Lecture 2: Smooth representations and Hecke algebras for p-adic groups

1. Introduction

Let F be a non-archimedean local field, with valuation ring $\mathscr{O}$ having maximal ideal $\mathfrak{m}$, uniformizer ϖ, and finite residue field k. Let $\mathbf{G}$ be a connected reductive F-group, with associated topological group of F-points $G = \mathbf{G}(F)$. By using an identification of $\mathbf{G}$ as a closed F-subgroup of GL_n, the group G inherits the following important property from $\mathrm{GL}_n(F)$:

Definition 1.1. A *locally profinite* group is a topological group that is Hausdorff and has an open neighborhood of 1 that is a profinite group (e.g., $G \cap \mathrm{GL}_n(\mathscr{O})$ inside $\mathrm{GL}_n(F)$).

In any locally profinite group, it is clear that all compact open subgroups are profinite. In our case of interest, $G = \mathbf{G}(F)$, there is visibly a countable base for the topology (since the same holds for the topological space of F-points of any affine F-scheme of finite type, as we deduce from the case of F^d), so for any compact open subgroup K in G the discrete quotient space G/K is *countable*. This latter countability is technically very useful; e.g., it underlies Jacquet's proof of a version of Schur's Lemma for representations of G that we will give later (see Theorem 3.8). To summarize these properties of $\mathbf{G}(F)$, we make another definition.

Definition 1.2. A *td-group* is a locally profinite group such that the quotient space modulo a compact open subgroup is countable. (Here, "td" is inspired by the property "totally disconnected".)

Example 1.3. We have seen above that $\mathbf{G}(F)$ is a td-group for any $\mathbf{G}$, and clearly any closed subgroup of a td-group is a td-group. Every profinite group is a td-group, though profinite groups need not have a countable base for the topology (e.g., $\prod_{i\in I} \mathbf{Z}/p\mathbf{Z}$ for an uncountable set I).

We will use many different kinds of closed subgroups of $\mathbf{G}(F)$ to build interesting representations, and these will not generally arise as the F-points of a closed F-subgroup of $\mathbf{G}$ or as the $\mathscr{O}$-points of an $\mathscr{O}$-model of $\mathbf{G}$ (e.g., the closed subgroup $F^\times \mathrm{GL}_n(\mathscr{O})$ inside $\mathrm{GL}_n(F)$ for $\mathbf{G} = \mathrm{GL}_n$), so the generality of td-groups is useful.

Our aim in this lecture is twofold: we will discuss smooth representations of td-groups (on $\mathbf{C}$-vector spaces, generally infinite-dimensional) and Hecke algebras that are used to turn representation-theoretic questions into module-theoretic questions. The basic reference for much of what we say is [BH, Ch. 1] (the relevant parts of which are generally written in a good degree of generality), and we provide references for a variety of other facts that arise in the course of our discussion.

2. Subgroups of $\mathbf{G}(F)$

For the remainder of these notes, G denotes $\mathbf{G}(F)$ with $\mathbf{G}$ understood from context unless we say otherwise. Since $\mathbf{G}$ is unirational (as for any connected reductive group over any field) and F is infinite, G is Zariski-dense in $\mathbf{G}$. Hence, any $g \in G$ that centralizes G also centralizes $\mathbf{G}$, so the group $Z_{\mathbf{G}}(F)$ of F-points of the schematic center $Z_{\mathbf{G}}$ of $\mathbf{G}$ coincides with the center Z of the group G. This equality of notions of "center" is used all the time without comment.

We will make extensive use of compact open subgroups of G, and a common source of such subgroups is the $\mathscr{O}$-points of integral models of $\mathbf{G}$. To be precise:

Definition 2.1. An *$\mathscr{O}$-model* of $\mathbf{G}$ is a pair $(\mathscr{G}, f)$ consisting of a flat affine finite type $\mathscr{O}$-group scheme $\mathscr{G}$ and an isomorphism of F-groups $f : \mathscr{G}_F \simeq \mathbf{G}$. Often we denote an $\mathscr{O}$-model as $\mathscr{G}$, with f understood from context. (Concretely, we can use f to view the coordinate ring of $\mathscr{G}$ as an $\mathscr{O}$-structure on $F[\mathbf{G}]$, making the specification of f irrelevant.)

Definition 2.2. A group scheme is *reductive* if it is smooth and affine with connected reductive fibers (in the sense of the theory over fields).

Example 2.3. Over any ring R the group schemes GL_n, SL_n, Sp_{2n}, and SO_n are reductive, where SO_n is the special orthogonal group of the standard "split" quadratic form q_n in n variables given by $q_{2m} = x_1x_2 + \cdots + x_{2m-1}x_{2m}$ and $q_{2m+1} = x_0^2 + q_{2m}$. Taking $R = \mathscr{O}$, these $\mathscr{O}$-groups are reductive $\mathscr{O}$-models of their generic fibers over F.

We will be especially interested in $\mathscr{O}$-models of $\mathbf{G}$ that are reductive (if any such $\mathscr{O}$-model exists). Over an irreducible base scheme, if a smooth affine group scheme has connected reductive generic fiber then any fiber whose identity component is reductive is necessarily connected [Co, Prop. 3.1.12]. Likewise, by slightly non-trivial "spreading out" arguments (see [Co, Prop. 3.1.9]), any connected reductive group over a global field extends to a reductive group scheme over a ring of S-integers and hence admits a reductive model at all but finitely many places.

Example 2.4. If $\mathbf{G} = \mathrm{GL}_n$ then $\mathscr{G} = \mathrm{GL}_{n,\mathscr{O}}$ is an $\mathscr{O}$-model in the usual way. By composing with F-automorphisms of $\mathbf{G}$ (i.e., applying conjugation by $g \in G$ and perhaps transpose-inverse if we wish) we get the $\mathscr{O}$-models $\mathrm{GL}(\Lambda)$ as Λ varies through all $\mathscr{O}$-lattices in F^n. By [Co, Thm. 7.2.16], all reductive $\mathscr{O}$-models of GL_n arise in this way. Thus, in this case the reductive $\mathscr{O}$-models are transitively permuted by G-conjugation.

One way to make an $\mathscr{O}$-model $\mathscr{G}$ of $\mathbf{G}$ in general is to choose an identification $j : \mathbf{G} \hookrightarrow \mathrm{GL}_n$ of $\mathbf{G}$ as a closed F-subgroup of some GL_n and define $\mathscr{G}$ to be the schematic closure of $\mathbf{G}$ in the $\mathscr{O}$-group $\mathrm{GL}_{n,\mathscr{O}}$. (This closure is $\mathscr{O}$-flat because $\mathscr{O}$ is Dedekind, so it is elementary to check – do it! – that this closure is an $\mathscr{O}$-subgroup scheme of $\mathrm{GL}_{n,\mathscr{O}}$.) Conversely, with more work involving the coordinate ring of an $\mathscr{O}$-model one can show via the Dedekind property of $\mathscr{O}$ that *every* $\mathscr{O}$-model $\mathscr{G}$ of $\mathbf{G}$ arises in this way for some j. Beware that $\mathscr{G}_k$ can *fail* to be connected, even if $\mathscr{G}$ is $\mathscr{O}$-smooth; see [SGA3, Exp. XIX, §5] for a counterexample (in which the open locus $\mathscr{G}^0$ complementary to the non-identity components in the special fiber is affine but not closed).

For any $\mathscr{O}$-model $\mathscr{G}$ of $\mathbf{G}$, the group $K := \mathscr{G}(\mathscr{O})$ is a compact open subgroup of $\mathscr{G}_F(F) = G$. Indeed, rather generally if X is any affine $\mathscr{O}$-scheme of finite type, a choice of closed immersion of X into an affine space over $\mathscr{O}$ shows that $X(\mathscr{O})$ is a compact open subset of $X(F)$. Given an $\mathscr{O}$-model $\mathscr{G}$, we get a base of open neighborhoods of 1 in $K = \mathscr{G}(\mathscr{O})$ via the congruence subgroups $K_r = \ker(\mathscr{G}(\mathscr{O}) \to \mathscr{G}(\mathscr{O}/\mathfrak{m}^r))$ for $r \geq 1$. Hence, working with small compact open subgroups of G is a way of imposing "congruence conditions" while remaining intrinsic to the F-group $\mathbf{G}$ (i.e., avoiding a choice of $\mathscr{O}$-model).

We say that $\mathscr{O}$-models $(\mathscr{G}, f)$ and $(\mathscr{G}', f')$ of $\mathbf{G}$ are *isomorphic* if there is an isomorphism $\mathscr{G} \simeq \mathscr{G}'$ that carries f to f' (equivalently, this $\mathscr{O}$-isomorphism induces the identity on $\mathbf{G}$ over F). Such an isomorphism is unique if it exists, and its existence amounts to *equality* of the corresponding $\mathscr{O}$-structures on $F[\mathbf{G}]$. Given an $\mathscr{O}$-model $(\mathscr{G}, f)$, we get more $\mathscr{O}$-models by composing f with an F-automorphism of $\mathbf{G}$; those automorphisms not arising from $\mathscr{O}$-automorphisms of $\mathscr{G}$ give $\mathscr{O}$-models not isomorphic to the initial $\mathscr{O}$-model.

Remark 2.5. The technique of group-smoothening [BLR, §7.1, Thm. 5] shows that any $\mathscr{O}$-model $\mathscr{G}$ admits an $\mathscr{O}$-homomorphism $\mathscr{G}' \to \mathscr{G}$ from a *smooth* affine $\mathscr{O}$-group $\mathscr{G}'$ such that the induced map between F-fibers is an isomorphism (thereby making $\mathscr{G}'$ an $\mathscr{O}$-model of $\mathbf{G}$) and the induced continuous map on $\mathscr{O}$-points is bijective. In other words, as far as the formation of $\mathscr{G}(\mathscr{O})$ as a compact open subgroup of G is concerned, nothing is lost by considering only *smooth* $\mathscr{O}$-models of $\mathbf{G}$. In the work of Bruhat and Tits on the structure of $\mathbf{G}(F)$, certain smooth $\mathscr{O}$-models of $\mathbf{G}$ (called *Bruhat–Tits group schemes*) play an essential role.

Although smooth $\mathscr{O}$-models always exist (as we may apply group -moothening to any $\mathscr{O}$-model), it is a rather nontrivial condition on $\mathbf{G}$ that it admits a reductive $\mathscr{O}$-model:

Theorem 2.6. *There exists a reductive $\mathscr{O}$-model of $\mathbf{G}$ if and only if $\mathbf{G}$ is quasi-split over F and splits over a finite unramified extension of F.*

Since we have seen that a connected reductive group over a global field extends to a reductive group over some ring of S-integers, by Theorem 2.6 such a group over a global field is quasi-split at all but finitely many places. It is generally *not* true that such a group is split at all but finitely many places (special unitary groups associated to ramified separable quadratic extensions give counterexamples).

Proof. First we prove necessity. The key ingredients are the existence and smoothness of schemes of Borel subgroups and schemes of maximal tori in the relative setting. Suppose $\mathbf{G}$ admits a reductive $\mathscr{O}$-model $\mathscr{G}$. By Lang's theorem over the finite field k, $\mathscr{G}_k$ admits a Borel k-subgroup B_0. The scheme of Borel subgroups of $\mathscr{G}$ is smooth [Co, Thm. 5.2.11(3)], so B_0 lifts to a Borel $\mathscr{O}$-subgroup scheme of $\mathscr{G}$. Its generic fiber is a Borel F-subgroup of $\mathbf{G}$, so $\mathbf{G}$ is quasi-split. By the same reasoning with the smooth $\mathscr{O}$-scheme of maximal tori [Co, Thm. 3.2.6], there is an $\mathscr{O}$-torus $\mathscr{T} \subset \mathscr{G}$ that is fiberwise maximal. Since $\mathscr{O}$ is complete, if k'/k is a finite extension splitting the special fiber of $\mathscr{T}$ then for the corresponding finite unramified extension $\mathscr{O}'$ of $\mathscr{O}$ the $\mathscr{O}'$-torus $\mathscr{T}_{\mathscr{O}'}$ is split [Co, Thm. B.3.2(2)]. The fraction field F' of $\mathscr{O}'$ is a finite unramified extension of F such that $\mathbf{G}_{F'}$ contains the split maximal F'-torus $\mathscr{T}_{F'}$, so $\mathbf{G}_{F'}$ is split.

Next, we prove sufficiency. For this the key ingredient is the existence and properties of the automorphism scheme in the relative setting (and the Existence and Isomorphism Theorems for split reductive group schemes). Let $\mathbf{G}_0$ denote the split F-form of $\mathbf{G}$, so this extends to a split reductive $\mathscr{O}$-group $\mathscr{G}_0$. The $\mathscr{O}$-forms of $\mathscr{G}_0$ are classified by the pointed cohomology set $\mathrm{H}^1_{\text{ét}}(\mathscr{O}, \mathrm{Aut}_{\mathscr{G}_0/\mathscr{O}})$, where the automorphism scheme $\mathrm{Aut}_{\mathscr{G}_0/\mathscr{O}}$ fits into a split exact sequence of smooth $\mathscr{O}$-groups

$$1 \to \mathscr{G}_0^{\mathrm{ad}} \to \mathrm{Aut}_{\mathscr{G}_0/\mathscr{O}} \to \mathrm{Out}_{\mathscr{G}_0/\mathscr{O}} \to 1$$

whose final term is the constant $\mathscr{O}$-group corresponding to the automorphism group Γ of the based root datum for $\mathscr{G}_0$ (see [Co, Thm. 7.1.9]); this splitting of the exact sequence over $\mathscr{O}$

rests on a choice of pinning. Consider the resulting map

$$\mathrm{H}^1(\mathscr{O}, \mathrm{Aut}_{\mathscr{G}_0/\mathscr{O}}) \to \mathrm{H}^1(\mathscr{O}, \Gamma) = \mathrm{Hom}_{\mathrm{cont}}(\pi_1(\mathrm{Spec}\,\mathscr{O}), \Gamma) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(F^{\mathrm{un}}/F), \Gamma).$$

This has a section σ arising from the semi-direct product structure on the automorphism scheme, and by hypothesis the class of $\mathbf{G}$ in $\mathrm{H}^1(F, \mathrm{Aut}_{\mathbf{G}_0/F})$ has image in $\mathrm{H}^1(F, \Gamma)$ that is an unramified class. That unramified class corresponds to an element in $\mathrm{H}^1(\mathscr{O}, \Gamma)$, so applying σ to this class gives an $\mathscr{O}$-form $\mathscr{G}$ of $\mathscr{G}_0$ such that the class of $\mathscr{G}_F$ in $\mathrm{H}^1(F, \mathrm{Aut}_{\mathbf{G}_0/F})$ has the same image in $\mathrm{H}^1(F, \Gamma)$ as the class of $\mathbf{G}$. But it is a standard fact in the theory over fields that each fiber of $\mathrm{H}^1(F, \mathrm{Aut}_{\mathbf{G}_0/F}) \to \mathrm{H}^1(F, \Gamma)$ contains a *unique* quasi-split form. Since $\mathbf{G}$ is quasi-split by hypothesis and $\mathscr{G}_F$ is quasi-split by the necessity verified above, it follows that $\mathbf{G} \simeq \mathscr{G}_F$. Hence, $\mathscr{G}$ is a reductive $\mathscr{O}$-model of $\mathbf{G}$. ■

Example 2.7. The F-group $\mathrm{SL}_1(\Delta)$ for a finite-dimensional central division algebra Δ over F is F-anisotropic, hence not quasi-split over F, so it has no reductive $\mathscr{O}$-model. Likewise, $\mathrm{SO}(q)$ for a non-degenerate finite-dimensional quadratic space (V, q) over F such that $\mathrm{rank}_F(q) < \dim V - 2$ is not quasi-split over F, so it has no reductive $\mathscr{O}$-model.

Likewise, if F'/F is a ramified separable quadratic extension then the associated special unitary groups $\mathrm{SU}_n(F'/F)$ are quasi-split but do not split over an unramified extension and hence do not admit a reductive $\mathscr{O}$-model.

For any reductive $\mathscr{O}$-model $\mathscr{G}$, $\mathscr{G}(\mathscr{O})$ is *maximal* as a compact open subgroup of G. This maximality is elementary to prove if $\mathscr{G} = \mathrm{GL}_{n,\mathscr{O}}$ but in general it is not obvious from the definitions (the proof rests on much of the work in [BTI] and [BTII]). Such maximal compact open subgroups $\mathscr{G}(\mathscr{O})$ are called *hyperspecial*. (This notion depends on $\mathbf{G}$, not just G.) For example, $\mathrm{GL}_n(\mathscr{O})$ is a hyperspecial maximal compact open subgroup of $\mathrm{GL}_n(F)$.

Example 2.8. Let $\mathbf{G} = \mathrm{SL}_n$ for $n \geq 2$. The "conjugates" of $\mathrm{SL}_{n,\mathscr{O}}$ by $\mathrm{diag}(\varpi^i, 1, \ldots, 1) \in \mathrm{GL}_n(F)$ for $0 \leq i \leq n-1$ are reductive $\mathscr{O}$-models of SL_n that are pairwise not related to each other through G-conjugation, and every reductive $\mathscr{O}$-model of SL_n is G-conjugate to one of these (see [Co, Ex. 7.2.15, Thm. 7.2.16]). Rather deeper, *every* maximal compact open subgroup of $\mathrm{SL}_n(F)$ is conjugate to the hyperspecial one arising from one of these n reductive $\mathscr{O}$-models.

In contrast, for $\mathbf{G} = \mathrm{PGL}_n$ the hyperspecial maximal compact open subgroups of G constitute a *single* G-conjugacy class (namely, that of $\mathrm{PGL}_n(\mathscr{O})$); see [Co, Thm. 7.2.16]. However, in such cases there are *other* conjugacy classes of maximal compact open subgroups. For example, with $\mathbf{G} = \mathrm{PGL}_2$, the normalizer in $G = \mathrm{PGL}_2(F)$ of the subgroup of elements

$$\begin{pmatrix} a & b \\ \varpi c & d \end{pmatrix}$$

for $a, b, c, d \in \mathscr{O}$ is a maximal compact open subgroup of G whose volume (relative to a Haar measure of G) is *distinct* from that of the hyperspecial $\mathrm{PGL}_2(\mathscr{O})$. Much insight into the nature of this additional conjugacy class of maximal compact open subgroups is obtained by studying the action of $\mathrm{PGL}_2(F)$ on the the Bruhat–Tits tree for PGL_2 over F.

By deep work with buildings ([BTI], [BTII]) one shows:

Theorem 2.9 (Bruhat–Tits)**.** *Every compact subgroup of G lies in a maximal compact open subgroup of G, and if $\mathbf{G}$ is semisimple and simply connected then the maximal compact open subgroups of G constitute exactly $\prod(r_i+1)$ conjugacy classes, where $\{r_i\}$ is the set of F-ranks of the F-simple factors of $\mathbf{G}$.*

Note that F-simple factors with F-rank 0 make trivial contribution, consistent with the fact that a connected semisimple F-group with F-rank 0 has compact group of F-points. In the absence of a "simply connected" hypothesis, the description of conjugacy classes of maximal compact open subgroups is combinatorially a bit complicated; see [BTI, 3.3.5].

3. Smooth representations

The consideration of global automorphic representations in Akshay's Lecture 1 provides ample motivation for the following notion.

Definition 3.1. A *smooth* representation of a td-group H is a $\mathbf{C}$-linear representation (V, ρ) of H such that every $v \in V$ has open stabilizer in H, or equivalently every $v \in V$ is fixed by a compact open subgroup K of H; in other words, $V = \cup_K V^K$.

The preceding definition is equivalent to the continuity of the action map $H \times V \to V$ when V is given the discrete topology.

Example 3.2. The space $C^\infty(H)$ of smooth (i.e., locally constant) functions $H \to \mathbf{C}$ equipped with the right regular representation is *not* smooth when H is non-compact, since smooth functions that do not factor through a discrete coset space for H do not have open stabilizer in H. By contrast, the space $C_c^\infty(H)$ of compactly supported smooth functions is a smooth H-representation (check!).

Example 3.3. If V is an arbitrary representation of H then its subset V^∞ of *smooth vectors* (i.e., those $v \in V$ with open stabilizer in H) is an H-subrepresentation. In this generality, V^∞ could vanish even if $V \neq 0$. For example, let $H = F$ and define $f_0 : F \to \mathbf{C}$ to be the function given by $f_0(x) = 1/|x|$ for $x \neq 0$ and $f_0(0) = 0$. Let V be the vector space of functions $F \to \mathbf{C}$ of the form $\sum c_j f_0(x + a_j)$ for $c_j \in \mathbf{C}$ and $a_j \in F$, on which H acts via $(h.f)(x) = f(x + h)$. By considering points of F near which a vector $f \in V$ is unbounded, we see that $V^\infty = 0$.

If V is a smooth representation, then the space $V^\vee := (V^*)^\infty$ of smooth vectors in the linear dual is called the *contragredient* representation. There is an evident map of H-representations $V \to (V^\vee)^\vee$ but it is not obvious if this is either injective of surjective.

It is clear from the definitions that the formation of V^∞ is left-exact in V. A deeper property is that if V is a smooth representation then $V^\vee$ "separates points" in V (see [BH, §2.8]). This says exactly that $V \to (V^\vee)^\vee$ is injective. We will address surjectivity shortly, after introducing some relevant notions.

In the global theory with irreducible automorphic representations, if we fix the archimedean data (the "weight") then the space of invariant vectors under a compact open subgroup of the finite-adelic points (corresponding to specifying the "level") is finite-dimensional. In the local theory, the corresponding finiteness condition on representations is:

Definition 3.4. A smooth representation (V, ρ) of a td-group H is *admissible* if $\dim V^K < \infty$ for all compact open subgroups $K \subset H$.

In [BH, §2.8–§2.10] one finds the following important fact (whose proof amounts to artful manipulation with spaces of K-invariants for varying K):

Theorem 3.5. *If a smooth representation V of H is admissible then its contragredient $V^\vee$ is admissible, and formation of the smooth dual is an exact functor on the category of admissible smooth representations of H. Moreover, if V is an admissible smooth representation of H then the natural injective map $V \to (V^\vee)^\vee$ is an isomorphism.*

As evidence for the importance of admissibility, it is worthwhile now to record two deep results. The first rests on the theory of cuspidal representations, and will be addressed in Niccoló's lecture later (building on Iurie's lecture):

Theorem 3.6 (Bernstein)**.** *Every irreducible smooth representation of $\mathbf{G}(F)$ is admissible.*

This theorem is motivated by a result of Harish-Chandra in the archimedean case that will be discussed in Zhiwei's lecture. The next theorem is discussed in [Ca, §2]; it relates the classical viewpoint of Hilbert space representations to the algebraic viewpoint of smooth representations, and once again there is a precursor in the archimedean case (to be discussed in Zhiwei's lecture):

Theorem 3.7. *If $\mathscr{V}$ is a unitary representation of $G = \mathbf{G}(F)$ on a Hilbert space and it is irreducible in the sense of being nonzero and having no nonzero proper closed subrepresentations then the subspace $V := \mathscr{V}^\infty$ of smooth vectors is dense, and it is irreducible as a smooth G-representation (so V is admissible).*

Conversely, if V is an irreducible smooth representation of G then it has at most one G-equivariant unitary structure up to $\mathbf{C}^\times$-scaling and when such a structure exists then the completion $\mathscr{V}$ of V has space of smooth vectors $\mathscr{V}^\infty$ equal to V. In particular, $\mathscr{V}$ is irreducible in the sense of Hilbert space representations.

In view of the preceding theorem, we now forget about unitary structures and focus on the purely algebraic theory of smooth representations. The key fact at the start of the theory is the following result of Jacquet that underlies the uniqueness (up to scaling) of the unitary structure in the preceding theorem:

Theorem 3.8 (Schur's Lemma)**.** *For a td-group H, any irreducible smooth H-representation V satisfies $\mathrm{End}_H(V) = \mathbf{C}$.*

Proof. Choose a nonzero $v_0 \in V$, so $v_0 \in V^K$ for some compact open subgroup $K \subset H$. By irreducibility, the orbit $(H/K)v_0$ spans V over $\mathbf{C}$. But this orbit is a *countable* set, so $\dim_{\mathbf{C}} V$ is countable.

Any $f \in \mathrm{End}_H(V)$ is determined by $f(v_0) \in V$ due to irreducibility, so $\dim_{\mathbf{C}} \mathrm{End}_H(V)$ is countable. But $\mathrm{End}_H(V)$ is a division algebra (with $\mathbf{C}$ in the center) due to irreducibility of V. Thus, if there exists $f \in \mathrm{End}_H(V)$ not in $\mathbf{C}$ then since $\mathbf{C}$ is algebraically closed we get an injection $\mathbf{C}(t) \hookrightarrow \mathrm{End}_H(V)$ induced by $\mathbf{C}[t] \to \mathrm{End}_H(V)$ as $\mathbf{C}$-algebras with $t \mapsto f$. Hence, $\dim_{\mathbf{C}} \mathbf{C}(t)$ would be countable, a contradiction. ∎

As an application of Schur's Lemma, we have:

Proposition 3.9. *If the F-simple factors of $\mathscr{D}(\mathbf{G})$ are F-isotropic then any irreducible smooth representation V of $G = \mathbf{G}(F)$ is 1-dimensional or infinite-dimensional.*

Proof. We assume $\dim V < \infty$ and seek to show that G acts through an abelian quotient. Then $\rho : G \to \mathrm{Aut}_{\mathbf{C}}(V)$ would land inside $\mathrm{Aut}_G(V) = \mathbf{C}^\times$ by Schur's Lemma, giving the desired 1-dimensionality due to irreducibility.

Let $\widetilde{\mathbf{G}}' \to \mathscr{D}(\mathbf{G})$ be the simply connected central cover. I claim that

$$q : \widetilde{\mathbf{G}}'(F) \to G$$

has normal image with commutative cokernel, so it would then suffice to prove that $\widetilde{\mathbf{G}}'(F)$ acts trivially on V.

The conjugation action of the smooth $\mathbf{G}$ on $\mathscr{D}(\mathbf{G})$ lifts uniquely to an action of $\mathbf{G}$ on $\widetilde{\mathbf{G}}'$ due to canonicity of the simply connected central cover of a connected semisismple group over a field, so normality of the image of q follows. To prove commutativity of the cokernel of q, consider the commutator morphism $\mathbf{G} \times \mathbf{G} \to \mathbf{G}$. This lands inside $\mathscr{D}(\mathbf{G})$ and factors through $\mathbf{G}^{\mathrm{ad}} \times \mathbf{G}^{\mathrm{ad}}$, where $\mathbf{G}^{\mathrm{ad}} := \mathbf{G}/Z_{\mathbf{G}}$. But $\mathbf{G}^{\mathrm{ad}} = \mathscr{D}(\mathbf{G})^{\mathrm{ad}} = (\widetilde{\mathbf{G}}')^{\mathrm{ad}}$ (!), so $\mathbf{G}^{\mathrm{ad}} \times \mathbf{G}^{\mathrm{ad}} \to \mathscr{D}(\mathbf{G})$ factors through $\widetilde{\mathbf{G}}' \to \mathscr{D}(\mathbf{G})$ via the commutator for $\widetilde{\mathbf{G}}'$. Now passing to F-points, we conclude that all commutators in G lie in the image of $\widetilde{\mathbf{G}}'(F)$, so the cokernel of q is commutative.

Since $\dim V$ is finite and G acts smoothly on V, $\ker(\widetilde{\mathbf{G}}'(F) \to \mathrm{GL}(V))$ is an open normal subgroup. Hence, by renaming $\widetilde{\mathbf{G}}'$ as $\mathbf{G}$ (this preserves the F-isotropicity hypothesis!) it suffices to prove that if $\mathbf{G}$ is a connected semisimple F-group that is simply connected and every F-simple factor is F-isotropic then $\mathbf{G}(F)$ has no proper open normal subgroup. The simply connected hypothesis implies that $\mathbf{G} \simeq \prod \mathrm{R}_{F_i/F}(\mathbf{G}_i)$ for finite separable extensions F_i/F and *absolutely simple* connected semisimple F_i-groups $\mathbf{G}_i$ that are simply connected, with each $\mathbf{G}_i$ an F_i-isotropic group. Thus, $\mathbf{G}(F) = \prod \mathbf{G}_i(F_i)$, so it suffices to treat each $\mathbf{G}_i(F_i)$ separately. That is, we may work separately with the F_i-group $\mathbf{G}_i$ for each i, and so may assume $\mathbf{G}$ is absolutely simple and simply connected, as well as F-isotropic. In this case, it is a theorem of Tits proved in [Pra] that every proper open subgroup of $\mathbf{G}(F)$ is compact, so it suffices to show that any open normal subgroup N of $\mathbf{G}(F)$ is non-compact.

Let $\mathbf{S}$ be a maximal F-split torus, and a a non-multipliable root in the relative root system. The associated root group $\mathbf{U}_a$ has the form $\mathbf{G}_{\mathrm{a}}^d$ on which $\mathbf{S}$ acts linearly through a, so for any $\lambda \in F^\times$ the point $s := a^\vee(\lambda) \in \mathbf{S}(F)$ acts via conjugation on $\mathbf{U}_a(F)$ through scaling by λ^2 on each $\mathbf{G}_{\mathrm{a}}$-factor. Hence, the commutator map $\omega \mapsto (s, \omega) = (s\omega s^{-1})\omega^{-1}$ is given by componentwise scaling by $\lambda^2 - 1$. By choosing λ with sufficiently negative valuation, the image of $\omega \mapsto (s, \omega)$ on the open neighborhood $\mathbf{U}_a(F) \cap N$ of the identity has image containing any desired compact neighborhood of the identity in $\mathbf{U}_a(F)$ in $\mathbf{U}_a(F)$. But such commutators lie inside N by normality of N in $\mathbf{G}(F)$, so N is non-compact as desired. ■

The isotropicity condition in Proposition 3.9 cannot be dropped; e.g., if $\mathbf{G}$ is semisimple and F-anisotropic then G is a profinite group and hence its irreducible smooth representations are finite-dimensional (as every compact open subgroup has finite index and therefore contains a *normal* compact open subgroup of G). Also, by Bruhat–Tits theory (see [BTIII, 4.5]) the isotropicity condition on an F-simple factor can only fail for certain F-groups of type A (e.g., the F-group of norm-1 units in a central division algebra over a finite separable extension of F).

4. Induction

Let H be a td-group, and H' a closed subgroup of H. There are two "induction" operations carrying smooth H'-representations to smooth H-representations. To explain the origin of two notions, we first review the story of induction for finite groups.

Let Γ be a finite group, and Γ' a subgroup. Let (V', ρ') be a representation of Γ'. In [S, §3.3] one finds the following definition of induction:

$$\mathrm{Ind}_{\Gamma'}^{\Gamma}(\rho') = \mathbf{C}[\Gamma] \otimes_{\mathbf{C}[\Gamma']} V'.$$

If one reads other references, one sometimes finds another definition:

$$\mathrm{Ind}_{\Gamma'}^{\Gamma}(\rho') = \{f : \Gamma \to V' \mid f(\gamma'\gamma) = \rho'(\gamma')(f(\gamma)) \text{ for all } \gamma \in \Gamma, \gamma' \in \Gamma'\}$$

(on which Γ acts via $(\gamma.f)(x) = f(x\gamma)$). The isomorphism between these constructions is given by sending $[\gamma] \otimes v'$ to the function $f_{\gamma,v'}$ that kills all cosets $\Gamma' x$ for $x \neq \gamma^{-1}$ and sends $\gamma'\gamma^{-1}$ to $\rho'(\gamma')(v')$. These two recipes respectively generalize to the functors c-$\mathrm{Ind}_{H'}^{H}$ and $\mathrm{Ind}_{H'}^{H}$ in the setting of td-groups, as follows.

Definition 4.1. Let H be a td-group, H' a closed subgroup, and (V', ρ') a smooth representation of H'. Its *induction* to H is

$$\mathrm{Ind}_{H'}^{H}(V') = \{f : H \to V' \mid f(h'h) = \rho'(h')(f(h)) \text{ for all } h \in H, h' \in H'\}^{\infty};$$

this consists of the smooth vectors relative to the H-action $(h.f)(x) = f(xh)$.

Define *compact induction* c-$\mathrm{Ind}_{H'}^{H}(V')$ to be the subspace of $\mathrm{Ind}_{H'}^{H}(V')$ consisting of those f such that the open and closed subset $H' \backslash \mathrm{supp}(f) \subset H' \backslash H$ is compact.

Example 4.2. For the trivial representation of H', $\mathrm{Ind}_{H'}^{H}(1)$ is the space of smooth vectors in the space of $\mathbf{C}$-valued functions on $H' \backslash H$. This is not admissible unless H/H' is compact!

Remark 4.3. If H' has finite index in H then c-$\mathrm{Ind}_{H'}^{H}(V') = \mathbf{C}[H] \otimes_{\mathbf{C}[H']} V'$ by the same procedure as used in the case of finite groups. Also, obviously if H/H' is compact then the inclusion of c-$\mathrm{Ind}_{H'}^{H}(V')$ into $\mathrm{Ind}_{H'}^{H}(V')$ is an equality. A notable case when this occurs is $H = \mathbf{G}(F)$ for a connected reductive F-group $\mathbf{G}$ and $H' = \mathbf{P}(F)$ for a parabolic F-subgroup $\mathbf{P} \subset \mathbf{G}$, in which case $H/H' = (\mathbf{G}/\mathbf{P})(F)$ is compact. (By smoothness of $\mathbf{G} \to \mathbf{G}/\mathbf{P}$ and the Zariski-local structure theorem for smooth and especially étale morphisms in [EGA, IV_4, 17.11.4, 18.4.6(ii)], it is elementary to check that $\mathbf{G}(F)/\mathbf{P}(F) \to (\mathbf{G}/\mathbf{P})(F)$ is an open embedding and even an open immersion of F-analytic manifolds, but it is not obvious that this is an equality since in general $\mathrm{H}^1(F, \mathbf{P}) \neq 1$ except when $\mathbf{G}$ is split and $\mathbf{P}$ is a Borel F-subgroup. One has to use "open cell" considerations adapted to the dynamic description of $\mathbf{P}$ to establish the equality (as works over any field whatsoever). The compactness of $(\mathbf{G}/\mathbf{P})(F)$ is elementary, since $\mathbf{G}/\mathbf{P}$ is Zariski-closed in a projective space over F.)

Here are some basic properties of induction operations (all proved in [BH, Ch. 1]):

Theorem 4.4. *Let H be a td-group and H' a closed subgroup. Let (V', ρ') be a smooth representation of H' and (W, σ) a smooth representation of H.*

(1) *The H'-homomorphism $\mathrm{Ind}_{H'}^{H}(\rho') \to \rho'$ defined by $f \mapsto f(1)$ defines the $\mathbf{C}$-linear "Frobenius reciprocity" isomorphism*
$$\mathrm{Hom}_H(\sigma, \mathrm{Ind}_{H'}^{H}(\rho')) \simeq \mathrm{Hom}_{H'}(\sigma, \rho').$$
(2) *The functor $\mathrm{Ind}_{H'}^{H}$ is exact and commutes with finite direct sums. (It does not commute with arbitrary direct sums in general.)*

(3) *If H' is open in H then composition with the H'-homomorphism $\rho' \to \text{c-Ind}_{H'}^{H}(\rho')$ carrying v' to the function $H \to V'$ supported inside H' via $h' \mapsto \rho'(h')(v')$ defines the $\mathbf{C}$-linear "Frobenius reciprocity" isomorphism*
$$\mathrm{Hom}_H(\text{c-Ind}_{H'}^{H}(\rho'), \sigma) \simeq \mathrm{Hom}_H(\rho', \sigma).$$
(4) *The functor $\text{c-Ind}_{H'}^{H}$ is exact and commutes with arbitrary direct sums.*

It is natural to ask how the induction operations interact with the formation of smooth duals. The relationship involves the intervention of modulus characters of H and H', so we remind the reader that $\mathbf{G}(F)$ is unimodular (i.e., has trivial modulus character).

Theorem 4.5. *For smooth representations ρ' of H',*
$$(\text{c-Ind}_{H'}^{H}(\rho'))^{\vee} \simeq \mathrm{Ind}_{H'}^{H}((\delta_H/\delta_{H'})|_{H'} \otimes \rho'^{\vee}).$$

For a proof, see [BH, 3.5].

Example 4.6. Assume $\mathbf{G}$ is split, so choose a split maximal F-torus $\mathbf{T}$ and a Borel F-subgroup $\mathbf{B}$ of $\mathbf{G}$ containing $\mathbf{T}$. Thus, for the unipotent radical $\mathbf{U}$ of $\mathbf{B}$ we have $\mathbf{B} = \mathbf{T} \ltimes \mathbf{U}$. In particular, $\mathbf{T} \simeq \mathbf{B}/\mathbf{U}$. Let G, T, B, U denote the corresponding groups of F-points.

Let $\chi : T \to \mathbf{C}^{\times}$ be a smooth character. Explicitly, if we choose a basis of $\mathrm{X}^*(\mathbf{T})$ to identify $\mathbf{T}$ with GL_1^r then T is identified with $(F^{\times})^r$, so $\chi : T \to \mathbf{C}^{\times}$ is given by $(c_1, \dots, c_r) \mapsto \prod \chi_j(c_j)$ for smooth characters $\chi_j : F^{\times} \to \mathbf{C}^{\times}$. The identification $B/U \simeq T$ promotes χ to a smooth character $B \to \mathbf{C}^{\times}$ that we also denote as χ.

Since the quotient G/B is compact, we have $\text{c-Ind}_B^G(\chi) = \mathrm{Ind}_B^G(\chi)$. These representations are instances of *principal series* representations, and they satisfy
$$\mathrm{Ind}_B^G(\chi)^{\vee} \simeq \mathrm{Ind}_B^G(\delta_B^{-1} \otimes \chi^{-1}).$$
More generally, for any connected reductive F-group $\mathbf{G}$ (no F-split hypothesis), parabolic F-subgroup $\mathbf{P}$ of $\mathbf{G}$, and smooth representation ρ of $P = \mathbf{P}(F)$ trivial on $\mathscr{R}_u(\mathbf{P})(F)$, $\mathrm{Ind}_P^G(\rho)^{\vee} \simeq \mathrm{Ind}_P^G(\delta_P^{-1} \otimes \rho^{\vee})$ ("parabolic induction"). This construction will be studied in Iurie's lecture.

Proposition 4.7. *For a td-group H and closed subgroup H' such that H/H' is compact, if ρ' is a smooth admissible representation of H' then $\mathrm{Ind}_{H'}^{H}(\rho')$ is admissible.*

Proof. By admissibility, ρ' is its own double smooth dual. Thus, it is equivalent to treat the induction of duals of smooth admissible representations. For compact open subgroup K in H, a spanning set for the space of K-invariants in $\mathrm{Ind}_{H'}^{H}(\rho'^{\vee})$ is described explicitly in [BH, 3.5, Lemma 2], and if H/H' is compact (so the space of double cosets $H' \backslash H/K$ is finite) then this explicit spanning set is finite. ■

5. Matrix coefficients

For a finite-dimensional $\mathbf{C}$-linear representation $\rho : \Gamma \to \mathrm{GL}(V)$ of a group Γ and a basis $\{e_1, \ldots, e_n\}$ of V identifying it with $\mathbf{C}^n$, the resulting matrix representation $[\rho] : \Gamma \to \mathrm{GL}_n(\mathbf{C})$ has matrix entries $a_{ij} : \Gamma \to \mathbf{C}$ given by

$$a_{ij}(\gamma) = e_i^*(\rho(\gamma)(e_j)) = \langle \rho(\gamma_(e_j), e_i^* \rangle.$$

Thus, rather generally, the functions $f_{v,\ell} : \Gamma \to \mathbf{C}$ defined by

$$\gamma \mapsto \langle \rho(\gamma)(v), \ell \rangle$$

for $v \in V$ and $\ell \in V^*$ are called *matrix coefficients* of ρ. More generally, elements of the $\mathbf{C}$-span $\mathscr{C}(\rho) \subset \mathrm{Func}(\Gamma, \mathbf{C})$ of such functions are called *matrix coefficients*. We now adapt this notion to smooth representations of td-groups.

Definition 5.1. Let (V, π) be a smooth representation of a td-group H. A *matrix coefficient* of π is a finite $\mathbf{C}$-linear combination of functions of the form $c_{v,\ell} : H \to \mathbf{C}$ given by $h \mapsto \langle \pi(h)(v), \ell \rangle$ for $v \in V$ and $\ell \in V^\vee$. This space of functions constitutes a smooth $H \times H$-representation $\mathscr{C}(\pi)$ inside the space $\mathbf{C}$-valued functions on H.

Note that if $V \neq 0$ then when V is admissible (so $V^\vee$ separates points) there exists a nonzero matrix coefficient. Since $c_{\pi^\vee,\ell,v}(h) = c_{\pi,v,\ell}(h^{-1})$, π and its smooth dual $\pi^\vee$ have matrix coefficients swapped via inversion on H. Also, if π is irreducible and $\omega_\pi : Z \to \mathbf{C}^\times$ is the associated central character (via Schur's Lemma) then $f(zh) = \omega_\pi(z) f(h)$ for *all* matrix coefficients f of π. In particular, the support of a matrix coefficient of such a π is stable under Z-multiplication (so such matrix coefficients that are nonzero cannot be compactly supported when Z is non-compact).

Example 5.2. If V is an admissible irreducible smooth representation of H then $V \otimes V^\vee \simeq \mathscr{C}(\pi)$ via $v \otimes \ell \mapsto c_{v,\ell}$ (as $H \times H$-representations). Consequently, in such cases V is determined up to isomorphism by its space of matrix coefficients as an $H \times H$-representation.

The following nontrivial result is [BH, 10.1, Prop., Ch. 3] (see Remark 1 in loc. cit.), and its proof makes essential use of Hecke algebras, which we will introduce in §6.

Theorem 5.3. *Let π be an irreducible smooth representation of a td-group H with center Z such that H/Z is unimodular. If some nonzero matrix coefficient of π has support that is compact modulo Z then all matrix coefficients of π have the same property and π is admissible.*

The irreducible smooth π whose matrix coefficients satisfy the compactness condition modulo Z as in the preceding theorem are called *cuspidal* representations. The fact that cuspidal irreducible smooth representations are admissible is the starting point for the proof of Bernstein's theorem that all irreducible smooth representations of $\mathbf{G}(F)$ are admissible (Theorem 3.6). Cuspidal representations of $\mathbf{G}(F)$ will be the focus of Niccoló's lecture.

Example 5.4. Let H be a td-group with center Z, and $K \subset H$ a closed subgroup containing Z such that K/Z is compact (e.g., $K = K_0 Z$ for a compact open subgroup K_0 in H). Let π be an irreducible smooth representation of K and assume that all irreducible smooth representations of H are admissible (as is the case for $H = \mathbf{G}(F)$, by Theorem 3.6). Then $\text{c-Ind}_K^H(\pi)$ is a cuspidal irreducible representation of H provided that

$$\{h \in H \mid \text{Hom}_{h^{-1}Kh\cap K}(\pi^h, \pi) \neq 0\} = K. \tag{5.1}$$

For a proof, see [BH, 11.4, Thm., Remark 1].

As an example, for $K_0 := \text{GL}_2(\mathscr{O})$ let π_0 be the inflation to K_0 of an irreducible cuspidal representation of $\text{GL}_2(k)$. Extend π_0 to a representation π of K_0Z by extension of the central character. Then $\text{c-Ind}_{K_0Z}^{\text{GL}_2(F)}(\pi)$ is a cuspidal irreducible representation, since the hypothesis (5.1) holds by [BH, 11.5, Lemma (2)].

6. Hecke algebras

Just as the group algebra of a finite group provides a mechanism for turning representation-theoretic questions into module-theoretic questions, Hecke algebras provide an analogue for the study of smooth representations of a td-group. Our main interest is the td-group $\mathbf{G}(F)$, but we allow considerable generality: let H be a td-group, Z a central subgroup of H (not necessarily the entire center), and assume H/Z is *unimodular* with a fixed Haar measure μ.

Example 6.1. If H is unimodular then so is H/Z, by an application of Fubini's theorem in the form that is relevant to short exact sequences of locally compact Hausdorff topological groups.

For any irreducible smooth representation (V, π) of H, we get a (smooth) central character $\omega_\pi : Z \to \mathbf{C}^\times$. Rather generally, for *any* smooth character $\chi : Z \to \mathbf{C}^\times$, we define $C_c^\infty(H)_\chi$ to be the space of locally constant functions $f : H \to \mathbf{C}$ such that two conditions hold: (i) $f(zh) = \chi(z)^{-1}f(h)$ for all $h \in H$ and $z \in Z$ (so $\text{supp}(f)$ is stable under Z-translation), and (ii) the closed subset $\text{supp}(f)/Z \subset H/Z$ is compact. The space $C_c^\infty(H)_\chi$ is an associative $\mathbf{C}$-algebra (without unit unless H/Z is compact) via the convolution

$$(f_1 * f_2)(h) = \int_{H/Z} f_1(x)f_2(x^{-1}h)\text{d}\mu(x);$$

the integral makes sense because the integrand is Z-invariant and as such descends into $C_c^\infty(H/Z)$.

Definition 6.2. The *Hecke algebra* $\mathscr{H}_\chi(H)$ is the convolution algebra $C_c^\infty(H)_\chi$.

In the special (and not so interesting) case that H/Z is compact and $\chi = 1$, the Hecke algebra is $C^\infty(H/Z)$ equipped with the usual convolution. In general, we will use (left) $\mathscr{H}_\chi(H)$-modules to study smooth H-representations (V, π) on which Z acts through χ. Although the Hecke algebra generally has no unit, we will soon see that it has an abundant supply of idempotents.

To relate smooth representations to modules over a Hecke algebra, consider any smooth representation (V, π) on which Z acts through χ. We make V into a left $\mathscr{H}_\chi(H)$-module by defining $f.v$ for $f \in \mathscr{H}_\chi(H)$ and $v \in V$ via the formula

$$f.v = \int_{H/Z} f(h)(h.v)\text{d}\mu(h);$$

note that the integrand is Z-invariant and descends to a function in $C_c^\infty(H/Z)$. The pairing $(f,v)\mapsto f.v$ is readily checked to define a structure of left $\mathscr{H}_\chi(H)$-module on V. The Hecke modules arising in this way satisfy a "smoothness" property that rests on an abundant supply of idempotents. To explain this, we first introduce idempotents with the following construction.

Let K_0 be a compact open subgroup of H that is sufficiently small so that $K_0\cap Z\subset\ker\chi$. Hence, $K=K_0Z$ is an open subgroup of H containing Z such that

(i) K/Z is compact (which ensures that the K-action on Z is semisimple),

(ii) there is a smooth character $\psi : K\to\mathbf{C}^\times$ extending χ on Z, via $k_0z\mapsto\chi(z)$.

We define $e_{K,\psi} : H\to\mathbf{C}$ to be the function supported on K via $k\mapsto(1/\mu(K/Z))\psi(k)^{-1}$. The following lemma is elementary.

Lemma 6.3. *We have $e_{K,\psi}*e_{K,\psi}=e_{K,\psi}$; in particular, $\mathscr{H}_\chi(H,\psi):=e_{K,\psi}\mathscr{H}_\chi(H)e_{K,\psi}$ has unit $e_{K,\psi}$. Moreover, for $f\in\mathscr{H}_\chi(H)$ we have $f\in\mathscr{H}_\chi(H,\psi)$ if and only if*

$$f(k_1hk_2)=\psi(k_1k_2)^{-1}f(h)$$

for all $h\in H$ and $k_1,k_2\in K$.

In the special case χ and ψ are trivial, the Hecke algebra $\mathscr{H}_\chi(H,\psi)$ is denoted $\mathscr{H}(H,K)$ and is often called the *spherical Hecke algebra* of H with respect to K. The general significance of $\mathscr{H}_\chi(H,\psi)$ is revealed by the following result whose proof is an exercise.

Proposition 6.4. *For a smooth representation (V,π) of H such that Z acts on V through χ, let $V^{K,\psi}$ be the ψ-isotypic part of V as a semisimple K-representation. The endomorphism $\pi(e_{K,\psi})$ of V has image $V^{K,\psi}$.*

Note that as we vary the pairs (K,ψ) above, the subspaces $V^{K,\psi}$ exhaust V. Indeed, for any $v\in V$ we can choose K_0 as above small enough so that v is invariant by K_0, so for $K:=K_0Z$ we have $v\in V^{K,\psi}$. This exhaustion property motivates (via the preceding proposition) a condition on the Hecke module side:

Definition 6.5. A left $\mathscr{H}_\chi(H)$-module M is *smooth* if every $m\in M$ satisfies $e_{K,\psi}m=m$ for some (K,ψ) as above.

The payoff for these definitions is the following result:

Theorem 6.6. *There is an equivalence of categories between the category of smooth H-representations on which Z acts through χ and the category of smooth $\mathscr{H}_\chi(H)$-modules.*

Under this equivalence, duality on smooth representations relate smooth $\mathscr{H}_\chi(H)$-modules and smooth $\mathscr{H}_{\chi^{-1}}(H)$-modules via the algebra isomorphism $\mathscr{H}_\chi(H)\simeq\mathscr{H}_{\chi^{-1}}(H)$ defined by

$$f\mapsto(f^\vee : h\mapsto f(h^{-1})).$$

This is proved in [BH, 4.3, Prop. (2)] for the special case $Z=1$ (so $\chi=1$); the proof adapts to the more general situation considered above.

Example 6.7. Let $V=\mathscr{H}_\chi(H)$ with left H-action via $(h.v)(x)=v(h^{-1}x)$. This is a smooth representation (check!) and its associated smooth $\mathscr{H}_\chi(H)$-module is $V=\mathscr{H}_\chi(H)$ as a left module via left multiplication. In contrast, if we make it into an H-representation via the right H-action $(h.v)(x)=v(xh)$ then the associated left module structure is $f.v=v*f^\vee$.

Example 6.8. If H' is an open subgroup of H that contains Z then $\text{c-Ind}_{H'}^{H}(\cdot)$ corresponds to $\mathscr{H}_\chi(H)\otimes_{\mathscr{H}_\chi(H')}(\cdot)$ on (smooth) left Hecke modules. This recovers Serre's preferred description of (compact) induction in the case of finite groups (choosing $Z = 1$; recall that Z is allowed to be any central subgroup).

By setting $Z = 1$ and $\chi = 1$ (so $\mathscr{H}_\chi(H) = C_c^\infty(H)$ with the usual convolution, denoted $\mathscr{H}(H)$), Proposition 6.4 implies:

Corollary 6.9. *A smooth representation* (V, π) *of* H *is admissible if and only if for every* $f \in C_c^\infty(H) = \mathscr{H}(H)$ *the linear operator* $\pi(f) : v \mapsto f.v$ *on* V *is finite rank.*

In the setting of this corollary, it makes sense to define the function $\chi_\pi : \mathscr{H}(H) \to \mathbf{C}$ by $f \mapsto \mathrm{Tr}(\pi(f))$. We call this the *character* of (V, π).

The proof of Theorem 6.6 readily adapts to prove:

Theorem 6.10. *Let* (V, π) *be an irreducible smooth* H*-representation on which* Z *acts through* χ. *If* $V^{K,\psi} \neq 0$ *then it is simple as an* $\mathscr{H}_\chi(H, \psi)$*-module, and the operation* $V \rightsquigarrow V^{K,\psi}$ *defines a bijection between sets of isomorphism classes of irreducible smooth representations* V *satisfying* $V^{K,\psi} \neq 0$ *and simple smooth* $\mathscr{H}_\chi(H, \psi)$*-modules.*

Example 6.11. If $Z = 1$ and $\chi = 1$ (so $\psi = 1$ and K is a compact open subgroup of H) then we get a bijection between the set of isomorphism classes of irreducible smooth H-representations V such that $V^K \neq 0$ and the set of isomorphism classes of simple smooth $\mathscr{H}(H, K)$-modules.

As especially noteworthy case is when $H = \mathbf{G}(F)$ for $\mathbf{G} = \mathscr{G}_F$ with a reductive $\mathscr{O}$-group scheme $\mathscr{G}$ and $K := \mathscr{G}(\mathscr{O}) \subset G := \mathbf{G}(F)$. In this case irreducible smooth G-representations admitting a nonzero K-invariant vector are classified by simple smooth $\mathscr{H}(G, K)$-modules. This is interesting because $\mathscr{H}(G, K)$ is a *commutative* and finitely generated $\mathbf{C}$-algebra. As Makisumi will explain in his lecture, the Satake transform makes this explicit by identifying $\mathscr{H}(G, K)$ with $\mathbf{C}[\mathrm{X}(\mathbf{S})]^W$ where $\mathbf{S}$ is a split maximal F-torus in $\mathbf{G}$ and $W = W(\mathbf{G}, \mathbf{S})$ is the relative Weyl group. For instance, if $\mathbf{G} = \mathrm{GL}_n$ then the spherical Hecke algebra (using a hyperspecial K) is

$$\mathbf{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n} = \mathbf{C}[s_1, \ldots, s_n, 1/s_n].$$

Setting $n = 2$, we get the classical description $\mathbf{C}[T, R, 1/R]$ where $T = K\mathrm{diag}(\varpi, 1)K$ and $R = K\mathrm{diag}(\varpi, \varpi)K$.

In fact, the commutativity of such Hecke algebras $\mathscr{H}(G, K)$ can be proved without reference to the Satake isomorphism, by using the Cartan decomposition $G = KAK$ with a commutative subgroup A such that there is an anti-automorphism of G preserving K and acting as the identity on A (so the identity endomorphism of the Hecke algebra is an anti-automorphism, forcing the Hecke algebra to be commutative). For $\mathbf{G} = \mathrm{GL}_n$ this anti-automorphism can be built using the transpose operation. In the general split case, one can construct it by using a Chevalley involution.

7. A variant

We wrap up our discussion with a useful generalization of the Hecke algebras $\mathscr{H}_\chi(H,\psi)$. Rather than focusing on ψ-isotypic parts of V with ψ a 1-dimensional character of K (extending χ on Z), we now allow more general finite-dimensional irreducible smooth representations ρ of K on which Z acts through χ. That is, we consider smooth H-representations V such that the ρ-isotypic subspace $V^{K,\rho}$ of V as a (necessarily semisimple!) smooth K-representation is nonzero.

The idempotents $e_{K,\psi} \in \mathscr{H}_\chi(H)$ generalize as follows. Let $K' := \ker(\rho) \subset K$, an open subgroup not necessarily containing Z, so $K'Z$ has finite index in K. Hence, K/K' is an extension of the *finite* group $K/K'Z$ by the *central* subgroup $K'Z/K'$. We define the smooth character $\chi_{K'} : K'Z \to \mathbf{C}^\times$ by $k'z \mapsto \chi(z)$. The group K/K' is not finite but its smooth representation theory subject to fixing the action of $K'Z/K'$ via $\chi_{K'}$ behaves like that of a finite group.

Define the element $e_\rho \in \mathscr{H}_\chi(H,\chi_{K'}) \simeq \mathscr{H}_{\chi_{K'}}(K/K')$ to be the idempotent associated to the irreducible representation ρ of K/K'. (The idea, which we leave for the reader to work out, is that $\mathscr{H}_{\chi_{K'}}(K/K')$ behaves much like the group ring of a finite group.) The proofs in the case of 1-dimensional $\rho = \psi$ as considered above can be generalized to establish:

Theorem 7.1. *Let (V,π) be a smooth representation of (V,π) on which Z acts through χ. The operator $\pi(e_\rho) : V \to V$ is the K-equivariant projection onto the ρ-isotypic part of V as a K-representation. Moreover, if V is irreducible with $V^\rho \neq 0$ then V^ρ is a simple smooth $\mathscr{H}_\chi(H,\rho)$-module, and this defines a bijection between the sets of isomorphism classes of such irreducible V's and the set of isomorphism classes of simple smooth $\mathscr{H}_\chi(H,\rho)$-modules.*

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