p-adic L-function of an ordinary weight 2 cusp form

The following exposition essentially summarizes one section of the excellent article [MS]. For any fixed cusp form $S_2(\Gamma_0(N), \mathbb{Q})$, we will associate a p-adic modular form on weight space.
The idea is to produce a measure on weight space $Hom_{cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$ which interpolates on the characters $< k >$ to give the special values

$$\frac{1}{\alpha^n} \times \frac{\tau(\overline{\chi}, 1) L(f, \chi, 1)}{\Omega_f^{\chi(-1)}}.$$

This constructs “one half” of the 2-variable p-adic analytic L-function associated to any Hida ordinary family. We will not discuss Hida families here.

Measures on locally compact totally disconnected spaces

Let T be a locally compact totally disconnected space, e.g. $G(\mathbb{Q}_p)$ (resp. $G(\mathbb{Z}_p)$) for any linear algebraic group over $\mathbb{Q}_p$ (resp. $\mathbb{Z}_p$).

Let W be an abelian group.
Definition. A W-valued *distribution* is a finitely additive homomorphism

$$\mu : Step(T) \to W,$$

where $Step(T)$ denotes the locally constant $\mathbb{Z}$-valued functions. This is equivalent to a finitely additive W-valued function on compact open subsets of T. We denote the set of such distributions $Dist(T, W)$.

Suppose $T = \varprojlim T_n$ for some finite (discrete) spaces T_n with surjective transition maps $T_{n+1} \to T_n$. Define the norm map

$$N : Dist(T_{n+1}, W) \to Dist(T_n, W) : (N\mu)(t) = \sum_{t' \mapsto t} \mu(t').$$

We can express any distribution $\mu \in Dist(T, W)$ as an inverse limit of distributions $\mu_n : T_n \to W$ satisfying the “norm compatibility condition”

$$(N\mu_{n+1})(t) = \mu_n(t).$$

Remarks.

- It may help to think of the above compatibility condition in the context of the example $T = \mathbb{Z}_p, T_n = \mathbb{Z}/p^n\mathbb{Z}, W = \mathbb{R}$ and μ the distribution corresponding to Haar measure on $\mathbb{Z}_p$.

- Note how reminiscent the norm map N is of Hecke operators, which are other instances of "summing over the fiber". More on that later.

Distributions become more interesting when we endow W with an interesting topology and demand that our distributions be continuous. Namely, suppose that W is a p-adic Banach space; most useful to us will actually be the case where W is a finite dimensional vector space over a finite extension of $\mathbb{Q}_p$.

Definition. A W-valued *measure* on T is a continuous linear map from the space of W-valued continuous functions $C^0(T, W)$ to W.

By the above definition, we see that if μ is a measure, then for any compact open $U \subset T$,

$$|\mu(\mathbf{1}_U)| \leq ||\mu||_\infty < \infty.$$

On the other hand, if $\mu \in Dist(T, W)$ is any distribution for which $\mu(\mathbf{1}_U)$ is bounded for every compact open U, an approximating an arbitrary continuous W-valued function on T by a step function will show that μ extends uniquely to a measure.

Example. The Dirac measures $\delta_t(f) = f(t)$ are indeed W-valued measures.

Measures on groups

Suppose G is a topological group which can be expressed as an inverse limit of quotients by finite index, normal subgroups

$$G = \lim_{\leftarrow} G/G_n$$

where $G_1 \supset G_2 \supset G_3 \supset ...$ and $\bigcap G_n = 0$, which ensures that G is separated.
There are canonical isomorphisms

$$Dist(G/G_n, O_K) \xrightarrow{i} O_K[G/G_n]$$

$$\mu \mapsto \sum_{g \in G/G_n} \mu(g) g$$

which are compatible with the earlier norm maps, i.e. the diagram

$$\begin{tikzcd} Dist(G/G_{n+1}, O_K) \arrow[r, "i"] \arrow[d, "N"'] & O_K[G/G_{n+1}] \arrow[d] \\ Dist(G/G_n, O_K) \arrow[r, "i"] & O_K[G/G_n] \end{tikzcd}$$

commutes.
The (O_K-valued) *Iwasawa algebra* of G is the inverse limit

$$\Lambda = O_K[[G]] := lim_{\leftarrow} O_K[G/G_n].$$

It is a compact topological O_K-algebra and the above maps i define an isomorphism

$$Dist(G, O_K) \xrightarrow{i} \Lambda$$

where by definition, we let μ_α be the preimage of an element α of the Iwasawa algebra. For example, it is easy to check that for $g \in G$, the Dirac measure $\delta_g = \mu_g$, i.e. it corresponds to the element $g \in O_K[[G]]$.

The topology on the Iwasawa algebra is such that
Lemma. The evaluation map

$$C(G, K) \times \Lambda \to K$$
$$(f, \alpha) \mapsto \mu_\alpha(f)$$

is continuous.

Continuity of the above evaluation map implies that any continuous homomorphism $\chi : G \to O_K^\times$ extends to a continuous homomorphism $\rho : \Lambda \to O_K$ of the Iwasawa algebra, by the formula

$$\rho(\alpha) = \mu_\alpha(\chi).$$

The Iwasawa algebra on groups essentially $\cong \mathbb{Z}_p^n$

We will define p-adic L-functions on weight space $= Hom_{cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$. This group equals

$$(\mathbb{Z}/p\mathbb{Z})^\times \times (1 + p\mathbb{Z}_p),$$

and $1 + p\mathbb{Z}_p$ is isomorphic to $\mathbb{Z}_p$ by the logarithm map. This provides weight space with a *linear p-adic structure.* In general,

Definition. A *linear p-adic* structure on a compact group G is a decomposition

$$G = C \times H$$

together with an isomorphism of topological groups

$$\mathbb{Z}_p^n \xrightarrow{\gamma} H.$$

This linear structure gives an explicit coordinatization of the Iwasawa algebra of G. We let $h_i = \gamma(e_i)$ where e_i is the tuple with a 1 in the i^{th} coordinate and zeros in all other coordinates.

Lemma 1. *The linear structure γ provides a unique isomorphism*

$$\Lambda_G \to O_K[C] \otimes_{O_K} O_K[[T_1, ..., T_n]]$$
$$h_i \mapsto T_i + 1.$$

Thus, we appear to be identifying the Iwasawa algebra with an algebra of analytic functions. Though we won't define them here, elements of the Iwasawa algebra are *rigid analytic functions* on the unit polydisk (or a finite union of polydiscs).
Let $X = Hom_{cont}(G, O_K^\times)$ be weight space. Let $F(X, O_K)$ denote the O_K-algebra of (rigidt) analytic functions on X.

Proposition 1. *The map*

$$\Lambda_G \to C^0(X, O_{\mathbb{C}_p})$$
$$\alpha \mapsto (\widehat{\mu_\alpha} = \chi \mapsto \mu_\alpha(\chi))$$

is injective with image contained in $F(X, O_{\mathbb{C}_p})$.

Remark. The above map $\alpha \mapsto \widehat{\mu_\alpha}$ is a kind of Fourier transform, associating a function/measure on an abelian topological group to a function on the dual group.

Proof.

- This amounts to identifying the group algebra $O_{\mathbb{C}_p}[[\mathbb{Z}_p^n]]$ with $O_{\mathbb{C}_p}[[T_1, ..., T_n]]$.
- The injectivity statement follows from a p-adic approximation theorem, the key point being the following:
  Fact: Any function on $G/G_n \to O_K$ can be written as an O_K-linear combination of characters on G/G_n.

  From this, it is easy to show that we can uniformly approximate any function by (locally constant) characters. So fi $\alpha \in \Lambda_G$ maps to 0, then $\mu_\alpha(f) = 0$ for any $f \in C^0(G, O_K), i.e. \mu_\alpha = 0$. But since the association $\alpha \mapsto \mu_\alpha$ is an isomorphism, α must be zero.

□

Call the image of the homomorphism from this proposition, a subalgebra of the analytic functions $F(X, O_K)$, the *Iwasawa algebra.*

Serre's congruences

Not every function in $F(X, O_K)$ lies in the image of the Iwasawa algebra. For example, take the case of $X = Hom_{cont}(\mathbb{Z}_p^\times, O_K^\times)$. Serre wrote down congruence conditions which completely characterize the image of the Iwasawa algebra inside $F(X, \mathbb{Z}_p)$ for the case $X = Hom_{cont}(1 + p\mathbb{Z}p, \mathbb{Z}_p^\times)$. We proceed to describe these congruences below.

Let f be a $\mathbb{Z}_p$-valued function on $Hom_{cont}(1 + p\mathbb{Z}_p, \mathbb{Z}_p)$. Via the exponential map, we may view it as an analytic function on $\mathbb{Z}_p$.
We can express f uniquely as

$$f(s) = \sum_n \delta_n \binom{s}{n}.$$

Let c_{in} be the coefficient of Y^i in $Y(Y-1)...(Y-n+1)$. Then f lies in the image of the Iwasawa algebra iff the following congruences are satisfied:

- $\delta_n \equiv 0 \ (mod\ p^n)$ for all n.
- $val_p(\sum_{i=0}^n c_{in}\delta_i p^{-i}) \geq val_p(n!)$.

Ignoring the specifics, we should expect some result of this type in analogy with real analysis. There, Paley-Winer type theorems relate the decay of Fourier transform of a distribution to smoothness of the distribution itself. The above congruence conditions pin down the necessary decay on the Fourier transform in order for the distribution μ to be a measure.

Computation of special L-values

Let $f(q) = \sum a_n q^n$ be a weight 2 cusp form for some congruence subgroup γ. Let χ be a Dirichlet character of conductor N.
We have the following lemma from finite group theory.

Lemma 1. *For any character χ of $(\mathbb{Z}/m\mathbb{Z})^\times$, we have the realtion*

$$\sum_a \chi(\overline{a}) e^{2\pi i a n/m} = \chi(n)\tau(\overline{\chi}, 1).$$

Proof. Let V be the space of functions $G = (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}$.
Consider the function $f_0(x) = e^{2\pi i x/m} \in V$. Then

$$\sum_a f(\overline{a}) e^{2\pi i a n/m} = f * f_0(n).$$

Any convolution operator on a finite abelian group is semisimple. This right convolution operator commutes with left translations, and so its eigenspaces are representations of G. Since all representations of G are a sum of characters, it follows that the eigenfunctions of $*f_0$ are given by characters and so each character is an eigenfunction.
But then

$$\chi * f_0(n) = c_\chi \chi(n)$$

for some constant c_χ. Plugging in $n = 1$ gives $c_\chi = \chi * f_0(1) = \tau(\overline{\chi}, 1)$. $\square$

Now let $f_\chi(q) = \sum a_n \chi(n) q^n$ be the twist of any cusp form. Then by the above lemma,

$$f_\chi(z) = \frac{1}{\tau(\chi, 1)} \sum_{a \mod m} \overline{\chi}(a) f(z + a/m)$$

follows easily.
We can express $L(f, s)$ as the Mellin transform

$$L(f, s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f(it) t^s \frac{dt}{t}.$$

Therefore, for weight 2 cusp forms f, the above computation shows that

$$L(f, \chi, 1) = \frac{1}{\tau(\chi, 1)} \sum_{a \in (\mathbb{Z}/m\mathbb{Z})^\times} \overline{\chi}(a)\ 2\pi \int_{a/m}^\infty f(it) dt.$$

Thus, remembering that the modular symbols $I_f^\pm$ were defined as periods of f between fixed cusps and ∞ normalized by fixed (transcendental) periods $\Omega_f^\pm$ (see Brandon's notes), we get that

$$\tau(\overline{\chi}, 1) L(f, \chi, 1)/\Omega_f^{\chi(-1)} = \sum_{a \ mod\ m} \overline{\chi}(a) I_f^{\chi(-1)}(a/m, \infty).$$

Now let $m = p^n$. If we squint our eyes, the above formula looks like taking a Riemann sum of $\overline{\chi}$ with respect to some measure $I_f^\pm$ on $\mathbb{Z}_p^\times$, namely the measure

$$\mu_f^\pm(\{x \in \mathbb{Z}_p^\times : \overline{x} = a \ mod\ p^n\}) = I_f^\pm(a/p^n, \infty).$$

This almost works. For $\mu_f^{\pm}$ to be a distribution, it must be finitely additive. We can express that through the equation

$$“\mu_f^{\pm}(\{x \in \mathbb{Z}_p^{\times} : \overline{x} = a \; mod \; p^n\}) = \sum_{b \in (\mathbb{Z}/p^{n+1}\mathbb{Z})^{\times}} \mu_f^{\pm}(\{x \in \mathbb{Z}_p^{\times} : \overline{x} = b \; mod \; p^{n+1}\})”$$

or equivalently

$$“I_f^{\pm}(a/p^n, \infty) = \sum_{b \in (\mathbb{Z}/p^{n+1}\mathbb{Z})^{\times}} I_f^{\pm}(b/p^{n+1}, \infty). \;\; (*)”$$

We've used quotation marks above because both equalities are false, though only mildly so. [PS, p.18] explains this lucidly, and we consolidate their explanation here for convenience.
The right side of $(*)$ looks an awful lot like the action of the T_p Hecke operator on the modular symbol $I_f^{\pm}(a/p^n, \infty)$, and indeed, the last equation above almost says that $I_f^{\pm}(a/p^n, \infty)$ is an eigensymbol of T_p. But there are two problems with this.

(1) The right side is not literally the action of T_p on $I_f^{\pm}(a/p^n, \infty)$.
We first specialize the above discussion to the case where f is an ordinary eigenform. This means that

$$T_p I_f^{\pm} = a_p I_f^{\pm}$$

with a_p a p-adic unit. Then

$$\begin{aligned} a_p I_f^{\pm}(\frac{a}{p^n}, \infty) &= T_p I_f^{\pm}(\frac{a}{p^n}, \infty) \\ &= \left(I_f^{\pm} | \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right) (\frac{a}{p^n}, \infty) + \sum_{j \; mod \; p} \left(I_f^{\pm} | \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \right) (\frac{a}{p^n}, \infty) \\ &= I_f^{\pm}(\frac{a}{p^{n-1}}, \infty) + \sum_{j \; mod \; p} I_f^{\pm}(\frac{a + jp^n}{p^{n+1}}, \infty). \end{aligned}$$

The first summand in this last expression appears to get in the way of the distribution property for $\mu_f^{\pm}$. But it wouldn't be there if f had been a function of the U_p operator, which is a sum over all of the above cosets except for $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.

Unfortunately, when we view f as an old form for $\Gamma_0(Np)$, it is *not* a U_p-eigenfunction. But $span\{f(z), f(pz)\}$ is invariant under the U_p operator, and it happens to have characteristic polynomial the Hecke polynomial

$$x^2 - a_p x + p.$$

Since a_p is a p-adic unit, one of the roots α is a p-adic unit as well, and the second β is not.

By an explicit computation, we find that

$$f_{\alpha}(z) = f(z) - \beta f(pz), f_{\beta}(z) = f(z) - \alpha f(pz)$$

are eigenvectors of U_p with respective eigenvalues α and β. It thus follows that

$$\alpha I^{\pm}_{f_\alpha}(\frac{a}{p^n},\infty) = \sum_{j\ mod\ p} I^{\pm}_{f_\alpha}(\frac{a+jp^n}{p^{n+1}},\infty).$$

(2) The above equation would be the additivity we need, if not for the annoying α. We deal with this by setting

$$\mu_f^{\pm}(\{x \in \mathbb{Z}_p^{\times} : \overline{x} = a\ mod\ p^n\}) = \frac{1}{\alpha^n} I^{\pm}_{f_\alpha}(\frac{a}{p^n},\infty).$$

Finally, this satisfies the additivity property we need.

It is clear that integrating Dirichlet characters of p-power conductor against this modified character $\mu_f^{\chi(-1)}$ will interpolate something related to special values of $L(f,\chi,1)/\Omega_f^{\chi(-1)}$. We just need to figure out what.

We can compute that

$$I^{\pm}_{f_\alpha} = I^{\pm}_f - \frac{1}{\alpha} I^{\pm}_f | \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

This implies that for χ a primitive Dirichlet character modulo p^n, that

$$\begin{aligned}
\int_{\mathbb{Z}_p^{\times}} \chi d\mu_f^{\chi(-1)} &= \frac{1}{\alpha^n}\sum_{a\in\mathbb{Z}/p^n\mathbb{Z}^{\times}} \chi(\overline{a}) I_f^{\chi(-1)}(\frac{a}{p^n},\infty) - \frac{1}{\alpha^n}\sum_{a\in\mathbb{Z}/p^n\mathbb{Z}^{\times}} \frac{1}{\alpha}\chi(\overline{a}) I_f^{\chi(-1)} | \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} (\frac{a}{p^n},\infty) \\
&= \frac{1}{\alpha^n}\sum_{a\in\mathbb{Z}/p^n\mathbb{Z}^{\times}} \chi(\overline{a}) I_f^{\chi(-1)}(\frac{a}{p^n},\infty) - \frac{1}{\alpha^n}\sum_{a\in\mathbb{Z}/p^n\mathbb{Z}^{\times}} \frac{1}{\alpha}\chi(\overline{a}) I_f^{\chi(-1)}(\frac{a}{p^{n-1}},\infty) \\
&= \frac{1}{\alpha^n}\sum_{a\in\mathbb{Z}/p^n\mathbb{Z}^{\times}} \chi(\overline{a}) I_f^{\chi(-1)}(\frac{a}{p^n},\infty) - \sum_{b\in(\mathbb{Z}/p^{n-1}\mathbb{Z})^{\times}} \left(\sum_{a\equiv b\ mod\ p^{n-1}} \chi(\overline{a})\right) \frac{1}{\alpha} I_f^{\chi(-1)}(\frac{b}{p^{n-1}},\infty).
\end{aligned}$$

But since χ is primitive, the inner sum

$$\sum_{a\equiv b\ mod\ p^{n-1}} \chi(\overline{a}) = 0$$

for each $b \in (\mathbb{Z}/p^{n-1}\mathbb{Z})^{\times}$. Thus, the final line in the above equation simplifies tremendously to give

$$\frac{1}{\alpha^n}\sum_{a\in\mathbb{Z}/p^n\mathbb{Z}^{\times}} \chi(\overline{a}) I_f^{\chi(-1)}(\frac{a}{p^n},\infty) = \frac{1}{\alpha^n}\tau(\overline{\chi},1)L(f,\chi,1)/\Omega_f^{\chi(-1)}$$

by our earlier comptuations using Birch's Lemma.
Note the importance of having α be a p-adic unit. In order for $d\mu_f^{\pm}$ to be a measure, it is necessary and sufficient for its values to be p-adically bounded as we integrate finite order characters. But this is the case because

- α is a p-adic unit, and
- the $\mathbb{Z}$-module of (period-normalized) modular symbols is finitely generated. Thus, the p-adic absolute value of any modular symbol is no greater than the maximum over a finite generating set.

Appendix: Hecke Operators

Later on, we will encounter various expressions that "look like Hecke actions on $X_0(N)$", which can be somewhat tricky when p divides N. The goal of this section is to understand this properly, especially the distinction between T_p and U_p.

Suppose that p does not divide N.
For any positive integer M, $X_0(M)$ is a coarse moduli space for the functor

$$S \mapsto (E/S, C/S)/\cong$$

of isomorphism classes generalized elliptic curves together with a fintie flat cyclic subgroup of order $M \subset E^{sm}$.

Remark. There are subtleties with the notion of cyclicity. But we'll really only need Hecke operators in the case of $S = \mathbb{C}$, where cyclic subgroups are just vanilla cyclic groups.

Since C is cyclic of order Np, it has a unique subgroup, C_1, and a unique quotient C/C_2 which are order N. Then we can define the Hecke correspondence T_p through two degeneracy maps

$$X_0(N) \xleftarrow{\pi_1} X_0(Np) \xrightarrow{\pi_2} X_0(N)$$
$$(E, C_1) \xleftarrow{\pi_1} (E, C) \xrightarrow{\pi_2} (E/C_2, C/C_2).$$

The Hecke operator T_p is defined to be

$$T_p = (\pi_2)_* \circ \pi_1^*.$$

When p divides the level N, we rename this Hecke operator to be U_p.
Note that T_p always has degree $p+1$, whereas U_p has degree p. This comes down to some group theory.

- Suppose that $(N, p) = 1$. Then if (E, C) is an elliptic curve with a cyclic subgroup of order N, how many elements lie in its preimage by π_1?
  We must find all subgroups of $(\mathbb{Z}/Np\mathbb{Z})^2 = (\mathbb{Z}/N\mathbb{Z})^2 \times (\mathbb{Z}/p\mathbb{Z})^2$ which contain a fixed cyclic subgroup of order N. The cyclic subgroup must be contained entirely within the $(\mathbb{Z}/N\mathbb{Z})^2$ factor, and we need to count the number of cyclic subgroups of $(\mathbb{Z}/p\mathbb{Z})^2$ of order p. This is exactly $p+1$ (the number of lines in a 2 dimensional $\mathbb{F}_p$ vector space).

- Suppose that $N = p^r M$ for some $r > 0, (M, p) = 1$. Then we must find all cyclic subgroups of order $Np = p^{r+1}M$ contained in $(\mathbb{Z}/Np\mathbb{Z})^2 = (\mathbb{Z}/M\mathbb{Z})^2 \times (\mathbb{Z}/p^{r+1}\mathbb{Z})^2$ which contain a fixed cyclic group of order Mp^r. Equivalently, we must find the number of cyclic subgroups D of order p^{r+1} contained in the cyclic subgroup $(\mathbb{Z}/p^r\mathbb{Z})^2$ which contain a fixed cyclic subgroup C of order p^r.

  $GL_2(\mathbb{Z}/p^{r+1}\mathbb{Z})$ acts transitively on the set of such flags $C \subset D$. For the standard flag, $C \subset D$ consisting of appropriate multiples of the first standard basis vector, the elements of $GL_2(\mathbb{Z}/p^{r+1}\mathbb{Z})$ which stabilize C are

  $$\begin{pmatrix} unit & * \\ ap^r & unit \end{pmatrix}$$

and the elements which stabilize D are
$\begin{pmatrix} unit & * \\ 0 & unit \end{pmatrix}$
The ratio $Stab(C)/Stab(D) = p$, and so U_p has degree p.

Explicitly, with respect to the isomorphism

$$\Gamma_0(N)\backslash\mathbb{H} \to X_0(N)(\mathbb{C}) : \Gamma_0(N)z \mapsto (\langle z, 1\rangle, \langle 1/N\rangle)$$

we see that

$$T_p(z) = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} z + \sum_{j \ mod \ p} \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} z$$

and

$$U_p(z) = \sum_{j \ mod \ p} \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} z.$$

The Hecke action on a weight 2 cusp form $f(z)$ is then computed through its induced "pullback followed by pushforward" action on the differential $f(z)dz$:

$$T_p(f(z)dz) = f\left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} z\right) d\left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} z\right) + \sum_{j \ mod \ p} f\left(\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} z\right) d\left(\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} z\right)$$

and similarly for U_p (just excluding the first summand from the above expression). The action at the level of the q-expansion of f is thereby readily computed:

$$U_p(\sum a_n q^n) = \sum a_{np} q^n$$
$$T_p(\sum a_n q^n) = \sum a_{np} q^n + p \sum a_n q^{pn}.$$

Earlier in our exposition of p-stabilization, we made use of the following:

Lemma 1. *Suppose that f is an eigenform of T_p acting on $S_2(\Gamma_0(N))$ with $(N, p) = 1$. Then span$\{f(z), f(pz)\}$ is preserved by the U_p operator acting on $S_2(\Gamma_0(Np))$.*

Proof. Note that for $f \in S_2(\Gamma_0(N))$, we can express

$$T_p f(z) = U_p f(z) + \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} f = U_p f(z) + f(pz).$$

But we can directly express the U_p action on the q-expansion of f. If $f(z) = \sum a_n q^n$, then $f(pz) = \sum a_n q^{pn}$ and

$$U_p(f(pz)) = U_p(\sum a_n q^{pn}) = \sum a_n q^n = f(z).$$

So if $T_p f = a_p f$, then

$$U_p f(z) = a_p f(z) - f(pz), U_p f(pz) = f(z).$$

so the span is clearly stable. □

If α and β are the roots of the above characteristic polynomial $x^2 - a_p x + p$, then

$$f_\alpha = f(z) - \beta f(pz), f_\beta = f(z) - \alpha f(pz)$$

are eigenvectors of U_p with respective eigenvalues α and β. If f is an ordinary eigenform for T_p, then a_p is a p-adic unit and so one root α is a p-adic unit too. We call f_α the *ordinary p-stabilization* of f (and f_β the *non-ordinary p-stabilization* of f).

References

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