LOCAL-GLOBAL COMPATIBILITY AND APPLICATIONS TO THE ARITHMETIC OF MODULAR CURVES

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1. Introduction and Motivation

Let $\mathcal{M}_K$ be the (proper) modular curve over $\mathbb{Q}$ associated to a compact open subgroup K of $\mathrm{GL}_2(\mathbb{A}_f)$. $\mathcal{M}_K$ admits a natural proper model over $\mathrm{Spec}(\mathbb{Z})$ with a moduli interpretation. If $K = K^p K_p$ for compact open $K^p \subset \mathrm{GL}_2(\mathbb{A}_f^p)$ and $K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)$ maximal (i.e. conjugate to $\mathrm{GL}_2(\mathbb{Z}_p)$), then $\mathcal{M}_K$ has good reduction at p—otherwise (if p "divides the level"), the reduction $\mathcal{M}_{K,p}$ is somewhat mysterious. This note aims to exposit certain predictions the Langlands philosophy make about this situation.

Let F be a number field, and F_v the completion at some place v, with Weil group W_{F_v} (for a definition, see Definition 3 of Niccolo's notes [1]). The Langlands program for GL_2 predicts the existence a diagram of the following sort:

$$\begin{tikzcd}
\mathrm{Rep}^2(W_{F_v})^{F\text{-ss}} \arrow[r, leftrightarrow, "\text{Local Langlands}"] & \{\text{Irreducible admissible reps of } \mathrm{GL}_2(F_v)\} \\
\text{“}\mathrm{Rep}^2(G_F)\text{”} \arrow[u, "\mathrm{res}^{W_{F_v}}_{G_F}"] \arrow[r, leftrightarrow, "\text{Global Langlands}"] & \text{“}\{\text{Automorphic reps of } \mathrm{GL}(\mathbb{A}_F)\}\text{”} \arrow[u, "\otimes_w \pi_w \mapsto \pi_v"']
\end{tikzcd} \qquad (*)$$

Here $\mathrm{Rep}^2(W_{F_v})^{F-ss}$ is the collection of 2-dimensional Frobenius semi-simple continuous representations of W_{F_v} over $\overline{\mathbb{Q}}_\ell$ (v not dividing ℓ). $\mathrm{Rep}^2(G_F)$ is the collection of 2-dimensional continuous G_F-reps over $\overline{\mathbb{Q}}_\ell$.

Remark 1. The Local Langlands correspondence is often stated in terms of Weil-Deligne representations; it is a theorem of Grothendieck that these are the same as Frobenius semi-simple continuous W_{F_v}-reps.

I've placed the objects on the bottom line of $(*)$ in quotes because this is only approximately what is predicted by Global Langlands; the existence of the correspondence on the bottom line is still conjectural, even for 2-dimensional representations with $F = \mathbb{Q}$. That said, there are certain cases in which the bottom arrow is defined (for example, for automorphic/Galois representations corresponding to classical cusp forms). In this case, one may ask if the Local and Global correspondences may be normalized so that the diagram above commutes. The existence (and description) of such a normalization is local-global compatibility.

Example 1 (GL_1). What does local-global compatibility say in the case of GL_1 (that is, class field theory)? In this case there is a diagram

$$\begin{tikzcd} F_v^* \arrow[r, "\mathrm{rec}", "\sim"'] \arrow[d] & W_{K_v}^{\mathrm{ab}} \arrow[d] \\ C_F \arrow[r, "\mathrm{rec}"] & G_F^{\mathrm{ab}}. \end{tikzcd}$$

Here rec denotes the reciprocity maps from local and global class field theory, normalized as in Cassels-Fröhlich; the vertical arrows are evident. Local-global compatibility states that the diagram commutes. One may interpret this as saying that the global Langlands correspondence for characters restricts to the local one.

We will be interested in the following abstract situation: given a representation V of G_F arising from geometry, how do we compute its restriction to the decomposition group at a place v, that is, $\mathrm{res}_{G_F}^{W_{F_v}}(V)$? This is the left-most vertical arrow in the diagram ($*$). In our situation we will compute it by tracing the diagram around the other way—we will compute the composition

$$(\text{Local Langlands}) \circ (\otimes_w \pi_w \mapsto \pi_v) \circ (\text{Global Langlands}).$$

Here is the precise situation we will try to understand. Fix N a positive integer and p a prime with $(p, N) = 1$. As usual, let

$$\begin{aligned} \Gamma(N) &:= \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})), \\ \Gamma_0(p) &= \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod p \right\}, \\ \Gamma_1(p) &= \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod p \right\}, \\ \Gamma_0(p, N) &= \Gamma_0(p) \cap \Gamma(N), \text{and} \\ \Gamma_1(p, N) &= \Gamma_1(p) \cap \Gamma(N). \end{aligned}$$

Let $X_1(p, N)$ be the proper modular curve over $\mathbb{Q}$ associated to $\Gamma_1(p, N)$, $X_0(p, N)$ the proper modular curve over $\mathbb{Q}$ associated to $\Gamma_0(p, N)$, and $X(N)$ the proper modular curve over $\mathbb{Q}$ associated to $\Gamma(N)$. These curves are geometrically connected; as remarked above, they admit natural models over $\mathbb{Z}$. There are natural maps ("forgetting level structure")

$$X_1(p, N) \to X_0(p, N) \to X(N).$$

Let $J_1(p, N)$ be Pic^0 of $X_1(p, N)$, let $J_0(p, N)$ be Pic^0 of $X_0(p, N)$, and let $J(N)$ be Pic^0 of $X(N)$. Then the pullback of line bundles induces maps

$$J(N) \to J_0(p, N) \to J_1(p, N).$$

We define

$$A_0 := \mathrm{coker}(J(N) \to J_0(p, N))$$

and

$$A_1 := \mathrm{coker}(J_0(p, N) \to J_1(p, N)).$$

Deligne has proven the following amazing theorem:

Theorem 2 (Deligne)**.** *Let A_0, A_1, p be as above. Then,*

(1) *the identity component of the reduction of A_0 at p is a torus, and*
(2) *A_1 has good reduction at the unique prime of $\mathbb{Q}(\zeta_p)$ lying over (p).*

The rest of this note will be explain why this theorem is a prediction of local-global compatibility. It is not clear to me if, given the work of Carayol etc. on local-global compatibility, these methods can actually be turned into a proof—I don't know enough about Carayol's work to say whether or not the argument would be circular. That said, I hope that these predictions are a convincing demonstration of the utility of the Langlands philosophy.

In §2, I will briefly review the representation theory of $\mathrm{GL}_2(F_v)$ and the local Langlands correspondence; we will need a somewhat more explicit form of the correspondence than that sketched by Niccolo in [1]. In §3, I'll say a bit about the global Langlands correspondence, and give a statement of local-global compatibility. §4 will contain the computations necessary to understand how Deligne's Theorem 2 follows from the Langlands philosophy. In particular, §4.1 will explain the relationship between the inertia action on the Tate module of an Abelian variety and its reduction type; §4.2 will translate the Galois-theoretic information from the Tate modules of A_0 and A_1 through the local Langlands correspondence; and §4.3 will use local-global compatibility to compute the local components of the relevant automorphic representations and deduce Theorem 2.

To orient the reader, I will briefly sketch how the argument goes.

(1) By work of Serre-Tate [2] and Grothendieck [3, Exposé IX], we may in many cases determine the reduction type of an Abelian variety by the inertia action on its Tate modules. Thus we wish to understand the representations of $W_{\mathbb{Q}_p}$ on the Tate modules of A_0, A_1.

(2) The global Langlands correspondence associates automorphic representations π_0, π_1 to the irreducible constituents of the Galois representations

$$\rho_i : G_{\mathbb{Q}} \to \mathrm{GL}(V_\ell(A_i))$$

for fixed ℓ.

(3) The automorphic representations π_i uniquely arise as a tensor product over places w of $\mathbb{Q}$

$$\pi_i = \bigotimes_w \pi_{w,i};$$

we wish to determine $\pi_{p,i}$.

(4) We use local Langlands to associate representations

$$\psi_{p,i} : W_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Q}_\ell})$$

to the representations $\pi_{p,i}$.

(5) Finally, local-global compatibility implies that $\psi_{p,i} = \rho_i|_{W_{\mathbb{Q}_p}}$. The explicit description of $\psi_{p,i}$ as well as the work of Serre, Tate, and Grothendieck in (1) will imply the theorem.

2. $\mathrm{Rep}(\mathrm{GL}_2(F))$ and Local Langlands for GL_2

Let L be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_L$, maximal ideal $\mathfrak{m}$, and residue field k. First recall the classification of irreducible admissible representations of $\mathrm{GL}_2(L)$.

Theorem 3. *Let V be an irreducible admissible representation of $\mathrm{GL}_2(L)$ over an uncountable algebraically closed field of characteristic zero, e.g. $\mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$. Then V is isomorphic to exactly one of the following:*

(1) *An irreducible principal series $\pi(\alpha, \beta)$ (the irreducibility is exactly the condition that $\alpha\beta^{-1} = |\cdot|^{\pm 1}$),*
(2) *A one-dimensional representation $g \mapsto \omega \circ \det(g)$, where ω is a character of L^*,*
(3) *A twist of the Steinberg representation by a character, or*
(4) *A supercuspidal representation.*

Remark 4. There is a natural isomorphism $\pi(\alpha, \beta) \simeq \pi(\beta, \alpha)$.

Remark 5. For definitions of these representations, see Niccolo's notes [1] (though note—our normalizations for the principal series differ slightly from his). See [5] for details and the notation I use here.

Remark 6. The twists of Steinberg appear as the unique infinite-dimensional quotients or sub representations of $\pi(\alpha, \beta)$ with $\alpha\beta^{-1} = |\cdot|^{\pm 1}$.

Remark 7. The one-dimensional representations $g \mapsto \omega \circ \det(g)$ will never show up as local factors of cuspidal automorphic representations (this is a non-trivial theorem [7, Theorem 11.1]), so we won't have to worry about them in later computations.

Later we will wish to identify which of these representations arise as the local components of certain automorphic representations, in terms of very soft data: namely, what is the largest (compact open) subgroup of $\mathrm{GL}_2(L)$ stabilizing a given nonzero vector? This information—the conductor—will translate into information about ramification via the local Langlands correspondence. Thus we make the following definitions.

Definition 8. V is unramified if it has a nonzero vector fixed by $\mathrm{GL}_2(\mathcal{O}_L)$.

Remark 9. Of the irreducible admissible representations above, the only unramified ones are $\pi(\alpha, \beta)$ with α, β unramified (that is, trivial on $\mathcal{O}_L^*$) and $\alpha\beta^{-1} \neq |\cdot|^{\pm 1}$, and $\omega \circ \det$, with ω unramified. As the latter case does not show up in cuspidal automorphic representations (Remark 7), Galois representations unramified at v will correspond exactly to automorphic representations whose local factor at v is a $\pi(\alpha, \beta)$ as above.

Recall that if χ is a character for L^* (valued in an uncountable algebraically closed field of characteristic zero), we define the conductor $c(\chi)$ of χ to be the least n such that $\chi|_{1+\mathfrak{m}^n\mathcal{O}_L}$ is trivial. This filtration corresponds to the ramification filtration of W_L^{ab} via local class field theory. There is an analogous invariant for representations of $\mathrm{GL}_2(L)$:

Definition 10. $\mathrm{GL}_2(L)$ admits a natural filtration by subgroups $K(n)$, with $K(0) := \mathrm{GL}_2(\mathcal{O}_L)$, and

$$K(n) := \left\{ \gamma \in \mathrm{GL}_2(\mathcal{O}_L) \mid \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \bmod \mathfrak{m}^n \right\}$$

for $n \geq 1$. V is said to have conductor $c(V) = n$ if n is minimal such that $V^{K(n)} \neq \{0\}$.

The filtration $K(n)$ will correspond to the ramification filtration on the Galois side of the local Langlands correspondence. For the reader's convenience, we record some information about the conductors of the various irreducible admissible representations of $\mathrm{GL}_2(L)$.

Theorem 11 ([8, p. 8])**.** *The conductors of the representations discussed earlier are as follows:*

(1) *For an irreducible principal series $\pi(\alpha, \beta)$, $c(\pi(\alpha, \beta)) = c(\alpha) + c(\beta)$.*
(2) *For a twist of Steinberg, $\chi \otimes St$,*

$$c(\chi \otimes St) = \begin{cases} 1 & \textit{if } \chi \textit{ is unramified} \\ 2c(\chi) & \textit{otherwise} \end{cases}$$

(3) $c(\omega \circ \det) = c(\omega)$.
(4) *If V is supercuspidal, $c(V) \geq 2$.*

Remark 12. Of the infinite-dimensional representations above, the only ones with conductor equal to 1 are twists of Steinberg by unramified characters, and $\pi(\alpha, \beta)$ with (without loss of generality, by the symmetry of the irreducible principal series in α, β), $c(\alpha) = 0, c(\beta) = 1$.

Remark 13. Luckily, all of the representations we will encounter will have conductor ≤ 1, so we will be able to avoid having to worry about supercuspidals.

We will actually need slightly more specific data than the above. Namely, we define

$$\Gamma_0(\mathfrak{m}^n) := \left\{ \gamma \in \mathrm{GL}_2(\mathcal{O}_L) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod \mathfrak{m}^n \right\}$$

and

$$\Gamma_1(\mathfrak{m}^n) := \left\{ \gamma \in \mathrm{GL}_2(\mathcal{O}_L) \mid \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{m}^n \right\}.$$

Then we may refine the earlier classification of representations by their conductors.

Theorem 14. *Let V be an infinite-dimensional irreducible admissible $\mathrm{GL}_2(L)$-representation.*

(1) *If $V^{\mathrm{GL}_2(\mathcal{O}_L)} = \{0\}$, but V admits a non-zero vector fixed by $\Gamma_0(\mathfrak{m})$, then V is a twist of the Steinberg representation by an unramified character.*
(2) *If $V^{\Gamma_0(1)} = \{0\}$, but V admits a non-zero vector fixed by $\Gamma_1(\mathfrak{m})$, then V is a principal series $\pi(\alpha, \beta)$ with $c(\alpha) = 0, c(\beta) = 1$.*

Proof. This follows from [10, Proposition 2.8]. Self-contained proof to be added later. □

We are now ready to give a more explicit description of the local Langlands correspondence than that in Niccolo's notes, which we will need for later computation.

Theorem 15 (Local Langlands)**.** *There is a "natural" bijection between irreducible admissible representations of $\mathrm{GL}_2(L)$ and 2-dimensional Frobenius semi-simple continuous representations of W_L, such that:*

(1) *The irreducible principal series* $\pi(\alpha, \beta)$ *corresponds to the representation*

$$\begin{pmatrix} \alpha \circ \mathrm{rec}^{-1} & 0 \\ 0 & \beta \circ \mathrm{rec}^{-1} \end{pmatrix}$$

where rec *is the reciprocity map from local class field theory.*

(2) *A twist of Steinberg* $\chi \otimes St$ *corresponds to a non-trivial extension*

$$\begin{pmatrix} \chi \circ \mathrm{rec}^{-1} & * \\ 0 & \chi(-1) \circ \mathrm{rec}^{-1} \end{pmatrix}.$$

(3) *Supercuspidals correspond to irreducibles.*

(4) $\chi \circ \det$ *corresponds to*

$$\begin{pmatrix} \chi \cdot |\cdot|^{\frac{1}{2}} \circ \mathrm{rec}^{-1} & 0 \\ 0 & \chi \cdot |\cdot|^{-\frac{1}{2}} \circ \mathrm{rec}^{-1} \end{pmatrix}.$$

Remark 16. The word "natural" is in quotes because there are various ways to normalize the correspondence; one may uniquely determine the correspondence by demanding certain compatibility properties with respect to change of group, change of field, taking contragradients, etc. I've tried to at least normalize the correspondence so that it commutes with contragradients; any expert reading this: please let me know if I've made an error.

Remark 17. In (2) of the above theorem, I refer to "a" nontrivial extension of $\chi(-1)$ by χ. The fact that there is a unique isomorphism class of representation of this type (forgetting its extension structure) boils down to the fact that

$$\mathrm{Ext}^1(\chi(-1), \chi) = H^1(G_L, \overline{\mathbb{Q}_\ell}(1))$$

is one-dimensional, which one may compute via the Kummer exact sequence.

Remark 18. Observe that in the Galois representation associated to $\chi \otimes \mathrm{St}$, the image of inertia is infinite. It suffices to show this for St itself. Frobenius acts semisimply, so the extension is still non-trivial when restricted to inertia. As inertia acts unipotently, we have a non-trivial homomorphism $I_K \to \overline{\mathbb{Q}_\ell}^+$. But $\overline{\mathbb{Q}_\ell}^+$ has no finite subgroups, (as $\mathbb{Q}_\ell$ is characteristic zero), so the image of inertia is infinite as desired.

3. Global Langlands for GL_2 and Local-Global Compatibility

We now briefly discuss the aspects of the global Langlands correspondence which will be necessary (1) to give the statement of local-global compatibility and (2) for our applications to modular curves. We first recall the "classical" relationship between Galois representations and modular forms.

Theorem 19 (Shimura, Deligne)**.** *Let* f *be a cusp form of weight* k*, level* N*, which is an eigenform for* $T_p, T_{p,p}$ *for all* $p \nmid N$*, with Hecke eigenvalues* $a_p, a_{p,p}$*. Fix embeddings*

$$\overline{\mathbb{Q}} \hookrightarrow \mathbb{C},$$

$$\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell.$$

Then there exists a continuous Galois representation

$$\rho_{f,\ell} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}_\ell})$$

uniquely determined by, for $p \nmid N\ell$*:*

(1) $\rho_{f,\ell}$ *is unramified at* p*, and*

(2) *The characteristic polynomial of any lift of any Frobenius element* Frob_p *is*

$$T^2 - a_p T + a_{p,p}.$$

I will need to say a bit about the structure of the construction above in the case of weight 2, which is due to Shimura (explained well in [9]). Namely:

- Suppose f is a newform of weight 2 and level N, which is an eighenform for the Hecke algebra away from N; let F be the number field generated by the eigenvalues of the Hecke algebra acting on f. Then we may view f as an element of $\Gamma(X(N), \omega_{X(N)})$.

- The fact that f is an eigenform implies that f is pulled back from a differential form on a $\mathbb{Q}$-simple isogeny factor A_f of $\mathrm{Jac}(X(N))$. F acts on $V_\ell(A_f)$ through the (rational) Hecke algebra, turning $V_\ell(A_f)$ into a rank 2 free module over
$$F \otimes \mathbb{Q}_\ell \simeq \prod_{\lambda | \ell} F_\lambda.$$
- The Galois representation we associate to f is then
$$V_\lambda(A_f) \otimes_{F_\lambda} \overline{\mathbb{Q}_\ell},$$
where $V_\lambda(A_f)$ is the factor of V_ℓ (as an $F \otimes \mathbb{Q}_\ell$-module) corresponding to the λ induced by the chosen embedding $F \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$ chosen above. One checks that this representation satisfies the desired conditions via the Eichler-Shimura relations.

There is a better, if less classical, version of this theorem, phrased in terms of automorphic representations. The statement will include our main topic: local-global compatibility.

Theorem 20. *Let F be a totally real number field, and π an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ such that π_∞ is discrete series. Then there exists a continuous Galois representation*
$$\rho_{\pi,\ell} : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}_\ell})$$
uniquely determined by the condition that π_v corresponds to $\rho_{\pi,\ell}|_{G_{F_v}}$ under the local Langlands correspondence, for all v not lying above ℓ.

Remark 21. The discrete series representation π_∞ is the analogue of the notion of "weight" in the classical case.

Remark 22. The statement about the local components of π at the end of this theorem is exactly what is meant by "local-global compatibility."

Remark 23. This statement is better than the classical one in two ways. One is that it applies to arbitrary totally real fields, rather than just $\mathbb{Q}$. The other is that it determines the Galois representations at all primes away from ℓ; in particular, we have a much better understanding of primes dividing the level, even over $\mathbb{Q}$. This extra information is exactly what we will use in §4.

Remark 24. This theorem associates a Galois representation to an automorphic representation; in some cases, we may go the other way, assigning an automorphic representation to a Galois representation, e.g. if the Galois representation shows up in the cohomology of a modular curve (or certain Shimura varieties). A part of the Langlands program is thus to find broad classes of Galois representations inside of the cohomology of Shimura varieties.

I will briefly discuss this interpretation, and connect it a bit with the classical theory. We may interpret the work of Shimura above as saying that there is an isomorphism
$$\varinjlim H^1_{\text{ét}}(\mathcal{M}_{\mathcal{K}}, \overline{\mathbb{Q}_\ell}) \simeq \bigoplus \pi_f \otimes \sigma(f)$$
where the direct limit is taken over all compact open subgroups $\mathcal{K}$ of $\mathrm{GL}_2(\mathbb{A}_f)$, and $\mathcal{M}_{\mathcal{K}}$ denotes the associated proper modular curve. Here $\sigma(f)$ is a two-dimensional Galois representation associated to a modular form f as above, and π_f is the associated automorphic representation (by taking translates of the associated function on $\mathrm{GL}_2(\mathbb{A}_f)/\mathcal{K}$). That is, the Galois-isotypic pieces of this cohomology group are "indexed" by the associated automorphic representations. The theory of newforms may be recovered by considering the Hecke action on the automorphic representations above; this action (for appropriate $\mathcal{K}$) gives the π_f a distinguished line, and thus picks out a distinguished irreducible subrep of the isotypic piece corresponding to $\sigma(f)$.

Remark 25. We may also restate local-global compatibility directly in terms of modular forms (this is just a translation of what has come before). Let f be a cusp form as above, with associated Galois representation $\rho_{f,\ell}$ and automorphic representation π_f. Then local-global compatibility says that $\rho_{f,\ell}|_{G_{\mathbb{Q}_p}}^{\mathrm{Frob}-ss}$ corresponds to $\pi_{f,p}$ under the local Langlands correspondence. All of these versions of local-global compatibility are due to Carayol.

4. Applications to Jacobians of Modular Curves

We may now turn to our analysis of the Jacobians of modular curves. We first recall the situation. There are (quotient) maps

$$X_1(p,N) \to X_0(p,N) \to X(N)$$

inducing maps of Jacobians

$$J(N) \to J_0(p,N) \to J_1(p,N).$$

A_0 was the cokernel of the map

$$J(N) \to J_0(p,N),$$

and A_1 was the cokernel of the map

$$J_0(p,N) \to J_1(p,N).$$

We wish to understand how Theorem 2, stating that A_0 has toric reduction at p and A_1 has good reduction at the prime above p after base change to $\mathbb{Q}(\zeta_p)$, followed from local-global compatibility.

4.1. **Reduction Type and the Tate Module.** To understand the reduction of A_0, A_1, we must first try to understand the reduction type of an Abelian variety in terms of its ℓ-adic Tate module. To do this we use work of Serre-Tate [2] and Grothendieck [3, Exposé IX].

Theorem 26 (Neron-Ogg-Shafarevich Theorem, [2, Theorem 1])**.** *Suppose $\mathcal{O}_K$ is a discrete valuation ring with fraction field K and residue field k. Let A be an Abelian variety over K. Then A has good reduction if and only if there exists a prime ℓ (equivalently for all ℓ) different from* $\mathrm{char}(k)$ *such that the Galois representation $V_\ell(A)$ is unramified.*

Remark 27. Serre and Tate only state the theorem in the case that k is perfect, but this is unnecessary; their arguments work essentially verbatim without this hypothesis.

Remark 28. We say a representation of G_K is unramified if the inertia group I_K acts trivially.

Remark 29. The easy direction of this theorem is that good reduction of A implies that $V_\ell(A)$ is unramified. Namely, suppose A has good reduction, so that A is the general fiber of an Abelian scheme $\mathcal{A}$ over $\mathrm{Spec}(\mathcal{O}_K)$. Then if n is prime to $\mathrm{char}(k)$, $A[n]$ is the generic fiber of a finite étale group scheme over $\mathrm{Spec}(\mathcal{O}_K)$, namely $\mathcal{A}[n]$. As $\mathcal{A}[n]$ is étale over $\mathrm{Spec}(\mathcal{O}_K)$, and thus isomorphic to

$$\bigsqcup_i \mathrm{Spec}(A_i)$$

with A_i a domain equal to the integral closure of A on $\mathrm{Frac}(A_i)$, and $\mathrm{Frac}(A_i)$ a finite unramified (separable) extension of K. Thus the Galois action on the geometric generic points of $\mathcal{A}[n]$ is unramified, giving the claim.

The other direction—that $V_\ell(A)$ unramified implies A is of good reduction, is harder; one uses Chevalley's structure theorem for smooth connected commutative $\overline{k}$-group schemes to analyze the ℓ-torsion of the special fiber of the Néron model. This is well-explained in Serre-Tate [2], so I do not explain it here.

Grothendieck extended this statement to the case of semi-stable reduction; that is, he gives an ℓ-adic criterion for when A extends to a smooth commutative separated $\mathcal{O}_K$-group scheme of finite type whose special fiber is an extension of an Abelian variety by a torus.

Theorem 30 (Grothendieck Orthogonality Theorem, [3, Exposé IX], [6, Theorem 3.1])**.** *Let A, K, etc. be as before, and $(\ell, \mathrm{char}(k)) = 1$. Then A has semi-stable reduction if and only if the action of I_K on $V_\ell(A)$ is unipotent. The image of the toric part of the ℓ-adic Tate module of A_k is precisely*

$$V_\ell(A)^{I_K} \cap V_\ell(A^\vee)^{I_K,\perp},$$

where $\perp$ indicates the orthogonal subspace under the Weil pairing.

Remark 31. The proof of this theorem (in a slightly better form) is well-explained in Brian Conrad's notes from a previous number theory learning seminar [4, Theorem 5.5].

We will need a corollary of the Orthogonality Theorem, describing the Tate module of an Abelian variety of toric reduction (that is, it admits a semiabelian model over $\mathcal{O}_K$ whose special fiber is a torus).

Corollary 32. *Let A, K, etc. be as before, $\dim A = g$, and $(\ell, \mathrm{char}(k)) = 1$. Then A has toric reduction if and only if $\dim V_\ell(A)^{I_K} = g$ and the inertia action on $V_\ell(A)/V_\ell(A)^{I_K}$ is trivial. That is, there is a short exact sequence of I_K-modules*

$$0 \to V_\ell(A)^{I_K} \to V_\ell(A) \to V_\ell(A)/V_\ell(A)^{I_K} \to 0$$

with the first and last terms trivial I_K-modules of rank g.

Proof. The corollary is a consequence of the following linear algebra lemma, taking $V = V_\ell(A)$, ω to be the Weil pairing, and $G = I_K$.

Lemma 33. *Let V be a vector space of rank $2g$ with non-degenerate symplectic form ω, and G a group acting on V and preserving ω. If V^G is of rank g and G acts trivially on V/V^G, then V^G is Lagrangian for ω (that is, a maximal subspace on which ω vanishes).*

Proof. Consider the isomorphism of G-representations $V \to V^\vee$ induced by ω. Under this map V^G is sent to $V^{\vee,G}$, which contains $(V/V^G)^\vee$. As $V \simeq V^\vee$, $V^{\vee,G}$ is in fact equal to $(V/V^G)^\vee$, as both are of rank g. So V^G is sent isomorphically onto $(V/V^G)^\vee$. Thus the map $V \to V^{G,\vee}$ factors through V/V^G, and so V^G is Lagrangian as desired. □

Now we prove the corollary.

First suppose A has toric reduction; then by the Orthogonality theorem, $V_\ell(A)^{I_K} \cap V_\ell(A)^{I_K,\perp}$ has rank g. Thus $V_\ell(A)^{I_K}, V_\ell(A)^{I_K,\perp}$ have rank at least g; so the rank of each must be exactly g, and the two subspaces must be equal. Thus $V_\ell(A)^{I_K}$ is Lagrangian for the Weil pairing. It remains to show that the inertia action on $V_\ell(A)/V_\ell(A)^{I_K}$ is trivial. But $V_\ell(A)^{I_K}$ is Lagrangian for the Weil pairing, so $V_\ell(A)/V_\ell(A)^{I_K}$ is naturally identified with its dual under the Weil pairing. As the I_K-action preserves the Weil pairing, this implies the action on $V_\ell(A)/V_\ell(A)^{I_K}$ is trivial as well.

For the other direction, suppose $\dim V_\ell(A)^{I_K} = g$ and the inertia action on $V_\ell(A)/V_\ell(A)^{I_K}$ is trivial. Then the lemma implies $V_\ell(A)^{I_K}$ is Lagrangian, and so the rank of the Tate module of the toric part of $\mathcal{A}_k$ (where $\mathcal{A}$ is the identity component of the Néron model of A) is g. As A has dimension g, the torus part of the special fiber is the whole special fiber, and we're done. □

Thus we will wish to show that $V_\ell(A_0)$ satisfies the conditions of Corollary 32 and $V_\ell(A_1)$ satisfies the conditions of Theorem 26 after base changing to $\mathbb{Q}_p(\zeta_p)$.

4.2. **Translating the Problem via Local Langlands.** To do so, we will try to understand the automorphic representations π_0, π_1 associated to $V_\ell(A_0), V_\ell(A_1)$. In particular, if we know $\pi_{i,p}$, we may find the representations of $G_{\mathbb{Q}_p}$ associated to $\pi_{i,p}$ under the local Langlands correspondence; by local-global compatibility, this will tell us the action of $G_{\mathbb{Q}_p}$ on $V_\ell(A_i)$.

Let us first consider

$$\rho_1 : G_{\mathbb{Q}_p} \to \mathrm{GL}(V_\ell(A_1) \otimes \overline{\mathbb{Q}_\ell}).$$

As this representation shows up in the cohomology of a modular curve, it naturally breaks into (isotypic pieces corresponding to) two-dimensional constituents, by the discussion in §3 above (we'll discuss this more in the next subsection). By Theorem 26 it would suffice to show that each of these constituents is unramified upon restriction to $G_{\mathbb{Q}_p(\zeta_p)}$. We will achieve this by showing that under the local Langlands correspondence, these constituents are associated to $\pi(\alpha, \beta)$, with $c(\alpha) \le 1, c(\beta) = 0$. (**Exercise.** Show that this suffices using "class field theory" for $\mathbb{Q}_p$.)

Now we consider

$$\rho_0 : G_{\mathbb{Q}_p} \to \mathrm{GL}(V_\ell(A_0) \otimes \overline{\mathbb{Q}_\ell}).$$

Suppose $V_\ell(A_0)$ has rank $2g$. By Corollary 32 we wish to show that $\rho_0|_{I_{\mathbb{Q}_p}}$ is a non-trivial extension of a trivial representation of rank g by $\rho_0^{I_K}$. As ρ_0 again naturally breaks into (isotypic pieces) corresponding to two-dimensional constituents, it would suffice to show that each of these constituents is, as a $I_{\mathbb{Q}_p}$-rep, a non-trivial extension of the trivial representation by itself. Thus by Remark 18, it would be enough to show that under the local Langlands correspondence, these constituents are associated to $\chi \otimes \mathrm{St}$, where χ is unramified.

4.3. **Computing the Relevant Local Factors.** The idea of this computation will be to simply run through *all* irreducible admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, and use very soft information coming from their classification to decide which of these representations could possibly show up as a local factor of an automorphic representation π_0, π_1 associated to $V_\ell(A_0), V_\ell(A_1)$. Let's briefly discuss how to think about these automorphic representations.

For the purpose of this section, if $\Gamma' \subset \Gamma \subset SL_2(\mathbb{Z})$ are congruence subgroups with the induced map $\iota : X(\Gamma') \to X(\Gamma)$, we say a modular form for Γ' is a *newform* for ι if it is in the span of Hecke eigenforms on $X(\Gamma')$ not pulled back from $X(\Gamma)$. Thus cusp forms on $X(\Gamma')$ break up as

$$S(\Gamma', k) = S(\Gamma, k) \oplus \{\text{newforms for } \iota\}.$$

Let $\mathrm{Jac}(\iota) := \ker(\mathrm{Jac}(X(\Gamma')) \to \mathrm{Jac}(X(\Gamma)))$. By our discussion in §3, the two-dimensional constituents of $V_\ell(\mathrm{Jac}(\iota)) \otimes \overline{\mathbb{Q}_\ell}$ are precisely the Galois representations associated to newforms for ι which are eigenforms for the Hecke algebra (at almost all primes).

We now consider Theorem 2(1), which states that A_0 has toric reduction. Recall that

$$A_0 := \mathrm{coker}(J(N) \to J_0(p, N));$$

thus the 2-dimensional Galois representations arising as constituents of $V_\ell(A_0)$ are precisely those associated to newforms f for the map

$$X_0(p, N) \to X(N)$$

which are also Hecke eigenforms for almost all primes. Consider one of these two-dimensional representations ρ_f, with associated automorphic representation π_f. This automorphic representation π_f has the following property—its p-component $\pi_{f,p}$ has a nonzero vector fixed by $\Gamma_0(\mathfrak{m}) \subset \mathrm{GL}_2(\mathbb{Z}_p)$ but not by $\mathrm{GL}_2(\mathbb{Z}_p)$ (that is, the vector associated to the newform f). This follows by translating to the adelic viewpoint and noting that the compact open subgroup of $\mathrm{GL}_2(\mathbb{A})$ associated to $\Gamma_0(p, N)$ is $\Gamma_0(\mathfrak{m})$ (that is, f may be viewed as a function on $\mathrm{GL}_2(\mathbb{A})/K^p K_p$ where $K_p = \Gamma_0(\mathfrak{m})$).

But by Theorem 14, this is enough to show that $\pi_{f,p}$ is of the form $\chi \otimes \mathrm{St}$, with χ unramified, which suffices to show that A_0 is of toric reduction by §4.2.

As for Theorem 2(2), we repeat the argument in the case of

$$A_1 := \mathrm{coker}(J_0(p, N) \to J_1(p, N)).$$

We wish to understand the Tate module $V_\ell(A_1)$, which has 2-dimensional constituents exactly the Galois representations associated to newforms for the map

$$\iota : X_1(p, N) \to X_0(p, N);$$

we wish to show that these Galois representations are unramified upon restriction to $G_{\mathbb{Q}_p(\zeta_p)}$. As before, we do this by computing the local factors of the automorphic representations π_f, where f is a new Hecke eigenform for ι. In this case, the representations π_f associated to newforms for ι have components $\pi_{f,p}$ with vectors fixed by $\Gamma_1(\mathfrak{m}) \subset \mathrm{GL}_2(\mathbb{Z}_p)$ but not $\Gamma_0(\mathfrak{m}) \subset \mathrm{GL}_2(\mathbb{Z}_p)$. Thus by Theorem 14, $\pi_{f,p}$ is of the form $\pi(\alpha, \beta)$ with $c(\alpha) = 1, c(\beta) = 0$ (without loss of generality, I've broken the symmetry in α, β, because $\pi(\alpha, \beta) \simeq \pi(\beta, \alpha)$). By §4.2, this suffices to show that A_1 has good reduction over $\mathbb{Q}_p(\zeta_p)$, as desired.

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