sigma), \end{aligned}$$

as $\mu_0 = \Phi\{0 \to \infty\} = \mu_\Phi$.

For the second statement, setting $\mu_{a,m} = \Phi\{a/p^m \to \infty\}$, we can write

$$\begin{aligned} L_p^*(U_p^m\Phi,\kappa,\sigma) &= \int_{(\mathbb{Z}_p^2)'} \kappa(y)\sigma(-x/y)\mathbf{1}_{\mathbb{Z}_p\times\mathbb{Z}_p^\times}\ d\mu_{U_p^m\Phi} \\ &= \sum_{a=0}^{p^m-1}\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} \kappa(y)\sigma(\frac{-p^mx+ay}{y})\ d\mu_{a,m}(x,y) \\ &= \sum_{a=0}^{p^m-1}\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} \kappa(y)\epsilon(\frac{-p^mx+ay}{y})(\frac{-p^mx+ay}{y})^{s-1}\ d\mu_{a,m}(x,y) \\ &= \sum_{a=0}^{p^m-1}\epsilon(a)\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} \kappa(y)(\frac{-p^mx+ay}{y})^{s-1}\ d\mu_{a,m}(x,y). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} L(\Phi_\kappa,\epsilon,s) &= (-1)^{s-1}((\Phi_\kappa\otimes\epsilon)\{0\to\infty\})(X^{s-1}Y^{k-1-s}) \\ &= (-1)^{s-1}\sum_{a=0}^{p^m-1}\epsilon(a)(\begin{pmatrix} p^m & -a \\ 0 & 1\end{pmatrix}\Phi_\kappa)\{0\to\infty\}(X^{s-1}Y^{k-1-s}) \\ &= (-1)^{s-1}\sum_{a=0}^{p^m-1}\epsilon(a)\Phi_\kappa\{a/p^m\to\infty\}((p^mX-aY)^{s-1}Y^{k-1-s}) \\ &= (-1)^{s-1}\sum_{a=0}^{p^m-1}\epsilon(a)\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times}\delta(y)(p^mx-ay)^{s-1}y^{k-1-s}\ d\mu_{a,m}(x,y). \end{aligned}$$

To finish the proof, we just need to compare the two sides. □

5. The two-variable p-adic L-function of a Hida family

5.1. **The Λ-adic modular symbol attached to an ordinary eigenform.** Let $f \in S_{k_0}(\Gamma_1(N))$ be an ordinary eigenform. Let f_∞ be a Hida family

$$f_\infty(q) = \sum_{n=1}^{\infty} a_n q^n,$$

where a_n's are analytic functions on some open $k_0 \in \mathcal{U} \subset \mathcal{W}$ satisfying

(1) The weight k specialization $f_k(q) = \sum_{n=1}^{\infty} a_n(k) q^n$ is an eigenform of weight k and level $\Gamma_0(N)$, for all $k \in \mathcal{U} \cap \mathbb{Z}^{\geq 2}$.
(2) $f_{k_0}(q) = f(q)$.

We will use f_∞ to construct a Λ-adic modular symbol which specializes to a nonzero multiple of I_{f_k} at weight $k \in \mathcal{U} \cap \mathbb{Z}^{\geq 2}$.

Let $\mathcal{R}$ be the local ring of $k_0 \in \mathcal{U}$. Then $\mathcal{R}$ is a $\mathbb{Z}_p[[\mathbb{Z}_p^\times]] \otimes \mathbb{C}_p$-algebra, and we can form $\mathbb{D}_{\mathcal{R},*} := \mathbb{D}_* \otimes_{\mathbb{Z}_p[[\mathbb{Z}_p^\times]] \otimes \mathbb{C}_p} \mathcal{R}$. Set $\mathbb{W}_{\mathcal{R}} := \mathrm{Symb}_{\Gamma_0(M)}(\mathbb{D}_{\mathcal{R},*})$. An element $\Phi \in \mathbb{W}_{\mathcal{R}}$ will be of the form $\Phi_1 \otimes \lambda_1 + ... + \Phi_r \otimes \lambda_r$, where, for all i, $\Phi_i \in \mathbb{W}$, and λ_i is defined on an open set around k_0. In particular, there is an open $\mathcal{U}' \subset \mathcal{U}$ around k_0 on which all the λ_i's are defined. Therefore, for every $\kappa \in \mathcal{U}'$, one can define the specialization $\rho_\kappa(\Phi)$.

Theorem 5.2. *There is $\Phi_{f_\infty} \in \mathbb{W}_{\mathcal{R}}$ defined over $\mathcal{U}'$, such that*

(1) $\rho_{k_0}(\Phi_{f_\infty}) = I_f$*;*
(2) *For all $k \in \mathcal{U}' \cap \mathbb{Z}^{\geq 2}$, there is a scalar $\lambda(k) \in \mathbb{C}_p^\times$ such that $\rho_k(\Phi_{f_\infty}) = \lambda(k) I_{f_k}$.*

Proof. See [GS, §6] for a complete proof. Here, we will only give a brief sketch. Let $\Gamma_r = \Gamma_0(M) \cap \Gamma_1(p^r)$. First, we claim that

$$H^1_{par}(\Gamma_0(M), \mathbb{D}_*) \cong \varprojlim_r H^1_{par}(\Gamma_r, \mathbb{Z}_p).$$

For any $r \geq 1$, let $\mathbb{D}_r$ denote the space of $\mathbb{Z}_p$-valued functions on $((\mathbb{Z}/p^r\mathbb{Z})^2)'$, the set of primitive vectors of $(\mathbb{Z}/p^r\mathbb{Z})^2$. The $\mathbb{Z}_p$-module $\mathbb{D}_r$ has a natural action of $\Gamma_0(M)$. Since $\Gamma_r \backslash \Gamma_0(N) \cong ((\mathbb{Z}/p^r\mathbb{Z})^2)'$ via the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (c, d) \ mod \ p^r,$$

it follows that $\mathbb{D}_r \cong Ind_{\Gamma_r}^{\Gamma_0(M)} \mathbb{Z}_p$. By Shapiro's lemma, for all $r \geq 1$, we have

$$H^1_{par}(\Gamma_0(M), \mathbb{D}_r) \cong H^1_{par}(\Gamma_r, \mathbb{Z}_P).$$

The $\mathbb{Z}_p[\Gamma_0(M)]$-modules $\{\mathbb{D}_r : r \geq 1\}$ form a projective system via the maps $\mathbb{D}_{r+1} \to \mathbb{D}_r$, sending $\mu_{r+1} \mapsto \mu_r$, where $\mu_r(x) = \sum_{y \equiv_{p^r} x} \mu_{r+1}(y)$. It follows that

$$\mathbb{D}_* \cong \varprojlim_r \mathbb{D}_r,$$

and, hence, we have Hecke equivariant maps

$$H^1_{par}(\Gamma_0(M), \mathbb{D}_*) \cong \varprojlim_r H^1_{par}(\Gamma_0(M), \mathbb{D}_r) \cong \varprojlim_r H^1_{par}(\Gamma_r, \mathbb{Z}_p)$$

where, one can check that the transition maps on the right side are corestrictions. By Hida's results, the Hecke eigensystem of f_∞ appears in the ordinary part of $\varprojlim_r H^1_{par}(\Gamma_r, \mathbb{Z}_p)$, and, hence, in $H^1_{par}(\Gamma_0(M), \mathbb{D}_*)^{ord}$. On the other hand, we have a Hecke equivariant isomorphism

$$\mathrm{Symb}_{\Gamma_0(M)}(\mathbb{D}_*) \cong H^1_c(\Gamma_0(M), \mathbb{D}_*).$$

Consider the composite (surjective) map

$$\mathrm{Symb}_{\Gamma_0(M)}(\mathbb{D}_*)^{ord}_{\mathcal{L}} \cong H^1_c(\Gamma_0(M), \mathbb{D}_*)^{ord}_{\mathcal{L}} \to H^1_{par}(\Gamma_0(M), \mathbb{D}_*)^{ord}_{\mathcal{L}}$$

where, we have taken ordinary parts, and tensored (over Λ) with the fraction field $\mathcal{L}$ of Λ. The Kernel of this map is Eisenstein: for any prime $\ell \neq p$, such that $\ell \equiv_N 1$, the operator $\eta_l := T_l - l{<}l{>} - 1$ kills the kernel of the above map. On the other hand, using Weil bounds for cusp forms, we see that η_l acts invertibly on $H^1_{par}(\Gamma_0(M), \mathbb{D}_*)^{ord}_{\mathcal{L}}$. We claim that any Hecke eigenclass in $H^1_{par}(\Gamma_0(M), \mathbb{D}_*)^{ord}_{\mathcal{L}}$ lifts to a Hecke eigenclass in $\mathrm{Symb}_{\Gamma_0(M)}(\mathbb{D}_*)^{\mathrm{ord}}_{\mathcal{L}}$. Let $v \in H^1_{par}(\Gamma_0(M), \mathbb{D}_*)^{ord}_{\mathcal{L}}$. Let $\tilde{w} \in \mathrm{Symb}_{\Gamma_0(M)}(\mathbb{D}_*)^{ord}_{\mathcal{L}}$ be any element in the preimage of $\eta_\ell^{-1} v$. Then $w := \eta_\ell \tilde{w}$ maps to v, and it is independent of the choice of $\tilde{w}$, since η_ℓ annihilates the kernel of the above map. The commutativity of the Hecke algebra implies that w has the same Hecke eigensystem as v.

Let Φ_{f_∞} be an element of $\mathrm{Symb}_{\Gamma_0(M)}(\mathbb{D}_*)^{ord}_{\mathcal{L}}$ with the same Hecke system as f_∞. Then, for any integer $k \geq 2$, at which Φ_{f_∞} is defined, $\rho_k(\Phi_{f_\infty})$ has the same Hecke eigensystem as f_k, and, hence, as I_{f_k}. Since the space of modular symbols of the same Hecke eigensystem as I_{f_k} is one-dimensional, it follows that

$$\rho_k(\Phi_{f_\infty}) = \lambda(k) I_{f_k},$$

for some $\lambda(k) \in \mathbb{C}_p$. By shrinking the domain of regularity of Φ_{f_∞}, and after normalizing, we can ensure the properties required by Theorem 5.2 are satisfied. $\square$

5.3. **The two-variable p-adic L-function of f.** Let χ be a Dirichlet character of conductor m. We define the twist of a Λ-adic modular symbol Φ by χ to be

$$\Phi \otimes \chi = \sum_{a=0}^{m-1} \chi(a)(\begin{pmatrix} m & -a \\ 0 & 1 \end{pmatrix} \Phi).$$

Lemma 5.4. *Let $\Phi \in \mathbb{W}$ satisfy $U_p(\Phi) = a_p \Phi$. Let χ be a Dirichlet character of conductor m prime to p. Then, $U_p(\Phi \otimes \chi) = a_p \chi(p)^{-1}(\Phi \otimes \chi)$.*

Proof. Since $(m,p)=1$, we can write

$$\begin{aligned}
U_p(\Phi\otimes\chi) &= \sum_{b=0}^{p-1}\begin{pmatrix} p & -bm \\ 0 & 1\end{pmatrix}\sum_{a=0}^{m-1}\chi(a)\begin{pmatrix} m & -a \\ 0 & 1\end{pmatrix}\Phi \\
&= \sum_{a=0}^{m-1}\chi(a)\sum_{b=0}^{p-1}\begin{pmatrix} p & -bm \\ 0 & 1\end{pmatrix}\begin{pmatrix} m & -a \\ 0 & 1\end{pmatrix}\Phi \\
&= \sum_{a=0}^{m-1}\chi(a)\sum_{b=0}^{p-1}\begin{pmatrix} m & -ap \\ 0 & 1\end{pmatrix}\begin{pmatrix} p & -b \\ 0 & 1\end{pmatrix}\Phi \\
&= \chi(p)^{-1}\sum_{a=0}^{m-1}\chi(ap)\begin{pmatrix} m & -ap \\ 0 & 1\end{pmatrix}\sum_{b=0}^{p-1}\begin{pmatrix} p & -b \\ 0 & 1\end{pmatrix}\Phi \\
&= a_p\chi(p)^{-1}(\Phi\otimes\chi).
\end{aligned}$$

□

Definition 5.5. Let $f\in S_{k_0}(\Gamma_0(N))$ be an ordinary eigenform, f_∞ a Hida family through f as in the previous section, and $\Phi_{f_\infty}\in\mathbb{W}_{\mathcal{R}}$ a modular symbol as in Theorem 5.2. Let χ be a Dirichlet character with conductor m prime to p. We define the two-variable p-adic L-function associated to f to be

$$L_p(f_\infty,\chi,k,s):=L_p(\Phi_{f_\infty}\otimes\chi,k,s),$$

where $k\in\mathcal{U}'$, $s\in\mathcal{X}$, and the right side is given in Definition 4.3.

Remark 5.6. The definition of $L_p(f_\infty,\chi,k,s)$ here differs from that of [BD] by a factor of $(-1)^s m^{s-1}$.

We finally prove the interpolation property of the two-variable p-adic L-function of f.

Theorem 5.7. *Let $k\in\mathcal{U}'\cap\mathbb{Z}^{\geq 2}$, and $s\in\mathbb{Z}$ satisfy $1\leq s\leq k-1$. Set $w_\infty=(-1)^{s-1}\chi(-1)$. Then,*

$$L_p(f_\infty,\chi,k,s)=\lambda(k)(1-\chi(p)a_p(k)^{-1}p^{s-1})L(I_{f_k},\chi,s).$$

Proof. This follows immediately from Proposition 4.7, Theorem 5.2, and Lemma 5.4. □

Remark 5.8. In light of Remarks 2.2 and 5.6, Theorem 5.7 is the same as Theorem 1.12 of [BD].

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