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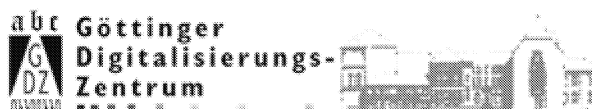
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## ***p*-adic *L*-functions and *p*-adic periods of modular forms <sup>★</sup>**

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### **Introduction**

Let  $E$  be an elliptic curve which is defined over  $\mathbf{Q}$  and has stable reduction modulo a given prime  $p$ . Assuming that  $E$  is modular, one can associate to  $E$  a  $p$ -adic  $L$ -function  $L_p(E, s)$ . (See [Mz-SwD, A-V, Vi, Mz-T-T] for its construction in various cases.) This function is defined by a certain interpolation property and is analytic for  $s \in \mathbf{Z}_p$ . In this paper, we will assume that  $E$  has split multiplicative reduction at  $p$ . Under this assumption the interpolation property implies that  $L_p(E, 1) = 0$ . We will prove a formula for  $L_p(E, 1)$  which was discovered experimentally by Mazur, Tate, and Teitelbaum [Mz-T-T].

By Tate's  $p$ -adic uniformization theory, there is a  $p$ -adic integer  $q_E \in p\mathbf{Z}_p$  (which we refer to as the Tate period for  $E$ ) and a  $p$ -adic analytic isomorphism

$$(0.1) \quad E(\bar{\mathbf{Q}}_p) \cong \bar{\mathbf{Q}}_p^*/q_E^{\mathbf{Z}}$$

which is defined over  $\mathbf{Q}_p$ . Let  $\log_p: \mathbf{Q}_p^* \rightarrow \mathbf{Z}_p$  be the usual  $p$ -adic logarithm on  $\mathbf{Z}_p^*$ , extended to  $\mathbf{Q}_p^*$  by the convention  $\log_p(p) = 0$ . Let  $\text{ord}_p: \mathbf{Q}_p^* \rightarrow \mathbf{Z}$  be the normalized valuation. We define the  $\Omega$ -invariant of  $E$  by

$$(0.2) \quad \Omega_p(E) = \frac{\log_p(q_E)}{\text{ord}_p(q_E)}.$$

Our main result (Theorem 7.1) specializes to the following.

(0.3) **Theorem.** *Let  $p$  be a prime  $\geq 5$  and let  $E$  be a modular elliptic curve with split multiplicative reduction at  $p$ . Then*

$$L_p(E, 1) = \Omega_p(E) \cdot \frac{L_\infty(E, 1)}{\Omega_E}.$$

<sup>★</sup> Partially supported by grants from the National Science Foundation

Here  $L_\infty(E, z)$  is the Hasse-Weil  $L$ -function for  $E/\mathbf{Q}$  and  $\Omega_E$  is the real period for  $E$ .

Let  $w_\infty = \pm 1$  denote the sign in the functional equation for  $L_\infty(E, z)$ . Then, under our assumption that  $E$  has split multiplicative reduction at  $p$ ,  $L_p(E, s)$  satisfies the functional equation

$$L_p(E, 2-s) = w_p \langle N \rangle^{s-1} L_p(E, s)$$

where  $w_p = -w_\infty$ . Here  $\langle \cdot \rangle$  is projection to the subgroup  $1 + p\mathbf{Z}_p$  of principal units in  $\mathbf{Z}_p^*$ . If  $w_\infty = -1$ , then  $w_p = +1$  and the above theorem is trivially true. Both sides of the equation vanish. If  $w_\infty = +1$ , then  $L_p(E, s)$  has a zero at  $s = 1$  of odd order. As a consequence of the above theorem, we see that this order is 1 if and only if both  $\log_p(q_E)$  and  $L_\infty(E, 1)$  are nonzero. Manin has conjectured that  $\log_p(q_E) \neq 0$  whenever  $E$  is a Tate curve with algebraic  $j$ -invariant (see [Man1], §4.12). The Birch and Swinnerton-Dyer conjecture predicts that  $L_\infty(E, 1) \neq 0$  precisely when the Mordell-Weil group  $E(\mathbf{Q})$  is finite. It is conjectured in [Mz-T-T] that

$$\text{ord}_{s=1}(L_p(E, s)) = 1 + \text{ord}_{z=1}(L_\infty(E, z)).$$

At the moment, all we can prove is that the order of vanishing of  $L_p(E, s)$  at  $s = 1$  is at least 2 if  $w_\infty = -1$  and at least 3 if  $w_\infty = +1$  and  $L_\infty(E, 1) = 0$ .

To explain the idea behind the proof of Theorem 0.3, we will give an outline in the special case  $E = X_0(11)$  and  $p = 11$ . For more details see (2.11) and (5.18). In this case  $E$  has split multiplicative reduction at  $p = 11$ ,  $L_\infty(E, 1) \neq 0$ , and  $w_\infty = +1$ . Let  $Ta_p(E)$  be the  $p$ -adic Tate module of  $E$  and let

$$\rho_E: G_{\mathbf{Q}} = \text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \rightarrow \text{Aut}(Ta_p(E))$$

be the associated Galois representation. Let  $f_E = q \prod (1 - q^n)^2 (1 - q^{11n})^2$  be the unique normalized weight two newform over  $\Gamma_0(11)$ . Then the Mellin transform of  $f_E$  is  $L_\infty(E, z)$ . We have  $f_E|T_p = f_E$  and, in particular,  $f_E$  is ordinary at  $p$ .

The basic ingredient in our proof of Theorem 0.3 is Hida's universal ordinary deformation of  $Ta_p(E)$ . In our example, the universal ordinary Hecke algebra turns out to be the completed group ring,  $\mathcal{A} = \mathbf{Z}_p[[1 + p\mathbf{Z}_p]]$ . Thus Hida's universal ordinary deformation of  $Ta_p(E)$  is a free rank two  $\mathcal{A}$ -module  $\mathbf{T}$  equipped with an action of the Galois group

$$\rho: \text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \rightarrow \text{Aut}_{\mathcal{A}}(\mathbf{T})$$

such that  $\mathbf{T}/P_0 \mathbf{T} \cong Ta_p(E)$  where  $P_0$  is the augmentation ideal in  $\mathcal{A}$ . The Galois module  $\mathbf{T}$  has a number of remarkable properties, which we will describe below.

For each  $k \in \mathbf{Z}_p$ , let  $\sigma_{k-2}: \mathcal{A} \rightarrow \mathbf{Z}_p$  be the unique continuous  $\mathbf{Z}_p$ -algebra homomorphism extending the character  $1 + p\mathbf{Z}_p \rightarrow \mathbf{Z}_p^*$ ,  $t \mapsto t^{k-2}$ . For  $\alpha \in \mathcal{A}$  we will write  $\alpha(k)$  instead of  $\sigma_{k-2}(\alpha)$  and refer to  $\sigma_{k-2}$  as specialization to weight  $k$ . Let  $P_k \subseteq \mathcal{A}$  be the kernel of  $\sigma_{k-2}$ . Let  $\mathbf{T}_k = \mathbf{T}/P_k \mathbf{T} \cong \mathbf{T} \otimes_{\mathcal{A}, \sigma_{k-2}} \mathbf{Z}_p$  and let

$$\rho_k: \text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \rightarrow \text{Aut}(\mathbf{T}_k)$$

be the reduction of  $\rho$  modulo  $P_k$ . Now fix, once and for all, an embedding

$$(0.4) \quad \bar{\mathbf{Q}} \subseteq \bar{\mathbf{Q}}_p.$$

Then  $\rho$  has the following properties.

$$(0.5)a. \quad \rho_2 = \rho_E.$$

(0.5)b. For each integer  $k \geq 2$ , there is a normalized newform  $f_k$  of weight  $k$  and conductor dividing  $p$  such that  $\rho_k$  is equivalent to Deligne's *p*-adic Galois representation [D] associated to  $f_k$  and our fixed embedding of  $\bar{\mathbf{Q}}$  into  $\bar{\mathbf{Q}}_p$ . The conductor of  $f_k$  is 1 precisely when  $k > 2$  and  $k \equiv 2$  modulo  $p - 1$ . For example,  $f_{1,2}$  = Ramanujan's  $\Delta$ -function. For other values of  $k$  the conductor is  $p$  and the Nebentype character is the Dirichlet character of conductor  $p$  associated to  $\omega^{2-k}$  where  $\omega: \mathbf{Z}_p^* \rightarrow \mu_{p-1} \subseteq \mathbf{Z}_p^*$  is the Teichmüller character.

(0.5)c. Let  $G_{\mathbf{Q}_p} = \text{Gal}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$  be a fixed decomposition group over  $p$ . Then the restriction  $\rho|_{G_{\mathbf{Q}_p}}$  of  $\rho$  to  $G_{\mathbf{Q}_p}$  is equivalent to an upper triangular representation

$$\rho|_{G_{\mathbf{Q}_p}} \sim \begin{pmatrix} \chi \varphi^{-1} & * \\ 0 & \varphi \end{pmatrix}$$

where  $\varphi: G_{\mathbf{Q}_p} \rightarrow A^*$  is an unramified character and  $\chi: G_{\mathbf{Q}_p} \rightarrow A^*$  is the unique character for which  $\sigma_{k-2} \circ \chi = \chi_0^{k-1} \omega^{2-k}$ , where  $\chi_0$  is the cyclotomic character and  $\omega$  is the Galois character associated to the Teichmüller character by class field theory.

The *A*-adic representation  $\rho$  can be used to construct *p*-adic analytic functions (in fact Iwasawa functions) which interpolate various data attached to the newforms  $f_k$ ,  $k \geq 2$ . For example, for each prime  $l \neq p$  let  $\text{Frob}_l$  be a Frobenius element at  $l$  and let  $a_l = \text{Tr}(\rho(\text{Frob}_l)) \in A$ . It follows from (0.5) b and the Eichler-Shimura relations that  $a_l(k)$  is the *l*-th Fourier coefficient of  $f_k$  for each integer  $k \geq 2$ .

The Euler factors at  $p$  will play an important role in our proof of Theorem 0.3. These factors can be described in terms of the representation  $\rho$ . Let  $a_p = \varphi(\text{Frob}_p) \in A^*$ . Then for each integer  $k \geq 2$ , the *p*-th Euler factor of the complex *L*-function  $L_\infty(f_k, z)$  of  $f_k$  has the form  $[(1 - \alpha_k p^{-z})(1 - \beta_k p^{-z})]^{-1}$ , where  $\alpha_k = a_p(k)$  and

$$\beta_k = \begin{cases} p^{k-1}/\alpha_k & \text{if } k > 2 \text{ and } k \equiv 2 \text{ modulo } (p-1), \\ 0 & \text{otherwise.} \end{cases}$$

These numbers satisfy the congruences  $\alpha_k \equiv \alpha_2 = 1$  and  $\beta_k \equiv 0$  modulo  $p$ . We will refer to  $\alpha_k$  as the unit root of Frobenius and to  $\beta_k$  as the non-unit root of Frobenius. From the above description we see that the family of unit roots of Frobenius  $\{\alpha_k\}_{k \geq 2}$  is interpolated by the Iwasawa function  $a_p(k)$ , but that the family of non-unit roots of Frobenius  $\{\beta_k\}$  cannot be interpolated by any *p*-adic analytic function of  $k$ .

The remarks of the last two paragraphs may be summarized by saying that there is a formal *q*-expansion

$$(0.6) \quad \mathbf{f} = \sum_{n=1}^{\infty} a_n q^n \in A[[q]]$$

such that for each integer  $k \geq 2$ , the specialization  $\mathbf{f}_k \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} a_n(k) q^n$  of  $\mathbf{f}$  to weight  $k$  is the  $q$ -expansion of the ‘ $p$ -stabilized newform’ associated to  $f_k$ . This is the cusp form  $f_k^*$  defined by

$$(0.7) \quad f_k^*(z) = f_k(z) - \beta_k f_k(pz)$$

for  $z$  in the upper half-plane  $[W]$ . In particular, the  $L$ -function of  $f_k^*$  is equal to the  $L$ -function of  $f_k$  with the Euler factor  $(1 - \beta_k p^{-z})^{-1}$  removed.

The final ingredient we require for the proof of Theorem 0.3 is a two variable  $p$ -adic  $L$ -function  $L_p(k, s)$ ,  $k, s \in \mathbf{Z}_p$ . In case  $E$  is an elliptic curve with complex multiplication, which has ordinary reduction at  $p$ , the construction of such a  $p$ -adic  $L$ -function having the four properties below is due to Katz [Kz]. When  $E$  is a modular elliptic curve with ordinary reduction at  $p$  (e.g. if  $E$  has multiplicative reduction at  $p$ ), this function was constructed in special cases by Mazur [Mz2]. Mazur’s construction was generalized by Kitagawa [K] for arbitrary ordinary  $A$ -adic cusp forms. Property (0.8)d was not treated. In this paper we will give another construction of the two variable  $p$ -adic  $L$ -function. The properties of it which we need for the case  $E = X_0(11)$  are as follows (see Theorem 5.15 for the general case, where we also prove an important functional equation for the ‘improved’  $p$ -adic  $L$ -function  $L_p^*$ ).

(0.8)a. (Analyticity)  $L_p(k, s)$  is analytic for  $k, s \in \mathbf{Z}_p$ .

(0.8)b. (Specialization to weight two)  $L_p(2, s) = L_p(E, s)$ .

(0.8)c. (Functional equation)  $L_p(k, k-s) = -L_p(k, s)$ .

(0.8)d. (Specialization to the critical value  $s = 1$ ) There is a factorization

$$L_p(k, 1) = (1 - a_p(k)^{-1}) L_p^*(k, 1)$$

where  $L_p^*(k, 1)$  is a  $p$ -adic analytic function of  $k$  for which

$$L_p^*(2, 1) = \frac{L_\infty(E, 1)}{\Omega_E}.$$

In fact, much more is true. The two variable  $p$ -adic  $L$ -function  $L_p(k, s)$  interpolates the one variable  $p$ -adic  $L$ -functions  $L_p(f_k, s)$  associated to the newforms  $f_k$ ,  $k \geq 2$ , as in [A-V, Vi]. More precisely, recall that the definition of  $L_p(f_k, s)$  depends on the choice of a complex period  $\Omega_{f_k} \in \mathbf{C}^*$ . This period is determined only up to multiplication by a non-zero element of the field generated by the Fourier coefficients of  $f_k$ . Fix, once and for all, a choice of complex periods  $\Omega_{f_k}$ ,  $k \geq 2$ , with  $\Omega_{f_2} = \Omega_E$ . Then the two variable  $p$ -adic  $L$ -function  $L_p(k, s)$  interpolates the functions  $L_p(f_k, s)$  in the following sense. For each integer  $k \geq 2$  there is a ‘period’  $\Omega_k \in \mathbf{Q}_p$  such that

$$L_p(k, s) = \Omega_k \cdot L_p(f_k, s).$$

Note that (0.8)b says that  $\Omega_2 = 1$ , and in particular that not all  $\Omega_k$  vanish. Note also that (0.8)c follows from the above interpolation property and the functional equation of  $L_p(f_k, s)$ .

The fourth property (0.8)d lies somewhat deeper. For each integer  $k \geq 2$ , and each integer  $s_0$  with  $0 < s_0 < k$  and  $s_0 \equiv 1 \pmod{p-1}$ , the  $p$ -adic  $L$ -function  $L_p(f_k, s)$  satisfies the following interpolation property

$$(0.9) \quad L_p(f_k, s_0) = (1 - \beta_k p^{-s_0})(1 - \alpha_k^{-1} p^{s_0-1}) \cdot \frac{L_\infty(f_k, s_0)}{\Omega_{f_k}}.$$

For a discussion of the Euler factors which occur in this expression, see [Gr]. For  $s_0 = 1$ , the second Euler factor is interpolated by an Iwasawa function, namely  $(1 - a_p(k)^{-1})$ . This vanishes at  $k = 2$  and so is a nonunit in the Iwasawa algebra  $\mathcal{A}$ . The function  $L_p(k, 1)$  is also an Iwasawa function in  $k$ . Moreover,  $L_p(k, 1)$  can be shown to be divisible in  $\mathcal{A}$  by  $(1 - a_p(k)^{-1})$ . The quotient  $L_p^*(k, 1)$  is an Iwasawa function in  $k$ , which we regard as an “improved”  $p$ -adic  $L$ -function. It satisfies the interpolation property

$$(0.10) \quad L_p^*(k, 1) = (1 - \beta_k p^{-1}) \frac{L_\infty(f_k, 1)}{\Omega_{f_k}} \cdot \Omega_k$$

for all integers  $k \geq 2$ . When  $k = 2$ , this reduces to (0.8)d since  $\beta_2 = 0$ ,  $\Omega_{f_2} = \Omega_E$ , and  $\Omega_2 = 1$ .

Theorem 0.3 is proved by calculating the linear term in the Taylor expansion of  $L_p(k, s)$  about  $(k, s) = (2, 1)$ . From the functional equation (0.8)c we see that  $L_p(k, k/2) = 0$  for all  $k \in \mathbf{Z}_p$ . Hence the linear term has the form  $c \cdot (-\frac{1}{2}(k-2) + (s-1))$  for some constant  $c \in \mathbf{Z}_p$ . We calculate  $c$  in two ways. From (0.8)b we see that  $c = L_p(E, 1)$ . But from (0.8)d and the fact that  $a_p(2) = \alpha_2 = 1$ , it follows that  $c = -2a'_p(2) \cdot L_\infty(E, 1) / \Omega_E$ . Comparing these expressions for  $c$  we obtain

$$L_p(E, 1) = -2a'_p(2) \cdot \frac{L_\infty(E, 1)}{\Omega_E}.$$

Theorem 0.3 is therefore a consequence of the following special case of Theorem 3.18.

$$(0.11) \quad \textbf{Proposition.} \quad \Omega_p(E) = -2a'_p(2).$$

This proposition can be interpreted as follows. The analytic isomorphism (0.1) leads to an exact sequence

$$(0.12) \quad 0 \rightarrow \mathbf{Q}_p(1) \rightarrow V \rightarrow \mathbf{Q}_p \rightarrow 0$$

where  $V = Ta_p(E) \otimes \mathbf{Q}_p$  is the Tate module of  $E$  tensored with  $\mathbf{Q}_p$ . For an arbitrary continuous  $G_{\mathbf{Q}_p}$ -module  $M$ , we let  $H^n(M)$  denote the continuous Galois cohomology  $H^n(G_{\mathbf{Q}_p}, M)$ . Then the isomorphism class of the exact sequence (0.12) is determined by a nontrivial extension class  $\xi \in H^1(\mathbf{Q}_p(1))$  and the isomorphism class of the middle term  $V$  is determined by the line spanned by  $\xi$  in  $H^1(\mathbf{Q}_p(1))$ . Now, by Kummer theory,  $\text{ord}_p$  and  $\log_p$  give rise to coordinates on  $H^1(\mathbf{Q}_p(1))$  in terms of which we get an isomorphism

$$(0.13) \quad H^1(\mathbf{Q}_p(1)) \cong \mathbf{Q}_p^2.$$

With respect to these coordinates, the  $\mathfrak{L}$ -invariant  $\mathfrak{L}_p(E)$  is the slope of the line spanned by  $\zeta$ . In this way the isomorphism class of  $V$  is determined by  $\mathfrak{L}_p(E)$  and vice-versa.

Now let  $\phi, \chi: G_{\mathbf{Q}_p} \rightarrow A^*$  be the characters of (0.5)c. Let  $\mathbf{T}(\varphi^{-1})$  denote the underlying free rank two  $A$ -module of the representation  $\rho$  with  $G_{\mathbf{Q}_p}$  acting on  $\mathbf{T}$  via  $\rho \otimes \varphi^{-1}$ . Then (0.5)c gives us an exact sequence  $0 \rightarrow A(\chi\psi) \rightarrow \mathbf{T}(\varphi^{-1}) \rightarrow A \rightarrow 0$  where  $\psi = \varphi^{-2}$ . For each  $k \in \mathbf{Z}_p$  we “specialize this exact sequence to weight  $k$ ” and tensor with  $\mathbf{Q}_p$  to obtain an exact sequence

$$(0.14) \quad 0 \rightarrow \mathbf{Q}_p(\chi_k \psi_k) \rightarrow \mathbf{V}_k \rightarrow \mathbf{Q}_p \rightarrow 0$$

where  $\chi_k = \sigma_{k-2} \circ \chi$  and  $\psi_k = \sigma_{k-2} \circ \psi$  and  $\mathbf{V}_k = \mathbf{T}(\varphi^{-1})_k \otimes \mathbf{Q}_p$ . Now, as before, the isomorphism class of  $\mathbf{V}_k$  is determined by an extension class  $\zeta_k \in H^1(\mathbf{Q}_p(\chi_k \psi_k))$  up to homothety. Clearly,  $\zeta_k \neq 0$  for  $k$  sufficiently close to 2. But for  $k \neq 2$  the cohomology group  $H^1(\mathbf{Q}_p(\chi_k \psi_k))$  is one-dimensional. Hence the isomorphism class of  $\mathbf{V}_k$  is completely determined by the character  $\chi_k \psi_k$ . As  $k$  approaches 2 in  $\mathbf{Z}_p$ , the sequence (0.14) flows into the sequence (0.12). Thus we should expect that the sequence (0.12) is completely determined by the characters  $\psi_k$  for  $k$  in a neighborhood of 2. More precisely, what we prove is

$$(0.15) \quad \left. \frac{d\psi_k(\text{Frob}_p)}{dk} \right|_{k=2} = \mathfrak{L}_p(E).$$

Since  $\psi_k(\text{Frob}_p) = a_p(k)^{-2}$ , this is equivalent to proposition (0.11). This completes our outline of the proof of Theorem 0.3 in the special case where  $E = X_0(11)$  and  $p = 11$ .

In general, we may start with an arbitrary newform  $f$  of weight 2 over  $\Gamma_1(Np)(p \nmid N)$  which is split multiplicative at  $p$  (i.e. the  $p$ th Hecke eigenvalue of  $f$  is  $+1$ ). In this case  $f$  corresponds to a simple quotient of the Jacobian variety of  $X_1(Np)$  which has multiplicative reduction at  $p$ . Our Theorem 7.1 is a strengthened form of Theorem 0.3 in which the  $\mathfrak{L}$ -invariant of  $f$  is defined as in [Mz-T-T]. For example, if  $E$  is a modular elliptic curve with split multiplicative reduction at  $p$  and if  $\psi$  is a primitive Dirichlet character (not necessarily quadratic) for which  $\psi(p) = 1$ , then we have  $L_p(E, \psi, 1) = 0$  and Theorem 7.1 implies  $L_p(E, \psi, 1) = \mathfrak{L}_p(E) \frac{L_\infty(E, \psi, 1)}{\Omega_{E, \psi}}$  where  $\Omega_{E, \psi} = \Omega_E^\pm / \tau(\bar{\psi})$  with  $\Omega_E^\pm$  being the real or imaginary period of  $E$  depending on the sign of  $\psi$ . This is the “local” property of the  $\mathfrak{L}$ -invariant discovered numerically by Mazur, Tate, and Teitelbaum in [Mz-T-T].

Some interesting problems arise in the general case which are not apparent in the special case  $E = X_0(11)$  described above. In the general case, as in the special case, Hida has constructed a finite  $A$ -algebra  $\mathcal{R}$  called the universal ordinary Hecke algebra and the newform  $f$  corresponds to a  $\mathbf{Q}_p$ -valued homomorphism  $\kappa$  of  $\mathcal{R}$ . But, in general,  $\mathcal{R}$  is larger than  $A$ , and we can no longer say that  $k \in \mathbf{Z}_p$  parametrizes an analytic family of  $\bar{\mathbf{Q}}_p$ -valued homomorphisms of  $\mathcal{R}$ . However, it turns out that by restricting  $k$  to a neighborhood of 2, we do get such a parametrization locally (see (2.7) and the remarks thereafter for a precise formulation). We then obtain, as before, an analytic family of ordinary

Galois representations  $\rho_k$  and a  $p$ -adic  $L$ -function  $L_p(k, s)$  which is defined for all  $s \in \mathbf{Z}_p$  and all  $k$  in a neighborhood of 2. In general, however, the functional equation relates the  $p$ -adic  $L$ -function to the contragredient family of Galois representations, which will differ from the original family if the Nebentypus is nontrivial. This means that the  $p$ -adic  $L$ -function need not vanish identically along the line  $s = \frac{k}{2}$  as it did in our special case. However, in Theorem 5.15e we will also prove a ‘functional equation’ for the improved two-variable  $p$ -adic  $L$ -function. This allows us to show that the restriction of the standard two-variable  $p$ -adic  $L$ -function to the line  $s = \frac{k}{2}$  vanishes to order  $\geq 2$  at the point  $(k, s) = (2, 1)$ . This is sufficient for our purposes.

We may also inquire about the ‘denominators’ in the various  $p$ -adic  $L$ -functions. In general, we expect the two-variable  $p$ -adic  $L$ -function to have no denominator at all. On the other hand, the improved  $p$ -adic  $L$ -function may have a denominator. This denominator is a ‘divisor’ of the characteristic power series of a certain torsion  $A$ -module – the  $A$ -adic cuspidal group – which arises in our calculations (see 6.12). It would be interesting to analyze the structure of this group and to perform a descent analogous to Mazur’s Eisenstein descent [Mz1].

We close this introduction by mentioning the following question. Assume that  $E$  is a modular elliptic curve with good, ordinary reduction or multiplicative reduction at  $p$ . Then there is an associated two-variable  $p$ -adic  $L$ -function  $L_p(k, s)$ . Let  $n = \text{ord}_{z=1} L_\infty(E, z)$ . Then it seems reasonable to believe that the expansion of  $L_p(k, s)$  at  $k=2, s=1$  should begin with the homogeneous term of degree  $n$  (or  $n+1$  in the case of split multiplicative reduction). If this degree is odd, then  $-\frac{1}{2}(k-2) + (s-1)$  will be a linear factor in this term. Can one determine the other linear factors?

### 1. Hecke operators and ordinary eigenforms

In this section we fix some of the terminology and conventions which will be used in the rest of the paper.

Following [Sh2] we define Hecke algebras as double coset algebras. Let  $\Sigma = GL_2(\mathbf{Q}) \cap M_2(\mathbf{Z})$  be the semigroup of  $2 \times 2$  integral matrices with nonzero determinant. For each arithmetic group  $\Gamma$  in  $SL_2(\mathbf{Z})$  we let  $D(\Gamma, \Sigma)$  denote the double coset algebra associated to the pair  $(\Gamma, \Sigma)$ . The elements of this algebra are the  $\mathbf{Z}$ -valued functions on  $\Sigma$  which are bi-invariant with respect to  $\Gamma$  and which are supported on the union of finitely many double cosets of  $\Gamma$ . Clearly  $D(\Gamma, \Sigma)$  is generated by the characteristic functions  $T(g) \in D(\Gamma, \Sigma)$  of the double cosets  $\Gamma g \Gamma$ , for  $g \in \Sigma$ . If  $\Sigma_1$  is a subsemigroup of  $\Sigma$  containing  $\Gamma$ , then we will denote by  $D(\Gamma, \Sigma_1)$  the subalgebra of  $D(\Gamma, \Sigma)$  consisting of functions supported on  $\Sigma_1$ . For each such  $\Sigma_1$  we will denote by  $\Sigma_1^+$  the subsemigroup of  $\Sigma_1$  consisting of matrices with positive determinant.

In this paper,  $\Sigma$ -modules will be *contravariant* (i.e.  $\Sigma$  acts on the right) unless otherwise stated. The algebra  $D(\Gamma, \Sigma)$  acts contravariantly and functorially on the cohomology of  $\Sigma$ -modules. If  $\Gamma$  is preserved by the anti-involution  $g \mapsto g^* = \det(g) g^{-1}$ , then  $*$  induces an anti-involution on  $D(\Gamma, \Sigma)$  by

$T(g) \mapsto T(g)^* = T(g^*)$ . In this case we can also define a covariant action of  $D(\Gamma, \Sigma)$  on the  $\Gamma$ -cohomology of  $\Sigma$ -modules by defining

$$(1.1) \quad T(g) \cdot \Phi \stackrel{\text{def}}{=} \Phi | T(g)^*$$

for a cohomology class  $\Phi$  and  $g \in \Sigma$ .

Fix a positive integer  $N$ , a prime  $p$  which does not divide  $N$ , and an integer  $r \geq 0$ . Let  $\Sigma_1(p^r) = \left\{ g \in \Sigma \mid g \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \text{ modulo } p^r \right\}$ . If  $N > 1$ , the algebra  $D(\Gamma_1(Np^r), \Sigma_1(p^r))$  is not commutative, but we can construct a central subalgebra as follows. Let  $\mathbf{Z}'$  denote the multiplicative set of integers which are prime to  $p$ . For each  $a \in \mathbf{Z}'$  choose  $\gamma_a \in \Gamma_1(N) \cap \Gamma_0(p^r)$  whose lower right hand entry is congruent to  $a$  modulo  $p^r$  and let  $[a]_p = T\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \gamma_a\right)$  in  $D(\Gamma_1(Np^r), \Sigma_1(p^r))$ . The map  $\mathbf{Z}' \rightarrow D(\Gamma_1(Np^r), \Sigma_1(p^r))$ ,  $a \mapsto [a]_p$ , is multiplicative, hence extends to a  $\mathbf{Z}$ -algebra morphism  $\mathbf{Z}[\mathbf{Z}'] \rightarrow D(\Gamma_1(Np^r), \Sigma_1(p^r))$ . The image of this map is a central subalgebra. On the other hand,  $\mathbf{Z}[\mathbf{Z}']$  embeds naturally in the completed group ring  $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$ . Hence we may form the  $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$ -algebra

$$(1.2) \quad D_p(Np^r) \stackrel{\text{def}}{=} D(\Gamma_1(Np^r), \Sigma_1(p^r)) \otimes_{\mathbf{Z}[\mathbf{Z}']} \mathbf{Z}_p[[\mathbf{Z}_p^*]].$$

If  $A$  is a  $\mathbf{Z}_p[[\Sigma_1(p^r)]]$ -module which satisfies Hypothesis P below, then the (contravariant or covariant) action of  $D(\Gamma_1(Np^r), \Sigma_1(p^r))$  on the  $\Gamma_1(Np^r)$ -cohomology of  $A$  extends uniquely to a continuous action of  $D_p(Np^r)$ .

(1.3) **Hypothesis P.** *The action of the scalar matrices  $aI$ ,  $a \in \mathbf{Z}'$ , extends to a continuous action of the scalar matrices  $aI$  for  $a \in \mathbf{Z}_p^*$ .*

We are going to view  $D_p(N)$  as a universal algebra which acts on the  $\Gamma_1(Np^r)$ -cohomology, for every  $r \geq 0$ , of every  $\mathbf{Z}_p[[\Sigma]]$ -module  $A$  satisfying Hypothesis P. To define this action we note that the Hecke pair  $(\Gamma_1(Np^r), \Sigma_1(p^r))$  is weakly compatible to  $(\Gamma_1(N), \Sigma)$  in the sense of [A-S]. Hence, as in [A-S], there is a natural surjective  $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$ -morphism

$$(1.4) \quad D_p(N) \rightarrow D_p(Np^r)$$

induced by restriction of functions on  $\Sigma$  to  $\Sigma_1(p^r)$ . Let  $D_p(N)$  act on the  $\Gamma_1(Np^r)$ -cohomology of  $A$  via the composition of this morphism with the natural action of  $D_p(Np^r)$ .

The elements of  $D(\Gamma_1(Np^r), \Sigma_1(p^r))$  supported on  $\Sigma_1^+(p^r)$  (elements of positive determinant) generate a  $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$ -subalgebra of  $D_p(Np^r)$  which we denote  $D_p^+(Np^r)$ . As before, the algebra  $D_p^+(N)$  acts naturally on the  $\Gamma_1(Np^r)$ -cohomology of any  $\mathbf{Z}_p[[\Sigma_1^+(p^r)]]$ -module  $A$  satisfying Hypothesis P.

(1.5) **Definition.** We define the following standard elements of  $D_p(N)$ .

a. For each positive integer  $n$ , let  $T_n = T\left(\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}\right)$ .

b. For each integer  $a$  prime to  $N$  choose an element  $\beta_a \in \Gamma_0(N)$  whose lower right entry is congruent to  $a$  modulo  $N$  and define  $[a]_N = T(\beta_a)$ .

c. We extend  $[\cdot]_p: \mathbf{Z}_p^* \rightarrow D_p(N)$  to a multiplicative function on all nonzero  $p$ -adic integers by defining  $[p]_p = T\left(\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\right)$ .

d.  $\iota = T\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$ .

e.  $W_N = T\left(\begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}\right)$ .

All of these elements, except  $\iota$ , are in the subalgebra  $D_p^+(N)$ . Moreover,  $D_p(N) = D_p^+(N)[\iota]$ . The elements  $[a]_N$ , for  $a \in \mathbf{Z}$  prime to  $N$ , generate a subgroup  $\Delta_N$  in  $D_p^+(N)$  isomorphic to  $(\mathbf{Z}/N\mathbf{Z})^*$ . Let  $\mathcal{H}$  be the  $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$ -subalgebra of  $D_p^+(N)$  given by

$$(1.6) \quad \mathcal{H} = \mathbf{Z}_p[[\mathbf{Z}_p^*]][\Delta_N, T_n(n \in \mathbf{Z}^+), [p]_p].$$

Then  $\mathcal{H}$  is commutative and is centralized by  $\iota$ . If we define  $[a] = [a]_N \cdot [a]_p \in \mathcal{H}$  for integers  $a$  which are prime to  $Np$ , then the map  $a \mapsto [a]$  extends uniquely to a continuous multiplicative map from  $\mathbf{Z}_{p,N}^* = \varprojlim (\mathbf{Z}/Np^r\mathbf{Z})^*$  to  $\mathcal{H}$ . In this way we obtain a  $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$ -morphism

$$(1.7) \quad \mathbf{Z}_p[[\mathbf{Z}_{p,N}^*]] \rightarrow \mathcal{H}$$

in terms of which we may view  $\mathcal{H}$  as a  $\mathbf{Z}_p[[\mathbf{Z}_{p,N}^*]]$ -algebra. The element  $W_N$  does not centralize  $\mathcal{H}[\iota]$ . Indeed,  $W_N$  does not even normalize  $\mathcal{H}$  (e.g.  $W_N T_n W_N^{-1} \notin \mathcal{H}$  if  $(n, N) > 1$ ). However, we do have the following relations in  $\mathcal{H}[\iota, W_N]$ :

$$(1.8) \quad \begin{aligned} W_n \cdot [a] \cdot W_n^{-1} &= [a]_N^{-1} \cdot [a]_p && \text{for all } a \in \mathbf{Z}_{p,N}^*; \\ W_N \cdot T_n \cdot W_N^{-1} &= [n]_N^{-1} \cdot T_n && \text{for every } n \text{ which is relative prime to } N; \\ W_N \cdot \iota \cdot W_N^{-1} &= [-1]_N \cdot \iota; \\ W_N^2 &= [-N]_p. \end{aligned}$$

Let  $k$  be an integer  $\geq 2$  and, for each congruence group  $\Gamma$ , let  $\mathcal{S}_k(\Gamma, \bar{\mathbf{Q}})$  denote the space of holomorphic weight  $k$  cusp forms over  $\Gamma$  whose  $q$ -expansions have algebraic coefficients. Let

$$(1.9) \quad \mathcal{S}_k(\bar{\mathbf{Q}}) = \bigcup \mathcal{S}_k(\Gamma, \bar{\mathbf{Q}})$$

be the union over all congruence groups  $\Gamma$ . We let the subsemigroup  $\Sigma^+ \subset \Sigma$  of elements with positive determinant act on  $\mathcal{S}_k(\bar{\mathbf{Q}})$  by the weight  $k$  action: if  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Sigma^+$  and  $f \in \mathcal{S}_k(\bar{\mathbf{Q}})$  then  $(f|g)(z) = \det(g)^{k-1} (cz + d)^{-k} f(gz)$  for  $z$  in the upper half plane. We extend this by linearity to an action of  $\Sigma^+$  on  $\mathcal{S}_k(\bar{\mathbf{O}}_p) = \mathcal{S}_k(\bar{\mathbf{Q}}) \otimes_{\bar{\mathbf{Q}}} \bar{\mathbf{Q}}_p$  and define

$$(1.10) \quad \mathcal{S}_k(\Gamma_1(Np^r), \bar{\mathbf{Q}}_p) = \mathcal{S}_k(\bar{\mathbf{Q}}_p)^{\Gamma_1(Np^r)}.$$

Since the nonzero scalar matrices over  $\mathbf{Z}$  act on  $\mathcal{S}_k(\bar{\mathbf{Q}}_p)$  via  $aI \mapsto a^{k-2}$ , Hypothesis P is satisfied and we obtain an action of  $\mathcal{H}[W_N]$  on  $\mathcal{S}_k(\Gamma_1(Np^r), \bar{\mathbf{Q}}_p)$ .

(1.11) **Definition. a.** If  $r$  is a nonnegative integer and  $\psi$  is a Dirichlet character whose conductor is a power of  $p$ , then we let  $\sigma_{r,\psi}: \mathbf{Z}_p^* \rightarrow \bar{\mathbf{Q}}_p^*$  be the character defined by  $a \mapsto \psi(a) a^r$ . Such a character will be called an *arithmetic character*. If  $\psi$  is trivial we will suppress it from the notation and write simply  $\sigma_r$  instead of  $\sigma_{r,\psi}$ .

**b.** If  $R$  is a commutative  $\mathbf{Z}_p[[\mathbf{Z}_{p,N}^*]]$ -algebra we let  $\mathcal{X}(R) = \text{Hom}_{\text{cont}}(R, \bar{\mathbf{Q}}_p)$  denote the set of continuous  $\bar{\mathbf{Q}}_p$ -valued homomorphisms on  $R$ . We will refer to the elements of  $\mathcal{X}(R)$  as the  $\bar{\mathbf{Q}}_p$ -valued *points* on  $R$ . A continuous homomorphism  $\kappa: R \rightarrow \bar{\mathbf{Q}}_p$  will be called an *arithmetic point* if its restriction to  $\mathbf{Z}_p^*$  is an arithmetic character. In that case we say that  $\kappa$  has weight  $r+2$  and character  $\varepsilon$  if  $\kappa([a]) = \varepsilon(a) a^r$  for every rational integer  $a$  prime to  $Np$ . Let

$$\mathcal{X}^{\text{arith}}(R) = \text{the arithmetic points on } R.$$

**c.** We will write  $\mathcal{X}_0$  and  $\mathcal{X}_0^{\text{arith}}$  for the points and the arithmetic points, respectively, of our base algebra  $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$ .

It will often be convenient to view the elements of  $R$  as functions on  $\mathcal{X}(R)$ . When we wish to emphasize this point of view, we will write  $\alpha(\kappa)$  instead of  $\kappa(\alpha)$  for  $\alpha \in R$  and  $\kappa \in \mathcal{X}(R)$ .

The homomorphism  $\kappa: \mathcal{H} \rightarrow \bar{\mathbf{Q}}_p$  associated to any eigenform  $f$  is easily seen to be an arithmetic point on the  $\mathbf{Z}_p[[\mathbf{Z}_{p,N}^*]]$ -algebra  $\mathcal{H}$  (1.7) whose weight is the weight of  $f$  and whose character is the nebentype character of  $f$ .

## 2. Deformations of ordinary Galois representations

Fix a positive integer  $N$  and a prime  $p \geq 5$  which does not divide  $N$ . For each integer  $n \geq 0$ , let  $J_{n/\mathbf{Q}}$  be the Jacobian of the modular curve  $X_1(Np^n)$  equipped with Shimura's canonical model [Sh2] associated to the adelic group

$$\left\{ g \in GL_2(\mathbf{A}_f) \mid g \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \pmod{Np^n} \right\}.$$

There is a natural covariant action of the Hecke algebra  $\mathcal{H}[\iota, W_N]$  on  $J_n(\mathbf{C})$  induced by the action of the double coset algebra  $D(\Gamma_1(Np^n), \Sigma_1^+(p^n))$  via algebraic correspondences and letting  $\iota$  act by complex conjugation. The elements  $[a]$ , for  $a \in \mathbf{Z}_{p,N}^*$  operate through the nebentype.

Let  $Ta_p(J_n)$  denote the  $p$ -adic Tate module of  $J_n$ . The action of  $\mathcal{H}$  on  $J_n$  is defined over  $\mathbf{Q}$ , and therefore induces an action of  $\mathcal{H}$  on  $Ta_p(J_n)$  which commutes with the action of the Galois group  $G_{\mathbf{Q}}$ . For each pair of nonnegative integers  $m \geq n$ , the natural projection  $X_1(Np^m) \rightarrow X_1(Np^n)$  is defined over  $\mathbf{Q}$ , hence induces a Galois equivariant map of Tate modules  $Ta_p(J_m) \rightarrow Ta_p(J_n)$ . If  $m, n$  are positive then this projection also commutes with  $D_p(N)$  and in particular with  $\mathcal{H}$ . (When  $n=0$  and  $m>0$ , it respects all of the generators  $T_q, [a]$  of  $\mathcal{H}$  except  $T_p$  and  $[p]_p$ .) We may therefore form the projective limit over  $n > 0$  and define an  $\mathcal{H}[G_{\mathbf{Q}}]$ -module

$$(2.1) \quad Ta_p(J_\infty) = \varprojlim_n Ta_p(J_n).$$

The importance of  $Ta_p(J_\infty)$  for the study of  $p$ -adic Galois representations attached to modular forms was first recognized by Shimura [Sh1] (see [O] for a published account). More recently, Hida defined a certain factor of  $Ta_p(J_\infty)$  called the ordinary part and made a careful and beautiful analysis of its structure. The following discussion is based on his works [H1, H2].

(2.2) **Definition.** Let  $\mathbf{A}$  be a profinite abelian group and  $T_p: \mathbf{A} \rightarrow \mathbf{A}$  be a continuous homomorphism. The ordinary submodule of  $\mathbf{A}$  is defined to be

$$\mathbf{A}^0 = \bigcap_{n=1}^{\infty} T_p^n(\mathbf{A}).$$

(2.3) **Proposition.** Let  $\mathbf{A} = \varprojlim \mathbf{A}_n$  be a profinite abelian group and let  $T_p$  be an operator on  $\mathbf{A}$  which is equal to a limit of operators on the finite quotients  $\mathbf{A}_n$ . Then  $T_p$  acts invertibly on  $\mathbf{A}^0$  and there is a canonical decomposition  $\mathbf{A} = \mathbf{A}^0 \oplus \mathbf{A}^{\text{nil}}$  where  $\mathbf{A}^{\text{nil}} = \{a \in \mathbf{A} \mid \lim_{n \rightarrow \infty} T_p^n(a) = 0\}$  is the subgroup on which  $T_p$  acts topologically nilpotently.

*Proof.* In case  $\mathbf{A}$  is finite,  $\mathbf{A}^0$  is the subgroup on which  $T_p$  acts periodically. Clearly,  $T_p$  acts invertibly on  $\mathbf{A}^0$  in this case. The asserted decomposition then follows in the finite case from the fact that every orbit of  $T_p$  is eventually periodic. The general case follows from the finite case by a simple compactness argument.

Since the Tate modules  $Ta_p(J_n)$  are profinite, so also is  $Ta_p(J_\infty)$ . Moreover,  $Ta_p(J_\infty)$  satisfies the hypotheses of Proposition 2.3, so we may define the ordinary part  $Ta_p(J_\infty)^0$ . Since the operator  $T_p$  commutes with  $\mathcal{H}[G_{\mathbf{Q}}]$ , we see that  $Ta_p(J_\infty)^0$  is a direct factor of the  $\mathcal{H}[G_{\mathbf{Q}}]$ -module  $Ta_p(J_\infty)$ .

Let  $A$  be the Iwasawa algebra  $\mathbf{Z}_p[[1 + p\mathbf{Z}_p]]$  embedded in the natural way in  $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$  and let  $\mathcal{L}$  be the fraction field of  $A$ . Let  $H^0$  be the image of  $\mathcal{H}$  in the endomorphism ring of  $Ta_p(J_\infty)^0$ . Hida has proven ([H1], Thm. 3.1) that  $Ta_p(J_\infty)^0$  is a free  $A$ -module of finite rank. Moreover, he has constructed an idempotent  $e_{\text{prim}} \in H^0_{\mathcal{L}} = H^0 \otimes_A \mathcal{L}$  ([H2], pp. 250, 252) analogous to projection to the space of  $N$ -primitive eigenforms in Atkin-Lehner theory. We define the primitive part of  $Ta_p(J_\infty)^0$  to be the  $\mathcal{H}[G_{\mathbf{Q}}]$ -submodule  $\mathbf{T} = Ta_p(J_\infty)^0_{\text{prim}}$  obtained as the intersection of  $Ta_p(J_\infty)^0$  and  $e_{\text{prim}} \cdot Ta_p(J_\infty)^0 \otimes_A \mathcal{L}$ . Then  $\mathbf{T}$  is a reflexive  $A$ -module and is therefore free of finite rank. Since  $\iota$  and  $W_N$  preserve the  $N$ -primitive part of  $Ta_p(J_n)$ , they induce operators on  $\mathbf{T}$ . Note, however, that  $W_N$  does not in general commute with  $\mathcal{H}$  and that neither  $\iota$  nor  $W_N$  commutes with the action of the Galois group.

We associate to  $\mathbf{T}$  the following data which will be used throughout the paper.

(2.4) **Definition. a.** The universal ordinary  $p$ -adic Hecke algebra of tame conductor  $N$  is defined to be the image  $\mathcal{R}$  of  $\mathcal{H}$  in  $\text{End}_A(\mathbf{T})$ . The natural map  $h: \mathcal{H} \rightarrow \mathcal{R}$  endows  $\mathcal{R}$  with the structure of  $\mathbf{Z}_p[[\mathbf{Z}_{p,N}^*]]$ -algebra inherited from  $\mathcal{H}$ . The induced character  $\eta: \mathbf{Z}_{p,N}^* \rightarrow \mathcal{R}^*$  will be called the canonical character. Let  $\mathcal{K} = \mathcal{R} \otimes_A \mathcal{L}$  where  $\mathcal{L}$  is the fraction field of  $A$  and let  $\tilde{\mathcal{R}}$  be the normalization of  $\mathcal{R}$  in  $\mathcal{K}$ . Let  $\mathcal{X} = \mathcal{X}(\tilde{\mathcal{R}}) = \text{Hom}_{\text{cont}}(\tilde{\mathcal{R}}, \mathbf{Q}_p)$  and set

$$\mathcal{X}^{\text{arith}} = \mathcal{X}^{\text{arith}}(\tilde{\mathcal{R}}) = \text{the arithmetic points on } \tilde{\mathcal{R}} \text{ (see (1.11)).}$$

b. Let  $\mathbf{T}^{\pm}$  denote the  $\pm$  eigenmodules of  $\iota$ .

- c. The *universal ordinary  $p$ -stabilized newform* of tame conductor  $N$  is defined to be the formal  $q$ -expansion  $\mathbf{f} \in \mathcal{R}[[q]]$  given by  $\mathbf{f} = \sum_{n=1}^{\infty} a_n q^n$  where the coefficients are given by  $a_n = h(T_n)$  for each integer  $n > 0$ .
- d. The *universal ordinary  $p$ -adic Galois representation* of tame conductor  $N$  is defined to be the representation  $\rho: G_{\mathbf{Q}} \rightarrow \text{Aut}_{\mathcal{R}}(\mathbf{T})$ .

(2.5) **Definition. a.** We will say that an eigenform  $f \in \mathcal{L}_k(\Gamma_1(Np^m), \bar{\mathbf{Q}}_p)$  ( $k \geq 2$ ) is *ordinary* if the eigenvalue  $a_p$  of  $T_p$  on  $f$  is a unit, that is, if  $|a_p|_p = 1$ .

**b.** An ordinary eigenform  $f$  will be called a  *$p$ -stabilized newform* of tame conductor  $N$  if it is normalized (its leading Fourier coefficient is 1) and the following two conditions hold.

- (1) The conductor of  $f$  is divisible by  $N$ .
- (2) The level of  $f$  is divisible by  $p$ .

It is not hard to see that an ordinary  $p$ -stabilized newform  $f$  is either already a newform, or is related to a newform  $g$  of conductor  $N$  as described in (0.7). In the latter case,  $f$  has level  $Np$ .

(2.6) **Theorem.** Let  $p$  be a prime  $\geq 5$  and suppose  $p \nmid N$ . Let  $r$  be the number of ordinary  $p$ -stabilized newforms of tame conductor  $N$  in  $\mathcal{L}_2(\Gamma_1(Np), \bar{\mathbf{Q}}_p)$ .

**a.** (Hida) The  $\mathcal{L}$ -algebra  $\mathcal{K} = \mathcal{R} \otimes_A \mathcal{L}$  is a finite product of finite field extensions of  $\mathcal{L}$  ([H2], Thm. 3.5). Moreover,  $\dim_{\mathcal{L}} \mathcal{K} = r$ . For each  $\kappa \in \mathcal{K}^{\text{arith}}$ , the localization  $\mathcal{R}_{(\kappa)}$  of  $\mathcal{R}$  at  $\kappa$  is a discrete valuation ring which is unramified over  $A$  ([H1], Cor. 1.4).

**b.** (Hida) The map  $\kappa \mapsto \mathbf{f}_{\kappa}$  establishes a one-one correspondence

$$\left\{ \begin{array}{c} \text{Arithmetic points} \\ \text{on } \mathcal{R} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Ordinary } p\text{-stabilized newforms} \\ \text{of tame conductor } N \end{array} \right\}.$$

**c.** (Hida) The  $A$ -modules  $\mathbf{T}^{\pm}$  are free of rank  $r$ . As  $\mathcal{K}$ -modules,  $\mathbf{T}_{\mathcal{L}}^{\pm} = \mathbf{T}^{\pm} \otimes_A \mathcal{L}$  are free of rank one. We may therefore regard  $\rho$  as a two-dimensional Galois representation over  $\mathcal{K}$ . This representation is unramified outside  $Np$  and for each prime  $l$  outside  $Np$ , the characteristic polynomial of  $\rho(\text{Frob}_l)$  is  $X^2 - a_l X + l\eta(l)$  where  $\eta: \mathbf{Z}_{p,N}^* \rightarrow \mathcal{R}^*$  is the canonical character (2.4) a.

**d.** (Mazur, Wiles) ([Mz-W], and [W] Theorem 2.2.2) Let  $\varphi: G_{\mathbf{Q}_p} \rightarrow \mathcal{R}^*$  be the unramified character for which  $\varphi(\text{Frob}_p) = a_p$ . Let  $\chi_0: G_{\mathbf{Q}_p} \rightarrow \mathbf{Z}_p^*$  be the cyclotomic character and let  $\eta: G_{\mathbf{Q}_p} \rightarrow \mathbf{Z}_{p,N}^* \subseteq \mathcal{R}^*$  be the Galois character associated to the canonical character  $\eta$  (2.4) a by class field theory (i.e. compose  $\eta$  with the homomorphism  $G_{\mathbf{Q}_p} \rightarrow \mathbf{Z}_{p,N}^*$  induced by the action on  $Np^{\infty}$  roots of unity). Then, as a  $G_{\mathbf{Q}_p}$ -module,  $\mathbf{T}_{\mathcal{L}}$  has a filtration

$$0 \rightarrow \mathcal{K}(\chi_0 \eta \varphi^{-1}) \rightarrow \mathbf{T}_{\mathcal{L}} \rightarrow \mathcal{K}(\varphi) \rightarrow 0.$$

These results can be interpreted analytically as follows. For each arithmetic point  $\kappa \in \mathcal{X}^{\text{arith}}$  of weight  $k_0$  and character  $\varepsilon$ , let  $\mathcal{A}(\kappa)$  be the subring of  $\bar{\mathbf{Q}}_p[[x - k_0]]$  consisting of formal power series in  $x - k_0$  with a positive radius of convergence and let  $\mathcal{M}(\kappa)$  be the field of fractions of  $\mathcal{A}(\kappa)$ . We endow  $\mathcal{A}(\kappa)$  with an  $\mathcal{R}_{(\kappa)}$ -structure  $\tilde{\kappa}: \mathcal{R} \rightarrow \mathcal{A}(\kappa)$  as follows. On the image of  $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$  in  $\mathcal{R}$  we define  $\tilde{\kappa}$  by associating to each  $t \in \mathbf{Z}_p^*$  the power series in  $\mathcal{A}(\kappa)$  representing the analytic function  $k \mapsto \varepsilon \omega^{k_0}(t) \langle t \rangle^k$ , where  $\omega$  is the Teichmüller character and

$\langle \rangle$  is projection to the principal units  $1+p\mathbf{Z}_p$ . Now it is well-known that the ring of convergent power series is Henselian ([N], Thm. 45.5) and, from (2.6) we have  $\mathcal{R}_{(\kappa)}$  is unramified over  $A$ . Hence there is a unique extension of this map to a morphism

$$(2.7) \quad \tilde{\kappa}: \mathcal{R}_{(\kappa)} \rightarrow \mathcal{A}(\kappa)$$

such that  $\tilde{\kappa}(a)(k_0) = \kappa(a)$  for every  $a \in \mathcal{R}$ . Moreover, we can extend  $\tilde{\kappa}$  by linearity to an  $\mathcal{L}$ -homomorphism  $\tilde{\kappa}: \mathcal{H} \rightarrow \mathcal{M}(\kappa)$ . We define the *domain of convergence* about  $\kappa$  to be the intersection of the disks of convergence of the power series  $\tilde{\kappa}(a) \in \mathcal{M}(\kappa)$  where  $a$  ranges over  $\mathcal{R}$ . Since  $\mathcal{R}$  is finite over  $A$ ,  $U_\kappa$  is an open disk centered at  $k_0$ .

(2.8) **Notational Conventions.** Let  $\kappa$  be an arithmetic point on  $\mathcal{R}$  and let  $a \in \mathcal{H}$ .

a. Define  $a(\kappa, k)$  to be the meromorphic function of  $k$  about  $k_0$  represented by  $\tilde{\kappa}(a) \in \mathcal{M}(\kappa)$ . For each  $k \in U_\kappa$ , let  $\kappa^{(k)} \in \mathcal{X}$  be the point defined by  $\kappa^{(k)}(a) = a(\kappa, k)$  for  $a \in \mathcal{H}$ .

b. We say that  $a$  is regular at  $\kappa$  if  $a(\kappa, k)$  does not have a pole at  $k = k_0$ . In that case, we will write  $a(\kappa)$ ,  $a'(\kappa)$  for the value, respectively the derivative, of  $a(\kappa, k)$  at  $k = k_0$ .

For each  $\kappa \in \mathcal{X}^{\text{arith}}$  and  $k \in U_\kappa$ , we let  $\mathbf{f}_{\kappa, k}$  denote the specialization  $\Sigma a_n(\kappa, k) q^n$  of  $\mathbf{f}$  to  $\kappa^{(k)}$ . As a function of  $k \in U_\kappa$  this is an analytic family of formal  $q$ -expansions which interpolates the  $q$ -expansions of ordinary  $p$ -stabilized newforms at integers  $k \geq 2$  in  $U_\kappa$ . Similarly, we can specialize  $\rho$  to obtain an analytic family  $\rho_{\kappa, k}$ ,  $k \in U_\kappa$ , of Galois representations interpolating the Galois representations associated to the forms  $\mathbf{f}_{\kappa, k}$  at integers  $k \geq 2$  in  $U_\kappa$ .

We close this section by describing an involution on  $\mathcal{H}$  which will play an important role later.

(2.9) **Proposition.** *Conjugation by  $W_N$  in  $\text{End}_{\mathcal{F}}(\mathbf{T}_{\mathcal{F}})$  preserves the subalgebra  $\mathcal{H}$  and induces an involution  $*$  on  $\mathcal{H}$  satisfying the following properties.*

- a.  $[t]_N^* = [t]_N^{-1}$  for all  $t \in \Delta_N$ ; and
- b.  $a_l^* = [l]_N^{-1} a_l$  for all primes  $l \nmid N$ .

*Proof.* Since, by Atkin-Lehner theory, the actions of  $\mathcal{H}$  and of  $W_N \mathcal{H} W_N^{-1}$  commute with one another on the  $N$ -primitive part of  $Ta_p(J_n)$  for each  $n$ , they also commute on  $\mathbf{T}$ . In particular,  $W_N \mathcal{H} W_N^{-1}$  centralizes  $\mathcal{H}$  in  $\text{End}_{\mathcal{F}}(\mathbf{T}_{\mathcal{F}})$ . Since  $\mathbf{T}_{\mathcal{F}}^{\pm}$  are free of rank one as  $\mathcal{H}$ -modules and are preserved by  $W_N \mathcal{H} W_N^{-1}$ , there are involutions  $i_{\pm}: \mathcal{H} \rightarrow \mathcal{H}$  over  $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$  such that, for every  $a \in \mathcal{H}$ ,  $i_{\pm}(a) = W_N a W_N^{-1}$  on  $\mathbf{T}_{\mathcal{F}}^{\pm}$ . By (1.8), each of these involutions satisfies properties a and b of the proposition. We need to show  $i_+ = i_-$ . Let  $\kappa \in \mathcal{X}^{\text{arith}}$  be an arbitrary arithmetic point on  $\mathcal{R}$ , and let  $\kappa_{\pm} = \kappa \circ i_{\pm}$ . Then  $\mathbf{f}_{\kappa_+}$  and  $\mathbf{f}_{\kappa_-}$  are ordinary  $p$ -stabilized newforms which, according to b, have identical eigenvalues for the Hecke operators  $T_n$ ,  $(n, N) = 1$ . Hence, by the strong multiplicity one theorem, we have  $\mathbf{f}_{\kappa_+} = \mathbf{f}_{\kappa_-}$ . By Theorem 2.6b we conclude that  $\kappa_+ = \kappa_-$  and, since  $\kappa$  was arbitrary, that  $i_+ = i_-$ . This completes the proof.

The involution  $*$  on  $\mathcal{H}$  induces an involution on  $\mathcal{X}$  which we will denote  $\kappa \mapsto \kappa^*$ . If  $\kappa$  is an arithmetic point of weight  $k_0$  and Nebentype character  $\varepsilon = \varepsilon_N \varepsilon_p$ , then  $\kappa^*$  is an arithmetic point of the same weight  $k_0$ , but with character  $\varepsilon_N^{-1} \varepsilon_p$ .

Moreover, for any  $a \in \mathcal{K}$  we have the following identity of meromorphic functions in a neighborhood of  $k = k_0$ :

$$(2.10) \quad a^*(\kappa^*, k) = a(\kappa, k).$$

(2.11) **Example.** In the example of the introduction, where  $E = X_0(11)$ ,  $p = 11$ , and  $N = 1$ , the space  $\mathcal{S}_2(\Gamma_1(11))$  is one dimensional and is spanned by  $f_E$ . Since  $f_E$  is ordinary at  $p = 11$  we have  $r = 1$  in Theorem 2.6. Hence the universal ordinary Hecke algebra (2.4)a is given by  $\mathcal{H} = A$ . Moreover, we have  $\mathbf{T} = Ta_p(J_\infty)_{\text{prim}}^0 = Ta_p(J_\infty)^0$ . If we set

$$\mathbf{f} = \sum a_n q^n \in A[[q]], \quad \rho: G_{\mathbf{Q}} \rightarrow \text{Aut}_A(\mathbf{T})$$

as in (2.4)c, d, then  $\mathbf{f}$ , and  $\rho$  are the deformations of  $f_E$  and  $Ta_p(E)$ , respectively, whose properties are described in (0.5). For each integer  $k \geq 2$  let  $\mathbf{f}_k$  (respectively,  $\rho_k$ ) be the specialization of  $\mathbf{f}$  (respectively,  $\rho$ ) to weight  $k$  and trivial character and let  $f_k$  be the associated normalized newform. The assertions (0.5) are then immediate consequences of Theorem 2.6. The precise prescription of the conductor of each  $f_k$  given in (0.5)b follows from [A-L]. Indeed, in [A-L] it is proven that if  $f$  is a newform of weight  $k \geq 2$  with prime level  $p$  and trivial character, then  $a_p(f) = \pm p^{\frac{k-2}{2}}$ . Hence  $f$  can be ordinary at  $p$  only if  $k = 2$ .

### 3. The $\mathfrak{Q}$ -invariant

*The  $\mathfrak{Q}$ -invariant of an abelian variety with split multiplicative reduction*

Let  $p$  be a rational prime and let  $A_{/\mathbf{Q}_p}$  be an abelian variety over  $\mathbf{Q}_p$  with split multiplicative reduction. Then the dual abelian variety  $B_{/\mathbf{Q}_p}$  also has split multiplicative reduction. Let  $X, Y$  be the character groups of  $B_{\mathbf{F}_p}^0, A_{\mathbf{F}_p}^0$ , respectively. Then  $X$  and  $Y$  are free abelian groups of rank  $\dim(A)$  on which the local Galois group  $G_{\mathbf{Q}_p} = \text{Gal}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$  acts trivially. From the theory of  $p$ -adic uniformization [M<sup>c</sup>C, Mo] we obtain a bi-multiplicative pairing

$$(3.1) \quad j: X \times Y \rightarrow \mathbf{Q}_p^*$$

and exact sequences of  $G_{\mathbf{Q}_p}$ -modules

$$(3.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{j} & \text{Hom}(Y, \bar{\mathbf{Q}}_p^*) & \longrightarrow & A(\bar{\mathbf{Q}}_p) \longrightarrow 0 \\ 0 & \longrightarrow & Y & \xrightarrow{j} & \text{Hom}(X, \bar{\mathbf{Q}}_p^*) & \longrightarrow & B(\bar{\mathbf{Q}}_p) \longrightarrow 0 \end{array}$$

where the maps labeled  $j$  are induced by (3.1). Moreover, the pairing  $\alpha = \text{ord}_p \circ j: X \times Y \rightarrow \mathbf{Z}$  is nondegenerate. Let  $X_p = X \otimes \mathbf{Q}_p, Y_p = Y \otimes \mathbf{Q}_p$  and let  $\alpha_p$  be the non-degenerate pairing of  $\mathbf{Q}_p$ -vector spaces induced by  $\alpha$ :

$$(3.3) \quad \alpha_p: X_p \times Y_p \rightarrow \mathbf{Q}_p.$$

Recall that  $\log_p: \mathbf{Q}_p^* \rightarrow \mathbf{Z}_p$  is the unique group homomorphism for which  $\log_p p = 0$  and  $\log_p(1+x) = x - x^2/2 + x^3/3 - \dots + (-1)^{n+1} x^n/n + \dots$  whenever  $x \in p\mathbf{Z}_p$ . The composition of  $j$  with  $\log_p$  induces another pairing

$$(3.4) \quad \beta_p: X_p \times Y_p \rightarrow \mathbf{Q}_p.$$

(3.5) **Definition.** The  $\mathfrak{L}$ -invariant of  $A$  is the  $\mathbf{Q}_p$ -endomorphism  $\mathfrak{L}_p(A): X_p \rightarrow X_p$  for which  $\beta_p(x, y) = \alpha_p(\mathfrak{L}_p(A)x, y)$  for all  $x \in X_p, y \in Y_p$ .

(3.6) **Example.** If  $A$  is an elliptic curve then  $A$  is canonically isomorphic to its dual abelian variety. Hence we may take  $A = B$  and  $X = Y$ . Moreover,  $X$  is free of rank one over  $\mathbf{Z}$ . Tate's multiplicative period  $q_A$  is given by  $q_A = j(x_0, x_0)$  where  $j$  is the pairing (3.1) and  $x_0$  is a generator of  $X = Y$ . Hence  $\beta_p(x_0, x_0) = \log_p(q_A)$ ,  $\alpha(x_0, x_0) = \text{ord}_p(q_A)$  and it follows that the  $\mathfrak{L}$ -invariant defined in (3.5) agrees with the  $\mathfrak{L}$ -invariant defined in the introduction for elliptic curves over  $\mathbf{Q}_p$  with split multiplicative reduction.

Returning now to the general case we will show how the  $\mathfrak{L}$ -invariant can also be described in terms of the  $p$ -adic Galois representation associated to  $A$ . Indeed, we will generalize the above definition by first introducing the notion of a *split multiplicative* Galois representation and then associating an  $\mathfrak{L}$ -invariant to an arbitrary split multiplicative Galois representation.

*The  $\mathfrak{L}$ -invariant of a split multiplicative representation*

Let  $\text{art}: \mathbf{Q}_p^* \rightarrow G_{\mathbf{Q}_p}^{ab}$  be the Artin symbol, where we observe the conventions of [Ser]. Thus, if  $\chi_0$  is the cyclotomic character then  $\chi_0(\text{art}(u)) = u$  for all  $u \in \mathbf{Z}_p^*$ . We will write  $\text{Frob}_p$  for  $\text{art}(p)^{-1}$ . This is a lifting to  $G_{\mathbf{Q}_p}^{ab}$  of the Frobenius element on the maximal unramified extension of  $\mathbf{Q}_p$ . If  $W$  is a finite dimensional vector space over a finite extension  $K$  of  $\mathbf{Q}_p$  and if  $W$  is equipped with the trivial action of  $G_{\mathbf{Q}_p}$ , then there is a canonical isomorphism  $H^1(W) \cong \text{Hom}(G_{\mathbf{Q}_p}^{ab}, W)$  of  $K$ -vector spaces. Moreover, for any nontrivial principal unit  $u \in 1 + p\mathbf{Z}_p$  the map

$$H^1(W) \rightarrow W \times W$$

$$\xi \mapsto \left( \xi(\text{Frob}_p), \frac{1}{\log_p u} \xi(\text{art}(u)) \right)$$

is an isomorphism whose definition is independent of the choice of  $u$ . The space  $H^1(W)$  therefore decomposes into a corresponding direct sum

$$H^1(W) = H^1(W)_{\text{unr}} \oplus H^1(W)_{\text{cyc}}$$

where  $H^1(W)_{\text{unr}}$  is the space of unramified homomorphisms and  $H^1(W)_{\text{cyc}}$  is the space of homomorphisms which factor through the basic cyclotomic  $\mathbf{Z}_p$ -extension of  $\mathbf{Q}_p$ . Let

$$(3.7) \quad \lambda_{\text{unr}}: W \rightarrow H^1(W)_{\text{unr}} \quad \text{and} \quad \lambda_{\text{cyc}}: W \rightarrow H^1(W)_{\text{cyc}}$$

be the induced linear isomorphisms. Hence for each  $w \in W$ ,  $\lambda_{\text{unr}}(w)$  is the unique unramified homomorphism for which  $\lambda_{\text{unr}}(w)(\text{Frob}_p) = w$  and  $\lambda_{\text{cyc}}(w)$  is the

unique cyclotomic homomorphism for which  $\lambda_{\text{cyc}}(w)(\text{art}(u)) = \log_p(u) \cdot w$  for every  $u \in \mathbf{Z}_p^*$ .

(3.8) **Definition.** A finite dimensional  $G_{\mathbf{Q}_p}$ -representation  $V$  over  $K$  will be called split multiplicative if the following conditions are satisfied.

**a.** There is an exact sequence of  $G_{\mathbf{Q}_p}$ -representations

$$0 \rightarrow V^0(1) \rightarrow V \rightarrow V^{et} \rightarrow 0$$

where  $G_{\mathbf{Q}_p}$  acts trivially on  $V^{et}$  and  $V^0$  (hence via the cyclotomic character on  $V^0(1)$ ).

**b.** The degree one coboundary map  $\delta: H^1(V^{et}) \rightarrow H^2(V^0(1))$  associated to the long exact cohomology sequence of **a** induces an isomorphism  $\delta: H^1(V^{et})_{\text{unr}} \xrightarrow{\sim} H^2(V^0(1))$ .

Since  $H^2(V^0(1))$  is canonically isomorphic to  $V^0$ , the composition of  $\delta$  with  $\lambda_{\text{unr}}$  and  $\lambda_{\text{cyc}}$  (3.7) gives rise to maps  $\delta_{\text{unr}}, \delta_{\text{cyc}}: V^{et} \rightarrow V^0$ . Condition **b** of (3.8) is equivalent to the assertion that  $\delta_{\text{unr}}$  is an isomorphism. In particular we see that a split multiplicative representation must be even dimensional.

(3.9) **Definition.** Let  $V$  be a split multiplicative  $G_{\mathbf{Q}_p}$ -representation. Then the  $\mathfrak{L}$ -invariant of  $V$  is defined to be the endomorphism  $\mathfrak{L}_p(V) \in \text{End}_K(V^{et})$  given by

$$\mathfrak{L}_p(V) = -\delta_{\text{unr}}^{-1} \circ \delta_{\text{cyc}}.$$

To compare the definitions (3.5) and (3.9) we will need some well known facts from Kummer theory and Tate duality theory. For each integer  $n \geq 0$ , let  $\gamma^{(n)}: \mathbf{Q}_p^* \rightarrow H^1(\mu_{p^n})$  be the Kummer homomorphism. This sends  $q \in \mathbf{Q}_p^*$  to the cohomology class  $\gamma_q^{(n)} \in H^1(\mu_{p^n})$  represented by the 1-cocycle which sends  $\sigma \in G_{\mathbf{Q}_p}$  to  $(q^{1/p^n})^{\sigma-1}$  where  $q^{1/p^n}$  is a fixed choice of a  $p^n$ -th root of  $q$  in  $\overline{\mathbf{Q}_p^*}$ . The family  $\{\gamma_q^{(n)}\}_{n \geq 0}$  corresponds to an element of  $\varinjlim_n H^1(\mu_{p^n}) = H^1(\mathbf{Z}_p(1))$ . Let  $\gamma_q$  denote

the image of this element in  $H^1(\mathbf{Z}_p(1)) \otimes \mathbf{Q}_p = H^1(\mathbf{Q}_p(1))$ . Then the map  $q \mapsto \gamma_q$  defines a continuous group homomorphism,  $\gamma: \mathbf{Q}_p^* \rightarrow H^1(\mathbf{Q}_p(1))$  whose image spans  $H^1(\mathbf{Q}_p(1))$ .

To a finite dimensional continuous  $\mathbf{Q}_p[G_{\mathbf{Q}_p}]$ -representation  $V$  we associate the contragredient representation  $V^*$ . From Tate duality we know that cup product induces perfect pairings

$$H^i(V) \times H^{2-i}(V^*(1)) \rightarrow \mathbf{Q}_p = H^2(\mathbf{Q}_p(1))$$

for  $i=0, 1, 2$ . In the important special case when  $V = \mathbf{Q}_p$  and  $i=1$ , the pairing is explicitly given by  $(\xi, \gamma_q) \mapsto \xi(\text{art}(q))$  for  $\xi \in H^1(\mathbf{Q}_p)$  and  $\gamma_q \in H^1(\mathbf{Q}_p(1))$ . It follows that the transposes of the maps  $\lambda_{\text{unr}}: \mathbf{Q}_p \rightarrow H^1(\mathbf{Q}_p)$  and  $\lambda_{\text{cyc}}: \mathbf{Q}_p \rightarrow H^1(\mathbf{Q}_p)$  are given by

$$(3.10) \quad \begin{array}{ccc} H^1(\mathbf{Q}_p(1)) & \xrightarrow{\lambda_{\text{unr}}^*} & \mathbf{Q}_p \\ \gamma_q \longmapsto & \text{ord}_p(q) & \end{array} \quad \text{and} \quad \begin{array}{ccc} H^1(\mathbf{Q}_p(1)) & \xrightarrow{\lambda_{\text{cyc}}^*} & \mathbf{Q}_p \\ \gamma_q \longmapsto & \log_p(q). & \end{array}$$

(3.11) **Theorem.** Let  $Ta_p(A)$  be the Tate module of an abelian variety  $A/\mathbf{Q}_p$  with split multiplicative reduction and let  $V = Ta_p(A) \otimes \mathbf{Q}_p$ . Then  $V$  is a split multiplicative Galois representation and  $\mathfrak{L}_p(A) = \mathfrak{L}_p(V)$ .

*Proof.* For each integer  $n \geq 0$  a simple application of the snake lemma to the  $p^n$ -power map acting on the exact sequences (3.2) gives us exact sequences

$$\begin{aligned} 0 &\longrightarrow \text{Hom}(Y, \mu_{p^n}) \xrightarrow{t_n} A[p^n] \xrightarrow{\delta} X/p^n X \longrightarrow 0 \\ 0 &\longrightarrow \text{Hom}(X, \mu_{p^n}) \xrightarrow{t_n} B[p^n] \xrightarrow{\delta} Y/p^n Y \longrightarrow 0. \end{aligned}$$

Passing to the projective limit over  $n$ , and tensoring with  $\mathbf{Q}_p$  we obtain the exact sequences

$$\begin{aligned} 0 &\rightarrow Y_p^*(1) \rightarrow V \rightarrow X_p \rightarrow 0 \\ 0 &\rightarrow X_p^*(1) \rightarrow V^* \rightarrow Y_p \rightarrow 0. \end{aligned}$$

These sequences are  $\mathbf{Q}_p(1)$ -dual to each other, hence Tate duality gives us a perfect pairing between the associated long exact cohomology sequences. In particular, the coboundary map  $\delta: H^1(X_p) \rightarrow H^2(Y_p^*(1)) = Y_p^*$  of degree one associated to the first of these sequences is the transpose of the coboundary map  $\delta^*: H^0(Y_p) \rightarrow H^1(X_p^*(1))$  of degree zero associated to the second sequence. If we identify  $H^0(Y_p)$  with  $Y_p$  and  $H^1(X_p^*(1))$  with  $X_p^* \otimes H^1(\mathbf{Q}_p(1))$  in the natural way, then a simple calculation shows that  $\delta^*$  is given by

$$(3.12) \quad \begin{aligned} \delta^*: Y_p &\rightarrow X_p^* \otimes H^1(\mathbf{Q}_p(1)) \\ y &\mapsto (x \mapsto \gamma_{j(x, y)}). \end{aligned}$$

Let  $\lambda_{\text{unr}}, \lambda_{\text{cyc}}: X_p \rightarrow H^1(X_p)$  be as in (3.7) and define  $\delta_{\text{unr}}, \delta_{\text{cyc}}: X_p \rightarrow Y_p^*$  by setting  $\delta_{\text{unr}} = \delta \circ \lambda_{\text{unr}}$  and  $\delta_{\text{cyc}} = \delta \circ \lambda_{\text{cyc}}$ . We will show that  $\delta_{\text{unr}}$  and  $\delta_{\text{cyc}}$  induce the pairings  $-\alpha_p$  (3.3) and  $\beta_p$  (3.4). Indeed, their duals are given by  $\delta_{\text{unr}}^* = \lambda_{\text{unr}}^* \circ \delta^*$  and  $\delta_{\text{cyc}}^* = \lambda_{\text{cyc}}^* \circ \delta^*$ , where the maps  $\lambda_{\text{unr}}^*, \lambda_{\text{cyc}}^*: H^1(X_p^*(1)) = X_p^* \otimes H^1(\mathbf{Q}_p(1)) \rightarrow X_p^*$  are induced by (3.10). Hence, using (3.12) we find that  $\delta_{\text{unr}}^*, \delta_{\text{cyc}}^*: Y_p \rightarrow X_p^*$  are given by  $\delta_{\text{unr}}^*(y)(x) = -\text{ord}_p(j(x, y)) = -\alpha_p(x, y)$  and  $\delta_{\text{cyc}}^*(y)(x) = \log_p(j(x, y)) = \beta_p(x, y)$ . By duality we therefore have

$$\delta_{\text{unr}}(x)(y) = -\alpha_p(x, y) \quad \text{and} \quad \delta_{\text{cyc}}(x)(y) = \beta_p(x, y)$$

for all  $x \in X_p$  and  $y \in Y_p$ . From the nondegeneracy of  $\alpha_p$  we see that  $\delta_{\text{unr}}$  is an isomorphism, hence  $V$  is split multiplicative. The above identities together with the definition of  $\mathfrak{Q}_p(A)$  (3.5) imply  $\delta_{\text{unr}} \circ \mathfrak{Q}_p(A) = -\delta_{\text{cyc}}$ . Hence, by the definition of  $\mathfrak{Q}_p(V)$  (3.9) we conclude that  $\mathfrak{Q}_p(A) = -\delta_{\text{unr}}^{-1} \circ \delta_{\text{cyc}} = \mathfrak{Q}_p(V)$  and the theorem is proved.

### The $\mathfrak{Q}$ -invariant and infinitesimal deformations

The  $\mathfrak{Q}$ -invariant exerts a strong influence on the  $p$ -adic deformations of a split multiplicative representation. To explain this, we begin with a few remarks about infinitesimal deformations. Let  $\tilde{\mathbf{Q}}_p = \mathbf{Q}_p[T]/T^2$ .

(3.13) **Definition.** An infinitesimal deformation of a  $\mathbf{Q}_p[G_{\mathbf{Q}_p}]$ -module  $W$  is a  $\tilde{\mathbf{Q}}_p[G_{\mathbf{Q}_p}]$ -module  $\mathbf{W}$  such that  $\mathbf{W}/T\mathbf{W} \cong W$ .

If  $W$  is a trivial  $\mathbf{Q}_p[G_{\mathbf{Q}_p}]$ -module, we let  $\tilde{W} = W \otimes \tilde{\mathbf{Q}}_p$  denote the trivial infinitesimal deformation of  $W$ . If  $\psi: G_{\mathbf{Q}_p} \rightarrow \text{Aut}_{\mathbf{Q}_p}(\tilde{W})$  is a continuous Galois representation then  $\tilde{W}(\psi)$  will denote  $\tilde{W}$  with  $G_{\mathbf{Q}_p}$  acting via  $\psi$ . Clearly,  $\tilde{W}(\psi)$  is an infinitesimal deformation of  $W$  if and only if  $\psi \equiv I$  modulo  $T$ . Assume now that  $\psi$  is such a character. Then  $\psi$  factors through  $G_{\mathbf{Q}_p}^{ab}$  and after differentiation with respect to  $T$  gives rise to an additive homomorphism

$$\frac{d\psi}{dT}: G_{\mathbf{Q}_p}^{ab} \rightarrow \text{End}(W)$$

for which  $\psi = I + \frac{d\psi}{dT} \cdot T$ .

(3.14) **Theorem.** *Let  $V$  be a split multiplicative Galois representation and fix a homomorphism  $\psi: G_{\mathbf{Q}_p} \rightarrow \text{Aut}_{\mathbf{Q}_p}(\tilde{V}^{et})$  and an exact sequence  $0 \rightarrow \tilde{V}^0(1) \rightarrow \mathbf{V} \rightarrow \tilde{V}^{et}(\psi) \rightarrow 0$  whose terms are infinitesimal deformations of the terms in the exact sequence (3.8)a. Then for any nontrivial principal unit  $u \in 1 + p\mathbf{Z}_p$*

$$\frac{d\psi}{dT}(\text{Frob}_p) = \mathfrak{L}_p(V) \circ \frac{1}{\log_p(u)} \frac{d\psi}{dT}(\text{art}(u)).$$

*Proof.* Multiplication by  $T$  gives rise to a Galois equivariant map  $V^{et} \rightarrow \tilde{V}^{et}(\psi)$  whose cokernel is  $V^{et}$ . Thus we obtain an exact sequence

$$(3.15) \quad 0 \rightarrow V^{et} \rightarrow \tilde{V}^{et}(\psi) \rightarrow V^{et} \rightarrow 0.$$

Let  $\delta_\psi: V^{et} = H^0(V^{et}) \rightarrow H^1(V^{et})$  be the degree zero coboundary map of the associated long exact cohomology sequence. A simple calculation shows that for each  $x \in V^{et}$ ,  $\delta_\psi(x)$  is the homomorphism whose value on an element  $\sigma \in G_{\mathbf{Q}_p}$  is given by  $\delta_\psi(x)(\sigma) = \frac{d\psi}{dT}(\sigma)(x)$ . Hence the homomorphism  $\delta_\psi: V^{et} \rightarrow H^1(V^{et})$  is given by

$$(3.16) \quad \delta_\psi = \lambda_{\text{unr}} \circ \frac{d\psi}{dT}(\text{Frob}_p) + \lambda_{\text{cyc}} \circ \frac{1}{\log_p(u)} \frac{d\psi}{dT}(\text{art}(u)).$$

Now consider the following diagram.

$$\begin{array}{ccccccc}
 \longrightarrow & H^1(\mathbf{V}) & \longrightarrow & H^1(\tilde{V}^{et}) & \longrightarrow & H^2(\tilde{V}^0(1)) & \longrightarrow & 0 \\
 & \uparrow & & \uparrow & & \uparrow & & \\
 \longrightarrow & H^1(V) & \longrightarrow & H^1(V^{et}) & \xrightarrow{\delta} & H^2(V^0(1)) & \longrightarrow & 0 \\
 & & & \uparrow & & & & \\
 & & & H^0(V^{et}) \cong V^{et} & & & & 
 \end{array}$$

Since  $i_2 \circ \delta_\psi = 0$  and the diagram is commutative, we have  $i_1 \circ \delta \circ \delta_\psi = 0$ . By a simple application of Tate-duality it follows that  $i_1$  is injective. Hence  $\delta \circ \delta_\psi = 0$ . The theorem now follows by composing  $\delta$  on the left with (3.16), and using the identities  $\delta_{\text{unr}} = \delta \circ \lambda_{\text{unr}}$ ,  $\delta_{\text{cyc}} = \delta \circ \lambda_{\text{cyc}}$ , and  $\mathfrak{L}_p(V) = -\delta_{\text{unr}}^{-1} \circ \delta_{\text{cyc}}$ .

*The  $\mathfrak{L}$ -invariant of a split multiplicative newform*

Let  $f$  be a classical weight two newform. Let  $K_f$  be the completion in  $\overline{\mathbf{Q}}_p$  of the field of Hecke eigenvalues and let  $V_f$  be the two dimensional  $G_{\mathbf{Q}}$ -representation over  $K_f$  constructed by Deligne [D]. We will say that  $f$  is split multiplicative at  $p$  if  $V_f$  is split multiplicative at  $p$ . From the work of Deligne and Rapoport [D-R] we know that  $f$  is split multiplicative at  $p$  if and only if (1) the conductor of  $f$  is  $Np$  where  $p \nmid N$  and (2)  $f|_{T_p} = f$ . In particular, these two conditions show that if  $f$  is split multiplicative at  $p$  then  $f$  is an ordinary  $p$ -stabilized newform of tame conductor  $N$ .

**(3.17) Definition.** The  $\mathfrak{L}$ -invariant of a weight two split multiplicative newform  $f$  is defined to be the  $\mathfrak{L}$ -invariant of its Galois representation  $V_f$ . Hence  $\mathfrak{L}_p(f) = \mathfrak{L}_p(V_f) \in K_f$ .

**(3.18) Theorem.** Let  $f$  be a weight two newform of conductor  $Np$  with  $p \geq 5$ , and suppose  $f$  is split multiplicative at  $p$ . Let  $\mathcal{R}$  be the universal ordinary Hecke algebra of tame conductor  $N$  and let  $a_p = h(T_p) \in \mathcal{R}$ . If  $\kappa$  is the arithmetic point on  $\mathcal{R}$  which corresponds to  $f$  by Theorem 2.6 then

$$a'_p(\kappa) = -\frac{1}{2} \mathfrak{L}_p(f).$$

*Proof.* Let  $\varphi: G_{\mathbf{Q}_p} \rightarrow \mathcal{R}^*$  be the unramified character with  $\varphi(\text{Frob}_p) = a_p$ . Let  $\mathcal{R}_{(\kappa)}$  be the localization of  $\mathcal{R}$  at  $\kappa$ . From Theorem 2.6d we obtain an exact sequence

$$(3.19) \quad 0 \rightarrow \mathcal{R}_{(\kappa)}(\chi_0 \eta \varphi^{-1}) \rightarrow \mathbf{T}_{\text{prim}}^0 \otimes_{\mathcal{R}} \mathcal{R}_{(\kappa)} \rightarrow \mathcal{R}_{(\kappa)}(\varphi) \rightarrow 0.$$

Since the specialization of this to  $\kappa$  is the sequence  $0 \rightarrow V_f^0(1) \rightarrow V_f \rightarrow V_f^{et} \rightarrow 0$ , we see that  $\varphi$  and  $\eta$  are congruent to 1 modulo the maximal ideal  $P_{\kappa}$  in  $\mathcal{R}_{(\kappa)}$ . In particular we see that  $\kappa(a_p) = 1$ , that  $\eta(\text{art}(u)) = [u]_p$  for any principle unit  $u \in 1 + p\mathbf{Z}_p$ , and that  $\eta(\text{Frob}_p) = 1$ . Now tensor (3.19) with  $\varphi \eta^{-1}$  and reduce modulo  $P_{\kappa}^2$ . This gives us an exact sequence

$$0 \rightarrow \tilde{K}_f(1) \rightarrow \tilde{V} \rightarrow \tilde{K}_f(\varphi^2 \eta^{-1}) \rightarrow 0$$

where  $\tilde{V}$  is an infinitesimal deformation of  $V_f$ . Since  $\varphi \eta^{-1} \equiv 1$  modulo  $P_{\kappa}$ , we may apply Theorem 3.14 with  $\psi = \varphi^2 \eta^{-1}$  to obtain the identity

$$(3.20) \quad \psi(\text{Frob}_p)'(\kappa) = \mathfrak{L}_p(f) \cdot \frac{\psi(\text{art}(u))'(\kappa)}{\log_p(u)}$$

for any nontrivial principal unit  $u \in 1 + p\mathbf{Z}_p$ . But  $\varphi(\text{Frob}_p) = a_p$ ,  $\eta(\text{Frob}_p) = \varphi(\text{art}(u)) = 1$ , and  $\eta(\text{art}(u)) = [u]_p \in \mathcal{A}$ . Hence  $\psi(\text{Frob}_p) = a_p^2$  and  $\psi(\text{art}(u)) = [u]_p^{-1}$ . The theorem now follows from (3.20) and the simple fact that  $[u]_p'(\kappa) = \log_p(u)$ .

### 4. Modular symbols and special values of $L$ -functions

In this section we review the basic definitions and properties of modular symbols. Modular symbols will be useful to us in two ways. First of all they give us a concrete realization of the one-dimensional compactly supported cohomology groups of a congruence subgroup of  $SL_2(\mathbf{Z})$  as well as a concrete description of the action of the Hecke operators on these groups. On the other hand they provide a powerful tool for studying the arithmetic properties of critical values of  $L$ -functions associated to modular forms. We will recall how modular symbols are used to attach one-variable  $p$ -adic  $L$ -functions to  $p$ -stabilized ordinary newforms as in [Mz-T-T].

#### Definitions and first properties

Fix a commutative ring  $R$  and let  $A$  be a contravariant  $R[\Sigma]$ -module where  $\Sigma = M_2(\mathbf{Z}) \cap GL_2(\mathbf{Q})$  as in “section 1”.

(4.1) **Definition.** Let  $\mathcal{D} \stackrel{\text{def}}{=} \text{Div}(\mathbf{P}^1(\mathbf{Q}))$  denote the group of divisors supported on the rational cusps  $\mathbf{P}^1(\mathbf{Q}) = \mathbf{Q} \cup \{\infty\}$  of the upper half plane  $\mathbf{H}$ . Let  $\mathcal{D}_0 \subseteq \mathcal{D}$  be the subgroup of divisors of degree zero. Note that  $\Sigma$  acts by fractional linear transformations on  $\mathcal{D}$  and on  $\mathcal{D}_0$ . Let  $\Gamma$  be a congruence group.

- a. An additive homomorphism  $\Phi: \mathcal{D}_0 \rightarrow A$  will be called *modular symbol* over  $\Gamma$  if  $\Phi$  is a  $\Gamma$ -homomorphism, i.e. if  $\Phi(\gamma D)|\gamma = \Phi(D)$  for all  $D \in \mathcal{D}_0$  and  $\gamma \in \Gamma$ . We will denote the  $R$ -module of all  $A$ -valued modular symbols over  $\Gamma$  by  $\text{Symb}_\Gamma(A)$ .
- b. An  $A$ -valued *boundary symbol* over  $\Gamma$  is a  $\Gamma$ -homomorphism  $\Phi: \mathcal{D} \rightarrow A$ . We will denote the group of all  $A$ -valued boundary symbols over  $\Gamma$  by  $\text{Bound}_\Gamma(A)$ .
- c. More generally, we define

$$\text{Symb}(A) = \bigcup_{\Gamma} \text{Symb}_\Gamma(A), \quad \text{Bound}(A) = \bigcup_{\Gamma} \text{Bound}_\Gamma(A)$$

where  $\Gamma$  runs over all congruence groups. We let  $\Sigma$  act on these groups according to the formula  $\Phi|g: D \mapsto \Phi(gD)|g$ , for  $g \in \Sigma$  and  $\Phi \in \text{Symb}(A)$ ,  $D \in \mathcal{D}_0$  (respectively  $\Phi \in \text{Bound}(A)$ ,  $D \in \mathcal{D}$ ).

Our interest in  $\text{Symb}_\Gamma(A)$  and  $\text{Bound}_\Gamma(A)$  is motivated by the following theorem which allows us to relate these groups to the cohomology of  $\Gamma$  (see 4.3). Let  $\tilde{\mathbf{H}}$  be the Borel-Serre completion of  $\mathbf{H}$ . Also, let  $t(\Gamma)$  be the least common multiple of the orders of the torsion elements of  $\Gamma$ .

(4.2) **Theorem.** *Let  $\Gamma$  be a congruence subgroup of  $SL_2(\mathbf{Z})$  and suppose  $t(\Gamma)$  is invertible in  $R$ . Then the long exact cohomology sequence of the pair  $(\Gamma \backslash \tilde{\mathbf{H}}, \partial(\Gamma \backslash \tilde{\mathbf{H}}))$  with coefficients in  $A$  is isomorphic to the right-shift of the long exact  $\Gamma$ -cohomology sequence of the exact sequence  $0 \rightarrow A \rightarrow \text{Hom}_{\mathbf{Z}}(\mathcal{D}, A) \rightarrow \text{Hom}_{\mathbf{Z}}(\mathcal{D}_0, A) \rightarrow 0$ . More precisely, for each integer  $i \geq 0$  we have the following commutative diagram:*

$$\begin{array}{ccccccc} \longrightarrow & H^{i-1}(\Gamma, \text{Hom}(\mathcal{D}_0, A)) & \longrightarrow & H^i(\Gamma, A) & \longrightarrow & H^i(\Gamma, \text{Hom}(\mathcal{D}, A)) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & H^i_c(\Gamma \backslash \mathbf{H}, A) & \longrightarrow & H^i(\Gamma \backslash \mathbf{H}, A) & \longrightarrow & H^i(\partial(\Gamma \backslash \tilde{\mathbf{H}}), A) & \longrightarrow \end{array}$$

where the vertical arrows are isomorphisms.

For a proof of this, see [A-S]. Recall that the parabolic cohomology group  $H_{\text{par}}^1(\Gamma, A)$  is defined to be the image of the map  $H_c^1(\Gamma \backslash \mathbf{H}, A) \rightarrow H^1(\Gamma \backslash \mathbf{H}, A)$ . The following theorem is an immediate consequence of Theorem 4.2.

(4.3) **Theorem.** *Suppose  $t(\Gamma)$  is invertible in  $R$ . Then there is a canonical isomorphism  $\text{Symb}_\Gamma(A) \cong H_c^1(\Gamma \backslash \mathbf{H}, A)$ . Moreover, there is a canonical exact sequence*

$$0 \rightarrow H^0(\Gamma, A) \rightarrow \text{Bound}_\Gamma(A) \rightarrow \text{Symb}_\Gamma(A) \rightarrow H_{\text{par}}^1(\Gamma, A) \rightarrow 0.$$

Now let  $\Gamma$  be a congruence group and let  $D(\Gamma, \Sigma)$  be the double coset algebra associated to the pair  $(\Gamma, \Sigma)$ . The action of the algebra  $D(\Gamma, \Sigma)$  on  $A$ -valued modular symbols over  $\Gamma$  can be made explicit as follows. If  $T(g) \in D(\Gamma, \Sigma)$  is the element associated to the double coset  $\Gamma g \Gamma$ ,  $g \in \Sigma$ , then we can write  $\Gamma g \Gamma$  as a finite disjoint union of right cosets,  $\bigcup_i \Gamma g_i$ . For a modular symbol

$\Phi \in \text{Symb}_\Gamma(A)$  we then have

$$(4.4) \quad \Phi | T(g) = \sum_i \Phi | g_i \in \text{Symb}_\Gamma(A).$$

The matrix  $\iota = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \Sigma$  induces an involution  $\Phi \mapsto \Phi | \iota$  on modular symbols.

If 2 is invertible in  $R$ , then we can decompose any modular symbol  $\Phi$  in a unique way as a sum

$$(4.5) \quad \Phi = \Phi^+ + \Phi^-$$

where  $\Phi^\pm | \iota = \pm \Phi^\pm$ . Let  $\text{Symb}_\Gamma(A) = \text{Symb}_\Gamma(A)^+ \oplus \text{Symb}_\Gamma(A)^-$  be the corresponding decomposition of the space of modular symbols.

### Modular symbols of integral weight $k \geq 2$

For each non-negative integer  $r \geq 0$ , let  $\text{Sym}^r(R^2)$  denote the  $R$ -module of homogeneous polynomials of degree  $r$  in two variables  $X, Y$  with coefficients in  $R$ . We let  $\Sigma$  act on  $\text{Sym}^r(R^2)$  by the formula  $(F|g)(X, Y) = F((X, Y)g^*)$  for  $g \in \Sigma$  and  $F \in \text{Sym}^r(R^2)$ , where  $*$  is the adjoint involution defined in “section 1”.

(4.6) **Definition.** Fix an integer  $k \geq 2$ . Then the  $R[\Sigma]$ -module  $\text{Symb}(\text{Sym}^{k-2}(R^2))$  is called the module of modular symbols of weight  $k$  over  $R$ .

This terminology is motivated by the following well known example of Eichler and Shimura. Recall from “section 1” that  $\mathcal{S}_k(\mathbf{Q})$  is the space of weight  $k$  cusp forms of all levels having algebraic  $q$ -expansions and that  $\Sigma^+$  acts on  $\mathcal{S}_k(\mathbf{Q})$  via the weight  $k$  action.

(4.7) **Definition.** The standard weight  $k$  modular symbol associated to a cusp form  $f \in \mathcal{S}_k(\mathbf{Q})$  is the modular symbol  $\Phi_f \in \text{Symb}(\text{Sym}^{k-2}(\mathbf{C}^2))$  defined on divisors  $\{c_2\} - \{c_1\} \in \mathcal{D}_0$ ,  $c_1, c_2 \in \mathbf{P}^1(\mathbf{Q})$  by

$$\Phi_f(\{c_2\} - \{c_1\}) = 2\pi i \int_{c_1}^{c_2} f(z)(zX + Y)^{k-2} dz$$

where the integral is over the geodesic in the upper half-plane joining  $c_1$  to  $c_2$ .

A straightforward calculation shows that the map  $f \mapsto \Phi_f$  commutes with the action of  $\Sigma^+$ . We have the following theorem of Shimura [Sh].

(4.8) **Theorem.** *Let  $f \in \mathcal{L}_k(\Gamma_1(Np^r))$  be a Hecke eigenform of weight  $k \geq 2$  and let  $K(f)$  be the field generated by the Hecke eigenvalues of  $f$ . Then for either choice of sign  $\pm$ , the Hecke eigenspace associated to  $f$  in  $\text{Symb}_{\Gamma_1(Np^r)}(\text{Sym}^{k-2}(K(f)^2))^\pm$  is one dimensional over  $K(f)$ . Moreover, there are ‘periods’  $\Omega_f^\pm \in \mathbf{C}^*$  such that the modular symbols  $\Psi_f^\pm = (\Omega_f^\pm)^{-1} \Phi_f^\pm$  are defined over  $K(f)$  and span the associated eigenspaces, that is*

$$0 \neq \Psi_f^\pm \in \text{Symb}_{\Gamma_1(Np^r)}(\text{Sym}^{k-2}(K(f)^2))^\pm.$$

Recall from Theorem 2.6b that  $\mathcal{X}^{\text{arith}} = \mathcal{X}^{\text{arith}}(\tilde{\mathcal{H}})$  parametrizes the ordinary  $p$ -stabilized newforms of level  $Np^r$ ,  $r > 0$ .

(4.9) **Definition.** For each  $\kappa \in \mathcal{X}^{\text{arith}}$  we fix the following data and notations.

- a.  $K_\kappa$  is the  $p$ -adic completion of  $K(\mathbf{f}_\kappa)$  with respect to our fixed embedding (0.4).
- b.  $\mathbf{W}_\kappa$  is the Hecke eigenspace in  $\text{Symb}_{\Gamma_1(Np^r)}(\text{Sym}^{k-2}(K_\kappa^2))$  associated to  $\mathbf{f}_\kappa$ . Here  $k$  is the weight of  $\mathbf{f}_\kappa$ .
- c. We fix, once and for all, two periods  $\Omega_{\mathbf{f}_\kappa}^\pm \in \mathbf{C}^*$  as in Theorem 4.8 and let  $\Psi_{\mathbf{f}_\kappa}^\pm \in \mathbf{W}_\kappa^\pm$  be the associated generators.

Consistent with the notational conventions of “section 2” (see (2.8)) we will write  $K_{\kappa,k}$  and  $\mathbf{W}_{\kappa,k}$  to denote  $K_{\kappa^{(k)}}$  and  $\mathbf{W}_{\kappa^{(k)}}$ , respectively.

*Special values of L-functions*

Modular symbols provide us with a convenient tool for studying values of  $L$ -functions. We will attach “special values of  $L$ -functions” to modular symbols and show how the critical values of the  $L$ -function of a cusp form of weight  $k \geq 2$  can be described in terms of the “special values of the  $L$ -function” of the associated modular symbol.

(4.10) **Definition.** Let  $\Phi \in \text{Symb}(A)$ . Then the *special value of the L-function of  $\Phi$*  is defined to be the element  $L(\Phi)$  of  $A$  given by  $L(\Phi) = \Phi(\{0\} - \{i\infty\})$ .

Let  $\psi: \mathbf{Z} \rightarrow R$  be a primitive Dirichlet character of conductor  $m \geq 1$ . Then we may also define “special values of  $L$ -functions twisted by  $\psi$ ”. For simplicity, suppose  $\Gamma = \Gamma_1(M)$  for some positive integer  $M$ . Define the *twist operator*  $R_\psi: \text{Symb}_\Gamma(A) \rightarrow \text{Symb}(A)$  by the formula

$$\Phi | R_\psi \stackrel{\text{def}}{=} \sum_{a=0}^{m-1} \psi(a) \Phi \left( \begin{pmatrix} 1 & a \\ 0 & m \end{pmatrix} \right).$$

(4.11) **Definition.** The *special value of the L-function of  $\Phi$  twisted by  $\psi$*  is  $L(\Phi, \psi) = L(\Phi | R_\psi)$ . In case  $\Phi$  is a modular symbol of weight  $k \geq 2$  over  $R$  then the special value  $L(\Phi, \psi)$  is a homogeneous polynomial of degree  $k-2$  in  $X$

and  $Y$ . If the binomial coefficients  $\binom{k-2}{r}$ ,  $0 \leq r \leq k-2$ , are not zero divisors in  $R$ , then we define the ‘special values’  $L(\Phi, \psi, s_0) \in R$ , for integers  $s_0$  with  $0 < s_0 < k$ , to be the unique elements of  $R$  for which

$$(4.12) \quad L(\Phi, \psi) = \sum_{s_0=1}^{k-1} \binom{k-2}{s_0-1} (-1)^{s_0-1} \cdot L(\Phi, \psi, s_0) \cdot X^{s_0-1} Y^{k-s_0-1}.$$

These definitions are motivated by the following well known example. If

$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$  is the Fourier expansion of a weight  $k$  cusp form  $f \in \mathcal{S}_k(\bar{\mathbf{Q}})$

and if  $\psi: \mathbf{Z} \rightarrow \mathbf{C}$  is a primitive Dirichlet character with conductor  $m > 0$ , then the complex  $L$ -function of  $f$  twisted by  $\psi$  is defined by the Dirichlet series

$$L_{\infty}(f, \psi, s) = \sum_{n=1}^{\infty} \psi(n) a_n n^{-s} \quad \text{for } \text{Re}(s) > \frac{k+1}{2}.$$

This function extends to an entire function in  $s$ . We are interested in its values at integers  $s = s_0$  in the ‘critical strip’  $0 < s_0 < k$ . For such values of  $s_0$  we define

$$(4.13) \quad A(f, \psi, s_0) \stackrel{\text{def}}{=} m^{s_0-1} (s_0-1)! \frac{\tau(\psi) L_{\infty}(f, \bar{\psi}, s_0)}{(2\pi i)^{s_0-1}}$$

where  $\tau(\psi)$  is the Gauss sum  $\sum_{a=0}^{m-1} \psi(a) e^{2\pi i a/m}$ . If  $\psi$  is the trivial Dirichlet character,

then we will suppress it from the notation and write simply  $L_{\infty}(f, s)$  and  $A(f, s_0)$  instead of  $L_{\infty}(f, \psi, s)$  and  $A(f, \psi, s_0)$  respectively.

(4.14) **Theorem.** *For every primitive Dirichlet character  $\psi$  and each integer  $s_0$  with  $0 < s_0 < k$  we have*

$$L(\Phi_f, \psi, s_0) = A(f, \psi, s_0).$$

The proof is a straightforward calculation.

*$p$ -adic  $L$ -functions attached to  $p$ -stabilized ordinary newforms*

For an arbitrary topological  $\mathbf{Z}_p$ -algebra  $R$ , we let  $\text{Meas}(\mathbf{Z}_p^*, R)$  denote the  $R$ -module of all bounded  $R$ -valued distributions on  $\mathbf{Z}_p^*$  (see [Mz-SwD]). For fixed  $\mu \in \text{Meas}(\mathbf{Z}_p^*, R)$  we associate to each continuous character  $\sigma: \mathbf{Z}_p^* \rightarrow R^*$  the element

$$(4.15) \quad L_p(\mu, \sigma) = \int_{\mathbf{Z}_p^*} \sigma(t) d\mu(t) \in R$$

in the usual way (see, for example, [Mz-SwD]). The resulting function of  $\sigma$  will be called the  $R$ -valued Iwasawa function associated to  $\mu$ .

For each  $\kappa \in \mathcal{X}^{\text{arith}}$  we follow [Mz-T-T] and define measures  $\mu_{\kappa}^{\pm} \in \text{Meas}(\mathbf{Z}_p^*, K_{\kappa})$  by setting

$$(4.16) \quad \mu_{\kappa}^{\pm}(a + p^m \mathbf{Z}_p) = a_p(\kappa)^{-m} \Psi_{\mathbf{f}_{\kappa}}^{\pm} \left( \left\{ \frac{a}{p^m} \right\} - \{i\infty\} \right) \Big|_{x=0, y=1}$$

for each  $a \in \mathbf{Z}$  prime to  $p$ , and each  $m > 0$ . We then define the  $p$ -adic  $L$ -function

$$(4.17) \quad L_p(\mathbf{f}_{\kappa}, \psi, s) = L_p(\mu_{\kappa}^{\text{sgn}(\psi)}, \psi \langle \cdot \rangle^{s-1})$$

for each  $\kappa \in \mathcal{X}^{\text{arith}}$ . If  $\psi$  is the trivial character, then we will suppress it from the notation and write simply  $L_p(\mathbf{f}_{\kappa}, s)$ . We have the following theorem.

(4.18) **Theorem.** *Let  $\kappa \in \mathcal{X}^{\text{arith}}$  be an arithmetic point of weight  $k$ . Let  $\psi$  be a finite character of  $\mathbf{Z}_p^*$  of conductor  $p^m$ ,  $m \geq 0$  and let  $s_0$  be an integer with  $0 < s_0 < k$ . Then*

$$L_p(\mathbf{f}_{\kappa}, \psi, s_0) = a_p(\kappa)^{-m} \cdot (1 - a_p(\kappa)^{-1} \psi \omega^{1-s_0}(p) p^{s_0-1}) \cdot \frac{A(\mathbf{f}_{\kappa}, \psi \omega^{1-s_0}, s_0)}{\Omega_{\mathbf{f}_{\kappa}}^{\text{sgn}(\psi)}}.$$

For more details of the construction of the  $p$ -adic  $L$ -function and a proof of Theorem 4.18, see [Mz-T-T].

## 5. $A$ -adic modular symbols and two-variable $p$ -adic $L$ -functions

Fix a prime number  $p > 0$  and a positive integer  $N$  which is not divisible by  $p$ . Let  $\Gamma = \Gamma_1(N)$ . In this section we examine the structure of the group of modular symbols over  $\Gamma$  which take values in the module  $\mathbf{D}$  of  $\mathbf{Z}_p$ -valued measures on  $(\mathbf{Z}_p^2)$  (=the set of primitive elements of  $\mathbf{Z}_p^2$ ). Such modular symbols will be referred to as  $A$ -adic modular symbols. Our interest in  $\text{Symb}_{\Gamma}(\mathbf{D})$  stems from two facts. First of all, the module  $\mathbf{D}$  is rich enough to admit non-trivial morphisms to each of the modules  $\text{Sym}^r(\mathbf{Z}_p^2)$ ,  $r \geq 0$ . Thus a measure gives rise to a family of elements in  $\text{Sym}^r$  as  $r$  varies, and correspondingly, a  $A$ -adic modular symbol gives rise to a family of modular symbols of varying weights. The second reason for our interest in  $\mathbf{D}$  rests on the fact that the elements of  $\mathbf{D}$  give rise in a natural way to two variable  $p$ -adic  $L$ -functions. The main results of this section are Theorems 5.13 and 5.15. Theorem 5.13, which is proved in the next section, asserts the existence of ordinary  $A$ -adic modular eigensymbols which are  $p$ -adic deformations of the modular symbols associated to any given ordinary  $p$ -stabilized newform. In Theorem 5.15 we describe the analytic properties of the two-variable  $p$ -adic  $L$ -function associated to such an eigensymbol.

Let  $\text{Cont}(\mathbf{Z}_p^2)$  denote the  $\mathbf{Z}_p$ -module of continuous  $\mathbf{Z}_p$ -valued functions on  $\mathbf{Z}_p^2$  and let  $\text{Step}(\mathbf{Z}_p^2)$  be the submodule of locally constant functions. The group  $\tilde{\mathbf{D}}$  of  $\mathbf{Z}_p$ -valued measures on  $\mathbf{Z}_p^2$  is defined to be  $\tilde{\mathbf{D}} = \text{Hom}_{\mathbf{Z}_p}(\text{Step}(\mathbf{Z}_p^2), \mathbf{Z}_p)$ . As is well known, every  $\mu \in \tilde{\mathbf{D}}$  has a unique extension to a  $\mathbf{Z}_p$ -homomorphism  $\text{Cont}(\mathbf{Z}_p^2) \rightarrow \mathbf{Z}_p$  which is continuous with respect to the supremum norm on

$\text{Cont}(\mathbf{Z}_p^2)$ . If  $\mu \in \tilde{\mathbf{D}}$ ,  $\varphi \in \text{Cont}(\mathbf{Z}_p^2)$  and  $K \subseteq \mathbf{Z}_p^2$  is a compact open set, we will use the integral notation and write

$$\int_K \varphi d\mu$$

for the value of  $\mu$  on the product of  $\varphi$  with the characteristic function of  $K$ . In case  $\varphi$  is identically 1, we will also write  $\mu(K)$  for this integral.

We will be particularly interested in a certain direct summand  $\mathbf{D}$  of  $\tilde{\mathbf{D}}$  which may be defined as follows. Let  $(\mathbf{Z}_p^2)'$  denote the primitive elements of  $\mathbf{Z}_p^2$ , i.e. those elements which are not divisible by  $p$ . Then  $\mathbf{D}$  is the submodule of  $\tilde{\mathbf{D}}$  consisting of measures which are supported on  $(\mathbf{Z}_p^2)'$ . Restriction of measures from  $\mathbf{Z}_p^2$  to  $(\mathbf{Z}_p^2)'$  gives a projection from  $\tilde{\mathbf{D}}$  to  $\mathbf{D}$ , hence  $\mathbf{D}$  is a direct summand of  $\tilde{\mathbf{D}}$ .

There is a natural continuous action of  $M_2(\mathbf{Z}_p)$  on  $\tilde{\mathbf{D}}$ . To describe this action we regard the elements of  $\mathbf{Z}_p^2$  as row vectors and let  $M_2(\mathbf{Z}_p)$  act by matrix multiplication on the right. Then  $M_2(\mathbf{Z}_p)$  acts covariantly on  $\text{Step}(\mathbf{Z}_p^2)$  by the formula  $\varphi \mapsto (g\varphi: \mathbf{x} \in \mathbf{Z}_p^2 \mapsto \varphi(\mathbf{x}g))$ . The contravariant action on  $\tilde{\mathbf{D}}$  is given by  $\mu \mapsto \mu|g$  where  $\mu|g$  is given by the integration formula

$$\int_{\mathbf{Z}_p^2} \varphi d(\mu|g) = \int_{\mathbf{Z}_p^2} g\varphi d\mu.$$

Since the kernel of the natural projection  $\tilde{\mathbf{D}} \rightarrow \mathbf{D}$  is preserved by this action of  $M_2(\mathbf{Z}_p)$  we also obtain an induced action of  $M_2(\mathbf{Z}_p)$  on  $\mathbf{D}$ . We will take this induced action as the natural action of  $M_2(\mathbf{Z}_p)$  on  $\mathbf{D}$ . Note, that while the action has been defined to commute with the natural surjection  $\tilde{\mathbf{D}} \rightarrow \mathbf{D}$ , it does not respect the natural inclusion  $\mathbf{D} \hookrightarrow \tilde{\mathbf{D}}$ .

The group  $\mathbf{Z}_p^*$  acts continuously on  $\mathbf{D}$  and  $\tilde{\mathbf{D}}$  via the scalar matrices in  $M_2(\mathbf{Z}_p)$ . We extend this to a continuous action of the algebra  $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$ . Note that the action of  $M_2(\mathbf{Z}_p)$  commutes with these  $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$ -structures on  $\mathbf{D}$  and  $\tilde{\mathbf{D}}$ . Now restrict the action of  $M_2(\mathbf{Z}_p)$  to an action of  $\Sigma$ . Since Hypothesis P (1.3) is satisfied, we may form the  $\mathcal{H}$ -modules

$$(5.1) \quad \mathbf{W} = \text{Symb}_\Gamma(\mathbf{D}) \quad \text{and} \quad \tilde{\mathbf{W}} = \text{Symb}_\Gamma(\tilde{\mathbf{D}}).$$

Restriction of measures induces a natural surjective  $\mathcal{H}$ -morphism  $\tilde{\mathbf{W}} \rightarrow \mathbf{W}$ .

We now describe a simple procedure for attaching  $p$ -adic  $L$ -functions to the elements of  $\mathbf{W}$ . Recall (1.11)c that  $\mathcal{X}_0$  is the space of  $\bar{\mathbf{Q}}_p$ -valued points on  $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$ . To each  $\Phi \in \mathbf{W}$  we attach its 'special value of the  $L$ -function'  $\mu_\Phi = L(\Phi) \in \mathbf{D}$  as in the last section (4.10) and define the *standard 2-variable  $p$ -adic  $L$ -function* associated to  $\Phi$  to be the  $\bar{\mathbf{Q}}_p$ -valued function  $L_p(\Phi)$  on  $\mathcal{X}_0 \times \mathcal{X}_0$  given by

$$(5.2) \quad L_p(\Phi, \kappa, \sigma) = \int_{\mathbf{Z}_p^* \times \mathbf{Z}_p^*} \kappa(x) \sigma(y/x) d\mu_\Phi(x, y)$$

for  $(\kappa, \sigma) \in \mathcal{X}_0 \times \mathcal{X}_0$ . If  $\sigma$  is an arithmetic point, then its restriction to  $\mathbf{Z}_p^*$  is an arithmetic character of the form  $\sigma_{r, \psi}$ . We can then extend  $\sigma$  to a continuous multiplicative function  $\mathbf{Z}_p \rightarrow \bar{\mathbf{Q}}_p$  by the convention  $\sigma(p) = p^r$  if  $\psi$  is the trivial character and  $\sigma(p) = 0$  otherwise. With this convention, we define the

improved 2-variable  $p$ -adic  $L$ -function associated to  $\Phi$  to be the function  $L_p^*(\Phi)$  on  $\mathcal{X}_0 \times \mathcal{X}_0^{\text{arith}}$  given by

$$(5.2)\text{b.} \quad L_p^*(\Phi, \kappa, \sigma) = \int_{\mathbf{Z}_p^* \times \mathbf{Z}_p} \kappa(x) \sigma(y/x) d\mu_\Phi(x, y)$$

for  $(\kappa, \sigma) \in \mathcal{X}_0 \times \mathcal{X}_0^{\text{arith}}$ .

It is clear from the definitions that  $L_p(\Phi, \kappa, \sigma)$  is analytic in  $(\kappa, \sigma)$  and that  $L_p^*(\Phi, \kappa, \sigma)$  is analytic in  $\kappa$  for each  $\sigma \in \mathcal{X}_0^{\text{arith}}$ . Moreover, for fixed  $\sigma$  there are unique  $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$ -morphisms

$$(5.3) \quad \begin{aligned} L_p(\cdot, \sigma) : \mathbf{W} &\rightarrow \mathbf{Z}_p[[\mathbf{Z}_p^*]] && \text{for fixed } \sigma \in \mathcal{X}_0, \text{ and} \\ L_p^*(\cdot, \sigma) : \mathbf{W} &\rightarrow \mathbf{Z}_p[[\mathbf{Z}_p^*]] && \text{for fixed } \sigma \in \mathcal{X}_0^{\text{arith}}, \end{aligned}$$

such that  $\kappa(L_p(\Phi, \sigma)) = L_p(\Phi, \kappa, \sigma)$  and  $\kappa(L_p^*(\Phi, \sigma)) = L_p^*(\Phi, \kappa, \sigma)$  for all  $\kappa \in \mathcal{X}_0$ .

Now fix a continuous  $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$ -algebra  $R$ . Let  $\mathbf{D}_R = \mathbf{D} \otimes_{\mathbf{Z}_p[[\mathbf{Z}_p^*]]} R$  and  $\mathbf{W}_R = \mathbf{W} \otimes_{\mathbf{Z}_p[[\mathbf{Z}_p^*]]} R$ . There is a natural isomorphism  $\mathbf{W}_R \cong \text{Symb}_\Gamma(\mathbf{D}_R)$  of  $\mathcal{H}$ -modules. Extending the maps (5.3) by  $R$ -linearity, we obtain  $R$ -homomorphisms  $L_p(\cdot, \sigma), L_p^*(\cdot, \sigma) : \mathbf{W}_R \rightarrow R$ . We may therefore extend our definitions (5.2) and associate to each  $\Phi \in \mathbf{W}_R$  a two-variable  $p$ -adic  $L$ -function  $L_p(\Phi)$  on  $\mathcal{X}(R) \times \mathcal{X}_0$  and an improved  $p$ -adic  $L$ -function  $L_p^*(\Phi)$  on  $\mathcal{X}(R) \times \mathcal{X}_0^{\text{arith}}$  by the formulas

$$(5.4) \quad \begin{aligned} L_p(\Phi, \kappa, \sigma) &= \kappa(L_p(\Phi, \sigma)) && \text{for } (\kappa, \sigma) \in \mathcal{X}(R) \times \mathcal{X}_0, \text{ and} \\ L_p^*(\Phi, \kappa, \sigma) &= \kappa(L_p^*(\Phi, \sigma)) && \text{for } (\kappa, \sigma) \in \mathcal{X}(R) \times \mathcal{X}_0^{\text{arith}}. \end{aligned}$$

In Proposition 5.8 we will prove a fundamental interpolation property of these  $p$ -adic  $L$ -functions. In particular we will show that they interpolate special values of  $L$ -functions attached to a certain family  $\Phi_\kappa, \kappa \in \mathcal{X}^{\text{arith}}(R)$ , of modular symbols of integral weights attached to  $\Phi$ . This family of modular symbols is defined as follows. For each  $\kappa \in \mathcal{X}^{\text{arith}}(R)$  with weight  $k \geq 2$  and character  $\varepsilon$  define the specialization map  $\phi_\kappa : \mathbf{D} \rightarrow \text{Sym}^{k-2}(\overline{\mathbf{Q}}_p^2)$  by the integration formula

$$(5.5)\text{a.} \quad \phi_\kappa(\mu) = \int_{\mathbf{Z}_p^* \times \mathbf{Z}_p} \varepsilon(x) \cdot (xY - yX)^{k-2} d\mu(x, y)$$

and extend this to the unique map  $\phi_\kappa : \mathbf{D}_R \rightarrow \text{Sym}^{k-2}(\overline{\mathbf{Q}}_p^2)$  which intertwines  $\kappa$ . If, moreover, the character  $\varepsilon$  is trivial, then we also define a map  $\tilde{\phi}_\kappa : \tilde{\mathbf{D}} \rightarrow \text{Sym}^{k-2}(\overline{\mathbf{Q}}_p^2)$  by

$$(5.5)\text{b.} \quad \tilde{\phi}_\kappa(\mu) = \int_{\mathbf{Z}_p^2} (xY - yX)^{k-2} d\mu(x, y)$$

and extend this to a map  $\tilde{\phi}_\kappa : \tilde{\mathbf{D}}_R \rightarrow \text{Sym}^{k-2}(\overline{\mathbf{Q}}_p^2)$  intertwining  $\kappa$  as well. A simple calculation shows that if the conductor of  $\varepsilon$  divides  $p^r$  and  $r > 0$  then  $\phi_\kappa$  commutes with the action of  $\Sigma_1(p^r)$ . Hence,  $\phi_\kappa$  induces an  $\mathcal{H}[t, W_N]$ -morphism  $\phi_{\kappa, *}: \mathbf{W}_R \rightarrow \text{Symb}_{\Gamma_1(Np^r)}(\text{Sym}^{k-2}(\overline{\mathbf{Q}}_p^2))$ . If  $\varepsilon$  is trivial, then  $\tilde{\phi}_\kappa$  commutes with all of  $\Sigma$ , hence  $\tilde{\phi}_\kappa$  induces an  $\mathcal{H}[t, W_N]$ -morphism

$$\tilde{\phi}_{\kappa, *}: \tilde{\mathbf{W}}_R \rightarrow \text{Symb}_{\Gamma_1(N)}(\text{Sym}^{k-2}(\overline{\mathbf{Q}}_p^2)).$$

We may therefore make the following definitions.

(5.6) **Definition.** Let  $\Phi \in \mathbf{W}_R$ .

- a. For each arithmetic point  $\kappa \in \mathcal{X}^{\text{arith}}(R)$  we define  $\Phi_\kappa$  to be the image of  $\Phi$  under  $\phi_{\kappa, *}$ .
- b. For each  $\kappa$  with trivial character, we also define  $\tilde{\Phi}_\kappa$  to be the image of  $\Phi$  under  $\tilde{\phi}_{\kappa, *}$ .

In Proposition 5.8 we will describe the behavior of the  $p$ -adic  $L$ -functions when  $\Phi$  is transformed by the operator  $T_p$ . For this purpose, it will be useful to first record a few facts about the Hecke operators at  $p$  acting on  $\mathbf{W}$ . We will say that a modular symbol  $\Phi \in \tilde{\mathbf{W}}$  is supported on a given compact open subset  $U$  of  $\mathbf{Z}_p^2$ , if, for every  $D \in \mathcal{G}_0$ , the measure  $\Phi(D) \in \tilde{\mathbf{D}}$  is supported on  $U$ . Hence,  $\mathbf{W}$  may be described as the set of modular symbols in  $\tilde{\mathbf{W}}$  which are supported on  $(\mathbf{Z}_p^2)'$ . Since the scalar matrix  $pI$  transforms any modular symbol  $\Phi \in \tilde{\mathbf{W}}$  to one supported on  $p\mathbf{Z}_p^2$ , we see that  $[p]_p$  annihilates  $\mathbf{W}$ .

The action of the operator  $T_p$  and its powers  $T_p^m, m > 0$ , on  $\mathbf{W}$  can be described as follows. Consider the reduction map  $(\mathbf{Z}_p^2)' \rightarrow \mathbf{P}^1(\mathbf{Z}/p^m\mathbf{Z})$ . For each  $\mathbf{x} \in \mathbf{P}^1(\mathbf{Z}/p^m\mathbf{Z})$  the preimage of  $\mathbf{x}$  in  $(\mathbf{Z}_p^2)'$  is a compact open set which we denote by  $U(\mathbf{x}, p^m)$ . Choose an element  $g_{\mathbf{x}, p^m} \in \Sigma_1(N)$  with determinant  $p^m$  for which  $U(\mathbf{x}, p^m) \subseteq ((\mathbf{Z}_p^2)')^{g_{\mathbf{x}, p^m}}$ . The coset  $\Gamma g_{\mathbf{x}, p^m}$  is independent of the choice of  $g_{\mathbf{x}, p^m}$  with this property. The  $m$ th power of  $T_p$  acting on  $\Phi \in \mathbf{W}$  is then given by

$$(5.7) \quad \Phi | T_p^m = \sum_{\mathbf{x} \in \mathbf{P}^1(\mathbf{Z}/p^m\mathbf{Z})} \Phi | g_{\mathbf{x}, p^m}$$

This decomposes  $\Phi | T_p^m$  into a sum of modular symbols which are supported on the disjoint compact open sets  $U(\mathbf{x}, p^m), \mathbf{x} \in \mathbf{P}^1(\mathbf{Z}/p^m\mathbf{Z})$ .

We will say that a pair of arithmetic points  $(\kappa, \sigma) \in \mathcal{X}^{\text{arith}}(R) \times \mathcal{X}_0^{\text{arith}}$  is *critical* if the weight of  $\kappa$  is greater than or equal to the weight of  $\sigma$ . This is equivalent to saying that  $\kappa \sigma^{-1}$  defines an arithmetic character on  $\mathbf{Z}_p^*$ .

(5.8) **Proposition.** *Let  $\Phi \in \mathbf{W}_R$  and  $(\kappa, \sigma) \in \mathcal{X}(R) \times \mathcal{X}_0$ .*

1. (Relation Between the Standard and Improved  $p$ -adic  $L$ -functions.) *If  $\sigma \in \mathcal{X}_0^{\text{arith}}$  then*

$$L_p(\Phi | T_p, \kappa, \sigma) = L_p^*(\Phi | T_p, \kappa, \sigma) - \sigma(p) \cdot L_p^*(\Phi, \kappa, \sigma).$$

Here  $\sigma(p)$  is defined as in the paragraph preceding (5.2)b.

2. (Interpolation.) *Suppose the pair  $(\kappa, \sigma)$  is critical and suppose  $\sigma = \sigma_{r, \psi}$  where  $r$  is an integer  $\geq 0$  and  $\psi$  is a finite character of  $\mathbf{Z}_p^*$  of conductor  $p^m$ , with  $m \geq 0$ . Then*

$$L_p^*(\Phi | T_p^m, \kappa, \sigma) = L(\Phi_\kappa, \psi, r + 1)$$

where  $\Phi_\kappa$  is the specialization defined by (5.6)a and  $L(\Phi_\kappa, \psi, r + 1)$  is defined by (4.12).

3. (Functional Equation for the standard  $p$ -adic  $L$ -function.) *Let  $W_N = \begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}$ . Then*

$$L_p(\Phi, \kappa, \sigma) = -\sigma^{-1}(-N) \cdot L_p(\Phi | W_N, \kappa, \kappa \sigma^{-1}).$$

4. (Functional Equation for the Improved  $p$ -adic  $L$ -functions.) *Suppose the pair  $(\kappa, \sigma)$  is critical and that both  $\kappa$  and  $\sigma$  have trivial characters and fix  $r \geq 0$  so that  $\sigma = \sigma_r$ . Then*

$$L_p^*(\Phi, \kappa, \sigma) - \sigma(-N)^{-1} L_p^*(\Phi | W_N, \kappa, \kappa \sigma^{-1}) = L_p(\Phi, \kappa, \sigma) + L(\tilde{\Phi}_\kappa, r + 1).$$

*Proof.* Each of these statements is verified by a straightforward calculation. We will only sketch the details. We restrict ourselves to the special case  $R = \mathbf{Z}_p[[\mathbf{Z}_p^*]]$ , since the general case then follows by linearity.

Let  $\mu = L(\Phi) \in \mathbf{D}$ . Then for each  $m > 0$  we can write  $L(\Phi|T_p^m)$  in the form

$$(5.9) \quad L(\Phi|T_p^m) = \sum_{\mathbf{x} \in \mathbf{P}^1(\mathbf{Z}/p^m\mathbf{Z})} \mu_{\mathbf{x}} |g_{\mathbf{x}, p^m}$$

where  $\mu_{\mathbf{x}} = \Phi(g_{\mathbf{x}, p^m} \cdot (\{0\} - \{i\infty\}))$ . As remarked above, the decomposition (5.9) exhibits  $L(\Phi|T_p^m)$  as a sum of measures supported on the disjoint compact open sets  $U(\mathbf{x}, p^m)$ , for  $\mathbf{x} \in \mathbf{P}^1(\mathbf{Z}/p^m\mathbf{Z})$ . Since the standard and improved  $p$ -adic  $L$ -functions are defined as integrals over  $\mathbf{Z}_p^* \times \mathbf{Z}_p^*$  and  $\mathbf{Z}_p^* \times \mathbf{Z}_p$ , respectively, only those terms in (5.9) associated to  $\mathbf{x}$  of the form  $\mathbf{x} = [1, a]$  with  $a \in \mathbf{Z}_p/p^m\mathbf{Z}_p$  will enter. Moreover, when  $\mathbf{x} = [1, a]$  we may choose  $g_{\mathbf{x}, p^m} = \begin{pmatrix} 1 & a \\ 0 & p^m \end{pmatrix}$ . Now a simple calculation shows that when  $\begin{pmatrix} 1 & a \\ 0 & p^m \end{pmatrix}$  operates on the characteristic function of  $U([1, a], p^m)$  the result is the characteristic function of  $\mathbf{Z}_p^* \times \mathbf{Z}_p$ . Hence we easily obtain

$$(5.10) \quad \begin{aligned} L_p(\Phi|T_p^m, \kappa, \sigma) &= \sum_{a \in (\mathbf{Z}/p^m\mathbf{Z})^*} \int_{\mathbf{Z}_p^* \times \mathbf{Z}_p} \kappa(x) \sigma(a + p^m y/x) d\mu_a(x, y) \\ L_p^*(\Phi|T_p^m, \kappa, \sigma) &= \sum_{a \in \mathbf{Z}/p^m\mathbf{Z}} \int_{\mathbf{Z}_p^* \times \mathbf{Z}_p} \kappa(x) \sigma(a + p^m y/x) d\mu_a(x, y) \end{aligned}$$

where we have written  $\mu_a$  for the measure  $\Phi(\{a/p^m\} - \{i\infty\})$ .

Now consider the first assertion of the proposition. Setting  $m = 1$  in (5.10) and calculating the difference of the two expressions occurring there we see that only the term corresponding to  $a = 0$  survives. Thus

$$L_p^*(\Phi|T_p, \kappa, \sigma) - L_p(\Phi|T_p, \kappa, \sigma) = \sigma(p) \int_{\mathbf{Z}_p^* \times \mathbf{Z}_p} \kappa(x) \sigma(y/x) d\mu_0(x, y).$$

But  $\mu_0 = L(\Phi)$  so **1** follows.

We now turn to the proof of **2**. Fix an arithmetic point  $\kappa \in \mathcal{X}^{\text{arith}}$  of weight  $k \geq 2$ . We must prove the identity  $L_p^*(\Phi|T_p^m, \kappa, \sigma_{r, \psi}) = L(\Phi_{\kappa}, \sigma_{r, \psi})$  for every finite character  $\psi$  on  $\mathbf{Z}_p^*$  of conductor  $p^m$  and every integer  $r$  with  $0 \leq r \leq k - 2$ . Fix the character  $\psi$  of conductor  $p^m$ . Using (5.10) one easily calculates

$$\begin{aligned} &\sum_{r=0}^{k-2} \binom{k-2}{r} (-1)^r \cdot L_p^*(\Phi|T_p^m, \kappa, \sigma_{r, \psi}) \cdot X^r Y^{k-2-r} \\ &= \sum_{a \in \mathbf{Z}/p^m\mathbf{Z}} \psi(a) \int_{\mathbf{Z}_p^* \times \mathbf{Z}_p} \varepsilon(x) (x(y - aX) - y(p^m X))^{k-2} d\mu_a(x, y). \end{aligned}$$

On the other hand, from the definition of  $\Phi_{\kappa}$  (5.6)a and the definition of  $L(\Phi_{\kappa}, \psi, r + 1)$  (4.12) we have

$$\begin{aligned} &\sum_{r=0}^{k-2} \binom{k-2}{r} (-1)^r \cdot L(\Phi_{\kappa}, \psi, r + 1) \cdot X^r Y^{k-2-r} \\ &= \sum_{a \in \mathbf{Z}/p^m\mathbf{Z}} \psi(a) \int_{\mathbf{Z}_p^* \times \mathbf{Z}_p} \varepsilon(x) (xY - yX)^{k-2} d\mu_a(x, y) \left| \begin{pmatrix} 1 & a \\ 0 & p^m \end{pmatrix} \right. \end{aligned}$$

where the last  $|\cdot$ -operator denotes the action on the given homogeneous polynomial of degree  $k-2$  in  $\text{Sym}^{k-2}(\overline{\mathbf{Q}}_p^2)$ . Now **2** follows easily by comparing the last two displayed equalities.

To prove the functional equation **3**, we just notice that since  $W_N$  interchanges the cusps  $\infty$  and  $0$ , we have  $\mu_\Phi|_{W_N} = -\mu_\Phi|_{W_N}$ . Now a simple calculation proves **3**.

The last property **4** follows from an application of the inclusion-exclusion principle. We have

$$\begin{aligned} L(\tilde{\Phi}_\kappa, \sigma) &= \int_{(\mathbf{Z}_p^2)'} \kappa \sigma^{-1}(x) \sigma(y) d\mu(x, y) \\ &= \int_{\mathbf{Z}_p \times \mathbf{Z}_p} + \int_{\mathbf{Z}_p \times \mathbf{Z}_p} - \int_{\mathbf{Z}_p \times \mathbf{Z}_p} \end{aligned}$$

The first of these three integrals is  $L_p^*(\Phi, \kappa, \sigma)$  and the last is  $L_p(\Phi, \kappa, \sigma)$ . The middle integral can easily be calculated as in the proof of the functional equation **3** and is equal to  $-\sigma([-N]_p)^{-1} L_p^*(\Phi|W_N, \kappa, \kappa\sigma^{-1})$ . This completes the proof of Proposition 5.8.

Recall the notations of (2.4) where  $\mathcal{R}$  is the universal ordinary Hecke algebra,  $\mathcal{X} = \mathcal{R} \otimes_A \mathcal{L}$ , and  $\mathcal{X} = \mathcal{X}(\tilde{\mathcal{R}})$ . For each  $\kappa \in \mathcal{X}$  let  $\mathcal{R}_{(\kappa)}$  be the localization of  $\tilde{\mathcal{R}}$  at  $\kappa$ . The fraction field  $\mathcal{K}_{(\kappa)}$  of  $\mathcal{R}_{(\kappa)}$  is a direct factor of  $\mathcal{K}$  and correspondingly, the  $\mathcal{K}_{(\kappa)}$ -space  $\mathbf{W}_{\mathcal{K}_{(\kappa)}}$  is a direct factor of  $\mathbf{W}_{\mathcal{X}}$ . We will say that an element  $\Phi \in \mathbf{W}_\kappa$  is *regular at  $\kappa$*  if the projection of  $\Phi$  to  $\mathbf{W}_{\mathcal{K}_{(\kappa)}}$  lies in the  $\mathcal{R}_{(\kappa)}$ -submodule  $\mathbf{W}_{\mathcal{R}_{(\kappa)}}$ . Every  $\Phi \in \mathbf{W}_{\mathcal{X}}$  is regular at all but finitely many  $\kappa$  in  $\mathcal{X}$ . To each  $\Phi \in \mathbf{W}_{\mathcal{X}}$  we associate elements  $L_p(\Phi, \sigma) \in \mathcal{K}$ , for  $\sigma \in \mathcal{X}_0$ , and  $L_p^*(\Phi, \sigma_0) \in \mathcal{K}$ , for arithmetic  $\sigma_0 \in \mathcal{X}_0^{\text{arith}}$ , as in (5.3). If  $\Phi$  is regular at  $\kappa$ , then  $L_p(\Phi, \kappa, \sigma)$  and  $L_p^*(\Phi, \kappa, \sigma_0)$  are defined as in (5.4).

We are going to recast Proposition 5.8 in terms of the local charts (2.8). Fix an arithmetic point  $\kappa \in \mathcal{X}^{\text{arith}}$  of weight  $k_0$  and character  $\varepsilon$ . Let  $\Phi \in \mathbf{W}_{\mathcal{X}}$  be a modular symbol which is regular at  $\kappa$ . Let  $U_\kappa$  be the domain of convergence about  $\kappa$  (see the remarks before (2.8)) and define the *domain of analyticity* of  $\Phi$  about  $\kappa$  to be the open set

$$U_{\kappa, \Phi} \stackrel{\text{def}}{=} \{k \in U_\kappa \mid \Phi \text{ is regular at } \kappa^{(k)}\}.$$

This is just  $U_\kappa$  minus a finite set. For each rational integer  $k \geq 2$  in  $U_{\kappa, \Phi}$ , let  $\Phi_{\kappa, k}$  denote the specialization of  $\Phi$  to  $\kappa^{(k)}$ . Then  $\Phi_{\kappa, k}$  is a modular symbol of weight  $k$  and character  $\varepsilon \omega^{k_0-k}$ . For arbitrary  $k \in U_{\kappa, \Phi}$ , and for  $\psi$  a finite character of  $\mathbf{Z}_p^*$ ,  $s \in \mathbf{Z}_p$ , and  $s_0 \in \mathbf{Z}^+$  we define

$$(5.11) \quad \begin{aligned} L_p(\Phi, \kappa, k, \psi, s) &\stackrel{\text{def}}{=} L_p(\Phi, \kappa^{(k)}, \psi \langle \cdot \rangle^{s-1}); \\ L_p^*(\Phi, \kappa, k, \psi, s_0) &\stackrel{\text{def}}{=} L_p^*(\Phi, \kappa^{(k)}, \psi \langle \cdot \rangle^{s_0-1}). \end{aligned}$$

We will be especially interested in these functions when  $\Phi$  is an eigensymbol for the Hecke operators.

(5.12) **Definition.** For each arithmetic point  $\kappa \in \mathcal{X}$ , let  $h_{(\kappa)}: \mathcal{H} \rightarrow \mathcal{R}_{(\kappa)}$  be the composition of the natural map  $h: \mathcal{H} \rightarrow \mathcal{R}$  defined in (2.4)a with the localization morphism. Define  $\mathbf{W}_{(\kappa)}$  to be the  $h_{(\kappa)}$ -eigenmodule in  $\mathbf{W}_{\mathcal{R}_{(\kappa)}}$ .

The involution  $\iota$  preserves these modules. Let  $\mathbf{W}_{(\kappa)}^{\pm}$  denote the  $\pm$  eigenmodules for  $\iota$ . The following version of Hida’s Control Theorem will be proved in the next section.

(5.13) **Theorem.** *Suppose  $p \geq 5$ . Then for every  $\kappa \in \mathcal{X}^{\text{arith}}$ , and for either choice of sign  $\pm$ , the following are true.*

- a.  $\mathbf{W}_{(\kappa)}^{\pm}$  is a free rank one  $\mathcal{R}_{(\kappa)}$ -module.
- b. The specialization map  $\phi_{\kappa, *}: \Phi \mapsto \Phi_{\kappa}$  (5.6) induces an isomorphism

$$\mathbf{W}_{(\kappa)}^{\pm} / P_{\kappa} \mathbf{W}_{(\kappa)}^{\pm} \cong \mathbf{W}_{\kappa}^{\pm}.$$

If  $\Phi$  is in one of the spaces  $\mathbf{W}_{(\kappa)}^{\pm}$ , then for each integer  $k \geq 2$  in the domain of analyticity of  $\Phi$  about  $\kappa$ , the specialization  $\Phi_{\kappa, k}$  lies in the Hecke eigenspace  $\mathbf{W}_{\kappa, k}^{\pm}$  associated to  $\kappa^{(k)}$  by (4.9). This is a one dimensional  $K_{\kappa, k}$ -vector space generated by the element  $\Psi_{\mathfrak{f}_{\kappa, k}}^{\pm}$  defined in (4.9). Thus there is a unique ‘period’  $\Omega_{\kappa, k}(\Phi) \in K_{\kappa, k}$  such that

$$(5.14) \quad \Phi_{\kappa, k} = \Omega_{\kappa, k}(\Phi) \cdot \Psi_{\mathfrak{f}_{\kappa, k}}^{\pm}.$$

If  $k$  is equal to the weight of  $\kappa$ , then we will suppress it from the notation and write simply  $\Omega_{\kappa}(\Phi)$ . Of course, our definition of the periods  $\Omega_{\kappa, k}$  depends on the choice of complex periods used to define  $\Psi_{\mathfrak{f}_{\kappa}}^{\pm}$ . It is interesting to ask whether there is a natural choice of complex periods and a choice of  $\Phi$  so that  $\Omega_{\kappa}(\Phi)$  extends to an analytic function of  $\kappa \in \mathcal{X}$ ? It follows from Theorem 5.13 that there is an element  $\Phi \in \mathbf{W}_{(\kappa)}^{\pm}$  for which  $\Omega_{\kappa}(\Phi) = 1$ . This is enough for our purposes.

Consider the sesquilinear map  $*: \mathbf{W}_{\mathcal{X}} \rightarrow \mathbf{W}_{\mathcal{X}}$  defined on  $\mathbf{W}$  by  $\Phi \mapsto \Phi^* \stackrel{\text{def}}{=} \Phi | W_N$  and extended to  $\mathbf{W}_{\mathcal{X}}$  by sesquilinearity:  $(a\Phi)^* = a^* \Phi^*$  where  $*$  is the involution on  $\mathcal{X}$  defined in (2.9). A simple calculation shows that, for  $\kappa \in \mathcal{X}^{\text{arith}}$ , if  $\Phi \in \mathbf{W}_{(\kappa)}$  then  $\Phi^* \in \mathbf{W}_{(\kappa^*)}$ . Note that  $*$  is not an involution though, by (1.8), we do have the simple relation  $(\Phi^*)^* = \Phi | [-N]_p$  for any  $\Phi \in \mathbf{W}_{\mathcal{X}}$ . In particular, since  $[-N]_p$  acts invertibly on  $\mathbf{W}$ , the map  $*: \mathbf{W}_{(\kappa)} \rightarrow \mathbf{W}_{(\kappa^*)}$  is an isomorphism.

(5.15) **Theorem.** *Let  $\kappa \in \mathcal{X}^{\text{arith}}$  be an arithmetic point of weight  $k_0$  and character  $\varepsilon$  and let  $\psi$  be a finite character of  $\mathbf{Z}_p^*$ . Fix an eigensymbol  $\Phi \in \mathbf{W}_{(\kappa)}^{\text{sgn}(\psi)}$  and let  $U = U_{\kappa, \Phi}$  be the domain of analyticity of  $\Phi$  about  $\kappa$ . Then the following assertions are true.*

- a. (Analyticity.)  $L_p(\Phi, \kappa, k, \psi, s)$  is analytic for  $(k, s) \in U \times \mathbf{Z}_p$  and is an Iwasawa function in the variable  $s$  (up to multiplication by a constant). For each positive integer  $s_0$ ,  $L_p^*(\Phi, \kappa, k, \psi, s_0)$  is analytic for  $k \in U$ .
- b. (Specialization of the weight variable.) For each rational integer  $k \geq 2$  in  $U$  we have the following identity of Iwasawa functions in  $s$ .

$$L_p(\Phi, \kappa, k, \psi, s) = \Omega_{\kappa, k}(\Phi) \cdot L_p(\mathfrak{f}_{\kappa, k}, \psi, s)$$

- c. (Functional Equation.) Let  $\varepsilon_p$  be the  $p$ -component of  $\varepsilon$ . Then for  $(k, s) \in U \times \mathbf{Z}_p$  we have

$$L_p(\Phi, \kappa, k, \psi, s) = -\psi^{-1}(-N) \langle -N \rangle^{1-s} \cdot L_p(\Phi^*, \kappa^*, k, \varepsilon_p \omega^{k_0-2} \psi^{-1}, k-s).$$

**d.** (Specialization to critical values.) For every  $k \in U$  and every positive integer  $s_0$  we have

$$(i) \quad L_p(\Phi, \kappa, k, \psi, s_0) = (1 - a_p(\kappa, k)^{-1} \psi \omega^{1-s_0}(p) p^{s_0-1}) \cdot L_p^*(\Phi, \kappa, k, \psi, s_0).$$

Moreover, if  $k$  is an integer  $\geq 2$  in  $U$  and if  $0 < s_0 < k$ , then

$$(ii) \quad L_p^*(\Phi, \kappa, k, \psi, s_0) = \Omega_\kappa(\Phi) \cdot a_p(\kappa, k)^{-m} \cdot \frac{A(\mathbf{f}_{\kappa, k}, \psi \omega^{1-s_0}, s_0)}{\Omega_{\mathbf{f}_{\kappa, k}}^{\text{sgn}(\psi)}}.$$

**e.** (Functional Equation for the improved  $p$ -adic  $L$ -function.) Suppose the  $p$ -component of  $\varepsilon$  is a power of the Teichmüller character, say  $\varepsilon_p = \omega^n$ . Then for every integer  $k \geq 2$  in  $U$  satisfying the congruence  $k \equiv n + k_0 - 2 \pmod{p-1}$  and for every integer  $s_0$  with  $0 < s_0 < k$  we have

$$\begin{aligned} &L_p^*(\Phi, \kappa, k, \omega^{s_0-1}, s_0) - (-N)^{1-s_0} L_p^*(\Phi^*, \kappa^*, k, \omega^{k-s_0-1}, k-s_0) \\ &= L_p(\Phi, \kappa, k, \omega^{s_0-1}, s_0) + L(\tilde{\Phi}_{\kappa, k, s_0}). \end{aligned}$$

*Proof.* The first assertion follows at once from the definitions. The assertions **c**, **d(i)**, and **e** follow at once from **1**, **3**, and **4** of Proposition 5.8. Assertion **d(ii)** follows from **2** of (5.8), together with (5.14) and Theorem 4.14. To prove **b** we note that from **d** and Theorem 4.18 the desired equality holds if  $s_0 = 1$  and  $\psi$  is any nontrivial character. Since both sides of the equation are Iwasawa functions they are determined by these values. This completes the proof of Theorem 5.15.

Note that since the map  $\Phi \mapsto \Phi^*$  is not an involution, the functional equations **c** and **e** are not symmetric in  $\Phi$  and  $\Phi^*$ . Using the identity  $(\Phi^*)^* = \Phi|[-N]_p$  and applying **c** with  $(\Phi^*, \kappa^*)$  replacing  $(\Phi, \kappa)$  we obtain

$$\begin{aligned} &L_p(\Phi^*, \kappa^*, k, \psi, s) \\ &= -\varepsilon_p \omega^{k_0-2} \psi^{-1}(-N) \langle -N \rangle^{k-s-1} \cdot L_p(\Phi, \kappa, k, \varepsilon_p \omega^{k_0-2} \psi^{-1}, k-s). \end{aligned}$$

A similar identity is easily derived for the functional equation of the improved  $p$ -adic  $L$ -function. In case the  $N$ -component of  $\varepsilon$  is trivial the next lemma shows that  $\Phi$  is actually an eigensymbol for the operator  $*$ . In that case the functional equations (5.15)**c** and **e** take on a simpler, more symmetric form which we will exhibit in Corollary 5.17 below.

(5.16) **Lemma.** Let  $\kappa \in \mathcal{X}$  be an arithmetic point of weight  $k_0$  and character  $\varepsilon$  and suppose the conductor of  $\varepsilon$  is prime to  $N$ . Then  $*$  acts on  $\mathbf{W}_{(\kappa)}$  as multiplication by an element  $\mathbf{w} \in \mathcal{R}_{(\kappa)}$  where  $\mathbf{w}^2 = h_{(\kappa)}([-N]_p)$ . Hence for  $k$  in the domain of convergence about  $\kappa$ , we have  $\mathbf{w}(\kappa, k) = w \cdot \langle -N \rangle^{\frac{k-2}{2}}$  where  $w \in \mathbf{Z}_p^*$  is a square root of  $\varepsilon \omega^{k_0-2}(-N)$ .

*Proof.* The condition that  $\varepsilon_N$  is trivial guarantees that the involution  $*$  fixes  $\mathcal{R}_{(\kappa)}$  elementwise. Hence  $*$  induces an automorphism of  $\mathbf{W}_{(\kappa)}$  and this automorphism is identical with  $W_N$ . Moreover, using (1.8) and the fact that  $\varepsilon_N(-1) = 1$ , we see that  $*$  preserves the  $\pm$  submodules  $\mathbf{W}_{(\kappa)}^\pm$ . Since these are free of rank one over  $\mathcal{R}_{(\kappa)}$ ,  $*$  acts on each of them as multiplication by a unit in  $\mathcal{R}_{(\kappa)}$ . Now specialize to  $\mathbf{W}_\kappa$  and use the fact that  $W_N$  acts by a scalar on  $\mathbf{W}_\kappa$  to deduce that  $\mathbf{w}$  acts on all of  $\mathbf{W}_{(\kappa)}$  by the same

unit in  $\mathcal{R}_{(\kappa)}$ . The identity  $\mathbf{w}^2 = h_{(\kappa)}([-N]_p)$  now follows from the fact that  $(\Phi^*)^* = \Phi|[-N]_p$  for all  $\Phi \in \mathbf{W}$ . The last assertion follows from the identity  $\tilde{\kappa}([-N]_p, k) = \varepsilon \omega^{k_0-2}(-N) \langle -N \rangle^{k-2}$ . This completes the proof of the lemma.

(5.17) **Corollary.** *Let  $\Phi \in \mathbf{W}_{(\kappa)}$  where  $\kappa \in \mathcal{X}^{\text{arith}}$  is an arithmetic point of weight  $k_0$  and character  $\varepsilon$  with conductor prime to  $N$ . Let  $w$  be the square root of  $\varepsilon \omega^{k_0-2}(-N)$  determined by  $\kappa$  as in Lemma 5.16. Then*

**a.**  $L_p(\Phi, \kappa, k, \psi, s) = -w \cdot \psi^{-1}(-N) \cdot \langle -N \rangle^{\frac{2-s}{2}}$ .  $L_p(\Phi, \kappa, k, \varepsilon \omega^{k_0-2} \psi^{-1}, k-s)$ ; and

**b.** *Suppose  $\varepsilon = \omega^n$ . Then for each integer  $k \geq 2$  in  $U$  satisfying  $k \equiv n + k_0 - 2 \pmod{p-1}$  and for each integer  $s_0$  satisfying  $0 < s_0 < k$  we have*

$$\begin{aligned} &L_p^*(\Phi, \kappa, k, \omega^{s_0-1}, s_0) - w \cdot \langle -N \rangle^{\frac{k-s_0}{2}} \cdot L_p^*(\Phi, \kappa, k, \omega^{k-s_0-1}, k-s_0) \\ &= L_p(\Phi, \kappa, k, \omega^{s_0-1}, s_0) + L(\tilde{\Phi}_{\kappa, k}, s_0). \end{aligned}$$

(5.18) **Example.** As an illustration we will construct the two-variable  $p$ -adic  $L$ -function described in the introduction associated to  $E = X_0(11)$ ,  $p = 11$ , and verify the properties (0.8). Let  $\Omega_E$  be the real period of  $E$  and let

$$\Psi_E^+ = \frac{1}{\Omega_E} \cdot \Phi_{f_E}^+ \in \text{Symb}_{\Gamma_1(11)}(\mathbf{Q})$$

where  $\Phi_{f_E}^+$  is the plus part of the modular symbol associated to  $f_E$  by (4.7). Then the  $p$ -adic  $L$ -function  $L_p(E, s)$ ,  $s \in \mathbf{Z}_p$ , is given by (4.16) and (4.17). In (2.11) we showed that  $\mathcal{R} = \mathcal{A}$  in this case. Let  $\kappa = \sigma_2$  be the unique arithmetic point on  $\mathcal{A}$  of weight two and trivial character and use Theorem 5.13 to choose a modular symbol  $\Phi \in \mathbf{W}_{(\kappa)}^+$  with nonzero specialization to  $\kappa$ . Since  $\mathcal{R} = \mathcal{A}$  we can also assume that  $\Phi^+$  is integral, i.e.  $\Phi \in \mathbf{W}^+$ , by ‘clearing denominators’. Now define  $L_p(k, s) = \Omega_\kappa(\Phi)^{-1} L_p(\Phi, \kappa, k, s)$  and  $L_p^*(, k, 1) = \Omega_\kappa(\Phi)^{-1} L_p^*(\Phi, \kappa, k, 1)$  for  $k, s \in \mathbf{Z}_p$ . It is clear from the definitions (5.2) that these are Iwasawa functions in both variables (up to multiplication by a scalar). Since the tame level is equal to 1, we have  $\Phi^* = \Phi|W_1$ . But  $\Phi$  is invariant with respect to  $SL(2, \mathbf{Z})$  and is therefore fixed by  $W_1$ . Thus  $\Phi^* = \Phi$ . The properties (0.8) now follow from Theorem 5.15 and its corollary 5.17.

### 6. Existence of $\mathcal{A}$ -adic eigensymbols

In this section we will prove Theorem 5.13. The proof is based on the following two propositions.

(6.1) **Proposition.** *The group  $\mathbf{W}^0$  of ordinary  $\mathcal{A}$ -adic modular symbols is a free  $\mathcal{A}$ -module of finite rank. For each arithmetic point  $\kappa \in \mathcal{X}_0^{\text{arith}}$  let  $\mathbf{P}_\kappa \subseteq \mathbf{Z}_p[[\mathbf{Z}_p^*]]$  be the prime ideal associated to  $\kappa$ . Then for  $\Phi \in \mathbf{W}^0$  we have  $\Phi_\kappa = 0 \Leftrightarrow \Phi \in \mathbf{P}_\kappa \mathbf{W}^0$ .*

(6.2) **Proposition.** *There is a natural injective  $\mathcal{H}$ -homomorphism  $Ta_p(J_\infty)^0 \rightarrow \mathbf{W}_\mathcal{A}^0$ .*

*Proof of Theorem 5.13.* Let  $\kappa \in \mathcal{X}^{\text{arith}}$ . Since  $h(T_p) = a_p$  is a unit in  $\mathcal{R}$ , the module  $\mathbf{W}_{(\kappa)}$  is contained in the ordinary part  $\mathbf{W}^0 \otimes_{\mathcal{A}} \mathcal{R}_{(\kappa)}$ . Since, by Proposition 6.1,  $\mathbf{W}^0$  is a free  $\mathcal{A}$ -module of finite rank,  $\mathbf{W}_{(\kappa)}$  is a free  $\mathcal{R}_{(\kappa)}$ -module of finite rank.

From the last assertion of Proposition 6.1 and the fact that  $\kappa$  is unramified over  $A$  (Hida's Theorem (2.6)a), we conclude that the kernel of  $\phi_{\kappa, *}$  in  $\mathbf{W}_{\mathcal{H}(\kappa)}^0$  is  $P_{\kappa} \mathbf{W}_{\mathcal{H}(\kappa)}^0$ . In particular we see that the linear map  $\mathbf{W}_{(\kappa)}/P_{\kappa} \mathbf{W}_{(\kappa)} \rightarrow \mathbf{W}_{\kappa}$  induced by  $\phi_{\kappa, *}$  is injective.

Since  $\mathbf{W}_{\kappa}$  has dimension two (Theorem 4.8), surjectivity of the map  $\mathbf{W}_{(\kappa)}/P_{\kappa} \mathbf{W}_{(\kappa)} \rightarrow \mathbf{W}_{\kappa}$  will follow if we show that  $\mathbf{W}_{(\kappa)}$  has  $\mathcal{H}(\kappa)$ -rank at least two. Recall the submodule  $\mathbf{T} = Ta_p(J_{\alpha})_{\text{prim}}^0$  of  $Ta_p(J_{\alpha})^0$  from "section 2". By Hida's Theorem 2.6c we know that  $\mathbf{T}_{\mathcal{F}}$  is a free  $\mathcal{H}$ -module of rank two. Using Proposition 6.2 we lift this to a free rank two  $\mathcal{H}$ -submodule of  $\mathbf{W}_{\mathcal{F}}^0$ . Since  $\mathcal{H}$  is a semisimple  $\mathcal{L}$ -algebra, this space projects injectively to the  $h$ -eigenspace in  $\mathbf{W}^0 \otimes_A \mathcal{H}$ . The intersection of this eigenspace with  $\mathbf{W}_{\mathcal{H}(\kappa)}$  is a rank two  $\mathcal{H}(\kappa)$ -submodule of  $\mathbf{W}_{(\kappa)}$ . Hence  $\mathbf{W}_{(\kappa)}$  has rank exactly 2 and the specialization morphism induces an isomorphism  $\mathbf{W}_{(\kappa)}/P_{\kappa} \mathbf{W}_{(\kappa)} \cong \mathbf{W}_{\kappa}$ . Since specialization commutes with the complex conjugation involution, (5.13)b follows. Assertion (5.13)a is a consequence of (5.13)b. This completes the proof.

*Proof of Proposition 6.1*

Our proof of Proposition 6.1 will be based on two simple lemmas. Fix  $\kappa \in \mathcal{A}_0^{\text{arith}}$ . We will say that a function  $\varphi: (\mathbf{Z}_p^2)^{\gamma} \rightarrow \overline{\mathbf{Q}}_p$  is homogeneous of degree  $\kappa$  if  $\varphi(t\mathbf{x}) = \kappa(t)\varphi(\mathbf{x})$  for every  $t \in \mathbf{Z}_p^*$  and every  $\mathbf{x} \in (\mathbf{Z}_p^2)^{\gamma}$ . The following lemma follows easily from the definitions.

(6.3) **Lemma.** *A measure  $\mu \in \mathbf{D}$  lies in  $P_{\kappa} \mathbf{D}$  if and only if  $\int \varphi d\mu = 0$  for every continuous function  $\varphi$  on  $(\mathbf{Z}_p^2)^{\gamma}$  which is homogeneous of degree  $\kappa$ .*

For each integer  $m > 0$  let  $\varphi_{\kappa}^{(m)}$  be the continuous function on  $(\mathbf{Z}_p^2)^{\gamma}$  given by

$$\varphi_{\kappa}^{(m)}(a, b) = \begin{cases} \kappa(a) & \text{if } b \equiv 0 \pmod{p^m}; \\ 0 & \text{otherwise.} \end{cases}$$

(6.4) **Lemma.** *Let  $\Phi \in \mathbf{W}$  be a  $A$ -adic modular symbol. Then the following are equivalent.*

- a.  $\Phi \in P_{\kappa} \mathbf{W}$ .
- b.  $\int \varphi d\Phi(D) = 0$  for all  $D \in \mathcal{D}_0$  and all continuous functions  $\varphi$  homogeneous of degree  $\kappa$ .
- c.  $\int \varphi_{\kappa}^{(m)} d\Phi(D) = 0$  for all  $D \in \mathcal{D}_0$  and all  $m > 0$ .

*Proof.* Since  $P_{\kappa}$  is a principal ideal, we have  $P_{\kappa} \mathbf{W} = \text{Symb}_{\Gamma}(P_{\kappa} \mathbf{D})$ . From this it follows that  $\Phi \in P_{\kappa} \mathbf{W} \Leftrightarrow \Phi(D) \in P_{\kappa} \mathbf{D}$  for all  $D \in \mathcal{D}_0$ . So the equivalence **a**  $\Leftrightarrow$  **b** follows from Lemma 6.3. The implication **b**  $\Rightarrow$  **c** follows *a priori*. Now assume **c** is true. Then for every  $\gamma \in \Gamma$ ,  $\int \gamma \varphi_{\kappa}^{(m)} d\Phi(D) = \int \varphi_{\kappa}^{(m)} d\Phi(\gamma D) = 0$ . So **b** follows from the fact that every continuous function  $\varphi$  which is homogeneous of degree  $\kappa$  is the uniform limit of a sequence of linear combinations of the functions  $\gamma \varphi_{\kappa}^{(m)}$ . This completes the proof of Lemma 6.4.

*Proof of Proposition 6.1.* We first prove the equality  $\ker(\phi_{\kappa, *}) = P_{\kappa} \mathbf{W}^0$ . Recall from (5.6)a that for  $\Phi \in \mathbf{W}$  the specialization  $\Phi_{\kappa}$  is the element of  $\text{Symb}_{\Gamma}(\text{Sym}^{k-2}(\overline{\mathbf{Q}}_p^2))$  whose value on a divisor  $D \in \mathcal{D}_0$  is given by

$$(6.5) \quad \Phi_{\kappa}(D) = \int_{\mathbf{Z}_p^{\times} \times \mathbf{Z}_p} \varepsilon(x)(xY - yX)^{k-2} d\Phi(D).$$

Since the integrand is homogeneous of degree  $\kappa$ , the inclusion  $\ker(\phi_{\kappa, *}) \supseteq P_{\kappa} \mathbf{W}$  follows from the implication  $\mathbf{a} \Rightarrow \mathbf{b}$  of (6.4). Conversely, suppose  $\Phi \in \mathbf{W}^0$  and that  $\Phi_{\kappa} = 0$ . We will show that  $\Phi \in P_{\kappa} \mathbf{W}^0$  by using  $\mathbf{c} \Rightarrow \mathbf{a}$  from the last lemma. Fix  $m > 0$  and  $D \in \mathcal{D}_0$ . Since  $\Phi$  is ordinary, there is a  $\Psi \in \mathbf{W}^0$  such that  $\Psi|T_p^m = \Phi$ . Using the notation of (5.7) we then have the identity

$$\int \varphi_{\kappa}^{(m)} d\Phi(D) = \sum_{\mathbf{x}} \int g_{\mathbf{x}, p^m} \varphi_{\kappa}^{(m)} d\Psi(g_{\mathbf{x}, p^m} \cdot D)$$

where the above sum runs over  $\mathbf{x} \in \mathbf{P}^1(\mathbf{Z}/p^m \mathbf{Z})$ . But  $g_{\mathbf{x}, p^m} \varphi_{\kappa}^{(m)} = 0$  unless  $\mathbf{x} = [1, 0]$ . Hence the above integral is equal to

$$\int g_{[1, 0], p^m} \varphi_{\kappa}^{(m)} d\Psi(g_{[1, 0], p^m} \cdot D) = \int_{\mathbf{Z}_p^{\times} \times \mathbf{Z}_p} \varepsilon(x) x^{k-2} d\Psi(g_{[1, 0], p^m} \cdot D).$$

But this vanishes since it is the coefficient of  $Y^r$  in  $\phi_{\kappa}^*(\Phi)(g_{[1, 0], p^m} \cdot D)$  (see (6.5)). We have therefore proven the equality  $\ker(\phi_{\kappa, *}) = P_{\kappa} \mathbf{W}^0$ , or equivalently, that  $\Phi_{\kappa} = 0 \Leftrightarrow \Phi \in P_{\kappa} \mathbf{W}^0$ . It follows from this and the compact version of Nakayama's lemma that  $\mathbf{W}^0$  is a free  $\mathcal{A}$ -module of finite rank. The proof of Proposition 6.1 is complete.

*Proof of Proposition 6.2*

The proof of Proposition 6.2 is based on a study of the cohomology exact sequence attached to  $\mathbf{D}$  by Theorem 4.3. It will be convenient to simplify the notation and write

$$(6.6) \quad \mathbf{B} = \text{Bound}_{\Gamma}(\mathbf{D}), \quad \mathbf{W} = \text{Symb}_{\Gamma}(\mathbf{D}), \quad \mathbf{V} = H_{\text{par}}^1(\Gamma, \mathbf{D}).$$

It is easy to see that there are no nonzero  $\Gamma$ -invariant measures in  $\mathbf{D}$ , hence Theorem 4.3 gives us an exact sequence

$$(6.7) \quad 0 \rightarrow \mathbf{B} \rightarrow \mathbf{W} \rightarrow \mathbf{V} \rightarrow 0.$$

(6.8) **Lemma.** *There is a natural  $\mathcal{H}$ -isomorphism  $Ta_p(J_{\infty}) \cong \mathbf{V}$ .*

*Proof.* Since the operator  $W_N$  intertwines the covariant and the contravariant actions of  $\mathcal{H}$  on  $\mathbf{V}$ , it will suffice to give an isomorphism  $\pi: \mathbf{V} \cong Ta_p(J_{\infty})$  for the covariant  $\mathcal{H}$ -structure on  $\mathbf{V}$ . For each integer  $n \geq 0$  let  $H_1(X_n(\mathbf{C}), \mathbf{Z}_p)$ ,  $H^1(X_n(\mathbf{C}), \mathbf{Z}_p)$  be the singular homology, respectively cohomology of the compact Riemann surface  $X_n(\mathbf{C})$ . Then the Albanese map gives us a canonical isomorphism  $\text{Alb}: H_1(X_n(\mathbf{C}), \mathbf{Z}_p) \cong Ta_p(J_n)$  of  $\mathcal{H}$ -modules. Moreover, by Poincaré duality the intersection pairing gives us an isomorphism  $H^1(X_n(\mathbf{C}), \mathbf{Z}_p) \cong H_1(X_n(\mathbf{C}), \mathbf{Z}_p)$  which intertwines the covariant action of  $\mathcal{H}$  on cohomology with the natural (covariant) action on homology. Now write  $\mathbf{V}_n$  for  $H_{\text{par}}^1(\Gamma_1(Np^n), \mathbf{Z}_p)$  equipped with the covariant action of  $\mathcal{H}$ . Then the Eichler-Shimura theorem gives us a canonical isomorphism  $ES: \mathbf{V}_n \cong H^1(X_n(\mathbf{C}), \mathbf{Z}_p)$  of  $\mathcal{H}$ -modules. We define  $\xi_n$  to be the composition

$$\xi_n: \mathbf{V}_n \xrightarrow{ES} H^1(X_n(\mathbf{C}), \mathbf{Z}_p) \xrightarrow{PD} H_1(X_n(\mathbf{C}), \mathbf{Z}_p) \xrightarrow{\text{Alb}} Ta_p(J_n).$$

For positive integers  $m, n$  with  $m > n$ , we have a commutative diagram of  $\mathcal{H}$ -modules

$$\begin{array}{ccc} \mathbf{V}_m & \xrightarrow{\xi_m} & Ta_p(J_m) \\ \downarrow & & \downarrow \\ \mathbf{V}_n & \xrightarrow{\xi_n} & Ta_p(J_n) \end{array}$$

where the vertical arrow on the left is the corestriction morphism  $\text{cores}_{m,n}$ . We may therefore construct the limit  $\mathbf{V}_\infty = \varprojlim \mathbf{V}_n$  and patch together an isomorphism

$$\zeta_\infty: \mathbf{V}_\infty \rightarrow Ta_p(J_\infty).$$

For each  $n > 0$ , consider the map  $\pi_n: \mathbf{D} \rightarrow \mathbf{Z}_p$ , given by  $\mu \mapsto \mu((0, 1) + p^n \mathbf{Z}_p^2)$ . This map commutes with the action of  $\Gamma_n$ , hence induces a map  $\pi_{n*}: \mathbf{V} \rightarrow \mathbf{V}_n$ . A simple verification shows  $\pi_{n*} = \text{cores}_{m,n} \circ \pi_{m*}$ . Thus there is a natural homomorphism

$$\pi_*: \mathbf{V} \rightarrow \mathbf{V}_\infty.$$

We will prove that  $\pi_*$  is an isomorphism by using Shapiro's Lemma and the simple observation that  $\mathbf{D}$  is naturally isomorphic to a projective limit of induced modules. For each integer  $n \geq 0$ , let  $\mathbf{M}_n = ((\mathbf{Z}/p^n \mathbf{Z})^2)'$  denote the primitive vectors in  $(\mathbf{Z}/p^n \mathbf{Z})^2$ . Let  $\mathbf{D}_n = \{\mu_n: \mathbf{M}_n \rightarrow \mathbf{Z}_p\}$  be the  $\mathbf{Z}_p$ -valued functions on  $\mathbf{M}_n$  and let  $\Gamma$  act on  $\mathbf{D}_n$  by the rule  $(\mu_n | \gamma)(v_n) = \mu_n(v_n \gamma^{-1})$ . Since  $\Gamma$  acts transitively on  $\mathbf{M}_n$  and  $\Gamma_n$  is the stabilizer of  $(0, 1)$  we see that  $\mathbf{D}_n$  is an induced  $\Gamma$ -module. Hence, by Shapiro's lemma, the map  $\mathbf{D}_n \rightarrow \mathbf{Z}_p, \mu_n \mapsto \mu_n((0, 1))$  induces an isomorphism  $H_{\text{par}}^1(\Gamma_1(N), \mathbf{D}_n) \cong H_{\text{par}}^1(\Gamma_n, \mathbf{Z}_p) = \mathbf{V}_n$ .

For each  $m \geq n$ , let  $\mathbf{M}_m \rightarrow \mathbf{M}_n$  be the natural projection and define the connecting homomorphism  $\delta_{m,n}: \mathbf{D}_m \rightarrow \mathbf{D}_n$  by  $\delta_{m,n}(\mu_m) = \mu_n$  where for each  $v_n \in \mathbf{M}_n$ ,  $\mu_n(v_n) = \sum \mu_m(v_m)$  where the sum is over all  $v_m \in \mathbf{M}_m$  lying over  $v_n$ . The maps  $\mathbf{D} \rightarrow \mathbf{D}_n$  given by  $\mu \mapsto \mu_n$  where  $\mu_n(v_n) = \mu(v_n + p^n \mathbf{M})$  for  $v_n \in \mathbf{M}_n$  commute with the connecting homomorphisms  $\delta_{m,n}$  and induce an isomorphism

$$\mathbf{D} \cong \varprojlim \mathbf{D}_n.$$

Since  $\Gamma$ -cohomology commutes with projective limits in the category of compact  $\Gamma$ -modules we conclude

$$\mathbf{V} \cong \varprojlim H^1(\Gamma(N), \mathbf{D}_n) \rightarrow \varprojlim \mathbf{V}_n.$$

This completes the proof of Lemma 6.8.

To complete the proof of Proposition 6.2 we need to construct a Hecke equivariant splitting of the exact sequence (6.7). In general such a splitting does not exist over  $A$ . The obstruction is a group analogous to the classical cuspidal divisor class group. We need to first extend scalars to the quotient field  $\mathcal{L}$  of  $A$ . Then, as in the classical setting, we will apply the Manin-Drinfeld principle to produce the desired splitting over  $\mathcal{L}$ . In order to use the Manin-Drinfeld principle we first need to analyze the structure of the  $\mathcal{H}$ -module  $\mathbf{B} = \text{Bound}_\Gamma(\mathbf{D})$ . The next lemma shows that  $\mathbf{B}$  is a free  $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$ -module of finite rank and

gives an explicit description of the action of the Hecke operators  $T_l$  for  $l \equiv 1 \pmod N$ .

For each  $\mathbf{x} \in \mathbf{P}^1(\mathbf{Q})$  let  $\Gamma_{\mathbf{x}} \subseteq \Gamma$  be the unipotent stabilizer of  $\mathbf{x}$  in  $\Gamma$ . Also, let  $\mathbf{M}_{\mathbf{x}}$  denote the ‘line’ in  $(\mathbf{Z}_p^2)'$  which is stabilized by  $\Gamma_{\mathbf{x}}$ .

(6.9) **Lemma.**

a. Let  $\text{cusps}(\Gamma) \subseteq \mathbf{P}^1(\mathbf{Q})$  be a complete set of representatives of the  $\Gamma$ -orbits in  $\mathbf{P}^1(\mathbf{Q})$ . Then there is a natural isomorphism of  $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$ -modules

$$\mathbf{B} \cong \bigoplus_{\mathbf{x} \in \text{cusps}(\Gamma)} \text{Dist}(\mathbf{M}_{\mathbf{x}}).$$

Hence  $\mathbf{B}$  is a free  $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$ -module of rank  $\# \text{cusps}(\Gamma)$ .

b. If  $l$  is a prime different from  $p$  which is congruent to 1 modulo  $N$  then  $T_l - l[l] - 1$  annihilates  $\mathbf{B}$ .

*Proof.* A function  $\Phi: \mathbf{P}^1(\mathbf{Q}) \rightarrow \mathbf{D}$  represents a boundary symbol  $\Phi \in \mathbf{B}$  if and only if  $\Phi(\gamma \mathbf{x}) = \Phi(\mathbf{x})|\gamma$  for all  $\mathbf{x} \in \mathbf{P}^1(\mathbf{Q})$  and  $\gamma \in \Gamma$ . Hence  $\Phi(\mathbf{x}) \in \mathbf{D}^{\Gamma_{\mathbf{x}}}$  and the map

$$\begin{aligned} \mathbf{B} &\rightarrow \bigoplus_{\mathbf{x} \in \text{cusps}(\Gamma)} \mathbf{D}^{\Gamma_{\mathbf{x}}} \\ \Phi &\mapsto \sum_{\mathbf{x} \in \text{cusps}(\Gamma)} \Phi(\mathbf{x}) \end{aligned}$$

is an isomorphism. Thus our problem is reduced to a determination of all measures on  $(\mathbf{Z}_p^2)'$  which are invariant under  $\Gamma_{\mathbf{x}}$ . We can obtain examples of such measures by ‘extending by zero’ measures on  $\mathbf{M}_{\mathbf{x}}$ . More precisely, define  $i_{\mathbf{x}}: \text{Dist}(\mathbf{M}_{\mathbf{x}}) \rightarrow \mathbf{D}^{\Gamma_{\mathbf{x}}}$  by  $i_{\mathbf{x}}(v) = \mu$  where  $\mu \in \mathbf{D}$  is given by the integration formulas

$$\int \varphi(x) d\mu(x) = \int_{\mathbf{M}_{\mathbf{x}}} \varphi(v) dv(v)$$

for all locally constant functions  $\varphi \in \text{Step}((\mathbf{Z}_p^2)')$ . Lemma 6.9a follows immediately from the following lemma.

(6.10) **Lemma.** The map  $i_{\mathbf{x}}: \text{Dist}(\mathbf{M}_{\mathbf{x}}) \rightarrow \mathbf{D}^{\Gamma_{\mathbf{x}}}$  is an isomorphism of  $\mathbf{Z}_p[[\mathbf{Z}_p^*]]$ -modules.

*Proof.* We will prove this assertion for any congruence group  $\Gamma$  whose level is prime to  $p$ . The collection of all such groups is closed under conjugation by elements of  $SL_2(\mathbf{Z})$ . Since  $\mathbf{P}^1(\mathbf{Q})$  is acted on transitively by  $SL_2(\mathbf{Z})$ , it suffices to prove the lemma in the special case  $\mathbf{x} = \infty$ . The map  $i_{\infty}$  is clearly injective. We must show that it is surjective. So let  $\mu \in \mathbf{D}^{\Gamma_{\infty}}$ . We need to show that  $\mu$  is supported on the line  $\mathbf{M}_{\infty} = \{(0, r) | r \in \mathbf{Z}_p^*\}$ . Let  $U$  be an arbitrary compact open subset of  $(\mathbf{Z}_p^2)'$  which is disjoint from  $\mathbf{M}_{\infty}$ . We will show  $\mu(U) = 0$ .

Choose a positive integer  $m_0$  such that  $v + p^{m_0} \mathbf{Z}_p^2 \subseteq U$  for every  $v \in U$ . Each  $v$  in  $U$  can be expressed as  $v = (p^r a, b)$  for some  $a \in \mathbf{Z}_p^*$ ,  $b \in \mathbf{Z}_p$ , and  $r \geq 0$ . Since the set  $v + p^{m_0} \mathbf{Z}_p^2$  is contained in  $U$  it does not intersect  $\mathbf{M}_{\infty}$ . Hence  $0 \notin p^r a + p^{m_0} \mathbf{Z}_p$  and consequently  $r < m_0$ . Now fix an arbitrary integer  $n \geq 0$ . Then  $U$  is the disjoint union of sets of the form  $V = v + p^{m_0+n} \mathbf{Z}_p^2$ . The open sets  $V_k = (p^r a + p^{m_0+n} \mathbf{Z}_p) \times (b + p^{n+m_0} k + p^{2m_0+2n-r} \mathbf{Z}_p)$ ,  $k = 0, 1, \dots, p^{m_0+n-r} - 1$ , are all

$\Gamma_\infty$ -equivalent to one another. Indeed  $V_k = V_0 \cdot \begin{pmatrix} 1 & p^{m_0+n-r}ka^{-1} \\ 0 & 1 \end{pmatrix}$ . Hence all of these sets have the same measure under  $\mu$ . Since  $V$  is the disjoint union of the  $V_k$  we conclude that  $\mu(V) = p^{m_0+n-r}\mu(U_0) \equiv 0 \pmod{p^n}$ . Since  $U$  is a disjoint union of such sets it follows that  $\mu(U) \equiv 0 \pmod{p^n}$ . But  $n$  is arbitrary. Thus  $\mu(U) = 0$ . This completes the proof of Lemma 6.10 and hence also the proof of assertion **a** of Lemma 6.9.

To prove **b** of Lemma 6.9, we let  $l$  be a prime which is congruent to one modulo  $Np$  and compute  $\Psi|T_l$  for an arbitrary element  $\Psi \in \mathbf{B}$ . Let  $\mathbf{x} \in \mathbf{P}^1(\mathbf{Q})$  and choose  $g \in SL_2(\mathbf{Z})$  so that  $g\mathbf{x} = \infty$ . For  $k=0, \dots, l-1$  let  $\sigma_k = g^{-1} \begin{pmatrix} 1 & k \\ 0 & l \end{pmatrix} g$  and let  $\sigma_l = g^{-1} \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} g$ . Using the fact that  $l \equiv 1 \pmod{N}$  it is easy to see that the Hecke operator  $T_l$  is represented by the double coset  $\Gamma\sigma_l\Gamma$ . Moreover this double coset can be expressed as the disjoint union of the right cosets  $\Gamma\sigma_k$ ,  $k=0, \dots, l$ . Thus

$$(\Phi|T_l)(\mathbf{x}) = \sum_{k=0}^l \Phi(\sigma_k \mathbf{x})| \sigma_k.$$

Since each  $\sigma_k$  fixes  $\mathbf{x}$  we have  $(\Phi|T_l)(\mathbf{x}) = \sum_{k=0}^l \Phi(\mathbf{x})| \sigma_k$ . A simple calculation shows that  $\sigma_l$  acts trivially on  $\mathbf{M}_\mathbf{x}$  and that for  $k=0, \dots, l-1$ ,  $\sigma_k$  acts on  $\mathbf{M}_\mathbf{x}$  by multiplication by  $l$ . Hence  $\Phi(\mathbf{x})| \sigma_k = [l] \Phi(\mathbf{x})$  for  $k=0, \dots, l-1$  and  $\Phi(\mathbf{x})| \sigma_l = \Phi(\mathbf{x})$ . Now **b** follows easily, and Lemma 6.9 is proved.

(6.11) **Lemma.** *Let  $l \neq p$  be a positive prime with  $l \equiv 1$  modulo  $N$  and let  $\eta_l = T_l - l[l] - 1 \in \mathcal{H}$ . Then  $\eta_l$  acts invertibly on  $\mathbf{V}_\mathcal{L}^0$  and annihilates  $\mathbf{B}$ . Moreover, the exact sequence*

$$0 \rightarrow \mathbf{B}_\mathcal{L}^0 \rightarrow \mathbf{W}_\mathcal{L}^0 \rightarrow \mathbf{V}_\mathcal{L}^0 \rightarrow 0$$

*admits a unique splitting which commutes with  $\mathcal{H}$ .*

*Proof.* The fact that  $\eta_l$  annihilates  $\mathbf{B}$  was proved in Lemma 6.9b. By the Weil bounds we see that the kernel of  $\eta_l$  acting on  $Ta_p(J_\infty)$  is trivial. So, by Lemma 6.8, the same is true of  $\mathbf{V}$ . Hence  $\eta_l$  acts invertibly on the finite dimensional  $\mathcal{L}$  vector space  $\mathbf{V}_\mathcal{L}^0$ . This proves the first assertion of the proposition. To construct a section  $s: \mathbf{V}_\mathcal{L}^0 \rightarrow \mathbf{W}_\mathcal{L}^0$  we proceed as follows. For each  $v \in \mathbf{V}_\mathcal{L}^0$  choose an element  $\tilde{w} \in \mathbf{W}_\mathcal{L}^0$  lying over  $\eta_l^{-1}v$  and set  $s(v) = \eta_l \tilde{w}$ . Clearly  $s(v)$  lies over  $v$ . It does not depend on the choice of  $\tilde{w}$  since  $\eta_l$  annihilates  $\mathbf{B}$ . This splitting is Hecke equivariant since  $\mathcal{H}$  is commutative. If  $s'$  is another such section, then  $s - s'$  is a morphism from  $\mathbf{V}_\mathcal{L}^0$  to  $\mathbf{B}_\mathcal{L}^0$  which intertwines  $\eta_l$ . Hence  $s = s'$  and Lemma 6.11 is proved.

*Proof of Proposition 6.2.* By composing the isomorphism (6.8)  $Ta_p(J_\infty)^0 \rightarrow \mathbf{V}^0$  with the section  $\mathbf{V}_\mathcal{L}^0 \rightarrow \mathbf{W}_\mathcal{L}^0$  constructed in Lemma 6.11, we obtain a natural injective  $\mathcal{H}$ -homomorphism  $Ta_p(J_\infty)^0 \rightarrow \mathbf{W}_\mathcal{L}^0$ . This completes the proof of Proposition 6.2.

(6.12) **Remark.** A more careful analysis of the proof of Lemma 6.9b reveals explicit formulas for the action of the Hecke operators  $T_l$  for any prime  $l$  not

dividing  $N$ , including  $l=p$ . In particular, we have  $\mathbf{B}=\mathbf{B}^0$ . As a consequence we find that the splitting of Lemma 6.11 extends to a splitting of the sequence

$$0 \rightarrow \mathbf{B}_{\mathcal{G}} \rightarrow \mathbf{W}_{\mathcal{G}} \rightarrow \mathbf{V}_{\mathcal{G}} \rightarrow 0.$$

Indeed, we have  $\mathbf{W}^{\text{nil}}=\mathbf{V}^{\text{nil}}$ . By analogy with the classical situation, it is natural to regard the  $p$ -adic  $L$ -functions defined in the last section as being associated to elements of  $Ta_p(J_{\infty})$ . More precisely, for each  $x \in Ta_p(J_{\infty})$ , we view  $x$  as an element of  $\mathbf{V}$  by Lemma 6.8 and then use the splitting  $s$  of the above sequence to lift  $x$  to a  $\mathcal{A}$ -adic modular symbol  $\Phi_x \in \mathbf{W}_{\mathcal{G}}$ . Since  $s$  is not in general defined over  $\mathcal{A}$ , we cannot say that  $\Phi_x \in \mathbf{W}$ . There will, in general, be denominators and we can ask what kind of poles these denominators will pass on to the  $p$ -adic  $L$ -functions associated to  $\Phi_x$ . We can say the following.

First, it is not difficult to see that the standard 2-variable  $p$ -adic  $L$ -function  $L_p(\Phi_x)$  has no poles. Indeed, we can find another modular symbol  $\Phi'_x \in \mathbf{W}$  which is congruent to  $\Phi_x$  modulo  $\mathbf{B}_{\mathcal{G}}$ , i.e.  $\Phi_x = \Phi'_x + \Psi$  for some  $\Psi \in \mathbf{B}_{\mathcal{G}}$ . But  $L(\Psi) \in \mathbf{D}$  is a measure which, by (6.10) and the definition of  $L(\Psi)$ , is supported on the set  $\{(a, b) \in \mathbf{Z}_p^2 \mid ab=0\}$ . Since  $L_p(\Psi)$  is defined by an integral over  $\mathbf{Z}_p^* \times \mathbf{Z}_p^*$  (see 5.2a) we have  $L_p(\Psi)=0$ . So  $L_p(\Phi_x)=L_p(\Phi'_x)$  and this is everywhere regular since  $\Phi'_x$  is everywhere regular.

Second, we can bound the denominators which arise in the  $\mathcal{A}$ -adic modular symbols  $\Phi_x$  for  $x \in Ta_p(J_{\infty})$  as follows. The submodule  $s^{-1}(\mathbf{W}) \subseteq \mathbf{V}$  is clearly preserved by the Hecke operators. In fact, it can be shown that  $s^{-1}(\mathbf{W})$  corresponds to a Galois invariant submodule  $S$  of  $Ta_p(J_{\infty})$ . The quotient  $C = Ta_p(J_{\infty})/S$  is a natural analog of the classical cuspidal divisor class group. It would be interesting to analyze the structure of  $C$  and to attempt an Eisenstein descent along the lines of [Mz1]. From the above discussion we see that  $C$  is annihilated by the operators  $T_l - 1 - l[l]$  for primes  $l \equiv 1$  modulo  $N$  with  $l \neq p$ . Hence  $C$  is a torsion  $\mathcal{A}$ -module which is annihilated by  $a_l - 1 - l[l]$ . From this it follows that the denominator in  $\Phi_x$  is a divisor of  $a_l - 1 - l[l]$  for every positive prime  $l \equiv 1$  modulo  $N$ . While the denominator in  $\Phi_x$  does not contribute poles to the standard 2-variable  $p$ -adic  $L$ -function (see last paragraph), we expect that it will contribute poles to the improved  $p$ -adic  $L$ -function  $L_p^*(\Phi_x)$ . Note that, by the Weil bounds,  $a_l - 1 - l[l]$  does not vanish at any arithmetic point  $\kappa \in \mathcal{X}^{\text{arith}}$ . Hence each  $\Phi_x$  is regular at these points, and correspondingly  $L_p^*(\Phi_x, \kappa, \sigma)$  is regular at arithmetic points.

### 7. The main theorem

We are now ready to prove our main theorem.

(7.1) **Theorem.** *Suppose  $f$  is a weight 2 newform which is split multiplicative at a prime  $p \geq 5$ . Then  $L_p(f, 1)=0$  and*

$$L_p(f, 1) = \mathfrak{L}_p(f) \cdot \frac{L_{\infty}(f, 1)}{\Omega_f^+}.$$

*Proof.* The assertion  $L_p(f, 1)=0$  follows from the interpolation properties of  $L_p(f, s)$  described in Theorem 4.18. Indeed, the eigenvalue of  $T_p$  acting on  $f$

is 1, hence the ‘Euler factor’ appearing in 4.18 vanishes when  $s_0=1$  and  $\psi$  is trivial.

Let  $N$  be the tame level of  $f$  and let  $\kappa \in \mathcal{X}^{\text{arith}}$  be the arithmetic point of weight two for which  $f = \mathbf{f}_\kappa$ . Using Theorem 5.13 we can choose an eigensymbol  $\Phi \in \mathbf{W}_{(\kappa)}^+$  so that  $\Phi_\kappa = \Psi_{\mathbf{f}_\kappa}^+$ . Let  $L_p(\Phi, \kappa, k, s)$ ,  $L_p(\Phi^*, \kappa^*, k, s)$ ,  $L_p^*(\Phi, \kappa, k, s)$  and  $L_p^*(\Phi^*, \kappa^*, k, s)$  be the functions defined in (5.11). They are defined and analytic for all  $(k, s) \in U \times \mathbf{Z}_p$  for some neighborhood  $U$  of 2 in  $\mathbf{Z}_p$ . Since  $f$  is split multiplicative at  $p$ , its nebentype character  $\varepsilon$  satisfies  $\varepsilon(p) = 1$ . Hence, by (2.9)b, we have  $a_p(\kappa, k) = a_p(\kappa^*, k)$ . We call this function  $a_p(k)$  and note that  $a_p(2) = 1$ . From 5.15d(i) we deduce the identities

$$(7.2) \quad \begin{aligned} L_p(\Phi, \kappa, k, 1) &= (1 - a_p(k)^{-1}) \cdot L_p^*(\Phi, \kappa, k, 1), \\ L_p(\Phi^*, \kappa^*, k, 1) &= (1 - a_p(k)^{-1}) \cdot L_p^*(\Phi^*, \kappa^*, k, 1). \end{aligned}$$

In particular, each of the functions  $L_p(\Phi, \kappa, k, s)$  and  $L_p(\Phi^*, \kappa^*, k, s)$  vanishes at  $(k, s) = (2, 1)$ . In fact, we will show that the Taylor expansions of these two functions have the same linear terms around the point  $(2, 1)$ . To see this we define constants  $c, d \in \mathbf{Q}_p$  (in fact, we have  $c, d \in K_\kappa$ ) for which

$$L_p(\Phi, \kappa, k, s) \sim c(-\frac{1}{2}(k-2) + (s-1)) + d(k-2)$$

where  $f(k, s) \sim g(k, s)$  means that  $f$  and its first partials agree with  $g$  and its first partials at the point  $(2, 1)$ . Replacing  $s$  by  $k-s$  in the functional equation 5.15c, we obtain

$$L_p(\Phi, \kappa, k, k-s) = -\langle -N \rangle^{1+s-k} L_p(\Phi^*, \kappa^*, k, s).$$

Hence  $L_p(\Phi^*, \kappa^*, k, s) \sim -L_p(\Phi, \kappa, k, k-s)$  and we easily calculate its linear terms

$$L_p(\Phi^*, \kappa^*, k, s) \sim c(-\frac{1}{2}(k-2) + (s-1)) - d(k-2).$$

Hence  $L_p(\Phi, \kappa, k, s) - L_p(\Phi^*, \kappa^*, k, s) \sim 2d(k-2)$ . To see that  $d=0$  we set  $s=1$  and calculate this difference using (7.2).

$$(7.3) \quad \begin{aligned} L_p(\Phi, \kappa, k, 1) - L_p(\Phi^*, \kappa^*, k, 1) \\ = (1 - a_p(k)^{-1}) \cdot (L_p^*(\Phi, \kappa, k, 1) - L_p^*(\Phi^*, \kappa^*, k, 1)). \end{aligned}$$

From 5.15e we have  $L_p^*(\Phi, \kappa, 2, 1) - L_p^*(\Phi^*, \kappa^*, 2, 1) = L_p(\Phi, \kappa, 2, 1) + L(\tilde{\Phi}_\kappa, 1)$  where  $\tilde{\Phi}_\kappa$  is the tame specialization of  $\Phi$  defined in (5.6)b. We have already seen  $L_p(\Phi, \kappa, 2, 1) = 0$ . As for  $L(\tilde{\Phi}_\kappa, 1)$ , we note that  $\tilde{\Phi}_\kappa$  is a weight 2 modular symbol over  $\Gamma_1(N)$  with the same eigenvalues as  $f$  for the Hecke operators  $T_l, l \nmid Np$ . But  $f$  is a newform whose level is divisible by  $p$ . Hence we must have  $\tilde{\Phi}_\kappa = 0$ . In particular,  $L(\tilde{\Phi}_\kappa, 1) = 0$ , and it follows that  $L_p^*(\Phi, \kappa, 2, 1) - L_p^*(\Phi^*, \kappa^*, 2, 1) = 0$ . Thus the expression in (7.3) has a double zero at  $k=2$ , and consequently  $d=0$ . We therefore have

$$(7.4) \quad L_p(\Phi, \kappa, k, s) \sim c(-\frac{1}{2}(k-2) + (s-1)).$$

We complete the proof, as in the introduction, by calculating the constant  $c$  in two ways. From 5.15 b we have  $L_p(\Phi, \kappa, 2, s) = L_p(f, s)$ . Hence, setting  $k=2$  in (7.4) we obtain

$$c = L_p(f, 1).$$

On the other hand, setting  $s=1$  in (7.4) and differentiating the first identity of (7.2) with respect to  $k$  we find

$$-\frac{1}{2} \cdot c = a'_p(2) \cdot L_p^*(\Phi, \kappa, 2, 1).$$

But from Theorem 3.18 we have  $a'_p(2) = -\frac{1}{2} \Omega_p(f)$  and since  $\Omega_\kappa^+(\Phi) = 1$ , Theorem 5.15 d(ii) gives us the identity  $L_p^*(\Phi, \kappa, 2, 1) = L_\infty(f, 1) / \Omega_f^+$ . Hence

$$c = \Omega_p(f) \cdot \frac{L_\infty(f, 1)}{\Omega_f^+}$$

and the proof is complete.

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