# HIDA THEORY

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Abstract.

Contents



## 1. Introduction

1.1. **The Eisenstein family.** The Bernoulli numbers $B_k$ are rational numbers defined by the generating series

$$\frac{X}{e^X-1} = \sum_{k\geq 0} B_k \frac{X^k}{k!}$$

Hence $B_0 = 1$, $B_1 = -1/2$, and since

$$\frac{X}{e^X-1} + \frac{X}{2} = \frac{X}{2}\left(\frac{e^{X/2}+e^{-X/2}}{e^{X/2}-e^{-X/2}}\right)$$

is an even function, it follows that $B_{2k+1} = 0$ for $k \geq 1$. For every integer $k \geq 1$, the holomorphic Eisenstein series of weight $2k$ can be described by its $q$-expansion:

$$G_{2k}(q) = \frac{-B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n, \quad \sigma_{2k-1}(n) = \sum_{d|n} d^{2k-1}.$$

Let $p$ be an odd prime. The Bernoulli numbers satisfy the *Kummer congruences*: if $2k \equiv 2l \pmod{p^a(p-1)}$ and $2k \not\equiv 0 \pmod{p-1}$ then

$$(1-p^{2k-1})\frac{B_{2k}}{2k} \equiv (1-p^{2l-1})\frac{B_{2l}}{2l} \pmod{p^{a+1}}.$$

Serre noticed [12] that if one defines

$$G_{2k}^*(q) = G_{2k}(q) - p^{2k-1}G_{2k}(q^p),$$

so that

$$G_{2k}^*(q) = (1-p^{2k-1})\left(\frac{-B_{2k}}{4k}\right) + \sum_{n\geq 1}\sigma_{k-1}^*(n)q^n, \quad \sigma_{k-1}^*(n) = \sum_{\substack{d\mid n\\(d,p)=1}} d^{k-1},$$

then the Kummer congruence, and the analogous elementary congruences between the higher coefficients $\sigma_{k-1}^*$, are equivalent with the congruence

$$G_{2k}^*(q) \equiv G_{2l}^*(q) \pmod{p^{a+1}}$$

if $2k \equiv 2l \not\equiv 0 \pmod{p^a(p-1)}$.

**Example 1.** Take $p = 5$, which is the minimal prime for which we can find a nonzero Eisenstein series of some weight not congruent to zero mod $p-1$. Then

$$G_2^*(q) = \frac{1}{6} + q + 3q^2 + 4q^3 + 7q^4 + q^5 + \cdots$$
$$G_{22}^*(q) = \frac{9260535240173320423}{138} + q + 2097153q^2 + 10460353204q^3 + 4398048608257q^4 + q^5 + \cdots,$$

and one checks that

$$G_{22}^*(q) - G_2^*(q) = \frac{2^3\cdot\mathbf{5^2}\cdot 11\cdot 2104667100039391}{3\cdot 23} + (2\cdot 3\cdot\mathbf{5^2}\cdot 11\cdot 31\cdot 41)q^2 + (2^4\cdot 3\cdot\mathbf{5^2}\cdot 11^2\cdot 61\cdot 1181)q^3 + (2\cdot 3\cdot\mathbf{5^3}\cdot 11\cdot 31\cdot 41\cdot 59\cdot 7109)q^4 + O(q^6).$$

In order to explain the $p$-adic continuity of Eisenstein series, Serre introduced a $p$-adic *weight space*

$$\mathcal{X} = \mathcal{X}(\mathbf{Z}_p) = \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Z}_p^\times, \mathbf{Z}_p^\times).$$

The additive ring of integers embeds into weight space by associating $k \in \mathbf{Z}$ to the map $x \mapsto x^k$ (one frequently uses the embedding $k \mapsto (x \mapsto x^{k-2})$, so that weight 2 corresponds to the trivial map). If we equip $\mathcal{X}$ with the topology of uniform convergence, then $\mathbf{Z}$ has dense image in $\mathcal{X}$. For an arbitrary weight $k \in \mathcal{X}$ one thus frequently writes $x^k$ for the image of $x \in \mathbf{Z}_p^\times$ under $k$. A weight $k \in \mathcal{X}$ is said to be *even* if $(-1)^k = 1$. Serre defined a $p$-adic family of Eisenstein series $G_k^*(q)$ for any even weight $k \in \mathcal{X}$ as follows: let $(k_i)$ be a sequence of positive even integers that converges to $\infty$ in the archimedean sense, and which converges to $k$ in the weight space. For example, if $k \in 2\mathbf{Z}$ and $p$ is odd, then one can take $k_i = k + p^i(p-1)$ for $i \geq 1$. Then set

$$G_k^*(q) = \lim_i G_{k_i}(q);$$

note that this agrees with the earlier definition when $k \geq 2$ is an even integer. The $p$-adic family of Eisenstein series $G_k^*(q)$ indexed by even weights $k \in \mathcal{X}$ varies continuously (in fact, analytically) with the weight. The weight 2 specialization $G_2^*(q)$ is a modular form of weight 2 on $\Gamma_0(p)$. If $2k$ is an even integer $\geq 4$, then $G_{2k}^*(q)$ is an oldform on $\Gamma_0(p)$ of weight $2k$.

*Remark* 2. What we have defined to be the weight space $\mathcal{X}$ is really the "set of $\mathbf{Z}_p$ points" of a rigid analytic variety which is represented by the following functor: for any affinoid algebra $A$ over $\mathbf{Q}_p$, $\mathcal{X}(A)$ denotes the set $\mathrm{Hom}_{\mathrm{cont}}(\mathbf{Z}_p^\times, A^\times)$. See Section 2 of [4] for details.

In fact, Serre [12] defined $p$-adic modular forms to be $p$-adic $q$-expansions which are limits of rational modular forms on $\mathrm{SL}_2(\mathbf{Z})$: for $f(q) \in \mathbf{Q}_p[\![q]\!]$ with bounded coefficients $a_n$, we write $v_p(f) = \inf_n v_p(a_n)$. Note that the Fourier expansion of a rational modular form on $\mathrm{SL}_2(\mathbf{Z})$ has $p$-adically bounded Fourier coefficients, so that $v_p(f)$ is finite for all formal series arising from such modular forms.

**Definition 3.** A (Serre) $p$-ADIC MODULAR FORM is a formal series $f(q) \in \mathbf{Q}_p[\![q]\!]$ such that there exists a sequence $(f_n)$ of modular forms on $\mathrm{SL}_2(\mathbf{Z})$ with *rational* Fourier coefficients, such that $v_p(f - f_n)$ tends to zero as $n$ tends to infinity.

Serre showed that such limits possess a well-defined weight in weight space, and he proved that the constant coefficients of $p$-adic modular forms inherit congruence properties from the other coefficients:

**Theorem 4** (Serre [12], Théorème 2)**.** *Let $f$ be a nonzero $p$-adic modular form. Let $(f_n)$ be a sequence of rational forms on $\mathrm{SL}_2(\mathbf{Z})$ which converges to $f$, and such that $f_n$ is of weight $k_n$, where $k_n \to \infty$ (in the archimdean sense) as $n \to \infty$. Then the sequence $(k_n)$ of weights has a limit in weight space. Moreover, this limit is intrinsic to $f$, being independent of the sequence $(f_n)$, and is called the* WEIGHT *of the $p$-adic modular form $f$.*

**Theorem 5** (Serre [12], Corollaire 2, p. 204)**.** *Let $f^{(i)} = \sum_{n=0}^{\infty} a_n^{(i)} q^n$ be a sequence of $p$-adic modular forms, where $f^{(i)}$ is of weight $k^{(i)} \in \mathcal{X}$. Suppose that*

(1) *the $a_n^{(i)}$, for $n \geq 1$, tend uniformely to elements $a_n \in \mathbf{Q}_p$;*
(2) *the $k^{(i)}$ tend to a nonzero limit $k \in \mathcal{X}$.*

*Then the sequence $(a_o^{(i)})$ of constant terms has a limit $a_0 \in \mathbf{Q}_p$ and the series $f = \sum_{n \geq 0} a_n q^n$ is a $p$-adic modular form of weight $k$.*

Serre used this last result to deduce the Kummer congruences from the elementary congruences between the coefficients $\sigma_{k-1}^*(n)$ of $G_k^*$ and his theory of $p$-adic modular forms. This allowed him to give a new definition of the $p$-adic $\zeta$-function in terms of constant terms of the $p$-adic Eisenstein family.

Serre also showed that every modular form on $\Gamma_0(p)$ with rational Fourier coefficients is a $p$-adic modular form. Suppose that $f$ is such a form; if it was known that $f$ lives in a $p$-adic family $f_k$ parameterized analytically by the weight $k \in \mathcal{X}$, with the property that all specializations $f_k$ to integer weights $k \geq 2$ are rational forms of weight $k$ on $\Gamma_0(p)$, then one could deduce Serre's result by considering the sequence of forms $f_{k+p^i(p-1)}$ for $i \geq 0$, where $k$ is the weight of $f$. Thus, Serre's work [12] strongly suggested that many modular forms should live in $p$-adic families.

1.2. **Some congruences between cusp forms.** The level $N = 11$ is the minimal level such that there exist nonzero modular forms of weight 2 on $\Gamma_0(N)$. In fact, $S_2(\Gamma_0(11))$ is 1-dimensional, spanned by the modular form with $q$-expansion described by

$$f(q) = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2 = \eta(\tau)^2 \eta(11\tau)^2, \quad q = e^{2\pi i \tau}.$$

Here $\eta(\tau) = q^{1/24} \prod_n (1 - q^n)$ is the Dedekind eta function. The preceding discussion of the Eisenstein family suggests that there might be a modular form on $\Gamma_0(N)$ of level $12 = 2 + (11 - 1)$ that is congruent to $f(q)$ modulo 11. There is a particularly "nice" form of weight 12 (which is old on $\Gamma_0(11)$): the modular discriminant

$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \eta(\tau)^{24}.$$

Note that

$$\begin{aligned} f(q) &= q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2 \\ &\equiv q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^n)^{22} \pmod{11} \\ &\equiv \Delta(q) \pmod{11}. \end{aligned}$$

At this point an optimist might suggest that $f(q)$ and $\Delta(q)$ are the weight 2 and 12 specializations of an 11-adic family of modular forms.

For our second example we let $\varepsilon$ denote the Kronecker symbol

$$\varepsilon(n) = \left(\frac{-23}{n}\right).$$

Thus for $p$ odd $\neq 23$, quadratic reciprocity shows that $\varepsilon(p)$ is the Legendre symbol $(p/23)$. The space $S_1(\Gamma_0(23), \varepsilon)$ is 1-dimensional and is spanned by the form whose $q$-expansion is given by the product

$$g'(q) = q\prod_{n=1}^{\infty}(1-q^n)(1-q^{23n}) = \eta(\tau)\eta(23\tau).$$

Thus $g = (g')^2$ is a cusp form of weight 2 on $\Gamma_0(23)$ (with trivial character since $\varepsilon$ is quadratic), $\Delta^2$ is an oldform of weight $24 = 2 + (23-1)$ on $\Gamma_0(23)$, and one has

$$\begin{aligned} g(q) &= q^2\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{23n})^2 \\ &\equiv q^2\prod_{n=1}^{\infty}(1-q^n)^2(1-q^n)^{46} \pmod{23} \\ &\equiv \Delta(q)^2 \pmod{23}. \end{aligned}$$

For more on this example see section 4.3 of Zagier's chapter in [3]. Note that $g$ and $\Delta^2$ are not eigenforms, and so these forms do not themselves live in a Hida family.

1.3. **Hida families.** Let $N$ be a positive integer and let $p$ be an *odd* prime such that $p$ divides $N$ exactly once. Since $p$ is odd, one has an identification of topological groups

$$\mathcal{X} \cong \mathbf{Z}/(p-1)\mathbf{Z} \times \mathbf{Z}_p,$$

and so we may use this identification to define analytic functions on weight space. If $U \subseteq \mathcal{X}$ is an open subset of $\mathcal{X}$, then we let $\mathcal{A}(U)$ denote the collection of analytic functions on $U$, that is, the collection of functions which can be expressed as a power series on each intersection $U \cap (\{a\} \times \mathbf{Z}_p)$. We will frequently assume that $U$ is contained in the residue disk of 2, and then $\mathcal{A}(U)$ is simply the ring of power series that converge on an open subset of $\mathbf{Z}_p$. A HIDA FAMILY is a formal $q$-expansion

$$f_\infty = \sum_{n=1}^{\infty} a_n q^n,$$

such that there exists a neighbourhood $U$ of 2 in $\mathcal{X}$ such that $a_n \in \mathcal{A}(U)$ for all $n$, and such that:

(*) if $k$ belongs to $U \cap \mathbf{Z}^{\geq 2}$, then the WEIGHT-$k$ SPECIALIZATION

$$f_k = \sum_{n=1}^{\infty} a_n(k) q^n$$

is a normalized ordinary eigenform of weight $k$ on $\Gamma_0(N)$;

We will define what it means for a form to be ordinary below. Weights in $\mathbf{Z}^{\geq 2} \subset \mathcal{X}$ are said to be CLASSICAL.

We say that a modular form $f$ of weight $k$ on $\Gamma_0(N)$ lives in a Hida family if it is the weight $k$ specialization of some $f_\infty$ as above. The conditions imposed on the classical specializations of $f_\infty$ are quite strict, and so not all eigenforms live in a Hida family. For example, the $p$th Fourier coefficient for the Eisenstein series $G_k(z)$ of level one is $1 + p^{k-1}$, and this is not a $p$-adically continuous function of $k$. However, consider the $p$-stabilized series

$$G_k^*(z) = G_k(z) - p^{k-1} G_k(pz) = \frac{-(1-p^{k-1})B_k}{2k} + \sum_{n \geq 0} \left( \sum_{\substack{d|n \\ (d,p)=1}} d^{k-1} \right) q^n,$$

which is an eigenform on $\Gamma_0(p)$. The Hecke operator $U_p$ is defined on modular forms of level $N$ with $p$ dividing $N$ via the formula

$$U_p \left( \sum_{n \geq 0} a_n q^n \right) = \sum_{n \geq 0} a_{pn} q^n,$$

and so one easily sees that $U_p(G_k^*(z)) = G_k^*(z)$. We observed in section 1.1 that $G_k^*(z)$ defines a $p$-adic analytic family. One is thus led to consider forms satisfying the following condition:

**Definition 6.** Let $N$ be a positive integer and let $p$ be a prime dividing $N$. An eigenform $f \in M_k(\Gamma_0(N))$ (for the Hecke operators $T_n$ with $(n, N) = 1$) is said to be ORDINARY at $p$ if $U_p f = \lambda_p f$ for some $\lambda_p \in \mathbf{Z}_p^\times$. Let $M_k(\Gamma_0(N))^{\mathrm{ord}}$ denote the subspace of $M_k(\Gamma_0(N))$ spanned by the $p$-ordinary forms.

Hida [10], [9] proved the following remarkable result:

**Theorem 7.** *Let $k \geq 2$ be an integer. Let $f \in S_k(\Gamma_0(N))$ be an eigenform and let $p$ be a prime divisor of $N$. Suppose $f$ is ordinary at $p$. Then there exists a unique Hida family $f_\infty$ which specializes to $f$ at weight $k$.*

*Remark* 8. Other types of families exist, e.g. due to Coleman and Mazur, which hit most of the non-ordinary forms missed by Hida.

1.4. **Outline of proof of Theorem 7.** Before explaining the method of the proof of Theorem 7, it is useful to recall the duality between spaces of cusp forms and Hecke algebras. Let $k \geq 2$ and $N \geq 1$ be integers. Let $S_k(N) = S_k(\Gamma_0(N))$ denote the space of cusp forms of weight $k$ for $\Gamma_0(N)$. Let $\mathbf{T}_k(N)$ denote the full Hecke algebra of $S_k(N)$ generated by all $T_l$ for $l$ not dividing $N$, and by $U_p$ for all $p$ dividing $N$. Then there is a natural bilinear pairing

$$\mathbf{T}_k(N) \times S_k(N) \to \mathbf{C}$$

defined by mapping $\langle T, f \rangle \mapsto a_1(Tf)$, where $a_1(g)$ denotes the coefficient of $q$ in the $q$-expansion of a modular form $g$.

**Theorem 9.** *The bilinear pairing defined above establishes a natural* $\mathbf{C}$*-linear isomorphism*

$$S_k(N) \cong \mathbf{T}_k(N)^\vee.$$

*A modular form* $f \in S_k(N)$ *corresponds to a* $\mathbf{C}$*-algebra homomorphism* $\mathbf{T}_k(N) \to \mathbf{C}$ *via the above identification if and only if* $f$ *is a normalized eigenform for all of the Hecke operators.*

If $f$ is an eigenform, then the corresponding algebra homomorphism $\mathbf{T}_k(N) \to \mathbf{C}$ maps $T \mapsto \lambda(T, f)$, where $\lambda(T, f)$ denotes the eigenvalue of $T$ acting on $f$.

Following Bertolini-Darmon [2], we let

$$\Lambda = \mathbf{Z}_p[\![1 + p\mathbf{Z}_p]\!]$$

and we let $\Lambda^\dagger$ denote the ring of power series over $\mathbf{Z}_p$ that converge on some neighbourhood of $2 \in \mathcal{X}$. The proof of the existence of Hida families consists of the following several steps:

(1) Define and study the ordinary Hecke algebra $\mathbf{T}_\infty^{\mathrm{ord}}$;
(2) Show that an ordinary weight $k$ eigenform $f$ determines a map $\eta_f \colon \mathbf{T}_\infty^{\mathrm{ord}} \to E$, where $E$ is the ring of integers in some finite extension $E/\mathbf{Q}_p$;
(3) Argue that this map lifts to a $\Lambda$-algebra homomorphism
$$\eta_{f_\infty} \colon \mathbf{T}_\infty^{\mathrm{ord}} \to \Lambda^\dagger;$$
(4) Show that one can define formal series $a_n(k) = \eta_{f_\infty}(T_n)$ that converge in a neighbourhood of $2 \in \mathcal{X}$, such that $f_\infty = \sum_{n=1}^\infty a_n(k) q^n$ has the desired properties of Theorem 7.

In Section 2 of this note, we will concentrate mainly on proving the first point above, and we will then outline proofs of the final three. As described very nicely by Emerton on page 21 of [5], there are at least three ways to construct the ordinary Hecke algebra. The historically first construction [10] of the ordinary Hecke algebra was as a space of endomorphisms of a space of $p$-adic modular forms à la Katz [11]. Hida [9] gave a second construction of the ordinary Hecke algebra as an algebra of endomorphisms of a certain "surrogate" (to quote Emerton) for the space of $p$-adic modular forms; this surrogate is constructed from the group cohomology of $\Gamma_1(N)$ and certain of its subgroups. Finally, Emerton [7] has given a third construction of the ordinary Hecke algebra using the $p$-adically completed cohomology of modular curves.

In the second and final section of this paper we will describe the group cohomological construction of the ordinary Hecke algebra, and then give a few more details on how one can use it to construct Hida families. Our exposition will largely follow the wonderful paper [6] of Emerton. Before jumping into this, however, we close this section with a final discussion of the role played by Hida families in [2].

1.5. **Periods of Hida families.** This subsection corresponds to subsection 2.3 of [2]; we will be rather brief and a little imprecise, as Payman will give more details in a future lecture, where he will connect this material to the construction of two-variable $p$-adic $L$-functions.

Let $N = pM$, where $M$ is coprime to $p$. Let $f_\infty$ be a Hida family of level $N$. Greenberg and Stevens [8] showed that modular symbols associated to the weight $k$ specialisations $f_k$ for all integers $k$ in some neighbourhood of 2 in weight space can be described in terms of a single modular symbol taking values in a space of measures. In [2], Bertolini and Darmon give a formula for Darmon's integral on $\mathcal{H}_p \times \mathcal{H}$ which appears in the construction of Stark-Heegner points, expressed in terms of the modular symbol associated to a Hida family. They are then able to use the relationship between this modular symbol and various two-variable $p$-adic $L$-functions to prove the main rationality theorem (Theorem 1) in [2]. In this section we describe how one can package together the modular symbols of the various specialisations of $f_\infty$ into a single measure valued modular symbol.

A LATTICE in $\mathbf{Q}_p^2$ is a free $\mathbf{Z}_p$-submodule of rank 2. If $L$ is a lattice then we write $L'$ for the set of PRIMITIVE vectors in the lattice: $v \in L$ is primitive if and only if $(1/p)v$ is not an element of $L$. Let $L_* = \mathbf{Z}_p^2$, so that $L'_* = \mathbf{Z}_p \times \mathbf{Z}_p^\times \cup \mathbf{Z}_p^\times \times \mathbf{Z}_p$. Let $\mathcal{W} = \mathbf{Q}_p^2 - \{0\}$ and let $\mathbf{D}$ denote the space of compactly supported measures on $\mathcal{W}$. Let $\mathbf{D}_* \subset \mathbf{D}$ denote the submodule of measures supported on $L'_*$. The module $\mathbf{D}$ is equipped with a right action of $\mathrm{GL}_2(\mathbf{Q}_p)$ defined by

$$\int_{\mathcal{W}} F d(\mu|\gamma) = \int_{\mathcal{W}} (F|\gamma^{-1}) d\mu,$$

where

$$(F|\gamma^{-1})(x,y) = F(ax+by, cx+dy), \qquad \text{for } \gamma^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $V_k(\mathbf{C}_p)$ denote the dual of the $\mathbf{C}_p$-vector space of homogenous polynomials of degree $k$ in two variables, equipped with a right action of $\mathrm{GL}_2(\mathbf{Q}_p)$ (induced by duality using the formula above for the action on homogenous polynomials). Then for every integer $k \geq 2$, there is a SPECIALIZATION MAP

$$\rho_k \colon \mathbf{D}_* \to V_k(\mathbf{C}_p)$$

defined by the formula: if $P(X,Y)$ is a homogeneous polynomial of degree $k$, then

$$\rho_k(\mu)(P) = \int_{\mathbf{Z}_p \times \mathbf{Z}_p^\times} P(X,Y) d\mu(X,Y).$$

Let $\Gamma_0(p\mathbf{Z}_p)$ denote the group of matrices in $\mathrm{GL}_2(\mathbf{Z}_p)$ that are upper triangular mod $p$:

$$\Gamma_0(p\mathbf{Z}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}_p) \;\middle|\; p|c, ad-bc \in \mathbf{Z}_p^\times \right\}.$$

Although $\mathbf{Z}_p \times \mathbf{Z}_p^\times$ is not stable under the action of $\mathrm{GL}_2(\mathbf{Z}_p)$, it is stable under the action of $\Gamma_0(p\mathbf{Z}_p)$: let $(x,y) \in \mathbf{Z}_p \times \mathbf{Z}_p^\times$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p\mathbf{Z}_p)$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x,y) = (ax+by, cx+dy),$$

and since $c$ is divisible by $p$ and $ad-bc \in \mathbf{Z}_p^\times$, it follows that $d$ is a unit and hence so is $cx+dy$.

**Lemma 10.** *The specialisation map $\rho_k$ is equivariant for the action of $\Gamma_0(p\mathbf{Z}_p)$.*

*Proof.* This is easy, and we only include it to check that our actions are compatible. Let $\gamma \in \Gamma_0(p\mathbf{Z}_p)$ and let $\chi(X,Y)$ denote the characteristic function of $\mathbf{Z}_p \times \mathbf{Z}_p^\times$ inside $\mathcal{W}$. Then we compute:

$$\begin{aligned}
\rho_k(\mu|\gamma)(P) &= (\mu|\gamma)(\chi(X,Y)P(X,Y)) \\
&= \mu\left([\chi(X,Y)P(X,Y)]|\gamma^{-1}\right) \\
&= \mu(\chi(\gamma^{-1}\cdot(X,Y))P(\gamma^{-1}\cdot(X,Y))) \\
&= \mu(\chi(X,Y)P(\gamma^{-1}\cdot(X,Y))) \qquad \text{since } \gamma^{-1}\cdot\left(\mathbf{Z}_p \times \mathbf{Z}_p^\times\right) = \mathbf{Z}_p \times \mathbf{Z}_p^\times \\
&= \rho_k(\mu)(P|\gamma^{-1}) \\
&= \left(\rho_k(\mu)|\gamma\right)(P).
\end{aligned}$$

□

The specialisation maps induce $\Gamma_0(p\mathbf{Z}_p)$-equivariant specialisation maps

$$\rho_k\colon \mathrm{MS}_{\Gamma_0(M)}(\mathbf{D}_*) \to \mathrm{MS}_{\Gamma_0(N)}(V_k(\mathbf{C}_p))$$

which are given simply by composition. The $\mathbf{D}_*$-valued symbols are endowed with an action of the Hecke algebra for $\Gamma_0(M)$, and furthermore with an action of an operator $U_p$ (in spite of the fact that $p$ does not divide the $M$). Let $\mathrm{MS}^{\mathrm{ord}}_{\Gamma_0(M)}(\mathbf{D}_*)$ denote the largest submodule on which $U_p$ acts invertibly – we call this the ordinary subspace, and we'll discuss ordinary parts of Hecke modules more in the next section.

We need one final piece of notation before we can state the main Theorem 5.13 of [8] which is used in [1]. The action of $\mathbf{Z}_p^\times$ on $\mathcal{W}$ by rescaling induces an action of the Iwasawa algebra $\Lambda$ on everything in sight. As before we let $\Lambda^\dagger$ denote the ring of analytic functions that converge on a neighbourhood of 2 in weight space. Write

$$\mathbf{D}_*^\dagger = \mathbf{D}_* \otimes_\Lambda \Lambda^\dagger,$$
$$\mathrm{MS}^{\mathrm{ord}}_{\Gamma_0(M)}(\mathbf{D}_*)^\dagger = \mathrm{MS}^{\mathrm{ord}}_{\Gamma_0(M)}(\mathbf{D}_*) \otimes_\Lambda \Lambda^\dagger.$$

Then one has the following wonderful theorem that packages together the periods of the Hida family $f_\infty$:

**Theorem 11** (Greenberg-Stevens, [8], Theorem 5.13)**.** *Let $f_\infty$ be a Hida family on $\Gamma_0(M)$. Then there exists a $\mathbf{D}_*^\dagger$-valued modular symbol $\mu_* \in \mathrm{MS}^{ord}_{\Gamma_0(M)}(\mathbf{D}_*)^\dagger$ such that:*

(1) $\rho_2(\mu_*) = I_f$;
(2) *there exists a neighbourhood $U$ of 2 in weight space, such that for all $k \in U \cap \mathbf{Z}^{\geq 2}$, there exist scalars $\lambda(k) \in \mathbf{C}_p$ such that $\rho_k(\mu_*) = \lambda(k) I_{f_k}$.*

Here $I_{f_k}$ is the modular symbol of the form $f_k$. The scalars $\lambda(k)$ depend on the periods chosen to construct $I_{f_k}$, and we make no claim regarding the coherency of the values $\lambda(k)$ as $k$ varies.

## 2. Existence of Hida families

### 2.1. Tower of modular curves.

### 2.2. Homology of the tower.

### 2.3. The ordinary part.

### 2.4. Structure of the ordinary part.

freeness plus control.

### 2.5. The ordinary Hecke algebra.

### 2.6. Existence of Hida families.

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