Seminar Hecke traces + counting mod p

Mar 13

We want to relate $\mathrm{Tr}(T_p, S_2(\Gamma_0(N)) \circlearrowleft)$ and $\# Y_0(N)(\mathbb{F}_p)$, $p \nmid N$

Warnings! The ideal ~~story~~ machinery work in this particular case, but only due to miracles, which usually do not work ~~[illegible]~~ in higher situations

(1) $Y_0(N)$ is a "coarse moduli space" ~~[illegible]~~

Usually base change, reduction mod p, and invariants for a group action do not commute

(i) $Y(\mathbb{F}_p) = Y(\overline{\mathbb{F}}_p)^{\mathrm{Gal}}$ with $Y(\overline{\mathbb{F}}_p) =$ ~~[illegible]~~ isom. classes of ...

(ii) Focusing on weight 2 means we get by using constant sheaves on Y (not higher symmetric powers)

(2)

$$\begin{tikzcd} & Y_0(Np) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ Y_0(N) & & Y_0(N) \end{tikzcd}$$

$$\begin{tikzcd} & (E, C) \arrow[dl, mapsto] \arrow[d, mapsto] \\ (E, C[N]) & (E/C[p], C/C[p]) \end{tikzcd}$$

C: cyclic subgroup scheme of order Np ($C \cong C[N] \times C[p]$)

so we need structure over $\mathbb{Z}[\frac{1}{N}]$

We need to go $\mathbb{C} \to \mathbb{Q} \to \mathbb{Z}[\frac{1}{N}] \to \mathbb{F}_p$

Recall Thm (Grothendieck-Lefschetz trace formula) K = finite field, Z = separated scheme finite type, $\ell \neq \mathrm{char}\, K$, then $\# Z(K) = \sum (-1)^i \mathrm{Tr}(F_{Z,q} \mid H^i_c(Z_{\overline{K}}, \mathbb{Q}_\ell) \circlearrowleft)$

For $F_{Z,q}$:

$$\begin{tikzcd} Z \arrow[rr] \arrow[dr] & & Z \arrow[dl] \\ & \mathrm{Spec}\, K & \end{tikzcd}$$

This is $(1 \otimes \mathrm{Frob}_q^{-1})$ geometric, use the Frobenius on coefficients

On constant coefficient cohomology the absolute Frobenius induce the identity map.

Note For affine curves $H^0_c(C, \mathbb{Q}_\ell) = 0$ compact cohomology
$H^2_c(C, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-1)$

We want to relate "T_p mod p" to $Fr_y : Y_{\mathbb{F}_p} \to Y_{\mathbb{F}_p}$
So we will relate ~~T_p~~ traces of T_p to the formula by Grothendieck, Lefschetz.
More specifically, we want to relate

$$\{T_p : S_2(\Gamma_0(N)) = H^0(\Omega^1_X) \circlearrowleft\} \longleftrightarrow \{Fr_y : H^1_c(Y_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell) \circlearrowleft\}$$

over $\mathbb{C}$ — coherent cohomology; étale cohom. $(\ell \nmid Np)$

Eichler-Shimura congruence relation

This will succeed in part b/c geometry of $Y_0(Np)_{\mathbb{F}_p}$ is not too bad.

Key input $Y_0(N)$ has a smooth compactification $X_0(N)$ over* $\mathbb{Z}[\frac{1}{N}]$ such that $(X \setminus Y)_{red} = X_\infty$ is smooth, and étale over $\mathbb{Z}[\frac{1}{N}]$

Moreover $Y_0(Np)_{\mathbb{F}_p}$ has mild singularities so we can apply "lemma of Deligne" to establish congruence relations of "in cohomology"* (Bourbaki 1960)

Def $Y = Y_0(N)$ = coarse moduli scheme over $\mathbb{Z}$ for moduli prime $S \longmapsto \{(E, C)/_S\}/\cong$
(This is smooth / $\mathbb{Z}[\frac{1}{N}]$ with geom connected fibers, dim 1)

with E elliptic curve$/S$, $C \subset E[N]$ a cyclic subgroup scheme of order N.

- If S over $\mathbb{Z}[\frac{1}{N}]$, just necessary $C \to S$ be finite étale with C – cyclic of order N
- S over $\mathbb{Z}[\frac{1}{N}]^c$ for $N = Mp$, just means $C = C_1 \times C_2$ with C_1 as in (*) for M and C_2 finite flat of order p

All the constructions will make sense with this definition, even if it is not a "nice" moduli space

~~Similarly~~ ~~$X = X_0(N)$~~ Now $Y \xrightarrow{j} \mathbb{A}^1_{\mathbb{Z}}$ j-invariant map proper (by valuative criterion) + ~~finite~~ quasifinite $\Rightarrow$ finite $\Rightarrow$ flat (by Miracle Flatness)

$$\begin{tikzcd} X \arrow[d] & \arrow[l, hook', "\text{open}"'] Y \arrow[d, "j"] \\ \mathbb{P}^1_{\mathbb{Z}} & \arrow[l, hook'] \mathbb{A}^1_{\mathbb{Z}} \end{tikzcd}$$

X is the normalization, and $X \to \mathbb{P}^1_{\mathbb{Z}}$ is also finite, proper, flat.

X is smooth$/\mathbb{Z}[\frac{1}{N}]$, with geom. conn'd fibers of dim. 1

$X_\infty = (X - Y)_{red}$ is finite étale $/\mathbb{Z}[\frac{1}{N}]$ $\coprod \operatorname{Spec} \mathcal{O}_{K_\alpha}[\frac{1}{N}]$ for $K_\alpha \subset \mathbb{Q}(\zeta_N)$

Proof (see Katz–Mazur)

Fact Formations of X, Y commutes with any base change

Remark In a natural way, $X^{an}_{\mathbb{C}} \cong$ usual $X_0(N)$

Now $S_2(\Gamma_0(N)) \overset{\text{by GAGA}}{\cong} H^0(X_{\mathbb{C}}, \Omega^1) = H^0(X^{\mathbb{Z}}_{\mathbb{Q}}, \Omega^1_{X_{\mathbb{Q}}}) \otimes_{\mathbb{Q}} \mathbb{C}$ since the curve is naturally defined over $\mathbb{Q}$

By Hodge theory (= weight 2 Eichler–Shimura isomorphism)

$S_2 \oplus \overline{S_2} \cong H^1(X^{an}, \mathbb{C}) \xrightarrow{\sim} \widetilde{H}^1(Y^{an}, \mathbb{C})$

$Y^{an} = X^{an} - \{\text{finite}\}$

where $\widetilde{H}^i = \operatorname{im}(H^i_c \to H^i)$ image of compactly supported cohomology

Now we can use that $\tilde{H}^i, \tilde{H}^{\dim X - i}$ are dual!

Now $T_p \oplus \overline{T_p} \cong S_2 \oplus \overline{S_2}$

$$\begin{tikzcd} & X_0(Np) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_0(N) & & X_0(N) \end{tikzcd} \quad \xrightarrow{\text{extend}} \quad \begin{tikzcd} & Y_0(Np) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ Y_0(N) & & Y_0(N) \end{tikzcd}$$

finite flat

unique extension!

defined as model as above $/ \mathbb{Z}[\frac{1}{N}]$

$$H^0(X_0(Np), \Omega^1)$$
$$\| \qquad \nwarrow \pi_2^*$$
$$H^0(X_0(N), \pi_{1*}\Omega^1_{X_0(Np)}) \qquad H^0(X_0(N)_K, \Omega^1)$$

char 0 field, to make sure ramification at branch locus $X' \to X$ is tame $X'_0(Np)$ $X''_0(N)$

$$\downarrow \mathrm{tr}\ \pi_1$$

$\cancel{H^0(X, \Omega}$

$$H^0(X_0(N), \Omega^1_{X_0(N)})$$

this over $\mathbb{C}$ recovers T_p, but works over $\mathbb{Q}$, so we can "define" T_p over $\mathbb{Q}$, and it respects the rational structure

So now $T_p \oplus \overline{T_p} \cong S_2 \oplus \overline{S_2}$, and we want to transport the action through.

$$\begin{tikzcd} & Y_0(Np) = Y' \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ Y_0(N) & & Y_0(N) = Y \end{tikzcd}$$

π_1 and π_2 are finite, so we can do pullback and push forward in the usual topology

$$H^1_c(Y^{an}, \mathbb{C}) \xrightarrow[\pi_1^*]{} H^1_c(Y'^{an}, \mathbb{C}) = H^1_c(Y^{an}, \pi_{1*}\mathbb{C})$$

One can check that this extends at the singular points.

So the same pullback + pushforward for H^1 have for $\tilde{H}^1$.

<u>Prop</u> $\tilde{H}^1(Y^{an}, \mathbb{C}) \xrightarrow[\pi_{1*}\pi_2^*]{} \tilde{H}^1(Y^{an}, \mathbb{C})$ is T_p via Eichler ~~Shimura~~ Shimura

we can replace $\mathbb{C}$ with F for any field of char 0, in particular $\mathbb{Q}_\ell$ coefficients

<u>Artin comparison thm</u>: relating cohomology of ℓ-adic sheaves

(can also be computed using $\tilde{H}^1_{et}(Y_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell) = \tilde{H}^1_{et}(Y_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$)

constant coeff. cohomology

$2\,\mathrm{Tr}(T_p) = $

$= \mathrm{Tr}(F_{\pi_{1*}\pi_2^*} ; \tilde{H}^1_{et}(Y_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)(\mathfrak{S}))$ via $\pi_{1*}\pi_2^*$

we now relate cohomology on geom fiber with cohomology on geom mod p fiber

We're working on nonsmooth stuff ($Y_0(Np)$ is not)

<u>Base change for $\tilde{H}^1$</u>: $Y \subset X \supset D$ relative normal crossing divisor = Zariski locally around D, we have an étale nbhd that are also étale over $\mathbb{A}^n_S - \{x_1=0, \dots x_r=0\}$

$$\begin{array}{c} Y \subset X \supset D \\ \bar{f} \downarrow \text{ proper smooth} \\ S \end{array}$$

S connected Dedekind base

This applies in our situation to $Y = X \setminus D$ and apply this to

let $\mathcal{G}$ = locally constant, constructible sheaf on Y_{et} whose torsion orders are invertible in S

(e.g. $\mathcal{G} = \mathbb{Z}/\ell^n\mathbb{Z}$, $S = \mathrm{Spec}\,\mathbb{Z}\left[\frac{1}{N\ell}\right]$)

We can form $\underbrace{R^n f_!(\mathcal{G})}_{\text{constructible}}$, $\underbrace{R^n f_*(\mathcal{G})}_{\text{by Deligne's generic base change theorem, also constructible}}$ and $R^n \bar{f}_*(\mathcal{G})$ all sheaves on S_{et}

Therefore ~~R~~ $R^n \tilde{f}(\mathcal{G})$ is also constructible

Thm Under the above assumptions, these 3 sheaves are lcc ~~and~~ and commutes with any base change

Again, since we erase the blackboard ...

$$\begin{tikzcd} Y = X - D \arrow[r, hook, "j"] \arrow[dr, "f"] & X \supset D \text{ relative NCD} \arrow[d, "\tilde{f} = \text{smooth, proper}"] \\ & S \arrow[u, bend right] \end{tikzcd}$$

we care about this f — very nice compactification of Y

let $\mathcal{G}$ be LCC on Y, torsion-orders invertible on S

S = connected, Dedekind base

Thm $R^n f_*(\mathcal{G})$, $R^n f_!(\mathcal{G})$ (and so $R^n \tilde{f}(\mathcal{G})$) are LCC and formation commutes with any base change on S

~~CONSTRUCTIBLE~~

Compatibility with base change for $R^n f_*(\mathcal{G})$ uses crucially the geometry of $X \supset D$, and its boundary.

Remark We only need case $\mathcal{G} = M_Y$ is constant sheaf

Proof (Sketch) Step 0: reduce to constant sheaf-case

As $\mathcal{G}$ is LCC$/Y$, it is represented by

$$\begin{tikzcd} Y' \arrow[d, "\text{finite Etale}"] & \\ Y \arrow[r, hook] & X \end{tikzcd}$$

Take X' to be normalization of X in Y':

$$\begin{tikzcd} Y' \arrow[r, hook] \arrow[d] & X' \arrow[d, "\text{finite}"] \\ Y \arrow[r, hook] & X \end{tikzcd}$$

Thanks to invertibility of torsion orders on the base + geometry of $X \supset D \Longrightarrow$ (Abhyankar's lemma)

X' is also smooth, and also

$D' = (X' - Y')_{red}$ is a ~~NCD~~ relative NCD

HARD FACT

see also SGA 1, XIII

(and likewise an $Y'' \longrightarrow Y'$ that splits $\mathcal{G}$, by taking higher powers ?)

In fact, we care principally for D a unique divisor

Step 1 We can now assume $\mathcal{G} = \underline{M}_Y$ constant. Both $R^n f_* \mathcal{G}$, $R^n f_! \mathcal{G}$ are then constructible $/S$, so to check that they're LCC it's enough to show that $\mathcal{F}_{\bar{s}} \longrightarrow \mathcal{F}_{\bar{\eta}}$ is isomorphism for closed $s \in S$, generic $\eta \in S$, i.e.

$\mathcal{F} = R^n f_* \mathcal{G}$

where $S = \mathrm{Spec}(R = $ str. henselian dvr$)$ we're saying

CLAIM $H^n(Y, M) \longrightarrow H^n(Y_{\bar{K}}, M)$

$H^n_c(Y, M) \longrightarrow H^n_c(Y_{\bar{K}}, M)$ } are isomorphisms

Proof This is technical, and the proof involves crucially geometry of X, where we have a constant sheaf with $X \xrightarrow{\bar{f}} S$ smooth proper. (hence the analogous statements for $\bar{f}$ are clear by smooth + proper base change)

The obstruction to the claim is contained in some "vanishing cycles" sheaves on D, so we'll use geometry of $(X \supset D)$ to get iso

We'll use the specialization criterion which work.

We want compatibility with base change, in particular the essential case is passing to geometric special fiber. That is if $S = \mathrm{Spec}(R =$ strictly henselian dvr$)$ we need $H^n(Y, M) \longrightarrow H^n(Y_0, M)$ be $\cong$

$H^n_c(Y, M) \longrightarrow H^n_c(Y_0, M)$

~~Lo~~ Leroy spectral sequence
$R^n \bar{f}_*(R^m j_* M) \Rightarrow R^{n+m} f_* M$ (For $f_!$ no problem)
We need to understand base change properties on X of $(R^m j_* M) \longrightarrow$ we use cohomological purety (and geometry of $X - D$)
The image sheaf also works well in this particular conditions, thanks to Deligne's result (SGA 4.5)

What to do in char p to compute trace T_p?
Setup in char p: $Y = Y_0(N)_{\mathbb{F}_p}$
Consider $\tilde{H}^1(Y_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell)$ which is self-dual up to $\mathbb{Q}_\ell(\pm 1)$, a Tate's twist
Frob $\curvearrowright \tilde{H}^1(\quad)$, but Frob is not self-dual, so

$$\tilde{H}^1 \times \tilde{H}^1 \longrightarrow H^2_c \cong \mathbb{Q}_\ell(\pm 1)$$
$$F \times F \qquad\qquad F = \text{mult. by } p$$

(up to multiplication by p, the Frob-action also is part of the perfect pairing)

$$T_p^{alg} := \mathrm{tr}_{\pi_1} \circ \pi_2^*$$

called TRACE $\pi_{1 !}$; in étale cohomology, we have a trace map defined for finite flat eqs.
FINITE FLAT

$Y_0(Np)_{\mathbb{F}_p}$ $\to$ $Y = Y_0(N)_{\mathbb{F}_p}$ finite flat