

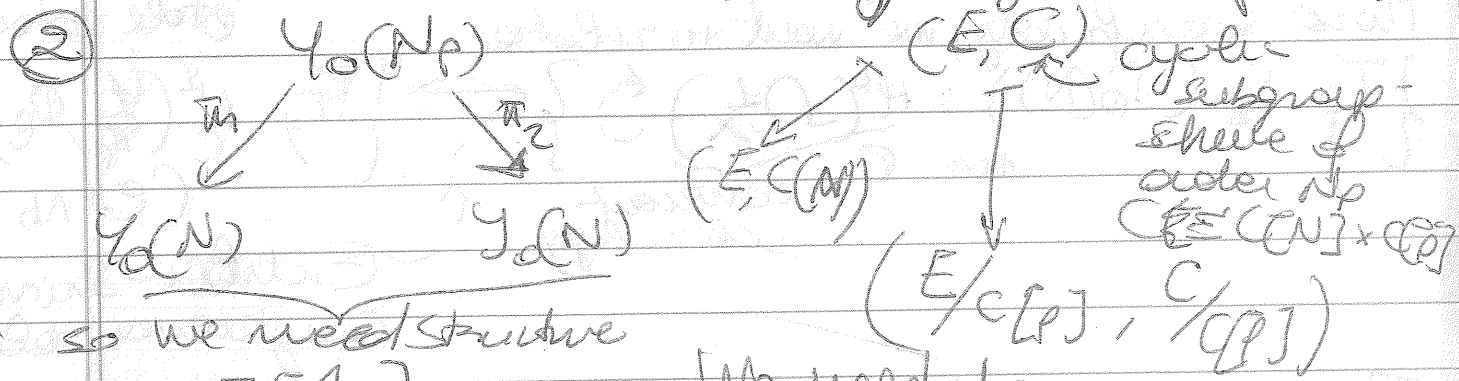
2L seminar Hecke traces + counting mod p

Star 13 We want to relate $\text{Tr}(\Gamma_p, \int_2 (\Gamma_0(N))_S)$ and $\# Y_0(N)(\mathbb{F}_p)$

Warmings! The ideal ~~story~~ ~~work~~ ~~with~~ this particular case, but only due to miracles, which usually do not work ~~in~~ in higher situations

(1) $Y_0(N)$ is a coarse moduli space ~~is~~ ~~usually~~ usually base change, reduction mod p, and invariants for a group action do not commute

- (i) $Y(\mathbb{F}_p) = Y(\overline{\mathbb{F}_p})$ with $Y(\overline{\mathbb{F}_p})$ ~~is~~ ~~of~~ ~~...~~ (same classes)
- (ii) Focusing on weight 2 means we get by using constant sheaves on Y (not hyper symmetric points)

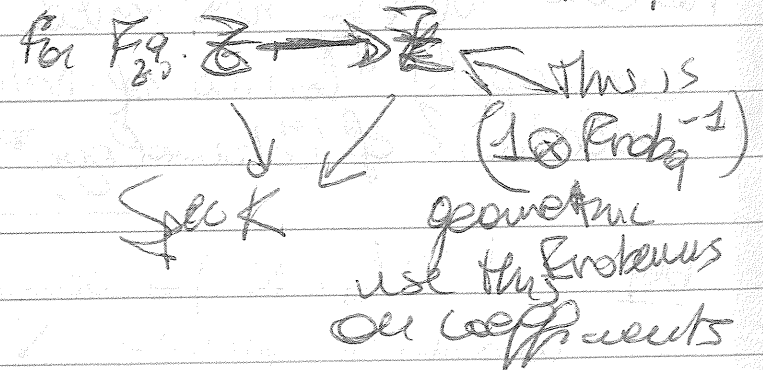


so we need structure over $\mathbb{Z}[\frac{1}{N}]$

We need to go $\mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Z}[\frac{1}{N}] \rightarrow \mathbb{F}_p$

Recall Thm (Grothendieck-Lefschetz trace formula) $K =$ finite field, $Z =$ separated scheme finite type $\ell \neq \text{char } K$, then $\# Z(K) = \sum (-1)^i \text{Tr}(H_{2i}^c(Z, \mathbb{Q}_\ell))$

At constant coefficient cohomology, the absolute Frobenius induce the identity map.



Note For affine curves $H_c^0(\mathbb{C}, \mathcal{O}_C) \cong \mathbb{C}$ compact
 $H_c^1(\mathbb{C}, \mathcal{O}_C) \cong \mathbb{C}(-1)$ cohomology

We want to relate " $T_p \text{ mod } p$ " to $\text{Fr}_Y: Y_{\mathbb{F}_p} \rightarrow Y_{\mathbb{F}_p}$
 So we will relate ~~T_p~~ Frobenius of T_p to the bundle
 by Grothendieck, Lefschetz.

More specifically, we want to relate étale cohom.
 $\{ T_p: S_2(\mathbb{P}^1(N)) = H^0(\mathbb{P}^1, \mathcal{O}(2N)) \} \xleftrightarrow{\text{over } \mathbb{C}} \{ \text{Fr}_Y: H_c^1(Y_{\mathbb{F}_p}, \mathcal{O}_Y) \}$
coherent cohomology ↑ Eichler-Shimura congruence relation

This will succeed in part b/c geometry of $Y_0(N)_{\mathbb{F}_p}$ is not too bad.

Key input $Y_0(N)$ has a smooth compactification $X_0(N)$ over $\mathbb{Z}[\frac{1}{N}]$ such that $(X \setminus Y)_{\text{red}} = X_\infty$ is smooth, and étale over $\mathbb{Z}[\frac{1}{N}]$

Moreover $Y_0(N)_{\mathbb{F}_p}$ has mild singularities so we can apply "theorem of Deligne" to establish congruence relations of cohomology. (Barthel 1980)

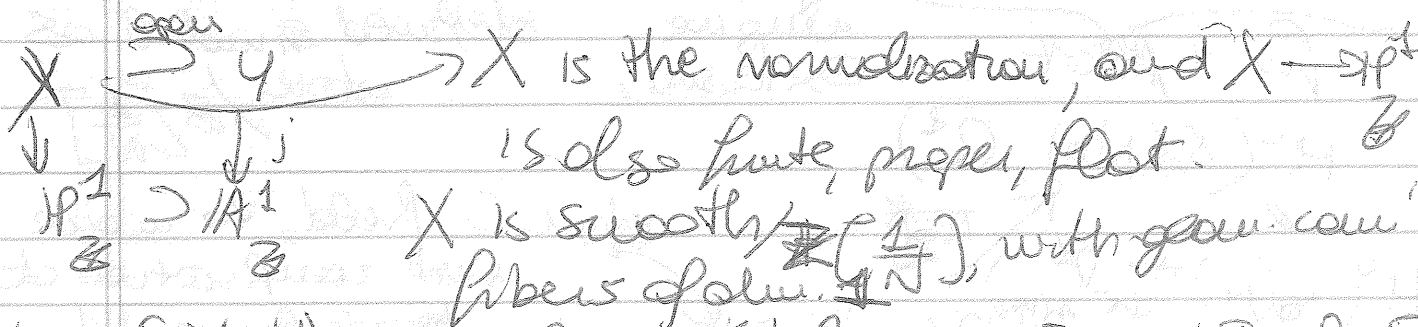
Def $Y = Y_0(N)$ = coarse moduli scheme over \mathbb{Z} for moduli prime $S \rightarrow \text{Hom}(\mathbb{C}^*, \mathbb{C}^*) / \mathbb{Z}/N$
 (This is smooth / $\mathbb{Z}[\frac{1}{N}]$ with geom. connected fibres, dim 1)

with E elliptic curves, $C \subset E[N]$ a cyclic subgroup scheme of order N .

- If S are $\mathbb{Z}[\frac{1}{N}]$, just necessary $C \rightarrow S$ be finite étale with $\sum_{i=1}^N C_i =$ cyclic of order N
- S over $\mathbb{Z}[\frac{1}{N}]^S$ for $k \neq \mathbb{F}_p$, just means $C = C_1 + C_2$ with C_1 as in (1) and C_2 finite flat of order p

All the constructions will make sense with this definition, even if it is not a "nice" moduli space

Similarly $X \rightarrow Y \rightarrow \mathbb{A}^1$ j -invariant map
 proper (by valuative criterion) + ~~finite~~ \Rightarrow quasi-finite
 \Rightarrow finite \Rightarrow flat (by miracle flatness)



$X_{00} = (X=Y)$ red is finite étale / $\mathbb{Z}[\frac{1}{N}]$ // $\mathbb{Z}[\frac{1}{N}] \subset \mathbb{Z}[\frac{1}{N}]$
 Prof (see Katz - Mazur) for $k \subset \mathbb{C} \subset \mathbb{Z}[\frac{1}{N}]$

Fact Formulas of X, Y computed with any base change

Remark In a natural way $X_{\mathbb{C}} \cong \text{used } X_0(N)$
 by GAGA

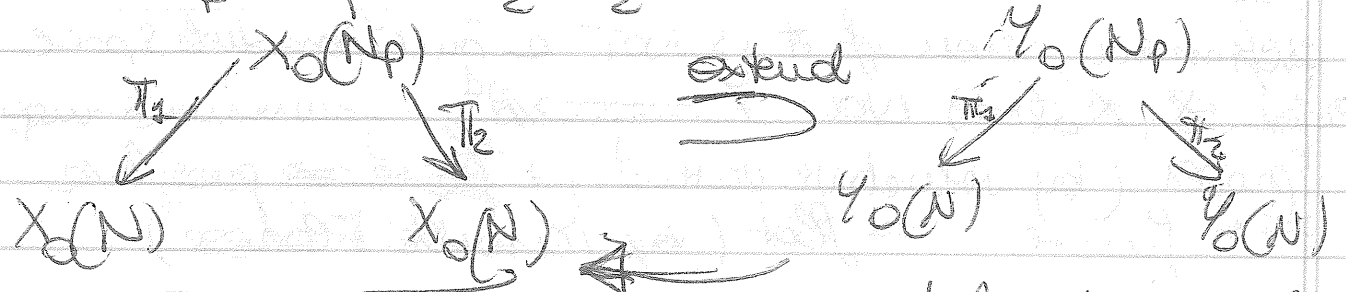
Now $S_2(\mathbb{F}_0(N)) \cong H^0(X_{\mathbb{C}}, \mathcal{O}^{\oplus 2}) = H^0(X_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{O}_{X_{\mathbb{C}}}^{\oplus 2}) \otimes_{\mathbb{C}} \mathbb{C}$
 since the curve is naturally defined over \mathbb{C}

By Hodge theory (= weight 2 Eichler-Shimura isomorphism)

$$S_2 \oplus \overline{S_2} \cong H^1(X^{\text{an}}, \mathbb{C}) \cong H^1(Y^{\text{an}}, \mathbb{C})$$

where $H^i = \text{im}(H_i^{\text{ét}} \rightarrow H_i^{\text{an}})$ $X^{\text{an}} = \text{finite } \mathbb{C}$
 image of compactly supported cohomology

Now we can use that H^1, H^2 (value $X-i$) are dual!
 Now $T_p \oplus T_p \cong \Sigma_1 \oplus \Sigma_2$



Prink plot $H^0(X_0(Np), \mathbb{R}^2)$ defined as module as above $\mathbb{Z}[\frac{1}{N}]$

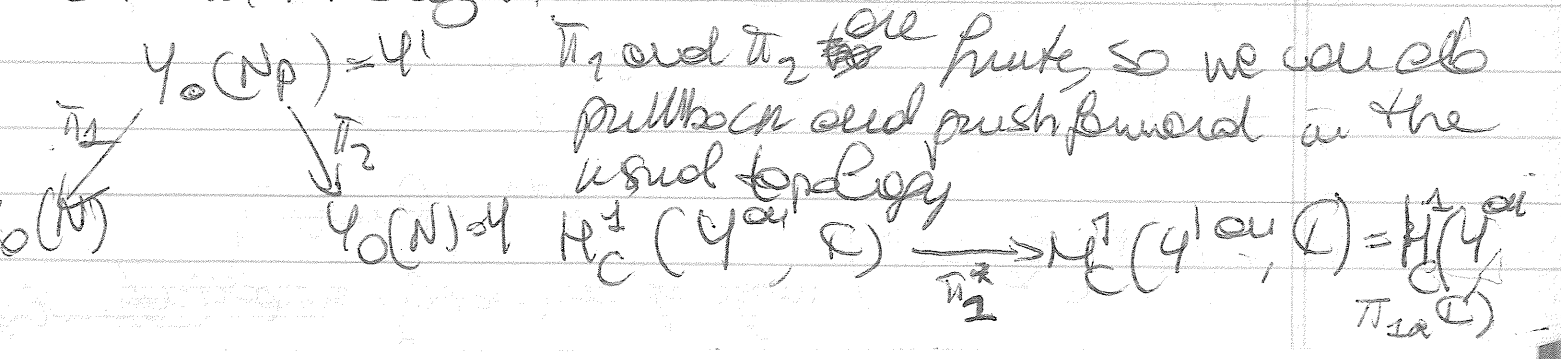
$H^0(X_0(N), \mathbb{R}^2)$ $\xrightarrow{\pi_1^*}$ $H^0(X_0(Np), \mathbb{R}^2)$ $\xrightarrow{\pi_2^*}$ $H^0(X_0(N), \mathbb{R}^2)$

char 0 field, to make sure ramification at branch locus $X' \rightarrow X$ is tame $X'_0(Np) \rightarrow X''_0(N)$

~~$H^0(X, \mathbb{R}^2)$~~
 $H^0(X_0(N), \mathbb{R}^2)$

this are \mathbb{C} vectors T_p , but works over \mathbb{Q} , so we can "define" T_p over \mathbb{Q} , and it respects the rotational structure

now $T_p \oplus T_p \cong \Sigma_1 \oplus \Sigma_2$, and we want to transport the action through.



One can check that this extends at the branch points.

to the same pullback + pushforward for H^1 have ~ 1
 Prop $H^1(Y_{\mathbb{C}}, \mathbb{C}) \xrightarrow{\pi_1 \oplus \pi_2} H^1(Y_{\mathbb{C}}, \mathbb{C})$ is $\mathbb{Z} \oplus \mathbb{Z}$ via
 Eichler-Shimura

we can replace \mathbb{C} with \mathbb{F} for any field of char 0, in particular \mathbb{Q} coefficients

Action comparison thm: relating cohomology of \mathbb{C} -coeff \rightarrow \mathbb{Q} -coeff
 can also be computed using $H_{\text{ét}}^1(Y_{\mathbb{Q}}, \mathbb{Q}) = H_{\text{ét}}^1(Y_{\mathbb{C}}, \mathbb{Q})$

$2 \text{Tr}(\rho) = \sum_{\pi_1 \oplus \pi_2} \text{Tr}(\rho)$ \rightarrow $H_{\text{ét}}^1(Y_{\mathbb{Q}}, \mathbb{Q})$ \leftarrow constant coeff. cohomology

we now relate cohomology on general fiber with cohomology on general moduli fiber
 We're working on non-singular stuff ($Y_0(N)$ is not)

Base change for $H^1(Y/X) \rightarrow D$ relative normal crossing divisor $=$ \mathbb{Z} usually locally around D , we have an étale neighborhood S covered by D and U \rightarrow S \leftarrow D

This applies in our situation to $Y = X \setminus D$ and apply this to \mathbb{Z}
 let $G =$ locally constant, constructible sheaf on $Y_{\text{ét}}$ whose torsion orders are invertible in S
 (e.g. $G = \mathbb{Z}/n\mathbb{Z}$, $S = \text{Spec } \mathbb{Z}[\frac{1}{n}]$)

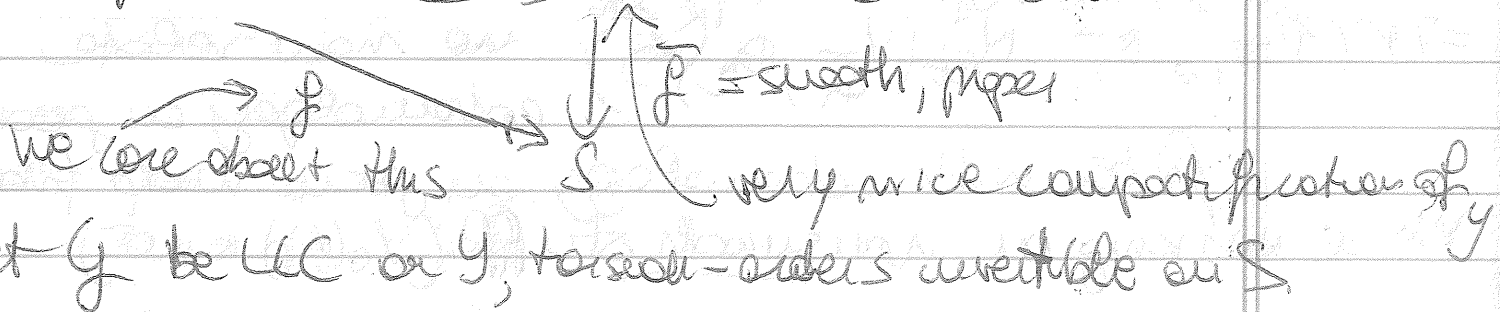
We can form $R^i f_* (G)$, $R^i f_*(G)$ and $R^i f_*(G)$ \rightarrow all sheaves on S are
 constructible by Deligne's general base change theorem, also construct

Therefore $R^n f_*(G)$ is also constructible

the above

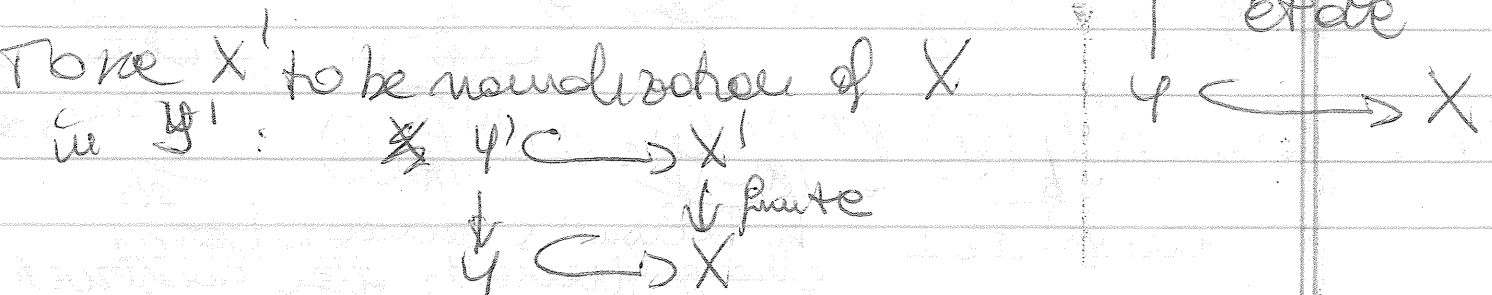
Thus Under assumptions, these 3 sheaves are lcc and commutes with any base change

Again, since we chose the blowboard $y = X - D \hookrightarrow X \supset D$ relative NCP



let G be lcc on y , torsion-orders available on S
 $S =$ connected, Dedekind base
 Thus $R^n f_*(G)$, $R^n f_!(G)$ (and $R^n f(G)$) are lcc and formation commutes with any base change on S
~~CONSTRUCTIBLE~~

Compatibility with base change for $R^n f_*(G)$ uses crucially the geometry of $X \supset D$, and its boundary.
 Remark We only used case $G = M_y$ is constant sheaf
 Proof (Sketch) Step 0: reduce to constant sheaf-case
 As G is lcc/ y , it is represented by y' finite étale



Thanks to universality of torsion orders in the base + geometry of $X \rightarrow D \implies$ (Abhyankar's Lemma)
 X' is also smooth, and also

$D' = (X' - Y')$ is a ~~relative~~ relative NCD \uparrow

(and likewise are $Y'' \rightarrow Y'$ that splits G , by taking higher powers?) HARD FACT

In fact, we care principally for D a unique divisor see also SGA 1, XVII

Step 1 We can now assume $G = \prod_y$ constant. Both

$R^i p_* G, R^i p'_* G$ are then constant sheaves / S , so

$\cong R^i p_* G$ to check that they're LC, it's enough to show that $\mathcal{F}_S \rightarrow \mathcal{F}_{\bar{S}}$ is isomorphic

for closed $S \in \bar{S}$, generic $\eta \in \bar{S}$, i.e.

where $S = \text{Spec}(R = \text{str. henselian DVR})$ we're saying

CLAIM $\left. \begin{array}{l} H^n(Y, M) \rightarrow H^n(Y_{\bar{S}}, M) \\ H^n_c(Y, M) \rightarrow H^n_c(Y_{\bar{S}}, M) \end{array} \right\}$ are isomorphisms

proof This is technical, and the proof involves crucially geometry of X , where we have a constant sheaf with $X \rightarrow S$ smooth proper.

(Check the analogous statements for \bar{p} are clear by smooth + proper base change)

The obstruction to the claim is contained in some "vanishing cycles" sheaves on D , so we'll use geometry of $(X \rightarrow D)$ to get iso

We'll use the specialization criterion which works.

We want compatibility with base change, in particular

the essential case is passing to geometric special fiber. That is if $S = \text{Spa}(R = \text{strictly henselian DVR})$ we need $\left. \begin{array}{l} H^n(Y, M) \rightarrow H^n(Y_{\bar{S}}, M) \\ H^n_c(Y, M) \rightarrow H^n_c(Y_{\bar{S}}, M) \end{array} \right\}$ be \cong

~~the Leray spectral sequence~~

$$R^n \tilde{p}_* (R^m \tilde{p}_* M) \rightarrow R^{n+m} p_* M \quad (\text{for } p: \text{no problem})$$

We need to understand base change properties
 on X of $(R^m \tilde{p}_* M) \rightarrow$ we use ~~the~~ standard
 properties (and geometry of X)

The image ~~of~~ also works well in this particular conditions, thanks to Deligne's result (SGA 4.5)

What to do in char p to compute trace T_p ?

~~Setup~~ Setup in char p : $Y = Y_0(N)$
 Consider $H^1(Y_{\overline{\mathbb{F}}_p}, \mathcal{O}_Y)$, which is self-dual
 up to $\mathcal{O}_Y(\pm 1)$, a Tate's twist
 Frob $\sim \tilde{H}^1(\quad)$, but Frob is not self-dual, so

$$\begin{array}{ccc} \tilde{H}^1 \times \tilde{H}^1 & \rightarrow & H^2 \cong \mathcal{O}_Y(\pm 1) \\ \uparrow & & \uparrow \\ \mathbb{F} \times \mathbb{F} & & \mathbb{F} = \text{mult. by } p \end{array}$$

(up to multiplication by p , the Frob-action also is part of the perfect pairing)

$T_p := \text{tr}_{\pi_2} \circ \pi_1^*$ in étale cohomology, we finite flat
 called ~~FINITE FLAT~~ TRACES have a trace map defined for finite flat

$Y_0(N)_p$
 \downarrow
 $Y = Y_0(N)_p$

\leftarrow
 \leftarrow