TRACES OF HECKE OPERATORS

1. Statement of main result - Evan

Let $\mathbf{A} = \mathbf{A}_{\mathbf{Q}}$ and let $G = \mathrm{GL}_2$. Given a nice function f on $G(\mathbf{A})$, we have derived a formula which tells us the trace of its convolution action $R(f)$ on the space $L^2_{\mathrm{cusp}}(G(\mathbf{A}), \omega)$ in terms of some geometric data expressed as orbital integrals and some spectral data expressed as information about Eisenstein series (related to the "missing" continuous and residual spectra). Classically, however, we care about only a few special actions on a few special functions in $L^2_{\mathrm{cusp}}(G(\mathbf{A}), \omega)$; namely, how the Hecke operators act on cusp forms of a given weight and level. The first goal of these two lectures is to pick a very special function f whose convolution action "is" precisely the action of a Hecke operator on the space of cusp forms: that is, we want $R(f)$ to act as the Hecke operator on cusp forms (embedded as usual into $L^2_{\mathrm{cusp}}(G(\mathbf{A}), \omega)$) and to act as the zero operator on the orthocomplement.

What does this give us? It means we can express the trace of the Hecke operators as geometric and spectral data associated with G. Actually, since $R(f)$ will act as zero off of the cuspidal part of the spectrum of $L^2(G(\mathbf{A}), \omega)$, we don't have to account for the "missing" continuous and residual spectra at all, because they were never there. Finally, we can simplify the geometric data in our case to get a formula in the classical (non-adelic) language, as follows:

Theorem 1.1 (Eichler-Selberg)**.** *Fix an integer weight $k > 2$, an integer level $N \geq 1$, a Dirichlet character ω' on $\mathbf{Z}/N\mathbf{Z}$, and an integer n such that $(n, N) = 1$. Let T_n be the Hecke operator on the space of cusp forms $S_k(N, \omega')$. Assume that $\omega'(-1) = (-1)^k$ (otherwise, $S_k(N, \omega')$ is trivial). Then*

$$\begin{aligned}
\mathrm{tr}(T_n) =& \frac{k-1}{12}\psi(N)\omega'(n^{1/2})^{-1} n^{\frac{k}{2}-1} \\
& - \frac{1}{2}\sum_{t^2 < 4n} \frac{\rho^{k-1} - \overline{\rho}^{k-1}}{\rho - \overline{\rho}} \left[\sum_{\substack{m>0 \\ m^2 \mid t^2-4n \\ \frac{t^2-4n}{m^2} \equiv 0,1 \bmod 4}} h_w\left(\frac{t^2-4n}{m^2}\right)\mu(t,m,n) \right] \\
& - \frac{1}{2}\sum_{\substack{d \mid n \\ d>0}} \min(d, n/d)^{k-1} \left[\sum_{\substack{\tau \mid N \\ \gcd(\tau, N/\tau) \mid \mathrm{lcm}\left(\frac{N}{N_{\omega'}}, d - \frac{n}{d}\right)}} \phi(\gcd(\tau, N/\tau))\omega'(y)^{-1} \right],
\end{aligned}$$

where

$$\psi(N) = [\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N)] = N \prod_{\textit{primes } p \mid N} \left(1 + \frac{1}{p}\right),$$

$\omega'(n^{1/2})^{-1}$ is defined to be zero if n is not a perfect square, ρ and $\overline{\rho}$ are the roots of the polynomial $X^2 - tX + n$, $h_w(d)$ is the weighted class number of the order in $\mathbf{Q}(\rho)$ with discriminant d,

$$\mu(t, m, n) = \frac{\psi(N)}{\psi(N/N_m)} \sum_c w'(c)^{-1}$$

with $N_m = \gcd(N, m)$ and c running over all elements of $(\mathbf{Z}/N\mathbf{Z})^*$ that lift to solutions of $c^2 - tc + n \equiv 0 \mod NN_m$, ϕ is the Euler totient function, and y is the unique integer modulo $N/\gcd(\tau, N/\tau)$ such that $y \equiv d \mod \tau$ and $y \equiv \frac{n}{d} \mod \frac{N}{\tau}$.

The first term corresponds to the identity term in the general trace formula, the second term to the elliptic terms, and the third term to the hyperbolic and unipotent terms. In particular, this is *not* a special case of Macky's "simple trace formula" from before; indeed, as we will see, only the local hyperbolic orbital integral at the archimedean place vanishes, and there is only one archimedean place of $\mathbf{Q}$. In fact, even at level one we keep all terms, although there is some simplification:

Corollary 1.2 (Eichler-Selberg at level 1)**.** *With notation as above, if $N = 1$ then*

$$\begin{aligned}\operatorname{tr}(T_n) = &\frac{d-1}{12} n^{\frac{k}{2}-1} \mathbf{1}_S(n) \\ &- \frac{1}{2} \sum_{t^2 < 4n} \frac{\rho^{k-1} - \overline{\rho}^{k-1}}{\rho - \overline{\rho}} \left[\sum_{\substack{m>0 \\ m^2 \mid t^2 - 4n \\ \frac{t^2-4n}{m^2} \equiv 0,1 \bmod 4}} h_w\left(\frac{t^2 - 4n}{m^2}\right) \right] \\ &- \frac{1}{2} \sum_{\substack{d \mid n \\ d > 0}} \min\left(d, \frac{n}{d}\right)^{k-1},\end{aligned}$$

where $\mathbf{1}_S$ is the indicator function of the set of squares.

The Eichler-Selberg formula is also true if $(n, N) \neq 1$ (due to Oesterlé [7] in his thesis), and also true if $k = 2$ and we make a slight modification. The problem with $k = 2$ is that the f that we will pick is then not absolutely integrable. Fortunately, the theory of psuedo-coefficients (in this instance due to Clozel and Delorme [2]) manufactures a certain non-explicit C_c^∞ function f' whose trace is the same as that of the archimedean place of f. Then we would use Arthur's invariant trace formula to employ f' in place of f.

Finally, note that the Eichler-Selberg trace formula can in principle be proven in a completely classical setting. See, for example, [10] (and later correction [11]) for a proof of 1.2 along those lines. The proof outlined here largely follows [5].

2. Cusp forms as automorphic forms - Evan

What follows is a quick reminder of how classical cusp forms naturally lie inside the space of automorphic forms, recalling Zeb's earlier talk. Let $h \in S_k(N, \omega')$, where ω' is a Dirichlet character modulo N such that $\omega'(-1) = (-1)^k$. By lifting from $(\mathbf{Z}/N\mathbf{Z})^*$ to $\hat{\mathbf{Z}}^*$ and using strong approximation on $\mathbf{A}^*$, we build a Hecke character $\omega : \mathbf{A}^* \to \mathbf{C}^*$ (which happens to be trivial on $\mathbf{R}_+^*$). This procedure preserves the conductor and gives a bijection

$$\{\text{Dirichlet chars. of conductor } M\} \leftrightarrow \{\text{ finite order Hecke chars. of conductor } M\}$$

for each M. Under this correspondence, $\omega'(d) = \omega(d_N)$, where d_N is the adele that agrees with d at all places $p|N$ and is set to 1 at all other places.

Recall that *strong approximation* for $G(\mathbf{A})$ implies that we can write

$$G(\mathbf{A}) = G(\mathbf{Q})\mathrm{GL}_2^+(\mathbf{R})K_0(N),$$

where $K_0(N) = \prod_{p<\infty} K_0(N)_p$,

$$K_0(N)_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_p, c \equiv 0 \mod N \right\},$$

and K_p is the maximal compact open subgroup $\mathrm{GL}_2(\mathbf{Z}_p)$.[1]

Promote ω to a character of $K_0(N)$ by setting

$$\omega\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \omega(d_N),$$

where the adele d_N is defined above.

Given our cusp form h, we can produce a function ϕ_h on $G(\mathbf{A})$ by

$$\phi_h(g) = h(g_\infty(i))j(g_\infty, i)^{-k}\omega(k_0),$$

where the factorization $g = \gamma g_\infty k_0$ is given by strong approximation and $j(g_\infty, z) = (c_\infty z + d_\infty)(\det g_\infty)^{-1/2}$ is the usual factor of automorphy. The basic result, proven in Zeb's earlier talk, is the following:

Proposition 2.1. *The map $h \mapsto \phi_h$ defines an isometric embedding of $S_k(N, \omega')$ into $L^2_{cusp}(G(\mathbf{A}), \omega)$. Its image consists precisely of the elements ϕ of $L^2_{cusp}(G(\mathbf{A}), \omega)$ such that*

- $\phi(gk) = \omega(k)\phi(k)$ *for all* $k \in K_0(N)$ *and* $g \in G(\mathbf{A})$
- $\phi\left(g\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = e^{ik\theta}\phi(g)$ *for all angles* θ *and all* $g \in G(\mathbf{A})$
- *If ϕ_∞ denotes the restriction of ϕ to the archimedean place, we have*

$$\Delta\phi_\infty = -\frac{k}{2}\left(\frac{k}{2} - 1\right)\phi_\infty.$$

We will explicitly need the first two properties (the transformation rules). From now on, we will identify $S_k(\omega', N)$ with its image under this map. Representation-theoretically, we can verify without too much difficulty that

$$S_k(N, \omega) \simeq \bigoplus_{\substack{\text{cuspidal } \pi \\ \pi_\infty \simeq \pi_k}} \mathbf{C}v_{\pi_\infty} \otimes \pi_{\mathrm{fin}}^{K_1(N)},$$

where π_k is the discrete series representation of $G(\mathbf{R})$ of lowest weight k.

We will need to recall a few basic facts about the π_k. For each weight k the discrete series are realizable in the following way: let

$$V_k^+ = \left\{ h \text{ holomorphic on } \mathcal{H} : ||h||^2 = \int_{\mathcal{H}} |h(x)|^2 y^{k-2}\, dx\, dy < \infty \right\},$$

[1] More generally (see page 41 of [3]) we can take in place of the groups $K_0(N)_p$ any choice of open subgroups K'_p such that $K'_p = K_p$ for almost all p and such that the determinant map $K_p \to \mathbf{Z}_p^*$ is surjective for every p.

and let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R})$ act on V_k^+ via

$$f(z) \mapsto (-bz+d)^{-k} f\left(\frac{az-c}{-bz+d}\right).$$

This is a unitary discrete series representation of $\mathrm{SL}_2(\mathbf{R})$ with lowest weight vector

$$f_0'(z) = \frac{1}{(z+i)^k}.$$

To promote this $\mathrm{SL}_2(\mathbf{R})$-representation to a discrete series representation of $G(\mathbf{R})$, we first let $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathrm{SL}_2(\mathbf{R})$ act in the same way but with a complex conjugate on V_k^-, where V_k^- is the analagous space of *antiholomorphic* functions satisfying the same L^2 property on the upper half plane.[2] Thus we have defined a $\mathrm{SL}_2^{\pm}(\mathbf{R})$-action on $V_k^+ \oplus V_k^-$. We let $G(\mathbf{R})$ act on $V_k^+ \oplus V_k^-$ by simply requiring that the positive determinant elements $Z^+(\mathbf{R})$ of the center act trivially. The lowest weight vector f_0 for this representation is $1/(z+i)^k$ on the holomorphic part and identically zero on the antiholomorphic part. We will normalize this vector for later use, defining

$$\tilde{f}_0 = \frac{f_0}{||f_0||}.$$

3. Constructing the non-archimedean places of f - Evan

We want to construct a test function f whose convolution action will mimic the Hecke operator on $S_k(N, \omega')$ and equal the zero operator on the orthocomplement in $L^2_{\mathrm{cusp}}(G(\mathbf{A}), \omega)$. Specifically, we want the following diagram to commute, where P is the orthogonal projection:

$$\begin{tikzcd}
L^2_{\mathrm{cusp}}(G(\mathbf{A}), \omega) \arrow[r, "R(f)"] \arrow[d, "P"'] & L^2_{\mathrm{cusp}}(G(\mathbf{A}), \omega) \\
S_k(N, \omega') \arrow[r, "n^{-(k/2-1)} T_n"] & S_k(N, \omega') \arrow[u, hook]
\end{tikzcd}$$

We will build up f place by place. The non-archimedean components will mimic the Hecke operator, while the archimedean component will be cooked up so as to kill the orthocomplement of $S_k(\omega', N)$.

To define the non-archimedean components of f, we first define

$$M(n,N)_p = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\hat{\mathbf{Z}}_p) : \det g \in n\hat{\mathbf{Z}}_p^* \text{ and } c \equiv 0 \mod N\hat{\mathbf{Z}}_p \right\}.$$

Note that $M(n,N)_p$ is equal to $K_0(N)_p$ if $p \nmid n$; i.e., for all but finitely many primes. Let

$$M(n,N) = \prod_{p<\infty} M(n,N)_p \subset M_2(\mathbf{A}).$$

Extend the character ω to a character on $M(n,N)$ by setting

$$\omega\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \omega(d_N),$$

[2]Alternatively, we could consider the action of $\mathrm{SL}_2(\mathbf{R}) \oplus \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathrm{SL}_2(\mathbf{R})$ on a certain set of holomorphic functions on $\mathbf{C} \setminus \mathbf{R}$; the resulting representation is the same.

just as for $K_0(N)$ above. Extend ω to $Z(\mathbf{A}_{\text{fin}})$ in a *different way*, setting

$$\tilde{\omega}\left(\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}\right) = \omega(d),$$

where d is the finite adele given by the usual embedding $\mathbf{Q} \to \mathbf{A}_{\text{fin}}$. In particular, this character is *not* the one given by the bijection between Dirichlet characters and finite order Hecke characters; we use the notation $\tilde{\omega}$ to distinguish it. Let $\overline{K_0(N)}$ denote the group $K_0(N)$ modulo its center. We define $f_{\text{fin}} : G(\mathbf{A}_{\text{fin}}) \to \mathbf{C}$ by

$$f_{\text{fin}}(g) = \begin{cases} \frac{\tilde{\omega}(z)^{-1}\omega(m)^{-1}}{\text{meas}(\overline{K_0(N)})} & \text{if } g = zm \text{ with } z \in Z(\mathbf{A}_{\text{fin}}), m \in M(n, N), \\ 0 & \text{otherwise.} \end{cases}$$

We have to check that this is well-defined, which we can easily do working locally. It is possible to massage this definition into a form that looks very close to the "double coset" definition of the Hecke operator in the classical setting, but we will not do this for lack of time. It turns out that the support of f_{fin} is equal to

$$\bigcup_{\substack{d_1, d_2 > 0 \\ d_2 | d_1, d_1 d_2 = n}} Z(\mathbf{A}_{\text{fin}}) K_0(N) \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} K_0(N).$$

In the case that $N = 1$ and $n = p$, f_{fin} actually turns out to be the characteristic function of

$$Z(\mathbf{A}_{\text{fin}}) K_{\text{fin}} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_{\text{fin}}.$$

For future reference, we check:

Lemma 3.1. *We have* $\text{meas}(\overline{K_0(N)}) = \frac{1}{\psi(N)}$, *where* $\psi(N) = [\text{SL}_2(\mathbf{Z}) : \Gamma_0(N)]$.

Proof. Let $K_{\text{fin}} = \prod_{p<\infty} K_p \subset G(\mathbf{A})$. Our measure is normalized so that the measure of K is one. We verify by easy local computation that

$$[K_{\text{fin}} : K_0(N)] = [\text{SL}_2(\mathbf{Z}) : \Gamma_0(N)],$$

and this suffices. □

Our first goal is the following, which proves that the above diagram commutes in the very special case that we started out in $S_k(N, \omega')$ to begin with:

Proposition 3.2. *Suppose n is a positive integer such that $(n, N) = 1$, and let $h \in S_k(N, \omega')$. Then*

$$R(f_{fin})\phi_h = \phi_{n^{-(k/2-1)} T_n h}.$$

Before proving this, we have to prove a couple of irritating decomposition lemmas in order to get a handle on what f_{fin} is actually doing.

Lemma 3.3. *Suppose $p | n$. Then the following is a disjoint union:*

$$M(n, N)_p = \bigcup_{j=0}^{v_p(n)} \bigcup_{a=0}^{p^j - 1} \begin{pmatrix} p^j & a \\ 0 & p^{v_p(n)-j} \end{pmatrix} K_p.$$

Proof. It's an easy check that the right hand side is contained in the left hand side. In the converse direction, let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(n, N)_p$. We are allowed to multiply on the right by K_p, and we will do so repeatedly in order to massage g into the correct form. First note that multiplying on the right by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in K_p$ swaps the columns of g, so we can assume without loss of generality that $v_p(c) \geq v_p(d)$. Then the matrix $\begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & 1 \end{pmatrix}$ is obviously in K_p, and multiplying on the right by it yields a matrix of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. Multiplying by a diagonal matrix with entries in $\mathbf{Z}_p$ brings us to a matrix with only powers of p on the diagonal, which must necessarily be of the form $\begin{pmatrix} p^j & b \\ 0 & p^{v_p(n)-j} \end{pmatrix}$ (with $b \in \mathbf{Z}_p$ and $0 \leq j \leq v_p(n)$) because the resulting matrix must still be in $M(n,N)_p$, hence must have determinant a unit multiple of n. Finally, we calculate

$$\begin{pmatrix} p^j & b \\ 0 & p^{v_p(n)-j} \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^j & p^j\alpha + b \\ 0 & p^{v_p(n)-j} \end{pmatrix}.$$

Therefore we can choose some $\alpha \in \mathbf{Z}_p$ to take b to its residue class modulo $p^j\mathbf{Z}_p$.

To prove disjointness of the decomposition, one checks straightforwardly that if

$$\begin{pmatrix} p^j & a \\ 0 & p^{v_p(n)-j} \end{pmatrix} \in \begin{pmatrix} p^{j'} & a' \\ 0 & p^{v_p(n)-j'} \end{pmatrix} K_p$$

then we must have $j = j'$ and $a \equiv a' \mod p^j$ (note that we *cannot* merely invert one of the matrices in $M(n,N)_p$ to resolve this, because the matrices in question are not invertible; this is an error in [5], Lemma 13.4). □

Lemma 3.4. *The following is a disjoint union:*

$$M(n,N) = \bigcup_{\substack{d_1,d_2>0 \\ d_1d_2=n}} \bigcup_{a \bmod d_1} \begin{pmatrix} d_1 & a \\ 0 & d_2 \end{pmatrix} K_0(N).$$

□

I will omit the proof for lack of time and relevance; one straightforwardly applies the above lemma at each place.

Proof of proposition. By definition,

$$R(f_{\mathrm{fin}})\phi_h(g) = \int_{Z(\mathbf{A}_{\mathrm{fin}})\backslash G(\mathbf{A}_{\mathrm{fin}})} f_{\mathrm{fin}}(x)\phi_h(gx)\,dx.$$

By the "double coset" description of the support of f_{fin} that we didn't prove above, the integrand has compact support modulo $Z(\mathbf{A}_{\mathrm{fin}})$, so convergence is clear.

Now we want to show that the integrand is right $K_0(N)$-invariant. In order to do this, we make a bit of a detour. Claim: if

$$K_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(N) : d \equiv 1 \mod N\hat{\mathbf{Z}} \right\},$$

then f_{fin} is bi-$K_1(N)$-invariant. This is proven by an easy check; multiplication of some $m \in M(n,N)$ on the left or right by an element of $K_1(N)$ preserves the determinant (obviously) and the lower right corner modulo N (calculation).

Going back to checking that the integrand is right $K_0(N)$-invariant, consider an element $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(N)$. Briefly, we can multiply by a scalar matrix to get it into $K_1(N)$, and we know how both f_{fin} and ϕ_h transform via the center, so we're in good shape. More precisely, let z be a scalar matrix such that $zk \in K_1(N)$. By the definition of f_{fin} and its $K_1(N)$-invariance, we have

$$f_{\text{fin}}(xk) = f_{\text{fin}}(xz^{-1}zk) = f_{\text{fin}}(xz^{-1}) = \tilde{\omega}(z^{-1})^{-1} f_{\text{fin}}(x) = \tilde{\omega}(z) f_{\text{fin}}(x).$$

By Proposition 2.1,

$$\phi_h(gxk) = \omega(k)\phi_h(gx).$$

Now note that the diagonal entries of z (which I will also denote by z) must be coprime to N, so in the notation of Section 2 we have $z_N = z$ and therefore $\tilde{\omega}(z) = \omega(z)$. Additionally, $zk \in K_1(N)$ and ω is trivial on $K_1(N)$, so all in all we have

$$f_{\text{fin}}(xk)\phi_h(gxk) = \tilde{\omega}(z)\omega(k) f_{\text{fin}}(x)\phi_h(gx) = \omega(zk) f_{\text{fin}}(x)\phi_h(gx) = f_{\text{fin}}(x)\phi_h(gx).$$

Therefore the integrand is right $K_0(N)$-invariant.

Looking at the decomposition in Lemma 3.4, therefore, we see that the integrand is actually *constant* on each coset, so the integral reduces to the sum

$$R(f_{\text{fin}})\phi_h(g) = \operatorname{meas}(\overline{K_0(N)}) \sum_{\substack{d_1, d_2 > 0 \\ d_1 d_2 = n}} \sum_{a \bmod d_1} f_{\text{fin}}\left(\begin{pmatrix} d_1 & a \\ 0 & d_2 \end{pmatrix}\right) \phi_h\left(g \begin{pmatrix} d_1 & a \\ 0 & d_2 \end{pmatrix}_{\text{fin}}\right).$$

But by the definition of f_{fin},

$$f_{\text{fin}}\left(\begin{pmatrix} d_1 & a \\ 0 & d_2 \end{pmatrix}\right) = \frac{\omega'(d_2)^{-1}}{\operatorname{meas}(\overline{K_0(N)})},$$

so the above simplifies to

$$R(f_{\text{fin}})\phi_h(g) = \sum_{\substack{d_1, d_2 > 0 \\ d_1 d_2 = n}} \sum_{a \bmod d_1} \omega'(d_2)^{-1} \phi_h\left(g \begin{pmatrix} d_1 & a \\ 0 & d_2 \end{pmatrix}_{\text{fin}}\right).$$

Claim: both $R(f_{\text{fin}})\phi_h$ and $\phi_{n^{-(k/2-1)}T_n h}$ are left $G(\mathbf{Q})$-invariant and right $K_1(N)$-invariant. In the former case this follows because f_{fin}; in the latter case this follows from Proposition 2.1. Strong approximation[3] gives us

$$G(\mathbf{A}) = G(\mathbf{Q}) \mathrm{GL}_2^+(\mathbf{R}) K_1(N),$$

so it suffices to check that the two functions agree on $\mathrm{GL}_2^+(\mathbf{R})$.

To this end, assume that $g = (g_\infty, 1, 1, \ldots)$, and let

$$\gamma = \begin{pmatrix} d_1 & a \\ 0 & d_2 \end{pmatrix}^{-1} = \begin{pmatrix} d_1^{-1} & -a(d_1 d_2)^{-1} \\ 0 & d_2^{-1} \end{pmatrix} \in G(\mathbf{Q}).$$

By $G(\mathbf{Q})$-invariance,

$$\phi_h\left(g \begin{pmatrix} d_1 & a \\ 0 & d_2 \end{pmatrix}_{\text{fin}}\right) = \phi_h\left(\gamma g \begin{pmatrix} d_1 & a \\ 0 & d_2 \end{pmatrix}_{\text{fin}}\right) = \phi_h(\gamma_\infty g_\infty \times 1_{\text{fin}}).$$

[3](in the somewhat greater generality afforded by the first footnote)

Therefore, if we let $z = g_\infty(i) \in \mathcal{H}$ and use the cocycle property of the usual factor of automorphy as well as an explicit calculation of $j(\gamma_\infty, z)$, we get

$$\begin{aligned}\phi_h\left(g\begin{pmatrix} d_1 & a \\ 0 & d_2\end{pmatrix}_{\text{fin}}\right) &= j(\gamma_\infty g_\infty, i)^{-k} h(\gamma_\infty g_\infty(i)) \\ &= j(\gamma_\infty, z)^{-k} j(g_\infty, i)^{-k} h(\gamma_\infty z) \\ &= (d_1 d_2)^{-k/2} d_2^k j(g_\infty, i)^{-k} h\left(\frac{d_1^{-2} z - a(d_1 d_2)^{-1}}{d_2^{-1}}\right) \\ &= n^{k/2} d_1^{-k} j(g_\infty, i)^{-k} h\left(\frac{d_2 z - a}{d_1}\right).\end{aligned}$$

After multiplying by $\omega'(d_2)^{-1}$ and summing up, we get

$$j(g_\infty, i)^{-k} n^{k/2} \sum_{\substack{d_1, d_2 > 0 \\ d_1 d_2 = n}} \sum_{a=0}^{d_1 - 1} \omega'(d_2)^{-1} d_1^{-k} h\left(\frac{d_2 z - a}{d_1}\right).$$

We find that this is precisely the "hands-on" definition of the Hecke operators on the space of cusp forms multiplied by the factor $n^{-(k/2-1)}$ (possibly modulo a sign of a, which is irrelevant as we are summing it over a cyclic group). □

4. Constructing the archimedean place of f - Evan

Define $f_k : G(\mathbf{R}) \to \mathbf{C}$ by

$$f_k(g) = \langle \pi_k(g)\tilde{f}_0, \tilde{f}_0\rangle,$$

where π_k is the discrete series representation on $V_k^+ \oplus V_k^-$ and $\tilde{f}_0$ is the lowest weight vector described above. In other words, f_k is the matrix coefficient π_k corresponding to the "diagonal" pair of vectors $(\tilde{f}_0, \tilde{f}_0)$. Recall that $\tilde{f}_0$ was complety explicit as a function, so f_k is as well. After some annoying integration that I will skip, it turns out that for $g \in G(\mathbf{R})$,

$$f_k(g) = \begin{cases} \frac{\det(g)^{k/2}(2i)^k}{(b-c+(a+d)i)^k} & \text{if } \det(g) > 0, \\ 0 & \text{if } \deg(g) < 0. \end{cases}$$

Another marginally less annoying calculation using this result shows that $f_k(g)$ is absolutely integrable over $Z(\mathbf{R})\backslash G(\mathbf{R})$ whenever $k > 2$. Unfortunately, as mentioned previously, it is not integrable when $k = 2$, so we exclude this case from our analysis from now on.

We now make a short digression into representation theory. Let d be the *formal degree* of π_k; that is, the element such that

$$\int_{Z(\mathbf{R})\backslash G(\mathbf{R})} |\langle \pi_k(g)v, w\rangle|^2 \, dg = \frac{1}{d}||v||^2||w||^2$$

for all vectors v, w. That is, it is the L^2 norm of any matrix coefficient formed from unit vectors. It is a theorem that this element always exists for a nonzero irreducible unitary *square-integrable* representation of a locally compact unimodular group. If the group in question were compact, d would just be the dimension of the representation.

Now I have to explain *square-integrable* more precisely. By a theorem of Godement, if χ is any unitary central character and (π, V) an irreducible unitary representation of a locally compact unimodular group G, then one matrix coefficient lies in $L^2(G, \chi)$ if and only if all do, if and only if (π, V) is an irreducible direct summand of the right regular representation of G on $L^2(G, \chi)$. Such a representation is called square-integrable. By our knowledge of the representation theory of $G(\mathbf{R})$, the discrete series representations are such.

We let

$$f_\infty = d_k \overline{f_k}$$

be the archimedean component of f.

Why did we pick a matrix coefficient of a lowest weight vector here? The goal is to kill all cuspidal automorphic functions that are not in $S_k(N, \omega')$, so we want to exploit the orthogonality of matrix coefficients. Specifically, we have the following strong version of Schur's lemma:

Proposition 4.1. *Let (π_0, V_0) be an irreducible unitary representation of a locally compact unimodular group G. Assume that the matrix coefficient ϕ_{v_0,v_0} is integrable for some vector $v_0 \in V_0$. Define $f(g) = d_{\pi_0} \overline{\langle \pi_0(g)v_0, v_0\rangle}$. For any unitary representation (π, V) of G with the same central character, $\pi(f)$ is the projection of V onto $W = \{Tv_0 : T \in \mathrm{Hom}_G(\pi_0, \pi)\} \subset V$. In particular, if $\mathrm{Hom}_G(\pi_0, \pi) = 0$, then $\pi(f) = 0$.* □

With the representation theory taken care of, we have two major steps remaining. Step one: we want to show that $R(f)$ kills the orthocomplement of $L^2_{\mathrm{cusp}}(G, \omega)$. This justifies ignoring the spectral terms in the trace formula. Step two: we want to show that $R(f)$ kills the orthocomplement of $S_k(N, \omega')$ and acts as (essentially) the Hecke operator on $S_k(N, \omega')$; i.e., the diagram at the beginning of Section 3 commutes.

Theorem 4.2 (Step one)**.** *$R(f)$ annihilates $L^2_{cusp}(G(\mathbf{A}), \omega)^\perp$.*

Proof. Let $\phi \in L^2(G(\mathbf{A}), \omega)$ be a function bounded (in the supremum norm) by M. We will use the properties of f_∞ to show that $R(f)\phi$ is a cusp form. Let $\overline{G}(\mathbf{A})$ denote the group $Z(\mathbf{A})\backslash G(\mathbf{A})$ (the adelic points of G modulo the center).

By definition, the constant term of $R(f)\phi$ (that we desire to show is zero) is

$$\int_{N(\mathbf{Q})\backslash N(\mathbf{A})} R(f)\phi(ng)\, dn = \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \left(\int_{\overline{G}(\mathbf{A})} f(x)\phi(ngx)\, dx \right) dn.$$

This integral is absolutely convergent because ϕ is bounded (specifically, by $M \cdot \mathrm{meas}(N(\mathbf{Q})\backslash N(\mathbf{A})) \cdot ||f||_1 < \infty$). Therefore we can rearrange things at will; the constant term is

$$
\begin{aligned}
&= \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \left(\int_{\overline{G}(\mathbf{A})} f(g^{-1}n^{-1}x)\phi(x)\, dx \right) dn \\
&= \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \left(\int_{N(\mathbf{Q})\backslash \overline{G}(\mathbf{A})} \sum_{\delta \in N(\mathbf{Q})} f(g^{-1}n^{-1}\delta x)\phi(x)\, dx \right) dn \\
&= \int_{N(\mathbf{Q})\backslash \overline{G}(\mathbf{A})} \left(\int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \sum_{\delta \in N(\mathbf{Q})} f(g^{-1}n^{-1}\delta x)\, dn \right) \phi(x)\, dx \\
&= \int_{N(\mathbf{Q})\backslash \overline{G}(\mathbf{A})} \left(\int_{N(\mathbf{A})} f(g^{-1}n^{-1}x)\, dn \right) \phi(x)\, dx.
\end{aligned}
$$

Therefore it suffices to show that the infinite place of the inner integral vanishes; that is, we want

$$\int_{N(\mathbf{R})} f_\infty(gnx)\, dn = 0$$

for all $g, x \in G(\mathbf{R})$. This follows by direct calculation; by the above calculation we know that f_k, hence f_∞, is a constant multiple of a function of the form $1/(At+B)^k$, where we have written $n = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Evaluating the integral of this function from $-\infty$ to ∞ yields zero, as desired (see Lemma 6.1 below). This calculation may seem lucky, but it is a specific case of a more general phenomenon noticed by Harish-Chandra that discrete series matrix coefficients, integrated over unipotent subgroups, vanish.

So we can conclude that $R(f)\phi$ is cuspidal if ϕ is bounded. But such ϕ are dense in $L^2(G(\mathbf{A}), \omega)$, $R(f)$ is a continuous operator, and the cuspidal subspace is closed. Therefore

$$R(f) : L^2(G(\mathbf{A}), \omega) \to L^2_{\text{cusp}}(G(\mathbf{A}), \omega).$$

Now we use a trick. I claim that $R(f)^*$, the adjoint, is equal to $R(f^*)$, where $f(g) = \overline{f(g^{-1})}$. This is an easy formal calculation. Similarly, by the construction of f_∞ as a matrix coefficient and a similarly easy formal argument, we see that $f_\infty^* = f_\infty$. The above argument about vanishing of constant terms only used properties of f_∞, so we can conclude that

$$R(f)^* : L^2(G(\mathbf{A}), \omega) \to L^2_{\text{cusp}}(G(\mathbf{A}), \omega)$$

as well. Now by a general easy fact about Hilbert spaces (if a continuous operator and its adjoint both carry a space into the same subspace, then the operator kills the orthocomplement of that subspace), we conclude that $R(f)$ kills $L^2_{\text{cusp}}(G(\mathbf{A}), \omega)^\perp$. □

Theorem 4.3 (Step two). *$R(f)$ annihilates $S_k(N, \omega')^\perp$, and the diagram from the beginning of Section 3 commutes.*

Proof. We play approximately the same game. First, I claim that $R(f) : L^2(\omega) \to S_k(N, \omega')$. Without loss of generality we can check this for v contained in an irreducible representation V_π, v cuspidal (by the above step) and v equal to a pure tensor $v_\infty \otimes v_{\text{fin}}$. Then, almost purely formally, we find that

$$R(f)v = \pi_\infty(f_\infty)v_\infty \otimes \pi_{\text{fin}}(f_{\text{fin}})v_{\text{fin}}.$$

By Proposition 4.1, $\pi_\infty(f_\infty)v_\infty = 0$ unless $\pi_\infty \simeq \pi_k$, and in this case we have $\pi_\infty(f_\infty)v_\infty \in \mathbf{C}v_{\pi_k}$, where v_{π_k} is the lowest weight vector for π_k. For the finite places, we note that f_{fin} is bi-$K_1(N)$-invariant, so $\pi_{\text{fin}}(f_{\text{fin}})v_{\text{fin}}$ is too. Therefore

$$R(f)v \in \mathbf{C}v_{\pi_k} \otimes \pi_{\text{fin}}^{K_1(N)} \subset S_k(N,\omega').$$

Now look at the adjoint, which is $R(f^*)$. The function f^* is still $K_1(N)$-invariant, so the same argument shows that $R(f)^* : L^2(G(\mathbf{A}),\omega) \to S_k(N,\omega')$. By the same general Hilbert space fact, we conclude that $R(f)$ kills $S_k(N,\omega')^\perp$.

Finally, we need to show that $R(f)\phi_h = \phi_{n^{-(k/2-1)}T_n h}$. We know that the finite part is correct by Proposition 3.2, so we just have to show that the infinite parts match. But again, this just follows by Schur orthogonality: an irreducible representation evaluated at the matrix coefficient of another irreducible representation is zero, and evaluation at its own diagonal matrix coefficient is a projection onto the given one-dimensional subspace. □

5. The trace formula for f - Zeb

Somewhere between the statement of Theorem 6.33 of [4] and Theorem 22.1 of [5] we have

Theorem 5.1. *For f as above,*

$$\begin{aligned}\operatorname{tr}R(f) = {}& \operatorname{Vol}(\bar{G}(\mathbf{Q})\backslash\bar{G}(\mathbf{A}))f(1)\\ &+ \int_{\bar{G}(\mathbf{Q})\backslash\bar{G}(\mathbf{A})} \sum_{\gamma \text{ elliptic}} f(x^{-1}\gamma x)\,dx\\ &+ \text{f.p.}_{s=1} Z_F(s)\\ &- \operatorname{Vol}(\mathbf{Q}^\times\backslash\mathbf{A}^1) \sum_{\substack{[\gamma]\subset\bar{G}(\mathbf{Q})\\ \text{hyperbolic}}} \int_{\bar{M}(\mathbf{A})\backslash\bar{G}(\mathbf{A})} f(g^{-1}\gamma_0 g)v(g)\,dg,\end{aligned}$$

where

$$F(y) = \int_K f\left(k^{-1}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}k\right)dk,$$

the zeta function $Z_F(s)$ is defined by

$$Z_F(s) = \int_{\mathbf{A}^\times} F(a)|A|^\times\, d^\times a,$$

the element γ_0 is chosen from $[\gamma]\cap\bar{M}(\mathbf{Q})$, and the height function $v(g)$ is determined by

$$v(g) = H(g) + H\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g\right),$$

and $H(g)$ is thought of as the height of $g(i)$ in the upper halfplane

$$H(g) = -\log\frac{||(0\,1)g||^2}{|\det g|},$$

which is characterized by

$$H\left(\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}nk\right) = \log\left|\frac{a}{b}\right|$$

for n unipotent and $k \in K$.

The first summand is the identity contribution, the second is the elliptic contribution, the third is the unipotent contribution, and the last is the hyperbolic contribution. Note that the contribution from the continuous spectrum is 0, since by construction this f projects to the discrete spectrum.

We can make an immediate simplification: with the standard normalization of the measures, we have

$$\mathrm{Vol}(\mathbf{Q}^\times \backslash \mathbf{A}^1) = 1.$$

6. Vanishing of the hyperbolic orbital integral at ∞ - Zeb

Recall that we explicitly have

$$f_k(g) = \begin{cases} \frac{\det(g)^{k/2}(2i)^k}{(b-c+(a+d)i)^k} & \det g > 0 \\ 0 & \det g < 0 \end{cases}.$$

Lemma 6.1. *For any $g, h \in G(\mathbf{R})$, and any integer $k > 2$, we have*

$$\int\limits_{N(\mathbf{R})} f_k(gnh)\, dn = 0.$$

Proof. For fixed g, h, it's clear from the definition of f_k that we can find constants $A, B \in \mathbf{C}$ such that

$$f_k\left(g \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} h\right) = \frac{1}{(At+B)^k}.$$

Since f_k takes finite values, we have $\frac{A}{B} \notin \mathbf{R}$, so we can apply the residue theorem to see that

$$\int_{\infty}^{\infty} \frac{dt}{(At+B)^k} = 0$$

as long as $k > 2$. $\square$

Corollary 6.2. *If $k > 2$, $g, h \in G(\mathbf{R})$, and $a, b, a-b \in \mathbf{R}^\times$ then we have*

$$\int_{N(\mathbf{A})} f\left(gn^{-1}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} nh\right) dn = 0.$$

Proof. This follows from the previous Lemma and the calculation

$$\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & t\left(\frac{a-b}{a}\right) \\ 0 & 1 \end{pmatrix}. \qquad \square$$

Proposition 6.3. *If $\gamma \in G(\mathbf{R})$ is hyperbolic, $G_\gamma(\mathbf{R})$ the centralizer of γ, then the orbital integral vanishes:*

$$\Phi(\gamma, f_\infty) = \int\limits_{\overline{G_\gamma(\mathbf{R})}\backslash \bar{G}(\mathbf{R})} f_\infty(g^{-1}\gamma g)\, dg = 0.$$

Proof. Without loss of generality we may take γ to be diagonal, so $\overline{G\gamma(\mathbf{R})} = \bar{M}(\mathbf{R})$, and

$$\int\limits_{\bar{M}(\mathbf{R})\backslash \bar{G}(\mathbf{R})} f_\infty(g^{-1}\gamma g)\, dg = \int_{K_\infty}\int_{N(\mathbf{R})} f_\infty(k^{-1}n^{-1}\gamma nk)\, dn\, dk = 0$$

by the Corollary and the fact that f_∞ is proprtional to $\bar{f}_k$. $\square$

Thus we have *one* vanishing orbital integral. This will be useful, but it will not simplify things as much as they simplified in Macky's talk.

7. Calculation: the identity term - Zeb

Recall the identity term from the trace formula:

$$\mathrm{Vol}(\bar{G}(\mathbf{Q})\backslash\bar{G}(\mathbf{A}))f(1).$$

With the standard measures, we have

$$\mathrm{Vol}(\bar{G}(\mathbf{Q})\backslash\bar{G}(\mathbf{A})) = \mathrm{Vol}(\Gamma(1)\backslash\mathbb{H}) = \int_{x=-\frac{1}{2}}^{\frac{1}{2}}\int_{y=\sqrt{1-x^2}}^{\infty}\frac{dy}{y^2}\,dx = \frac{\pi}{3}.$$

At the Archimedian place, we had $f_\infty = \bar{f}_k d_k$. By definition, we have

$$f_k(1) = \frac{1^{k/2}(2i)^k}{(0-0+(1+1)i)^k} = 1,$$

and

$$\frac{1}{d_k} = \int\limits_{\bar{G}(\mathbf{R})} |f_k(g)|^2\,dg,$$

and by the Cartan decomposition $\bar{G}(\mathbf{R}) = K_\infty A^+ K_\infty$ together with the fact that $|f_k|$ is invariant under K_∞ acting on either side of the argument (since f_k was defined to be a matrix coefficient), we have

$$\int\limits_{\bar{G}(\mathbf{R})} |f_k(g)|^2 dg = \pi\int_1^\infty \left|f_k\left(\begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2}\end{pmatrix}\right)\right|^2 (1-t^{-2})\,dt$$

$$= 4^k\pi\int_1^\infty \frac{1-t^{-2}}{(t+t^{-1}+2)^k}\,dt = 4^k\pi\int_4^\infty\frac{1}{s^k}\,ds = \frac{4\pi}{k-1},$$

where we have used the substitution $s = t+t^{-1}+2$. Thus we have

$$f_\infty(1) = \bar{f}_k(1)d_k = \frac{k-1}{4\pi}.$$

By our definition of f_{fin}, we have

$$f_{\mathrm{fin}}(1) = \begin{cases} \frac{\tilde{\omega}(z)^{-1}\omega(m)^{-1}}{\mathrm{meas}(\overline{K_0(N)})} & \text{if } 1 = zm \text{ with } z\in Z(\mathbf{A}_{\mathrm{fin}}), m\in M(n,N),\\ 0 & \text{otherwise.}\end{cases}$$

We easily have

$$\frac{1}{\mathrm{meas}(\overline{K_0(N)})} = \psi(N) = N\prod_{p|N}\left(1+\frac{1}{p}\right).$$

Now suppose that $1 = zm, z\in Z(\mathbf{A}_{\mathrm{fin}}), m\in M(n,N)$. Then by the definition of $M(n,N)$, $n\det(z)$ has an even p-valuation for every p, so since n is positive n must be the square of an integer. Thus we may as well take

$$z = \begin{pmatrix} n^{-1/2} & 0 \\ 0 & n^{-1/2}\end{pmatrix},\ m = \begin{pmatrix} n^{1/2} & 0 \\ 0 & n^{1/2}\end{pmatrix}.$$

We have

$$\tilde{\omega}(z) = \frac{\omega_\infty(1)}{\omega_\infty(n^{1/2})}\prod_p \omega_p(n^{1/2}) = 1,$$

since $n^{1/2} > 0$, and

$$\omega(m) = \omega'(n^{1/2}).$$

Thus, the identity term is

$$\mathrm{Vol}(\bar{G}(\mathbf{Q})\backslash\bar{G}(\mathbf{A}))f(1)=\frac{k-1}{12}\psi(N)\omega'(n^{1/2})^{-1},$$

where we take $\omega'(n^{1/2})$ to be 0 if n is not a square.

8. Calculation: the hyperbolic terms - Zeb

The hyperbolic term we would like to evaluate is the sum over hyperbolic conjugacy classes $[\gamma]\subset\bar{G}(\mathbf{Q})$ of

$$\int\limits_{\bar{M}(\mathbf{A})\backslash\bar{G}(\mathbf{A})} f(g^{-1}\gamma_0 g)v(g)\,dg.$$

At this point we will make our first use of the vanishing of hyperbolic orbital integrals at ∞: since $v(g)=v_\infty(g)+v_{\mathrm{fin}}(g)$, we have

$$\begin{aligned}\int\limits_{\bar{M}(\mathbf{A})\backslash\bar{G}(\mathbf{A})} f(g^{-1}\gamma_0 g)v(g)\,dg &= \left(\int\limits_{\bar{M}(\mathbf{R})\backslash\bar{G}(\mathbf{R})} f_\infty(g^{-1}\gamma_0 g)v_\infty(g)\,dg\right)\left(\int\limits_{\bar{M}(\mathbf{A}_{\mathrm{fin}})\backslash\bar{G}(\mathbf{A}_{\mathrm{fin}})} f_{\mathrm{fin}}(g^{-1}\gamma_0 g)\,dg\right)\\ &+\left(\int\limits_{\bar{M}(\mathbf{R})\backslash\bar{G}(\mathbf{R})} f_\infty(g^{-1}\gamma_0 g)\,dg\right)\left(\int\limits_{\bar{M}(\mathbf{A}_{\mathrm{fin}})\backslash\bar{G}(\mathbf{A}_{\mathrm{fin}})} f_{\mathrm{fin}}(g^{-1}\gamma_0 g)v_{\mathrm{fin}}(g)\,dg\right),\end{aligned}$$

and the second term is just 0 since it is a multiple of the orbital integral at ∞. We are left with the task of evaluating the first term. Suppose now that

$$\gamma=\begin{pmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{pmatrix},$$

with $\gamma_1>\gamma_2\in\mathbf{Z}^+$ and $\gamma_1\gamma_2=n$.

Proposition 8.1. *With γ as above and $k>2$ we have*

$$\int\limits_{\bar{M}(\mathbf{R})\backslash\bar{G}(\mathbf{R})} f_\infty(g^{-1}\gamma_0 g)v_\infty(g)\,dg=\frac{n^{1-\frac{k}{2}}\gamma_2^{k-1}}{\gamma_1-\gamma_2}.$$

Proof. Note that f_∞ is invariant under conjugation by K_∞ by a purely formal check, using the fact that it is defined as a matrix coefficient of the discrete series representation π_k, which acts via multiplication by some $e^{ik\theta}$ when applied to elements of K_∞:

$$\begin{aligned}\left\langle\pi_k(k_\theta^{-1}gk_\theta)\tilde{f}_0,\tilde{f}_0\right\rangle &=\left\langle\pi_k(g)\pi_k(k_\theta)\tilde{f}_0,\pi(k_\theta)\tilde{f}_0\right\rangle\\ &=e^{ik\theta}e^{-ik\theta}\left\langle\pi_k(g)\tilde{f}_0,\tilde{f}_0\right\rangle\\ &=\left\langle\pi_k(g)\tilde{f}_0,\tilde{f}_0\right\rangle.\end{aligned}$$

Thus we have $f_\infty(k^{-1}n^{-1}\gamma nk)=f_\infty(n^{-1}\gamma n)$. By the definition of v_∞, we also have $v_\infty(nk)=v_\infty(n)$. Also, with the standard normalization of measures we have

$\mathrm{Vol}(K_\infty) = 1$, so the integral we want to evaluate is just

$$\begin{aligned}\int\limits_{N(\mathbf{R})} f_\infty(n^{-1}\gamma n)v_\infty(n)\,dn &= d_k \int_{-\infty}^{\infty} \frac{n^{k/2}(2i)^k}{((\gamma_1-\gamma_2)t + i(\gamma_2+\gamma_1))^k} \cdot (-\log(1+t^2))\,dt \\ &= -\frac{d_k n^{k/2}(2i)^k}{(-1)^k(\gamma_1-\gamma_2)^k} \int_{-\infty}^{\infty} \frac{\log(1+t^2)}{(t - \frac{\gamma_1+\gamma_2}{\gamma_1-\gamma_2}i)^k}\,dt \\ &= -\frac{k-1}{4\pi}\frac{n^{k/2}(2i)^k(-1)^k}{(\gamma_1-\gamma_2)^k(k-1)} \int_{-\infty}^{\infty} \frac{2t}{1+t^2}\left(t - \frac{\gamma_1+\gamma_2}{\gamma_1-\gamma_2}i\right)^{-k+1} dt \\ &= \frac{1}{4\pi}\frac{n^{k/2}(2i)^k(-1)^{k+1}}{(\gamma_1-\gamma_2)^k} \cdot 2\pi i \cdot (-1) \cdot \left(-i - \frac{\gamma_1+\gamma_2}{\gamma_1-\gamma_2}i\right)^{-k+1} \\ &= \frac{n^{1-k/2}\gamma_2^{k-1}}{\gamma_1-\gamma_2},\end{aligned}$$

where the third equality followed from integration by parts and the fourth equality followed from

$$\frac{2t}{1+t^2} = \frac{1}{t+i} + \frac{1}{t-i}$$

and the residue theorem applied to a large semicircle not containing $\frac{\gamma_1+\gamma_2}{\gamma_1-\gamma_2}i$ (here we are making use of $k > 2$ and $\gamma_1 > \gamma_2$). $\square$

So far we have shown that

$$\int\limits_{\bar{M}(\mathbf{A})\backslash\bar{G}(\mathbf{A})} f(g^{-1}\gamma_0 g)v(g)\,dg = \frac{n^{1-k/2}\gamma_2^{k-1}}{\gamma_1-\gamma_2} \prod_{p<\infty} \int_{\bar{M}(\mathbf{Q}_p)\backslash\bar{G}(\mathbf{Q}_p)} f_p(g^{-1}\gamma g)\,dg.$$

If we write $\psi(N) = \prod_p \psi_p(N)$ in the obvious way, then we have

$$\begin{aligned}\int\limits_{\bar{M}(\mathbf{Q}_p)\backslash\bar{G}(\mathbf{Q}_p)} f_p(g^{-1}\gamma g)\,dg &= \psi_p(N)^{-1} \int\limits_{K_p/K_0(N)_p} \int_{N(\mathbf{Q}_p)} f_p(k^{-1}n^{-1}\gamma nk)\,dn\,dk \\ &= \psi_p(N)^{-1} \sum_\alpha \int_{N(\mathbf{Q}_p)} f_p(\alpha^{-1}n^{-1}\gamma n\alpha)\,dn,\end{aligned}$$

where α runs over a system of representatives of left cosets of $K_0(N)_p$ in K_p. We now split the evaluation of these local orbital integrals up into three cases.

(i) If $p \nmid N$ then $\psi_p(N) = 1$, we can take $\alpha = 1$, and we get

$$\begin{aligned}\int\limits_{\bar{M}(\mathbf{Q}_p)\backslash\bar{G}(\mathbf{Q}_p)} f_p(g^{-1}\gamma g)\,dg &= \int_{\mathbf{Q}_p} f_p\left(\begin{pmatrix} \gamma_1 & t(\gamma_1-\gamma_2) \\ 0 & \gamma_2 \end{pmatrix}\right) dt \\ &= \omega_p(\gamma_2)^{-1}\,\mathrm{Vol}\left(\frac{1}{\gamma_1-\gamma_2}\mathbf{Z}_p\right) \\ &= \frac{1}{|\gamma_1-\gamma_2|_p}.\end{aligned}$$

Lemma 8.2. *Suppose $p|N$. Then we can write*

$$K_p = \bigcup_{\delta\in\mathbf{Z}_p/N\mathbf{Z}_p} \begin{pmatrix} \delta & 1 \\ 1 & 0 \end{pmatrix} K_0(N)_p \cup \bigcup_{\tau\in p\mathbf{Z}_p/N\mathbf{Z}_p} \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix} K_0(N)_p$$

as a disjoint *union.*

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_p$.

Two cases: if $p|c$, then because the determinant is invertible we must have $a \in \mathbf{Z}_p^*$, so $h = \begin{pmatrix} a^{-1} & \frac{-b}{ad-bc} \\ 0 & \frac{a}{ad-bc} \end{pmatrix} \in K_0(N)_p$. Then $gh = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix}$, so we can express g as a product in the second union of the statement of the lemma. It is easy to see by the definition of $K_0(N)_p$ that c/a is then unique up to N.

In the other case, $p \nmid c$ so $c \in \mathbf{Z}_p^*$, and we multiply by $h' = \begin{pmatrix} c^{-1} & \frac{d}{ad-bc} \\ 0 & \frac{-c}{ad-bc} \end{pmatrix}$ and proceed in the same way.

Disjointness follows from the calculation

$$\begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix} \begin{pmatrix} w & x \\ Ny & z \end{pmatrix} = \begin{pmatrix} * & * \\ \tau w + Ny & * \end{pmatrix}$$

and the observation that $p|(\tau w + Ny)$, so the right hand side cannot be of the form $\begin{pmatrix} \delta & 1 \\ 1 & 0 \end{pmatrix}$. □

(ii) $p \mid N$ and $\alpha = \begin{pmatrix} \delta & 1 \\ 1 & 0 \end{pmatrix}$. We have

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \delta + t & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 \\ 1 & -t \end{pmatrix} \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \gamma_2 & 0 \\ t(\gamma_1 - \gamma_2) & \gamma_1 \end{pmatrix},$$

so

$$\begin{aligned} \psi_p(N)^{-1} \int_{N(\mathbf{Q}_p)} f_p(\alpha^{-1} n^{-1} \gamma n \alpha)\, dn &= \psi_p(N)^{-1} \int_{\mathbf{Q}_p} f_p\left(\begin{pmatrix} \gamma_2 & 0 \\ t(\gamma_1 - \gamma_2) & \gamma_1 \end{pmatrix}\right) dt \\ &= \psi_p(N)^{-1} \omega_p(\gamma_1)^{-1} \psi_p(N) \operatorname{Vol}\left(\frac{N}{\gamma_1 - \gamma_2} \mathbf{Z}_p\right) \\ &= \frac{|N|_p \omega_p(\gamma_1)^{-1}}{|\gamma_1 - \gamma_2|_p}. \end{aligned}$$

The sum of this over all choices of δ is

$$\frac{\omega_p(\gamma_1)^{-1}}{|\gamma_1 - \gamma_2|_p}.$$

(iii) $p \mid N, \alpha = \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix}$. If $\tau = 0$, we get $\frac{\omega_p(\gamma_2)^{-1}}{|\gamma_1 - \gamma_2|_p}$ as in case (i). Otherwise, we have

$$\alpha^{-1} \begin{pmatrix} 1 & t \\ 1 & 0 \end{pmatrix}^{-1} \gamma \begin{pmatrix} 1 & t \\ 1 & 0 \end{pmatrix} \alpha = \begin{pmatrix} \gamma_1 + \tau(\gamma_1 - \gamma_2)t & (\gamma_1 - \gamma_2)t \\ -\tau(\gamma_1 - \gamma_2)(1 + \tau t) & \gamma_2 - \tau(\gamma_1 - \gamma_2)t \end{pmatrix}.$$

Setting $y = \gamma_2 - \tau(\gamma_1 - \gamma_2)t$, we get

$$\psi_p(N)^{-1} \int_{N(\mathbf{Q}_p)} f_p(\alpha^{-1} n^{-1} \gamma n \alpha)\, dn = \frac{\psi_p(N)^{-1}}{|\tau|_p |\gamma_1 - \gamma_2|_p} \int_{\mathbf{Q}_p} f_p\left(\begin{pmatrix} \gamma_1 + \gamma_2 - y & \frac{\gamma_2 - y}{\tau} \\ -\tau(\gamma_1 - y) & y \end{pmatrix}\right) dy.$$

The integrand is nonzero if $y \in \mathbf{Z}_p$,

$$y \equiv \gamma_2 \mod \tau\mathbf{Z}_p,$$
$$y \equiv \gamma_1 \mod \frac{N}{\tau}\mathbf{Z}_p,$$

and then the value is $\omega_p(y)^{-1}\psi_p(N)$. For this to happen, we need $\gamma_1 - \gamma_2 \in (\tau, \frac{N}{\tau})\mathbf{Z}_p$, and then y is determined modulo $\operatorname{lcm}(\tau, \frac{N}{\tau})\mathbf{Z}_p$. Suppose y_p is a solution to this congruence. Then the integral we wish to evaluate becomes

$$\frac{|\operatorname{lcm}(\tau\frac{N}{\tau})|_p}{|\tau|_p|\gamma_1-\gamma_2|_p}\omega_p(y_p)^{-1}\int_{\mathbf{Z}_p}\omega_p(1+\operatorname{lcm}(\tau, \frac{N}{\tau})z)^{-1}\,dz.$$

This is 0 unless $\operatorname{lcm}(\tau, \frac{N}{\tau}) = \frac{N}{(\tau, \frac{N}{\tau})}$ is in $N_\omega\mathbf{Z}_p$, where N_ω is the conductor of ω. In that case, the integral above comes out to 1.

Summing over τ with p-adic valuation $k \geq 1$, we get

$$\frac{\varphi_p((p^k, \frac{N}{p^k}))\omega_p(y_p)^{-1}}{|\gamma_1-\gamma_2|_p},$$

when $\frac{N}{(p^k, \frac{N}{p^k})} \in N_\omega\mathbf{Z}_p$, $\gamma_1 - \gamma_2 \in (p^k, \frac{N}{p^k})\mathbf{Z}_p$.

Combining cases (ii) and (iii), we see that for $p \mid N$ the local orbital integral is

$$\frac{1}{|\gamma_1-\gamma_2|_p}\sum_{\substack{\tau=p^k|N\\ \frac{N}{(\tau,\frac{N}{\tau})}\in N_\omega\mathbf{Z}_p\\ \gamma_1-\gamma_2\in(\tau,\frac{N}{\tau})\mathbf{Z}_p}}\varphi_p\left(\left(\tau, \frac{N}{\tau}\right)\right)\omega_p(y_p)^{-1},$$

where

$$y_p \equiv \gamma_2 \mod \tau\mathbf{Z}_p,$$
$$y_p \equiv \gamma_1 \mod \frac{N}{\tau}\mathbf{Z}_p.$$

Multiplying out the local orbital integrals, we get

$$\begin{aligned}\int_{\bar{M}(\mathbf{A})\backslash\bar{G}(\mathbf{A})} f(g^{-1}\gamma_0 g)v(g)\,dg &= n^{1-k/2}\gamma_2^{k-1}\prod_{p|N}\sum_{\substack{\tau=p^k\\ \cdots}}\varphi_p\left(\left(\tau, \frac{N}{\tau}\right)\right)\omega_p(y_p)^{-1}\\ &= n^{1-k/2}\gamma_2^{k-1}\sum_{\substack{\tau|N\\ (\tau,\frac{N}{\tau})\ |\ (\frac{N}{N_\omega},\gamma_1-\gamma_2)}}\varphi\left(\left(\tau, \frac{N}{\tau}\right)\right)\omega'(y)^{-1}.\end{aligned}$$

Finally, we see that the hyperbolic term is

$$-n^{1-k/2}\sum_{\substack{d|n\\ d<\sqrt{n}}}d^{k-1}\sum_{\substack{\tau|N\\ (\tau,\frac{N}{\tau})\ |\ (\frac{N}{N_\omega},d-\frac{n}{d})}}\varphi\left(\left(\tau, \frac{N}{\tau}\right)\right)\omega'(y)^{-1},$$

where y satisfies

$$y \equiv d \pmod{\tau},$$
$$y \equiv \frac{n}{d} \pmod{\frac{N}{\tau}}.$$

Alternatively, since the inner sum is invariant under replacing d with $\frac{n}{d}$ (by swapping τ with $\frac{N}{\tau}$), we can also write this as

$$-\frac{1}{2}n^{1-k/2}\sum_{\substack{d|n\\ d\neq\sqrt{n}}}\min\left(d,\frac{n}{d}\right)^{k-1}\sum_{\substack{\tau|N\\ (\tau,\frac{N}{\tau})\ |\ (\frac{N}{N_\omega},d-\frac{n}{d})}}\varphi\left(\left(\tau,\frac{N}{\tau}\right)\right)\omega'(y)^{-1}.$$

9. Calculation: the unipotent term - Evan

For the unipotent term, we have to calculate the finite part of $Z_F(s)$ at $s=1$, where Z_F is the Tate zeta integral

$$Z_F(s)=\int_{\mathbf{A}^*}F(t)|t|^s\ d^*t$$

associated to the function

$$F(t)=\int_K f\left(k^{-1}\begin{pmatrix}1&t\\0&1\end{pmatrix}k\right)\ dk.$$

The obvious plan of attack, then, is to evaluate the zeta integrals explicitly, place by place. This is possible but for the nonarchimedean places it is very complicated and (fortunately) unnecessary, thanks to the following proposition:

Proposition 9.1. *Let $\phi:\mathbf{A}\to\mathbf{C}$ be Schwartz-Bruhat on $\mathbf{A}_{fin}$ and of quadratic decay on $\mathbf{R}$.*[4] *Let $\zeta_p(s)=\frac{1}{1-p^{-s}}$ be the p-local part of the usual Riemann zeta function, and let $Z_{\phi_p}(s)$ be the p-local part of $Z_\phi(s)$. Then the function*

$$\Omega_\phi(s)=\prod_{p<\infty}\frac{Z_{\phi_p}(s)}{\zeta_p(s)}$$

is well-defined and entire. If we further assume that $Z_{\phi_\infty}(1)=0$, then the finite part of $Z_\phi(s)$ at $s=1$ is equal to $Z'_{\phi_\infty}(1)\Omega_\phi(1)$.

Proof. As ϕ is Schwartz-Bruhat on the finite places, it must equal the characteristic function of $\mathbf{Z}_p$, $\chi_{\mathbf{Z}_p}$, at almost all places (this is easy and standard argument using that every Schwartz-Bruhat function is a finite sum of characteristic functions of compact sets). A quick calculation of the local zeta integral shows that

$$Z_{\chi_{\mathbf{Z}_p}}(s)=\int_{\mathbf{Z}_p^*}|t|^s\ d^*t=\sum_{n=0}^{\infty}\int_{p^n\mathbf{Z}_p^*}p^{-ns}\ d^*t=\sum_{n=0}^{\infty}p^{-ns}=\frac{1}{1-p^{-s}}=\zeta_p(s).$$

Therefore Ω_ϕ is well-defined as an infinite product. A similar but slightly more involved computation shows that at *every* place, the ratio $Z_{\phi_p}(s)/\zeta_p(s)$ is a rational function in p^{-s}, hence an entire function in s. (This boils down to noting that the integrand of $Z_{\phi_p}(s)$ is compact and constant on small enough balls. The "small enough balls" integrate to give some multiple of ζ_p, while the remainder is a finite sum yielding a rational function in p^{-s}.)

Trivially rearranging and multiplying out, we have

$$Z_\phi(s)=Z_{\phi_\infty}(s)\Omega_\phi(s)\zeta(s),$$

[4] Note that this is less stringent than the usual class of functions for which the Tate zeta functions are defined. For these functions the functional equation holds as usual but we can only conclude meromorphic continuation on some strip containing $0<\Re(s)<2$ whose width depends on the decay of ϕ_∞ and its Fourier transform.

so at $s = 1$ we have merely a simple pole contributed by $\zeta(s)$ (it is not difficult to show that the local zeta integral at the infinite place is analytic on some strip containing $s = 1$). Now it's just a matter of writing down some Laurent series.

$$Z_{\phi_\infty}(s) = Z_{\phi_\infty}(1) + Z'_{\phi_\infty}(1)(s-1) + \dots,$$
$$\Omega_\phi(s) = \Omega_\phi(1) + \Omega'_\phi(1)(s-1) + \dots,$$
$$\zeta(s) = \frac{1}{s-1} + \gamma + \dots$$

where γ is the usual Euler constant. Multiplying out, we get

$$Z_\phi(s) = \frac{Z_{\phi_\infty}(1)\Omega_\phi(1)}{s-1} + (Z_{\phi_\infty}(1)\Omega_\phi(1)\gamma + Z_{\phi_\infty}(1)\Omega'_\phi(1) + Z'_{\phi_\infty}(1)\Omega_\phi(1)) + \dots$$

If $Z_{\phi_\infty}(1) = 0$, we immediately get the desired result. $\square$

This proposition means that we only really have to evaluate $Z_{F_\infty}(s)$ as a function; at all the finite places we can get by evaluating just at $s = 1$. First, let's tackle the infinite place. Because f_∞ is invariant under conjugation by K_∞ (by the calculation in the previous section), it is trivially easy to evaluate F_∞ (recall that our measure is nicely normalized!):

$$F_\infty(t) = \int_{K_\infty} f_\infty\left(k^{-1}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}k\right)\,dk = f_\infty\left(\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\right) = \frac{d_k(2i)^k}{(-t+2i)^k}.$$

Now evaluation of the zeta integral is just a computation:

Proposition 9.2. *On some strip around $s = 1$,*

$$Z_{F_\infty}(s) = \frac{2^{s-1}\cos(\pi s/2)\Gamma(s)\Gamma(k-s)}{\pi(k-2)!}.$$

Extremely sketchy proof outline. Split up into two similar integrals from zero to ∞, change variables $w = -\frac{t}{2i}$ in the integrand, use contour integration to replace the integral from zero to $i\infty$ with an integral from zero to ∞, and recognize the result as a beta integral which can be evaluated in terms of gamma functions. $\square$

Due to the fortunate fact that $\cos(\pi/2) = 0$, we deduce the following from the product rule immediately:

Corollary 9.3.

$$Z'_{F_\infty}(1) = -\frac{1}{2}.$$

$\square$

By Proposition 9.1, it remains only to calculate the local zeta integrals evaluated at $s = 1$.

Claim: $F(t)$ is identically zero unless n is a perfect square. We have the following argument: by definition, f is supported at each p on $Z(\mathbf{Q}_p)M(n,N)_p$. Therefore if some g is such that $f(g) \neq 0$, the determinant of g must be a square multiple of n, as it must be in $(\mathbf{Q}_p^*)^2 n\mathbf{Z}_p^*$ for each p. But in the integral defining F, f is evaluated only on a matrix with determinant one. We conclude that if F is to be nonzero, n must be a square.

Second claim: if $p \nmid nN$, then $Z_{F_p}(s) = \zeta_p(s)$. I will omit this easy argument, noting only that it follows from the definition of f and the resulting conclusion that F_p is equal to $\chi_{\mathbf{Z}_p}$.

Third claim: if $p|n$, then

$$Z_{F_p}(s) = \omega_p(\sqrt{n})|\sqrt{n}|_p^{-s}\zeta_p(s).$$

Again, we use the definition of f_p, noting that if $k^{-1}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}k = zm$ for $z \in Z(\mathbf{Q}_p)$, $m \in M(n,N)_p$, we can assume by multiplying by a unit that z is the matrix with $\sqrt{n}$ in the diagonal entries. Then

$$m = k^{-1}\begin{pmatrix}\sqrt{n} & t\sqrt{n}\\ 0 & \sqrt{n}\end{pmatrix}k,$$

which implies that $t\sqrt{n}$ must lie in $\mathbf{Z}_p$. Evaluating, we get $f_p(zm) = \omega_p(\sqrt{n})$. We then integrate over all t such that $t\sqrt{n}\mathbf{Z}_p$ and get $F_p(t) = \omega_p(\sqrt{n})\chi_{\mathbf{Z}_p}(\sqrt{n}t)$. Evaluating the zeta integral is then easy.

It remains to tackle the case where $p|N$, for which we really do only want to calculate $Z_{F_p}(1)$. Note that

$$\begin{aligned} Z_{F_p}(1) &= \int_{\mathbf{Q}_p^*}\int_{K_p} f_p\left(k^{-1}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}k\right)\, dk\, |t|^1\, d^*t \\ &= \zeta_p(1)\int_{\mathbf{Q}_p}\int_{K_p} f_p\left(k^{-1}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}k\right)\, dk\, dt. \end{aligned}$$

But this is exactly the same as the local hyperbolic orbital integral that we have already calculated above, upon specializing appropriately. Therefore we conclude that

$$\frac{Z_{F_p}(1)}{\zeta_p(1)} = \sum_\tau \phi_p(\gcd(\tau, N/\tau)),$$

where the sum is taken over all powers of p dividing N satisfying $N/\gcd(\tau, N/\tau) \in N_\omega\mathbf{Z}_p$.

Putting all of this together, by multiplying out, we get the desired result: the finite part of $Z_F(s)$ at $s = 1$ is equal to

$$-\frac{1}{2}\sqrt{n}\omega'(\sqrt{n})^{-1}\sum_\tau \phi(\gcd(\tau, N/\tau)),$$

when n is a square, and zero otherwise. Here the sum is taken over all positive $\tau|N$ satisfying $N/\gcd(\tau, N/\tau) \in N_\omega\mathbf{Z}$. One can easily check that this precisely fills in the missing $d = \sqrt{n}$ term in the hyperbolic result.

10. Calculation: the elliptic terms - Zeb

We want to compute

$$\int_{\bar{G}(\mathbf{Q})\backslash\bar{G}(\mathbf{A})}\ \sum_{\gamma\in\bar{G}(\mathbf{Q})\text{ ell.}} f(g^{-1}\gamma g)\, dg.$$

If $f(g^{-1}\gamma g) \neq 0$ for any g, then we can find two lifts $\tilde{\gamma}$ of γ to $G(\mathbf{Q})$ with $\det\tilde{\gamma} = n$, so this becomes

$$\frac{1}{2}\int_{\bar{G}(\mathbf{Q})\backslash\bar{G}(\mathbf{A})}\ \sum_{\gamma\in G(\mathbf{Q})\text{ ell.}} f(g^{-1}\gamma g)\, dg = \sum_{\substack{[\gamma]\text{ ell.}\\ \det\gamma = n}} \frac{1}{2}\int_{\overline{G_\gamma(\mathbf{Q})}\backslash\bar{G}(\mathbf{A})} f(g^{-1}\gamma g)\, dg,$$

where $\overline{G_\gamma(\mathbf{Q})} = Z(\mathbf{Q})\backslash G_\gamma(\mathbf{Q})$, and $G_\gamma(\mathbf{Q})$ is the centralizer of γ in $G(\mathbf{Q})$ (note this is not the same as the centralizer of γ in $\bar{G}(\mathbf{Q})$).

Thus we just need to calculate $\Phi(\gamma, f) = \int_{\overline{G_\gamma(\mathbf{Q})}\backslash \bar{G}(\mathbf{A})} f(g^{-1}\gamma g)\, dg$. Note that $G_\gamma(\mathbf{Q}), G_\gamma(\mathbf{A})$ are both abelian, since we have

$$G_\gamma(\mathbf{Q}) = \mathbf{Q}[\gamma]^*,$$

and similarly for $\mathbf{A}$, for γ elliptic or hyperbolic.

Proposition 10.1. *If γ is elliptic in $G(\mathbf{Q})$ but hyperbolic in $G(\mathbf{R})$, then $\Phi(\gamma, f) = 0$.*

Proof. Since $G_\gamma(\mathbf{Q}), G_\gamma(\mathbf{A})$ are unimodular, we have

$$\begin{aligned}\Phi(\gamma, f) &= \mathrm{Vol}(\overline{G_\gamma(\mathbf{Q})}\backslash \overline{G_\gamma(\mathbf{A})} \int\limits_{\overline{G_\gamma(\mathbf{A})}\backslash \bar{G}(\mathbf{A})} f(g^{-1}\gamma g)\, dg \\ &= \mathrm{Vol}(\overline{G_\gamma(\mathbf{Q})}\backslash \overline{G_\gamma(\mathbf{A})} \prod_{p\le\infty} \int_{\overline{G_\gamma(\mathbf{Q}_p)}\backslash \bar{G}(\mathbf{Q}_p)} f(g^{-1}\gamma g)\, dg,\end{aligned}$$

and the $p = \infty$ term is 0. □

Proposition 10.2. *If $\gamma \in G(\mathbf{Q})$ is elliptic in $G(\mathbf{R})$, then there exists a fundamental domain $\mathcal{F} \subseteq \bar{G}(\mathbf{A}_{fin})$ for $\overline{G_\gamma(\mathbf{Q})}\backslash \bar{G}(\mathbf{A}_{fin})$, i.e. $\overline{G_\gamma(\mathbf{Q})}$ is discrete in $\bar{G}(\mathbf{A}_{fin})$.*

Proof. We need to show that $Z(\mathbf{A}_{\mathrm{fin}})\backslash(G_\gamma(\mathbf{Q}) \cap Z(\mathbf{A}_{\mathrm{fin}})K_{\mathrm{fin}})$ is a finite set. By considering the valuation of the determinant modulo 2, we see that we just need to show that $G_\gamma(\mathbf{Q}) \cap SL_2^{\pm}(\mathbf{Z})$ is a finite set. For $g \in G_\gamma(\mathbf{Q}) \cap SL_2^{\pm}(\mathbf{Z})$ we can write $g = a + b\gamma$, and since $g \in SL_2^{\pm}(\mathbf{Z})$ we have $a, b \in \frac{1}{d}\mathbf{Z}$ for some d depending only on γ. Finally, since $\det(a + b\gamma)$ is a positive definite quadratic form (since γ is elliptic in $G(\mathbf{R})$) it can only take on the values ± 1 for finitely many pairs of $a, b \in \frac{1}{d}\mathbf{Z}$. □

Corollary 10.3. *If $\gamma \in G(\mathbf{Q})$ is elliptic in $G(\mathbf{R})$, we have*

$$\Phi(\gamma, f) = \int_{\bar{G}(\mathbf{R})} f_\infty(g^{-1}\gamma g)\, dg \int\limits_{\overline{G_\gamma(\mathbf{Q})}\backslash \bar{G}(\mathbf{A}_{fin})} f_{fin}(g^{-1}\gamma g)\, dg = \Phi(\gamma, f_\infty)\Phi(\gamma, f_{fin}).$$

Proof. The second equality follows from the fact that $\overline{G_\gamma(\mathbf{R})}$ is conjugate to $\bar{K}_\infty$, which has volume 1. □ □

Proposition 10.4. *If gamma is elliptic in $G(\mathbf{R})$ with eigenvalues γ_1, γ_2, then*

$$\Phi(\gamma, f_\infty) = -n^{1-k/2}\frac{\gamma_1^{k-1} - \gamma_2^{k-1}}{\gamma_1 - \gamma_2}.$$

Proof. By conjugating γ, we may assume that we have $\gamma = \begin{pmatrix} n^{\frac12} & 0 \\ 0 & n^{\frac12}\end{pmatrix} k_\theta$, where $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, for some $\theta \in [0, 2\pi)$. Since $\omega_\infty(n^{\frac12}) = 1$, we have $\Phi(\gamma, f_\infty) = \Phi(k_\theta, f_\infty)$. Using $\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} k_\theta \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} = k_{-\theta}$ we have

$$\Phi(k_\theta, f_\infty) = \int_{SL_2(\mathbf{R})} f_\infty(g^{-1}k_\theta g)\, dg + \int_{SL_2(\mathbf{R})} f_\infty(g^{-1}k_{-\theta} g)\, dg.$$

By the Cartan decomposition we can take $g = k_1 \begin{pmatrix} t^{\frac{1}{2}} & 0 \\ 0 & t^{-\frac{1}{2}} \end{pmatrix} k_2$. Since f_∞ is invariant under conjugation by elements of K_∞ and $k_1, k_{-\theta}$ commute, we get

$$\begin{aligned}
\int_{SL_2(\mathbf{R})} f_\infty(g^{-1}k_{-\theta}g)\, dg &= \pi \int_1^\infty f_\infty\left(\begin{pmatrix} t^{-\frac{1}{2}} & 0 \\ 0 & t^{\frac{1}{2}} \end{pmatrix} k_{-\theta} \begin{pmatrix} t^{\frac{1}{2}} & 0 \\ 0 & t^{-\frac{1}{2}} \end{pmatrix}\right)(1-t^{-2})\, dt \\
&= \pi \int_1^\infty f_\infty\left(\begin{pmatrix} \cos\theta & -t^{-1}\sin\theta \\ t\sin\theta & \cos\theta \end{pmatrix}\right)(1-t^{-2})\, dt \\
&= \frac{k-1}{4}\int_1^\infty \frac{(2i)^k}{(\sin\theta(t+\frac{1}{t}) + 2i\cos\theta)^k}(1-t^{-2})\, dt \\
&= \frac{k-1}{2}\int_1^\infty \frac{i^k}{(s\sin\theta + i\cos\theta)^k}\, ds \\
&= \frac{-e^{i(k-1)\theta}}{e^{i\theta}-e^{-i\theta}},
\end{aligned}$$

where the second to last equality came from the substitution $s = \frac{t+1/t}{2}$. □

Thus the elliptic term is

$$-\frac{1}{2}n^{1-k/2}\sum_{\substack{[\gamma] \text{ ell.} \\ t^2<4n}} \frac{\gamma_1^{k-1}-\gamma_2^{k-1}}{\gamma_1-\gamma_2}\Phi(\gamma, f_{\text{fin}}),$$

where t is the trace of γ and the determinant of γ is n.

Proposition 10.5. *We have bijections*

$$\begin{aligned}
\overline{G_\gamma(\mathbf{Q})}\backslash\bar{G}(\mathbf{A}_{fin})/\bar{K}_{fin} &\leftrightarrow G_\gamma(\mathbf{Q})\backslash G(\mathbf{A}_{fin})/K_{fin} \\
&\leftrightarrow G_\gamma(\mathbf{Q})\backslash\{\textit{lattices in } \mathbf{Q}^2\} \\
&\leftrightarrow \{\textit{classes of lattices in } \mathbf{Q}[\gamma]\}.
\end{aligned}$$

Proof. The only nontrivial bijection is the second one, and this follows from the local-global principle for lattices: a lattice in $\mathbf{Q}^2$ is the same as a collection of rank 2 $\mathbf{Z}_p$-submodules of Q_p^2, almost all of which are trivial. □

Let L be a lattice contained in $\mathbf{Q}[\gamma]$. We define $\mathcal{O}_L$ to be the ring of elements $g \in \mathbf{Q}[\gamma]$ satisfying $gL \subseteq L$ (note that $\mathcal{O}_L$ is then automatically an order of $\mathbf{Q}[\gamma]$). Let $\mathcal{O}_\gamma$ be the order $\mathbf{Z}[\gamma]$. For any order $\mathcal{O}$ of $\mathbf{Q}[\gamma]$, we define the weighted class number to be

$$h_w(\mathcal{O}) = \frac{2h(\mathcal{O})}{|\mathcal{O}^\times|},$$

where $h(\mathcal{O})$ is the usual class number (i.e. the number of invertible ideals of $\mathcal{O}$ modulo invertible principal ideals). If d is the discriminant of an order $\mathcal{O}$, we also use $h_w(d)$ to denote $h_w(\mathcal{O})$

First we work out the local orbital integral in the case $N = 1$. Then f_{fin} is the characteristic function of $Z(\mathbf{A}_{\text{fin}})K_{\text{fin}}\{g \in M_2(\hat{\mathbf{Z}}) \mid \det g \in n\hat{\mathbf{Z}}\}$. If γ has trace t

and determinant n, we get

$$\begin{aligned}
\Phi(\gamma, f_{\text{fin}}) &= \sum_{g\in G_\gamma(\mathbf{Q})\backslash G(\mathbf{A}_{\text{fin}})/K_{\text{fin}}} \frac{\operatorname{Vol}(\bar{K}_{\text{fin}})}{\operatorname{Vol}(\bar{K}_{\text{fin}} \cap g^{-1}\overline{G_\gamma(\mathbf{Q})}g)} f_{\text{fin}}(g^{-1}\gamma g) \\
&= \sum_{\substack{\text{classes of lattices } L \\ \gamma L \subseteq L}} \frac{2}{|\mathcal{O}_L^\times|} \cdot 1 \\
&= \sum_{\text{orders } \mathcal{O}\supseteq\mathcal{O}_\gamma} h(\mathcal{O}) \frac{2}{|\mathcal{O}^\times|} \\
&= \sum_{\substack{m\geq 1 \\ m^2 \mid \frac{t^2-4n}{\text{disc}}}} h_w\left(\frac{t^2-4n}{m^2}\right).
\end{aligned}$$

In general, a similar (but much more tedious) calculation gives

Theorem 10.6. *The elliptic term is*

$$-n^{1-\frac{k}{2}} \sum_{t^2<4n} \frac{\rho^{k-1}-\bar{\rho}^{k-1}}{\rho-\bar{\rho}} \sum_{\substack{m^2 \mid t^2-4n \\ \frac{t^2-4n}{m^2}\equiv 0,1 \pmod 4}} h_w\left(\frac{t^2-4n}{m^2}\right)\mu(t,m,n),$$

where $\rho, \bar{\rho}$ are the roots of the polynomial $x^2 - tx + n$ and

$$\mu(t,m,n) = \frac{\psi(N)}{\psi\left(\frac{N}{(N,m)}\right)} \sum_{\substack{c \pmod N \text{ lifting to} \\ c^2-tc+n\equiv 0 \pmod N}} \omega'(c)^{-1}.$$

□

11. Application: dimensions and eigenvalues - Zeb

If we plug in $n = 1$ to the trace formula, then we will get a formula for $\operatorname{tr}(T_1) = \operatorname{tr}(1) = \dim S_k(N, \omega')$. In this case there will be no hyperbolic term, and in the elliptic term the sum will include only $t = 0, \pm 1$ and $m = 1$. Furthermore, the elliptic term corresponding to $t = 1$ and the term corresponding to $t = -1$ will be equal. The result is the following formula for the dimension.

Theorem 11.1.

$$\dim S_k(N,\omega') = \frac{k-1}{12}\psi(N) - \frac{1}{2}s_0 - s_1 - \frac{1}{2}\prod_{p\mid N}\operatorname{par}(p),$$

where

$$s_0 = \begin{cases} \frac{1}{2}(-1)^{\frac{k}{2}-1}\omega'(x_0)2^r & \textit{if } x_0^2 \equiv -1 \pmod N, r = \#\textit{odd prime factors of } N, \omega'(x_0) \textit{ well-defined} \\ 0 & \textit{if } k \textit{ odd}, 4 \mid N, \exists p \mid N\ p \equiv -1 \pmod 4,\ \textit{or } \exists p \mid N\ \omega_p'(-1) = -1, \end{cases}$$

$$s_1 = \begin{cases} \frac{\alpha}{3}\omega'(2)^{-1} \prod_{p\mid N, p\neq 3}(\omega_p'(1+x_1) + \omega_p'(1-x_1)) & x_1^2 \equiv -3 \pmod{N \textit{ or } N/3}, \alpha = \begin{cases} 1 & k \equiv 2,3 \pmod 6 \\ -1 & k \equiv 0,5 \pmod 6 \end{cases} \\ 0 & k \equiv 1 \pmod 3, 2 \mid N, 9 \mid N,\ \textit{or } \exists p \mid N\ p \equiv -1 \pmod 3, \end{cases}$$

and

$$\mathrm{par}(p) = \begin{cases} 2p^{v_p(N)-v_p(N_{\omega'})} & \text{if } \lfloor \frac{v_p(N)}{2} \rfloor < v_p(N_{\omega'}) \\ p^{\frac{v_p(N)}{2}} + p^{\frac{v_p(N)}{2}-1} & \text{if } \frac{v_p(N)}{2} \rfloor \geq v_p(N_{\omega'}) \text{ and } v_p(N) \text{ even} \\ 2p^{\lfloor \frac{v_p(N)}{2} \rfloor} & \text{if } \frac{v_p(N)}{2} \rfloor \geq v_p(N_{\omega'}) \text{ and } v_p(N) \text{ odd.} \end{cases}$$

□

Corollary 11.2.

$$\dim S_k(N, \omega') = \frac{k-1}{12}\psi(N) + O\left(\frac{\psi(N)}{N^{\frac{1}{2}}}\right).$$

□

The second easy application of the trace formula is to show that the Hecke eigenvalues are integral.

Theorem 11.3. *For every n relatively prime to N, the eigenvalues of T_n are algebraic integers.*

Proof. By multiplicativity, it's enough to prove this for n a power of a prime p, and by the recurrences connecting the T_{p^j}s it's enough to prove this for T_p. By the form of the trace formula, we have $\mathrm{tr}(T_{p^j}) \in \frac{1}{12}\bar{\mathbf{Z}}$ for all j, from which we can immediately conclude that $\mathrm{tr}(T_p^j) \in \frac{1}{12}\bar{\mathbf{Z}}$ for all j. Now let λ be an eigenvalue of T_p, and let $d = \dim S_k(N, \omega')$. Now by Newton's identities, we have $\lambda^j \in \frac{1}{12^d d!}\bar{\mathbf{Z}}$ for every j, and from this we can conclude that in fact $\lambda \in \bar{\mathbf{Z}}$. □

12. Application: equidistribution of eigenvalues - Evan

Here we summarize a result on the "vertical" distribution of Hecke eigenvalues, due to Serre in ([8]). Fix a prime p and think of the weight k and the level N as varying. Let $T_p(N, k)$ denote the pth Hecke operator on $S_k(N, \mathrm{Id})$ (we have taken the nebentypus to be the trivial character for simplicity's sake). The Hecke operators are self-adjoint with respect to the Petersson inner product, so their eigenvalues are real. By the proof of the Weil conjectures (and the consequent proof of the Ramanujan-Petersson conjecture), we know that the Hecke eigenvalues of $T_p(N, k)$ lie in the interval $[-2p^{(k-1)/2}, 2p^{(k-1)/2}]$. For convenience, therefore, we shall normalize as follows:

$$T'_p(N, k) = \frac{T_p(N, k)}{p^{(k-1)/2}}.$$

Thus the eigenvalues of $T'_p(N, k)$ will lie in $[-2, 2]$.

The natural question, therefore, is whether these eigenvalues are equidistributed with respect to some natural probability measure on $[-2, 2]$ in some limit.[5] The "vertical" situation considered by Serre, which can be resolved using the Eichler-Selberg trace formula, is where p is held fixed and $N+k$ is allowed to tend to infinity

[5]Recall that a sequence x_ℓ on a compact space X is equidistributed with respect to a Radon probability measure μ if $\frac{1}{\ell}\sum_{i=1}^{\ell} \delta_{x_i} \to \mu$ in the weak topology as $\ell \to \infty$, where δ_x is the "delta distribution" supported at x. In other words, we require that

$$\lim_{\ell\to\infty} \frac{1}{\ell}\sum_{i=1}^{\ell} f(x_\ell) = \int_X f(x)\, d\mu(x)$$

for all continuous functions f on X.

in any way, so long as k is always even and $p \nmid N$. In particular, we get results for the limit in large weight only (keeping N fixed), in large level only (keeping k fixed), or in an arbitrary combination of the two. The related question of the limiting distribution obtained while holding N and k fixed and letting p tend to infinity is (a generalization of) the Sato-Tate conjecture and much more difficult; however, the $k = 2$ and N squarefree case has been established ([1]).

In our case, the measure μ_p that we want has the following definition:

$$d\mu_p = \frac{p+1}{\pi} \frac{(1-x^2/4)^{1/2}}{(p^{1/2}+p^{-1/2})^2 - x^2}\, dx,$$

where dx is the Lebesgue measure on $[-2, 2]$. It has an interpretation as follows: we can write

$$\mu_p = f_p \mu_\infty,$$

where μ_∞ is the Sato-Tate measure (which comes from the pushforward of the Haar measure on conjugacy classes of the Lie group SU(2)) and f_p is a function which comes from the Plancherel measure on the spectrum of the Bruhat-Tits tree associated to $\mathrm{PGL}_2(\mathbf{Q}_p)$. More preciasely, if X_n is the polynomial in x given by

$$X_n(x) = e^{in\phi} + e^{i(n-2)\phi} + \ldots + e^{-in\phi}, \quad x = 2\cos\phi,$$

then f_p is the corresponding generating series given by

$$f_p(x) = \sum_{m=0}^{\infty} q^{-m} X_{2m}(x) = \frac{q+1}{(q^{1/2}+q^{-1/2})^2 - x^2}.$$

When p is small (for instance, $p = 3$ or $p = 5$), the graph of $\frac{d\mu_p}{dx}$ looks like a symmetric hill with two peaks and a valley in between at $x = 0$; as p gets larger μ_p flattens somewhat and tends towards the Sato-Tate distribution μ_∞, which looks like a semicircle.

Theorem 12.1 (Serre)**.** *Let $\{k_\lambda\}$ and $\{N_\lambda\}$ be sequences of positive integers such that k_λ is always even, p never divides N_λ, and $\lim_{\lambda\to\infty}(k_\lambda + N_\lambda) = \infty$. Let $\{x_\ell\}$ be the sequence formed by concatenating the sets of eigenvalues of each operator $T_p'(N_\lambda, k_\lambda)$ (where each finite set of eigenvalues of a single operator can be ordered arbitrarily). Then the sequence $\{x_\ell\}$ is equidistributed with respect to the measure μ_p.*

Proof sketch. A priori the statement of equidistribution needs to be checked for every continuous function f on $[-2, 2]$, but it is clear by linearity and continuity that it actually suffices to check that the weak limit holds for any sequence of functions whose span is dense in the space of continuous functions. We will pick our test functions carefully to take advantage of the multiplicativity of the Hecke operators and then use the trace formula.

More specifically, as the trace is the sum of the eigenvalues it suffices to check that

$$\frac{\mathrm{Tr}(P(T_p'(N,k)))}{\dim S_k(N, \mathrm{Id})} \to \int_{-2}^{2} P(x)\, d\mu_p(x)$$

for a set of polynomials P with dense span. We take these P to be the polynomials X_m from above, for which the normalized Hecke operators obey the recurrence relation $T'_{p^m} = X_m(T'_p)$. For these polynomials, an easy calculation shows that

$$\int_{-2}^{2} X_m(x)\, d\mu_p(x) = \begin{cases} p^{-m/2} & \text{if } m \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

To calculate the limit on the right hand side, we employ the Eichler-Selberg trace formula. After some easy and unenlightening analysis bounding the hyperbolic, unipotent, and elliptic terms, we find that in the large weight and/or level limit,

$$\mathrm{Tr}(T'_n(N,k)) \sim \begin{cases} \frac{k-1}{12}\Psi(N)n^{1/2} & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

Specializing to $n = 1$, we get $\dim S_k(N) \sim \frac{k-1}{12}\Psi(N)$. Putting these two asymptotic results together, we conclude that $\frac{\mathrm{Tr}(T'_{p^m}(N,k))}{\dim S_k(N)}$ tends to $p^{-m/2}$ if m is even and zero otherwise, which exactly matches the above calculation of the left hand side. $\square$

Variants of this proof show that the same basic result holds in more generality; for instance, we can introduce a nontrivial nebentypus or restrict to only newforms and get similar results. More interestingly, we can consider the related problem of simultaneous equidistribution of eigenvalues of the operators $T'_p(N_k)$ as p ranges over a finite set S of primes and ask what the limiting distribution in $[-2,2]^{|S|}$ is. As expected, it turns out to be the tensor product $\bigotimes_{p \in S} d\mu_p$, which implies in particular that for any finite set of primes and fixed weight k we can pick a level N for which one of the eigenvalues of $T_p(N,k)$ is arbitrarily close to the Ramanujan-Weil bound for each $p \in S$ (alternatively, fix N and pick a k such that the same statement holds).

By specializing to the $k = 2$ case, Serre's equidistribution results have the corollary that the maximum of the dimensions of the $\mathbf{Q}$-simple factors of the modular Jacobian $J_0(N)$ must tend to infinity as $N \to \infty$. In particular, there are only finitely many $J_0(N)$ isogenous to a product of elliptic curves. All of these results can be made effective (see [6]); for example, it is known that the largest N for which $J_0(N)$ is isogenous to a product of elliptic curves is $N = 1200$, due to Yamauchi in [9].

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