1 The Jacquet functor and parabolic induction

Retain the notations of Brian's notes: $G = \mathbf{G}(\mathbf{F})$ is the groups of $\mathbf{F}$-rational points of a connected reductive $\mathbf{F}$-group. Fix $B = \mathbf{B}(\mathbf{F})$ be a Borel, and also fix a maximal split torus A in B.

Let $P \supset B$ a parabolic subgroup of G containing B (not as fixed as B). If $N \subset P$ is the unipotent radical of P (called U in Brian's lecture), it is a normal subgroup and we have the Levi decomposition $P = MN$ where M is a reductive group that contains A, M is called a Levi subgroup. We have $M \bigcap N = \{1\}$ hence $P = M \rtimes N$ is a semidirect product and $M \cong P/N$. (Note also that thus M normalizes N)

For all such M we obtain (dependent on which $P \supset B$ we choose) we write $M < G$, and call M *standard*.

Example 1.1. If $\mathbf{G} = \mathbf{GL}_n$, let B be the subgroup of upper-diagonal matrices. Then all standard subgroups correspond to an (ordered) partition $\underline{n} = (n_1, \ldots, n_r)$ of n i.e. a solution in $\mathbf{N}$ of the equation $n_1 + \ldots + n_r = n$.

For such a partition, M is the set of block diagonal matrices (with respect to the corresponding r blocks), P is the set of block upper-triangular matrices and N is the set of block unipotent matrices.

This will be the example to have in mind for the entirety of the lecture.

Definition 1.2 (Jacquet module)**.** Let (π, V) be an admissible representation of G. Let $V(N) = V(N, 1)$ be the subspace of V generated by vectors of form $\pi(n)v - v$.

Let $V_N = V/V(N)$ and (π_N, V_N) the corresponding M-representation obtained by either restriction from G to M, or restricting from G to P and taking quotients $M \cong P/N$. This is called the *Jacquet module* of V (with respect to P), and $V \mapsto V_N$ is called the *Jacquet functor*.

More generally, let χ be a character of N normalized by M. Set $V(N, \chi)$ be the subspace of V generated by vectors of form $\pi(n)v - \chi(n)v$ and set $V_{N,\chi} = V/V(N, \chi)$ be the corresponding Jacquet module acted upon by M.

Proposition 1.3 (Frobenius Reciprocity)**.** *The Jacquet functor is left adjoint to Induction:*

$$Hom_M(\pi_N, \rho) \cong Hom_G(\pi, Ind_P^G(\rho))$$

where π is a representation of G as above, and ρ is a representation of M, regarded as a representation on P on the right via $P/N \cong M$.

More generally

$$Hom_M(\pi_{N,\chi}, \rho) \cong Hom_G(\pi, Ind_P^G(\rho \otimes \chi))$$

where $\rho \otimes \chi$ is regarded as a representation of P via $P = MN$ (here we use that M normalizes χ).

The space of $Ind_P^G(\rho \otimes \chi)$ is the space of functions $f \colon G \to W$ that satisfy $f(nmg) = \chi(n)\rho(m) \cdot f(g)$. The action of G, as usual, is by composition to the right.

Proof. The proof is standard: by the standard Frobenius reciprocity

$$\mathrm{Hom}_P(\pi, \rho) \cong \mathrm{Hom}_G(\mathrm{Ind}_P^G(\rho))$$

and the first is canonically isomorphic to $\mathrm{Hom}_M(\pi_N, \rho)$ as N acs trivially on ρ.

More explicitly: let $\phi\colon V_N \to W$ be an intertwining map on the left-hand side. Then it corresponds to $\psi\colon V \to \mathrm{Ind}_P^G(\rho)$ which sends $v \in V$ to the function $f\colon G \to W, f(g) = \phi(\overline{g \cdot v})$.

The inverse map sends $\psi\colon V \to \mathrm{Ind}_P^G(\rho)$ to the map ϕ that satisfies $\phi(\overline{v}) = (\psi(v))(1)$. This is independent, because $\psi(nv)(1) = \psi(v)(n) = n \cdot \psi(v)(1) = \psi(v)(1)$.

Similarly, $\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\rho \otimes \chi)) \cong \mathrm{Hom}_P(\pi, \rho \otimes \chi)$ and the latter factors as $Hom_M(\pi_N, \rho)$ as N must annihilate the image of $V(N, \chi)$.

□

Proposition 1.4 (Jacquet's lemma). *If (π, V) is admissible, and let K is a compact open subgroup G admitting an Iwahori factorization with respect to $P = MN$, that is $K = (K \bigcap N)(K \bigcap M)(K \bigcap N^-)$ where $P = MN^-$ is the parabolic opposite to P.*

Then the projection map $V \to V_N$ maps V^K surjectively onto $(V_N)^{K \bigcap M}$.

Corollary 1.5. *The Jacquet functor takes admissible representations of G to admissible representations of M. It also takes finitely generated smooth representations of G to finitely generated smooth representations of M.*

Proof. The first part follows immediately from Jacquet's lemma and the fact that G has a neighborhood basis of the identity of compact open subgroups that posess Iwahori factorizations with respect to $P = MN$.

For the second part, let $v_1, \ldots, v_\ell$ generate π over G, and let K be a compact open that fixes $v_1, \ldots, v_\ell$. Then the set $P \setminus G/K$ is finite, and choosing representatives $g_1, \ldots, g_r$ for the double cosets means $\{g_i v_j\}$ generate V over P, and hence they generate V_N over $P/N \cong M$. For general χ, the proof is the same by choosing K on which χ is trivial.

□

Lemma 1.6 (Jacquet, Langlands). *$v \in V(N, \chi)$ if and only if there exists a compact subgroup $K \subset N$ such that*

$$\int_K \chi(h^{-1})\phi(h) \cdot v d\mu_K = 0$$

(the above operator was also met in the previous lecture under the name $e_{K,\chi}$)

Proof. We shall use the fact that N the unipotent radical of P, is exhausted by its compact subgroups, i.e. every compact subset of N is contain in a compact subgroup of N.

If $v = \sum \pi(n_i) \cdot v_i - \chi(n_i) v_i$ then we choose K be a compact containing $n_1, \ldots, n_k$. Clearly, "χ-twisted averaging over K" will then take $\pi(n_i) \cdot v_i$ and $\chi(n_i) \cdot v_i$ to the same thing, which shows $e_{K,\chi} v = 0$.

Conversely, assume $e_{K,\chi}(v) = 0$.

Choose $K' \subset K$ compact, open in K and small enough such that χ acts trivially on K' and k' acts trivially on v (by smoothness). Let $g_1, \ldots, g_r$ be representatives of $K\mathcal{K}'$.

We then immediately decompose $e_{K,\chi}(v)$ as $\sum_{i=1}^r \frac{1}{r} \chi(g_i^{-1}) \pi(g_i) \cdot v$.

This is zero, so adding v to it yields $v = \sum_{i=1}^{r} \frac{1}{r}(v - \chi(g_i^{-1})\pi(g_i) \cdot v))$.

The interior terms have the for $\pi(g_i) \cdot v_i - \chi(g_i)v_i$ for $v_i = \frac{\chi(g_i^{-1})}{r}v$, hence $v \in V(N, \chi)$

□

Corollary 1.7. *The Jacquet functor is exact.*

Proof. Right exactness follows from adjointness, but it is also obvious by inspection. It remains to show injectivity: if $V_1 \subset V_2$ then $V_1(N, \chi) = V_2(N, \chi) \bigcap V_1$. This, however, is immediate from the description in the above lemma. □

Proposition 1.8. *Induction from a parabolic subgroup takes admissible representations to admissible representations.*

Proof. Let K be a compact open, and consider $\mathrm{Ind}_P^G(\rho)$ for an admissible representation ρ of P. Choose representatives g_i for each double coset $P \setminus G/K$; note that this is a finite set because $P \setminus G$ is compact and K is open.

Then $f \in (\mathrm{Ind}_P^G(\rho))^K$ one Pg_iK is uniquely determined by $f(g_i)$.

Even better, $f(g_i)$ must be fixed by $P \bigcap g_iKg_i^{-1}$: if $g_ikg_i^{-1} \in P$ then $f(g_i) = f(g_ik)\rho(g_ikg_i^{-1}) \cdot f(g_i)$.

We then identify $(\mathrm{Ind}_P^G(\rho))^K \cong \oplus_{\overline{g}_i \in P \setminus G/K} \rho^{P \bigcap g_iKg_i^{-1}}$ which is finite.

□

To simplify certain formulas it is convenient to tensor up the above two formulas with the module character.

Definition 1.9. Let δ_H be the module character of the group H, i.e. $(Rh)^*(\mu_H) = \delta_H(h) \cdot \mu_H$ where μ_H is any left-invariant Haar measure on H.

For M, N, P, χ as above, define

$r_{N,\chi}(\pi) = \delta_P^{-1/2} \otimes \pi_N$

$I_{N,\chi}(\rho) = \delta_P^{1/2} \otimes \mathrm{Ind}_P^G(\rho \otimes \chi)$

The latter has as underlying space the function $f \colon G \to W$ that satisfy $f(nmg) = \chi(n)\delta_N^{1/2}(m)\rho(m) \cdot f(g)$. [Note that M is unimodular because reductive so $\delta_P(nm) = \delta_N(m)$ the latter using the conjugation action of M on N. N is also unimodular because it is exhausted by its compact subgroups.]

These will be the functors we will be working with, the Jacquet functor and respectively parabolic induction.

Proposition 1.10. *(a) Parabolic induction and the Jacquet functors are exact, and $r_{N,\chi}$ is left adjoint to $I_{N,\chi}$.*

(b) Let $M' < M$ corresponding to N', and χ' a character of N' normalized by M'. Then

$$I_{N,\chi} \circ I_{N',\chi'} = I_{NN',\chi\chi'}, r_{N',\chi'} \circ r_{N,\chi} = r_{NN',\chi'}$$

(c) Parabolic induction commutes with contragredients:

$\widetilde{I_{N,\chi}(\rho)} = I_{N,\chi}(\widetilde{\rho})$

Proof. (a) The two functors being adjoint to each other follows immediately from the fact that they were obtained by tensoring two adjoint functors with $\delta_P^{-1/2}, \delta_P^{1/2}$.

In particular, $r_{N,\chi}$ is right exact and $I_{N,\chi}$ is left exact.

Let's prove the other two statements. The character $\delta_P^{\pm 1/2}$ may be dropped. Then exactness of the Jacquet functor was shown before.

Right exactness of induction follows immediately from the tensor product description.

(b) The module components are $\delta_N \delta(N')$ respectively $\delta(NN')$ which coincide.

The rest follows from transitivity of the usual induction. For the Jacquet functor, use adjunction.

(c) Let $f \in \mathrm{Hom}(G, \delta_P^{1/2} \otimes \rho), \tilde{f} \in \mathrm{Hom}(G, \delta_P^{-1/2} \otimes \tilde{\rho})$.

Consider the function $g \to \langle \tilde{f}(g), f(g) \rangle$.

Then pg gets mapped to $\langle \tilde{f}(pg), f(pg) \rangle = \delta_P^{-1/2}(p) \langle \tilde{f}(g), p^{-1} f(pg) \rangle = \langle \tilde{f}(g), f(g) \rangle$.

Thus this function is invariant under right action by P and we define

$(f, \tilde{f}) = \int_{P \backslash G} \langle \tilde{f}(g), f(g) \rangle d_{P \backslash G}(\overline{g})$

This is the inner product that realizes the required isomorphism.

□

2 The composition of Jacquet and Parabolic Induction

Definition 2.1. If π is a smooth representation of G, let $JH(\pi)$ be the set of all irreducible subquotients of π. Let $\ell(\pi)$ be the length of π as a G-module; if $\ell(\pi)$ is finite then we know it has a Jordan-Holder series whose members we denote $JH^0(\pi)$. Each element of $JH(\pi)$ is then contained in $JH^0(\pi)$ with some multiplicity.

Our aim is to describe the Jordan-Holder series of a parabolic induction, and how different such things relate to each other.

Definition 2.2. A representation π of G is called quasicuspidal if $r_{M,G}(\pi) = 0$ for any standard subgroup $M < G, M \neq G$. In that case, by exactness of $r_{M,G}$, any of its subquotients is quasicuspidal too.

A quasicuspidal admissible representation is called cuspidal.

Cuspidal representations are building blocks of G-representations, in the sense that every admissible G-representation is induced from a cuspidal one.

Proposition 2.3. *Let π be an irreducible admissible G-representation. Then π is a subrepresentation of $Ind_N^G(\rho)$ for some $M < G$ and a cuspidal irreducible representation ρ of M.*

Sketch. We will consider a minimal $M < G$ for which $r_{M,G}(\pi) \neq 0$; it exists and $r_{M,G}(\pi)$ is then cuspidal. The Jacquet functor takes finitely generated modules to finitely generated module, which implies $r_{M,G}(\pi)$ has an irreducible factor representation. By adjunction, there is a non-zero map from π to $\mathrm{Ind}_N^G(\rho)$ which must then be an embedding. □

The aim of this section is to study how the representations $I_{M,G}(\rho)$ decompose into subfactors and how they relate to each other as the irreducible cuspidals ρ and standard groups M are allowed to vary.

Proposition 2.4. *(a) Every smooth representation π of G decomposes into $\pi = \pi_c \oplus \pi_C^\perp$ where π_c is quasicuspidal and $\pi_c^\perp$ has no non-zero quasicuspidal subquotients.*

(b) If π is admissible and ω is a cuspidal subquotient of π, then π has a submodule and a factormodule equivalent to ω.

(c) If π is cuspidal then so is $\tilde{\pi}$.

(d) If M is a proper standard subgroup of G and ρ is a smooth M-representation then $\pi = I_{G,M}(\rho)$ then $\pi_c = 0$.

Proof. (a),(b), (c) Skip.

(d) $\mathrm{Hom}(\pi_c, \pi) = \mathrm{Hom}(r_{G,M}(\pi_c), \rho) = \mathrm{Hom}(0, \rho) = 0$.

□

Definition 2.5. Let $W = N_G(A)/Z_G(A)$ be the Weyl group of G. For $w \in W$, lift w to $\overline{w} \in N_G(A)$ and consider the action of $\overline{w}$ on G by conjugation: $w(g) = \overline{w} g \overline{w}^{-1}$. This action is not well-defined, but images of subgroups that contain $Z_G(A)$ are well-defined. In particular, for standard $M < G$ that contain B, $Z_G(A) \subseteq M$ which means $w(M)$ is another subgroup of G that does not depend on the choice of lift $\overline{w}$. It is standard with respect to the Borel $w(B)$, but not necessarily B.

Let W_M be the Weyl group of M, $W_M \subseteq W$ and it corresponds to $\overline{w} \in M \bigcap N_G(A)$.

If M' is another standard subgroup of G, we set $W(M, M') = \{w \in W \mid w(M) = M'\}$.

If this is non-empty, we call M, M' associated and write $M \sim M'$. Similarly, if $w(\rho) = \rho'$ we write $\rho \sim \rho'$.

Clearly, $W(M, M') = W_{M'} W(M, M') W_M$.

Also, set $W(M, \star) = \bigcup_{M'} W(M, M')$ for M' standard groups (with respect to B) associated to M, and let $\ell(M)$ be the cardinality of the set $W(M, \star)/W_M$.

Note that W_M has finite index in W, hence all relevant quotients are finite.

Example 2.6. For $G = \mathrm{GL}_n$, the Weyl group is S_n. M, M' are associated if and only if the corresponding partitions are reorderings of each other. $W(M, M')$ is the set of permutations that takes one partition into the other.

Note, in particular, that $W(M, M)$ is not in general equal to $W(M)$: the first is the set of permutations that preserve the partition whereas the second is the set of permutations that preserve each block of the partition. If a partition has two blocks of equal length, these two definitions disagree.

The set $W(M, \star)$ consists of permutations that preserve the "consecutivity" of each block, and $W(M, star)/W_M$ is the set of permutations of the blocks. If M corresponds to the partition $\underline{n} = (n_1, \ldots, n_r)$ then $\ell(M) = r!$.

From now on, let's fix a cuspidal representation ρ of M, and let $\pi = I_{G,N}(\rho)$.

The main theorems are as follows:

Theorem 2.7. *The length $\ell(\pi)$ of the representation π is finite, and $\ell(\pi) \leq \ell(M)$.*

Theorem 2.8. *(a) The following are equivalent:*

(i) $M \sim M', \rho \sim \rho'$

(ii) $Hom(\pi, \pi') \neq 0$

(iii) $JH^0(\pi) = JH^0(\pi')$

(iv) $JH(\pi) \bigcap JH(\pi') \neq 0$

(b) Set $W(\rho, \rho') = \{w \in W(M, M') \mid w(\rho) = \rho'\}$

Then $\dim Hom(\pi, \pi') \leq |W(\rho, \rho')/W_M|$

The proof of these two theorems relies on the following three lemmas:

Lemma 2.9. *Let M, M' be standard subgroups of G. Let*

$$W^{M,M'} = \{w \in W \mid w(M \bigcap B) \subset B, w^{-1}(M' \bigcap B) \subset B\}$$

Then:

(a) Every double coset $W_{M'} \setminus W/W_M$ contains exactly one element of $W^{M,M'}$.

(b) If $w \in W^{M,M'}$ then $M \bigcap w^{-1}(M') < M, M' \bigcap w(M) < M'$.

(Note that part (a)) implies $W^{M,M'}$ is finite.

Lemma 2.10 (Geometric lemma of Langlands)**.** *Suppose $W^{M,M'}$ is non-empty. For $w \in W^{M,M'}$ define*

$$F_w = I_{M', M' \bigcap w(M)} \circ w \circ r_{M \bigcap w^{-1}(M'), M}$$

The functor $F = r_{G,M'} \circ I_{G,M}$ is glued from the functors $F_w, w \in W^{M,M'}$. (In particular if $W^{M,M'}$ is empty the functor is zero)

That is, there exists a numeration $w_1, \ldots, w_k$ of $W^{M,M'}$ and $F(\rho)$ has a filtration $0 \subset \tau_1 \subset \ldots \subset \tau_k = F(\rho)$ with $\tau_i/\tau_{i-1} \cong F_{w_i}(\rho)$

Corollary 2.11. *Let $M, M' < G$ and ρ be a cuspidal representation of M. Let $\tau = (r_{M',G} \circ I_{M,G})(\rho)$*

(a) If M' has no standard subgroups associated to M, then $\tau = 0$.

(b) If M' is not associated to M, then τ has no non-zero cuspidal quotients.

(c) If $M \sim M'$ then τ is glued from representations $w(\rho)$ where $w \in W(M, M')/W_M$, in particular τ is cuspidal.

Proof. (a) If M' has no standard subgroups associated to M, then all functors F_{w_i} must be zero because they involve $r_{M \bigcap w_i^{-1}(M'), M}(\rho)$ which is 0, because $M \cap w_i^{-1}(M') \subsetneq M$ must be a proper subgroup and ρ is supercuspidal.

(b) Skip.

(c) If $M \sim M'$ then as in (a), F_w are non-zero only for $w \in W^{M,M'} \bigcap W(M, M')$.

By lemma 2.8., this is in bijection with $W_{M'} \setminus W(M, M')/W_M$ which equals $W(M, M')/W(M)$.

For $w \in W(M'M')$, $F_w(\rho)$ equals $w(\rho)$ because both induction and restriction are the identity functor.

□

Lemma 2.12. *Let $M \sim M' = \omega(M)$. Then there is a chain of standard subgroups $M_0 = M, M_1, \ldots, M_k = M'$ and $w_i \in W$ elementary maps such that $w_i \colon M_{i-1} \xrightarrow{\sim} M_i$, and $w = w_k \circ \ldots \circ w_1$.*

Here, a map $w\colon M \to M'$ is elementary if there exists $L < G$, $M, M' < L$ and $\ell(M) = 2$ inside L. Equivalently, w corresponds to a reflection of the two corresponding Weyl chambers with respect to a shared wall.

In the example of GL_n, elementary maps correspond to exchanging two consecutive blocks in the partition. Thus the statement becomes the theorem that any permutation can be obtained as a composition of transpositions of consecutive blocks.

Proof of theorem 2.7. We need to bound the length of any chain $0 = \pi_0 \subset \pi_1 \subset \ldots \subset \pi_r = \pi$. Let $\pi_i/\pi_{i-1} = \rho_i$.

By exactness, $r_{M',G}(\pi)$ obtains a chain $r_{M',G}(\pi_0) \subset r_{M',G}(\pi_1) \subset \ldots \subset r_{M',G}(\pi_r)$ with quotients $r_{M',G}(\rho_i)$.

We claim that for some $M' \sim M$, $r_{M',G}(\rho_i)$ is non-zero. Indeed, let us choose a minimal $L < G$ such that $r_{L,G}(\rho_i)$ is non-zero, then this is also cuspidal. This is a subquotient of $r_{L,G}(\mathscr{I}_{M,G}(\pi))$ hence by corollary 2.10. b), cuspidality implies $L \sim M$ as desired.

We then conclude that $r \leq \sum_{M' \sim M} \ell(r_{M',G}(\pi))$.

But by 2.11.c) we know the length of $r_{M',G}$ is $|W(M, M')/W_M|$ and summing over all M' yields $\ell(M)$, as desired.

□

Proof of theorem 2.8. (a) The implications (ii) to (iv), (iii) to (iv) are trivial.

Now we prove (iv) to (i). Assume $\tau_1 \subset \tau_2 \subset \pi$ with $\tau_2/\tau_1 \cong \rho$, and ρ is an irreducible subquotient of π' as well.

Then as above $r_{L,G}(\rho)$ is cuspidal for some L. Corollary 2.11. b) implies $L \sim M$ and 2.10. c) implies $r_{L,G}\rho$ is glued from representations of form $w(\pi)$. For the same reason, $L' \sim M$ and $r_{L,G}(\rho)$ is glued from representations of form $w(\pi')$. This implies $M \sim M', \pi \sim \pi'$ as desired.

Next, we show (i) implies (ii).

By adjunction, $\mathrm{Hom}(\pi, \pi') = \mathrm{Hom}(r_{M',G} \circ I_{G,M}(\rho), \rho') = \mathrm{Hom}(F(\rho), \rho')$.

By the geometric lemma, if $\rho \sim \rho'$ then $F(\rho)$ has a subquotient isomorphic to ρ' which implies that the above set is non-empty.

Finally, we show that (i) implies (iii).

The first step is to assume that $\ell(M) = 2$. In particular, by the previous theorem π, π' both have length at most 2 so they are either irreducible or extensions of irreducibles by one another.

By (ii), there are non-zero maps maps $A\colon \pi' \to \pi, A'\colon \pi' \to \pi'$.

If either A, A' are isomorphisms then (iii) is immediate. Otherwise, this is only possible when π, π' have length 2 and hence the kernels of A, A' are proper irreducible submodules π_0, π'_0 and the quotients $\pi/\pi_0, \pi'/\pi'_0$ are irreducible.

We claim that π_0, π'_0 are only irreducible submodules of π.

Indeed, we know $\ell(r_{M,G})(\pi) + \ell(r_{M',G}\pi) \leq \ell(M) = 2$.

If we have $\tilde{\pi}_0$ another irreducible submodule of M then $\ell(r_{M,G}(\pi_0)) + \ell(r_{M',G}(\pi_0) = 1$.

Since $\text{Hom}(\pi_0, \pi) = \text{Hom}(r_{M,G}(\tilde{\pi}_0), \rho) \neq 0$ it follows that $r_{M,G}(\pi_0) \neq 0$ and $r_{M',G}(\pi_0) = 0$.

This means that $\text{Hom}(\pi_0, \pi') = \text{Hom}_{r_{M',G}(\pi_0), \rho'} = 0$ so A kills $\tilde{\pi}_0$ which means $\tilde{\pi}_0 \subseteq \pi_0$ as desired.

Similarly, π'_0 is the only irreducible submodule of π'.

This implies $\pi/\pi_0 \cong \pi', \pi'/\pi'_0 \cong \pi$ so $JH^0(\pi) = JH^0(\pi_0) = \{\pi_0, \pi'_0\}$.

This finishes the proof for the case when $\ell(M) = 2$. The general case follows from applying the lemma. We may need to restrict to a smaller L instead of G, and then JH^0 will be induced from the L case.

□