J-L seminar
Supercuspidal representations

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1 Intro

This talk basically consists of 3 parts: we first analyze a little more closely supercuspidal representations and show how they are "building blocks" for the category of smooth representations of G. Then we give a concrete example, constructing almost from scratch a supercuspidal representation of $\mathrm{GL}_2(F)$; this will be done by following a general procedure that lets us create *many* supercuspidal representation under very mild conditions on the group G. Finally, we will state the Local Langlands Correspondance for GL_n, which will underline once again the importance of supercuspidal representations, and show how this relates to parabolic induction.

Throughout the talk, $\mathbb{G}$ is a connected reductive group defined over a non-archimedean local field F with ring of integers $\mathcal{O}$ and uniformizer ϖ. Denote $G = \mathbb{G}(F)$ its F-rational points; $P = \mathbb{P}(F)$ for a parabolic subgroups $\mathbb{P}$ of $\mathbb{G}$, with Levi decomposition $P = MN$. Sometimes $\mathbb{P}$ will be assumed to be minimal, and we will state when this is so.
We will consider smooth complex representations. Every representation of G is understood to be smooth, and the category of such representations is denoted $\mathrm{Rep}(G)$, with $\mathrm{Irrep}(G)$ the set of isomorphism classes of irreducible representations.

2 Supercuspidal representations

We recall a few definitions. Fix a minimal parabolic $\mathbb{P}_0 \subset \mathbb{G}$ and a maximal split k-torus $\mathbb{T}_0 \subset \mathbb{P}_0$. Any parabolic k-subgroup $\mathbb{P} \subsetneq \mathbb{G}$ is $\mathbb{G}$-conjugate to a unique $\mathbb{P}' \supset \mathbb{P}_0$; we say $\mathbb{P}$ is standard if $\mathbb{P} \supset \mathbb{P}_0$, and for any such parabolic there exists a unique Levi factor $\mathbb{M}$ containing $\mathbb{T}_0$(see [1], section 2.1), which is hence called a *standard* Levi. Notice that $P = P_0 M$ is determined by M, hence when we consider (standard) parabolics we will just mention the associated standard Levi's (and denote $r_{P,G}$ as $r_{M,G}$).

Definition 1. A representation $\pi \in \mathrm{Rep}(G)$ is supercuspidal if for every *proper* parabolic k-subgroup $\mathbb{P} \subset \mathbb{G}$, the Jacquet functor $r_{M,G}(\pi)$ is zero. In fact, it suffices to consider standard parabolics $\mathbb{P}$.

We now give a slightly different interpretation of supercuspidality in terms of a very important subgroup of G.

Let $X(\mathbb{G})$ denote the group of algebraic characters of G defined over F. We set

$$G^1 := \bigcap_{\chi \in X(\mathbb{G})} \ker |\chi|,$$

this is the subgroup of G consisting of elements sent to $\mathcal{O}^\times$ by every rational character χ.

Fact 1. *This subgroup has many interesting properties (see [2], remark 7.2.5):*

1. *G^1 contains every compact subgroup of G.*
2. *G^1 is an open, closed, normal, unimodular subgroup of G.*
3. *G/G^1 is isomorphic to $\mathbb{Z}^m$ where m is the k-rank of the maximal central k-torus of $\mathbb{G}$.*
4. *$G/(ZG^1)$ is finite, equivalently $Z/(ZG^1)$ is a finite-index subgroup in G/G^1.*
5. *$Z \cap G^1$ is compact.*

Example 1. When $G = \mathrm{GL}_n(F)$, we simply have $G^1 = \{g \in \mathrm{GL}_n(F) \mid \det g \in \mathcal{O}^\times\}$.

The following theorem partially underlines the importance of G^1.

Theorem 2. *Let $\pi \in \mathrm{Rep}(G)$. The following are equivalent:*

1. *π is supercuspidal.*
2. *$\mathrm{res}^G_{G^1}\pi$ is a finite representation, i.e. every matrix coefficient is compactly supported over G^1.*
3. *every matrix coefficient is compactly supported modulo the center $Z \subset G$.*

Conditions (2) and (3) are clearly very similar in spirit, and the proof of their equivalence boils down to the properties of G^1 mentioned above.
We will show a complete proof for (1) $\iff$ (3) only for $G = \mathrm{GL}_n$ as the general case, which is not much more complicated, needs a little bit of theory of root systems. For a general proof see [13], theorem VI.2.1.

Before we proceed to the proof, let's define an Hecke-algebra version of compact-support for matrix coefficients. Let $K \subset G$ be a compact open subgroup, $v \in V$ and consider

$$D_{v,K} : G \longrightarrow V \quad D_{v,K}(g) = \pi(e_K)\pi(g^{-1}).v = \int_G \pi(hg^{-1}).v \, d\mu_K(h)$$

where $d\mu_K$ is a Haar measure on K. Notice that $D_{v,K}$ has compact support if and only if $D_{v,K'}$ has compact support for any compact open subgroup K', so for the set of functions $\{D_{v,K}\}$ the notions of compact support and compact support modulo Z are independent of the choice of K.

Fact 3. *Every matrix coefficient is compactly supported (modulo the center Z) if and only if every $D_{v,K}$ is compactly supported (modulo Z).*

Proof. We prove the version for compact support, the other being very similar.
Let $m_{\lambda,v}$ be a matrix coefficient. Take a compact open subgroup K fixing λ, so it's easy to check that $\mathrm{Supp} m_{\lambda,v} \subset \mathrm{Supp} D_{v,K}$ which proves one direction.

Vice-versa, fix v and K; we want to find finitely many λ_i such that

$$\mathrm{Supp} D_{v,K} \subset \bigcup_{i=1}^{l} \mathrm{Supp} m_{\lambda_i,v}$$

. Obviously $D_{v,K}$ has image contained in V^K, hence for any nonzero vector in the image we can find $\lambda \in \widetilde{V}^K$ which does not kill it. Thus, it's enough to show that $\mathrm{Im}(D_{v,K})$ is finite-dimensional. Arguing by contradiction suppose it is not, so we can find a sequence $\{g_m\} \subset G$ such that $v_m = D_{v,K}(g_m)$ are countably many linearly independent vectors, and thus the set $\{g_m\}$ is not contained in any compact set.
Define a functional $\Lambda \in \widetilde{V}^K$ as $\Lambda(v_m) = 1$ and extend by zero on the rest of V. A quick check shows that

$$1 = \Lambda(v_m) = \mu(K) m_{\Lambda,v}(g_m)$$

and thus $\{g_m\} \subset \mathrm{Supp} m_{\Lambda,v}$, which contradicts the fact that $m_{\Lambda,v}$ is compactly supported. $\square$

Proof of the Theorem. We now show that π is supercuspidal if and only if each $D_{v,K}$ is compactly supported modulo Z.

Suppose first that π is supercuspidal and fix $v \in V$. We can pick a small enough congruence subgroup $K = K_i = \mathrm{id}_n + \varpi^i \mathrm{M}_n(\mathcal{O})$ such that K_i fixes v.
Recall the Cartan decomposition for $\mathrm{GL}_n(F)$: denoting $K_0 = \mathrm{GL}_n(\mathcal{O})$ we have

$$G := GL_n(F) = K_0 \Lambda^+ K_0 \text{ where } \Lambda^+ = \left\{ \mathrm{diag}(\varpi^{l_1}, \dots, \varpi^{l_n}) \text{ such that } l_1 \geqslant \dots \geqslant l_n \right\}.$$

If $\{x_1, \dots, x_r\}$ is a choice of representatives for $K \backslash K_0$, we have then

$$G = \bigcup_{i,j=1}^{r} x_i K \Lambda^+ K x_j,$$

and (using the notation $a(g) = \pi(e_K) * \pi(\delta_g) * \pi(e_K)$ so that $\pi(a(g)) = \pi(e_K)\pi(g)\pi(e_K))$ it suffices to show that

$$\forall i, j \quad \lambda \mapsto \pi(x_i)\pi(a(\lambda))\pi(x_j)v \text{ has compact support modulo } Z \text{ in } \Lambda^+$$

hence that

$$\lambda \mapsto \pi(a(\lambda))v \text{ has compact support modulo } Z \text{ in } \Lambda^+.$$

Now we make a brief digression concerning the structure of GL_n. Suppose that the minimal parabolic fixed is the standard Borel of upper triangular matrices. Every standard unipotent subgroup N admits a description

$$N = N_\lambda = \left\{ x \in G \,|\, \lim_{n\to\infty} \mathrm{Ad}(\lambda^n)x = 1 \right\}$$

for some $\lambda \in \Lambda^+$, and vice-versa any $\lambda = (l_1, \dots, l_n) \in \Lambda^+$ determines a partition of n (by means of "consecutive equal l_i's") and thus a unique standard unipotent subgroup, which is exactly N_λ. If we fix a standard parabolic $P = MN$, our congruence subgroup K admits an Iwahori factorization with respect to P: $K = (K \cap N)(K \cap M)(K \cap N^-)$ and we have then a filtration of $N = N_\lambda$ by compact subgroups

$$\bigcup_n \lambda^{-n}(K \cap N)\lambda^n = N$$

It is then not hard to show that

$$\bigcup_n \ker a(\lambda^n) \cap V^K = V(N) \cap V^K.$$

The latest equivalence is the main ingredient for the rest of the proof. Fix then a basis $\nu_1, \dots, \nu_n$ in Λ^+ and let $\lambda = \sum m_i \nu_i$. By the cuspidality assumption, for every nontrivial unipotent $N = N_\lambda \neq 1$ we have $V = V(N_\lambda)$ and hence

$$V^K = V^K \cap V(N_\lambda) = \bigcup_n \ker a(\lambda^n) \cap V^K.$$

Thus for every $w \in V^K$ there is a $k_{i,w}$ such that $\pi(a(\nu_i^k)).w = 0$ for every $k \geqslant k_{i,w}$. As $a(\lambda) = \prod_i a(\nu_i^{m_i})$, taking $L = \max_i k_{i,w}$ and $w = \pi(e_K)v$ gives that

$$\pi(a(\lambda)).v = 0 \text{ if any } m_i \geqslant L.$$

We bounded the m_i's above, now it remains to bound them below in order to get the claim. We can only get a bound below by using the 'modulo Z' assumption: in fact up to multiplication by an element of Z, each $\lambda \in \Lambda^+$ has every coefficient $m_i \geqslant 0$. This concludes the first part of the proof.

Vice-versa, assume that every $D_{v,K}$ is compactly supported modulo the center Z. By reversing the argument above, this is equivalent to

$$\lambda \mapsto \pi(a(\lambda)).v \text{ having compact-modulo-center support in } \Lambda^+.$$

But for any fixed non-central $\lambda \in \Lambda^+$, the sequence (λ^n) eventually leaves every compact-modulo-center subset C of Λ^+, because each such subset $C = C'Z$ has $C' \subset \Lambda^+$ compact, hence

$$\max_{\lambda \in C'}(l_i - l_j) < M = M(C') \quad \text{denoting } \lambda = \operatorname{diag}(\varpi^{l_1}, \dots, \varpi^{l_n}).$$

Z preserves such differences $l_i - l_j$, thus preserves $M(C') =: M(C)$, but the sequence (λ^n) does not admit such an upper bound.
Thus $\pi(a(\lambda^n)).v = 0$ for any n large enough, i.e. $V^K = \bigcup_n \ker a(\lambda^n)$. Paired with

$$\bigcup_n \ker a(\lambda^n) \cap V^K = V(N_\lambda) \cap V^K,$$

this shows

$$V(N_\lambda) \cap V^K = V^K.$$

This is true for every congruence subgroup K_i, because each such subgroup admits an Iwahori factorization with respect to $P = MN$.
By smoothness $V = \bigcup_i V^{K_i}$ and hence $V = V(N)$ whenever $N = N_\lambda$ for a non-central $\lambda \in \Lambda^+$. But central elements $\lambda \in \Lambda^+ \cap Z(G)$ give $N_\lambda = 1_G$, so that the only non-trivial Jacquet module is $r_{G,G}(\pi)$. This shows π is supercuspidal. □

We also have the following easy remark:

Remark. Suppose π and $\tilde{\pi}$ are irreducible. Then to prove that π is supercuspidal it's enough to show one nonzero matrix compactly supported modulo the center.
In fact, if $m_{\lambda,v}$ is compactly supported modulo the center and nonzero, then any other $m_{\lambda',v'}$ is a finite linear combination of matrix coefficients $m_{\tilde{\pi}(g_j')(\lambda),\pi(g_i)(v)}$: each of these is a translate of $m_{\lambda,v}$ and thus still has compact support modulo the center.

Definition 2. A representation $\pi \in \operatorname{Rep}(G)$ is cuspidal if it is admissible and supercuspidal.

Example 2. Every irreducible supercuspidal representation is cuspidal.

This follows from the following fact.

Theorem 4 (Jacquet, 1975)**.** *Let $\pi \in \mathrm{Rep}(G)$ be irreducible. Then it is admissibile.*

Proof. It has been proven before that π is a subrepresentation of some $\mathrm{Ind}_{G,M}\rho$ for ρ a supercuspidal representation of the Levi M. As parabolic induction preserves admissibility, it suffices to prove the claim when π is supercuspidal.
We use the characterization of supercuspidal via matrix coefficients. Arguing by contradiction, suppose it is not admissibile and hence $\dim_{\mathbb{C}} V^K = \infty$ for some open compact $K \subset G$. Then

$$(\widetilde{V})^K \cong (V^K)^* = \mathrm{Hom}_{\mathbb{C}}(V^K, \mathbb{C}) \text{ has uncountable dimension.}$$

Fix a nonzero $v \in V^K$ and (denoting by $C(\pi)$ the space of matrix coefficients of π) consider the map

$$(\widetilde{V})^K \xrightarrow{\Gamma_v} C(\pi) \quad \lambda \mapsto m_{\lambda,v}$$

which is injective as Gv spans V by irreducibility.
Denoting by ω_π the central character of π, the image is a space of functions $f : G \longrightarrow \mathbb{C}$ such that

$$f(zkgk') = \omega_\pi(z)f(g),$$

Thus on a coset $ZKgK$ the value of f is either constantly zero or always nonzero. Due to Cartan decomposition we have

$$Z\backslash G = \bigcup_g Z\backslash ZKgK,$$

a disjoint union of countably many such cosets (again by Cartan decomposition $G = K\Lambda^+K$ with Λ^+ countable). Moreover, f is compactly supported modulo the center, hence f is supported on finitely many such cosets. So the vector space of such f's has countable dimension, which gives a contradiction due to injectivity of Γ_v. □

Here's why we like supercuspidal representations:

Fact 5. *Let $\pi \in \mathrm{Irrep}(G)$, then there exists a parabolic $P = MN$ and an irreducible supercuspidal $\sigma \in \mathrm{Irrep}(M)$ such that π is a subrepresentation of $\mathrm{Ind}_{G,M}\sigma$.*

This has been shown before, except the fact that σ is irreducible. But now $r_{M,G}\pi$ is finitely generated, so it has an irreducible quotient σ by a Zorn's lemma argument, hence

$$0 \neq \mathrm{Hom}_M(r_{M,G}\pi, \sigma) = \mathrm{Hom}_G(\pi, \mathrm{Ind}_{G,M}\sigma).$$

Thus any irreducible representations embeds into a parabolically-induced representation from an irreducible supercuspidal from some Levi! So to "generate" representations for G it suffices to generate the supercuspidal ones for all its Levi subgroup (G included), and the general philosophy is that we should mainly worry about how to create supercuspidal reps.

As an example, consider $G = \mathrm{GL}_n$: if as minimal parabolic we take the Borel subgroup of upper triangular matrices, we have seen that choosing a standard Levi corresponds to choosing an ordered partition of n. Then we have the following result (see [1], theorem 4.2):

Theorem 6. *Let $(n_1, \ldots, n_k)$ be an ordered partition of n and $\rho_i \in \operatorname{Irrep}(\mathrm{GL}_{n_i})$ be cuspidal representations. Let M be the standard Levi corresponding to such partition and denote*

$$\pi = \operatorname{Ind}_{G,M}(\rho_1 \otimes \ldots \otimes \rho_k).$$

Then π is irreducible if and only if $\rho_i \not\cong \nu\rho_j$ for every i, j, where $\nu(g) = |\det g|$.

Remark. Clearly $\rho \not\cong \nu\rho$ because the center of GL_m has elements whose determinant is not a unit in $\mathcal{O}$, and thus the two representations have different central characters.

Example 3. Take $n = 6$. If we pick the partition $(1, 2, 3)$, or any permutation of it, then obviously any choice of ρ_i's will give rise to an irreducible representations, as the condition $\rho_i \cong \nu\rho_j$ implies in particular $n_i = n_j$.
Similarly, we can take the partition $(2, 2, 2)$ and $\rho_i = \rho$ for each i. By the remark, the induced representation is also irreducible. We will see later what an irreducible cuspidal representation of GL_2 looks like.

3 Construction of a supercuspidal representation

We will construct an irreducible supercuspidal representation for $\mathbb{G} = \mathrm{GL}_2$.

We start by proving a very general result.

Theorem 7. *Let H be an open subgroup of G, containing the center Z and compact modulo the center. Let $\sigma \in \operatorname{Rep}(H)$. If $\pi = \text{c-Ind}_H^G \sigma$ is irreducible and admissibile, then it is supercuspidal.*

Proof. Denote the representation space of σ by V.
As π is irreducible and admissibile, the contragredient representation $\widetilde{\pi}$ is also irreducible (since subrepresentations of π and $\widetilde{\pi}$ correspond to eachother by taking orthogonal complements, in case π, and thus $\widetilde{\pi}$, are admissibile).
It is then enough to find one nonzero matrix coefficient compactly supported modulo the center.
Let then $v \in V$ and $\lambda \in \widetilde{V}$ be nonzero vectors such that $\lambda(v) \neq 0$, define $f_v \in \text{c-Ind}_H^G V$ as

$$f_v(g) = \begin{cases} \sigma(h)v & \text{if } h = g \in H \\ 0 & \text{otherwise} \end{cases}$$

and similarly define $f_\lambda \in \text{c-Ind}_H^G \widetilde{V}$.
The composition map

$$\text{c-Ind}_H^G V \longrightarrow V \longrightarrow \mathbb{C} \quad g \mapsto f(1) \mapsto \langle f_\lambda(1), f(1)\rangle$$

is clearly a smooth functional on $\text{c-Ind}_H^G V$, hence f_λ can be regarded as an element of $\widetilde{\pi}$, and as such is nonzero because

$$\langle f_\lambda(1), f_v(1)\rangle = \langle \lambda, v\rangle \neq 0.$$

We thus get a matrix coefficient for π:

$$m_{f_\lambda, f_v}(g) = \langle f_\lambda, \pi(g) f_v\rangle = \langle f_\lambda(1), (\pi(g) f_v)(1)\rangle = \langle \lambda, f_v(g)\rangle.$$

This is nonzero when $g = 1$, and has support contained in the support of f_v, which is H that is compact modulo Z by assumption. Hence this matrix coefficient is compactly supported modulo the center. $\square$

Denoting by $k = \mathcal{O}/\mathfrak{p}$ the residue field of F, consider now an irreducible representation (σ, V) of the group $\mathrm{GL}_2(k)$. Notice that this group is finite. By means of the canonical quotient map

$$\mathrm{GL}_2(\mathcal{O}) \twoheadrightarrow \mathrm{GL}_2(k)$$

we inflate σ to a representation of the maximal compact subgroup $K = \mathrm{GL}_2(\mathcal{O})$.

The central character of the finite group representation is lifted to a character χ of $\mathcal{O}^\times$ which we extend to a unitary character of $F^\times$ (still denoted by χ). Then we can naturally consider σ as a representation of ZK where $z = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \in Z(G) \cong F^\times$ acts by $\chi(x)$. Define the compactly-induced representation $\pi = \text{c-Ind}_{ZK}^G \sigma$ and denote by $W = \text{c-Ind}_{ZK}^G(V)$ the representation space.

Theorem 8 (see [5], section 4.8)**.** *Suppose that the following assumption on σ holds:*

$$\sigma \text{ does not admit a nonzero vector fixed by } N(k) = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$$

the (unique) standard unipotent subgroup of $\mathrm{GL}_2(k)$*. Then π is an irreducible supercuspidal representation of* $\mathrm{GL}_2(F)$*.*

Proof. Notice that as σ is a finite-dimensional representation, it is unitarizable, so let $(,)$ be some positive-definite Hermitian inner product on V which is ZK-invariant. Define then

$$\langle f_1, f_2 \rangle := \int_G (f_1(g), f_2(g)) dg \quad \forall f_i \in W,$$

it is not hard to show that this gives a G-invariant, positive-definite Hermitian inner product on W, thus making π unitary.
It suffices to show that π is irreducible, and then from a theorem in the previous section it follows that π is admissible and from the previous theorem we get that π is supercuspidal. As we know that π is unitary, to show that it is irreducible it suffices to show that $\mathrm{Hom}_G(W, W)$ has dimension 1: then if $U \subset W$ were a nontrivial subrepresentation, the orthogonal projection onto U is an element of such a Hom-space, so it suffices to show that it's not the identity, i.e. that there's some element of W orthogonal to U.
Decompose (π, W), into its K-isotypic components. By compactness of K, each isotypic component is finite dimensional and we have algebraic direct sums

$$\bigoplus_\tau W(\tau) = W \supset U = \bigoplus_\tau U(\tau)$$

where $S(\tau)$ is the τ-isotypic component for a representation space S. As $W \neq U$, there is some $W(\tau)$ strictly containing $U(\tau)$, so pick a vector in the orthogonal complement of $U(\tau)$ with respect to $W(\tau)$. This is orthogonal to the whole U as the spaces $W(\tau)$ and $W(\tau')$ are mutually orthogonal if $\tau \ncong \tau'$.
We have $\text{c-Ind}_{ZK}^G(V) \subset \mathrm{Ind}_{ZK}^G(V)$ so it suffices to show that

$$\dim \mathrm{Hom}_G\left(W, \mathrm{Ind}_{ZK}^G(V)\right) = 1$$

and by Frobenius reciprocity

$$\mathrm{Hom}_G\left(W, \mathrm{Ind}_{ZK}^G(V)\right) \cong \mathrm{Hom}_{ZK}\left(\mathrm{res}\, W, V\right).$$

Now we use a Mackey theory argument: first of all, fixing a set of representatives for $K\backslash G/ZK$, the Cartan decomposition makes it clear that we can pick them to be

$$\gamma_n = \begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix} \quad \forall n \in \mathbb{N}.$$

Then consider the subgroups $S^n = K \cap \gamma_n^{-1} K \gamma_n$ acting on $V^n = V$ as $\sigma^n(g) := \sigma(\gamma_n g \gamma_n^{-1})$. We have then a decomposition

$$\mathrm{Hom}\left(\mathrm{res}\,(W), V\right) \cong \bigoplus_{n \in \mathbb{N}} \mathrm{Hom}_{S^n}\left(V, V^n\right)$$

so it suffices to show that all these spaces are zero-dimensional, except one which is 1-dimensional. Obviously for $n = 0$ we obtain $\mathrm{Hom}_K(V, V)$ which is one-dimensional as (σ, V) is irreducible. If $n \geqslant 1$, notice that $N(\mathcal{O}) \subset K \cap \gamma_n^{-1} K \gamma_n = S^n$ and then if $\phi \in \mathrm{Hom}_{S^n}(V, V^n)$ we have for each $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in N(\mathcal{O})$ that

$$\phi\left(\sigma\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right).v\right) = \sigma^n\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right).\phi(v) = \sigma\left(\begin{pmatrix} 1 & \varpi^n a \\ 0 & 1 \end{pmatrix}\right).\phi(v) = \phi(v)$$

because σ has been lifted from a representation of $\mathrm{GL}_2(F)$, and thus acts trivially on $N(\mathfrak{p})$, the subgroup of unipotent upper triangular matrices with upper-right element in $\mathfrak{p}$.
We finally use the assumption on σ, which tells us that $v \neq \sigma\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right).v$ for some $a \in \mathcal{O}$ (in fact, for some coset $a + \mathfrak{p} \in \mathcal{O}/\mathfrak{p} \cong k$). But every $\phi \in \mathrm{Hom}_{S^n}(V, V^n)$ acts in the same way on the two vectors, and this is only possible if each $\phi = 0$, ultimately proving the claim. $\square$

The only thing that remains to do is provide a representation σ of $\mathrm{GL}_2(k)$ having the required property:

$$\sigma \text{ does not admit a nonzero vector fixed by } N(k) = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}.$$

Consider the (unique) quadratic extension $k' \supset k$. Choosing a k-basis for k' makes it isomorphic to $k \oplus k$ as a vector space, and thus embeds $(k')^\times$ into $\mathrm{GL}_2(k)$ in such a way that $Z(G) \cong k^\times \subset (k')^\times$. Fix now θ a multiplicative character of $(k')^\times$ that is regular (i.e. *not* invariant by the nontrivial element of $\mathrm{Gal}(k'|k)$) and ψ a nontrivial additive character of k and define

$$\theta_\psi : ZN \longrightarrow \mathbb{C} \quad \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mapsto \theta(a)\psi(u).$$

Fact 9. *Consider the virtual representation*

$$\pi = \mathrm{Ind}_{ZN}^G \theta_\psi - \mathrm{Ind}_{(k')^\times}^G \theta.$$

Then this is an irreducible representation of $\mathrm{GL}_2(k)$ *and does not contain the trivial character of* $N(k)$ *(equivalently, does not admit nonzero vectors fixed under* $N(k)$*).*

The proof is just representation theory of finite groups: use Mackey's formula and the knowledge of conjugacy classes in $\mathrm{GL}_2(k)$ to compute the character of π and prove the two claims.

4 The Local Langlands Conjecture for GL_n

In this section, we give an overview of the Local Langlands conjecture for GL_n over a non-archimedean local field, and connect it to our study of the supercuspidal representations. From now on, $G = \mathrm{GL}_n(F)$ unless otherwise specified.

Suppose first of all that $n = 1$, then $\mathrm{GL}_n(F) = F^\times$. An admissibile irreducible representation is just an homomorphism $\phi : F^\times \longrightarrow \mathbb{C}^\times$ with kernel containing $1 + \varpi^m \mathcal{O}$ for some m. This is equivalent to saying that ϕ is continuous with respect to the discrete topology on $\mathbb{C}^\times$.
Now we notice that, due to local class field theory, $F^\times \cong W_F^{\mathrm{ab}}$ (the abelianization of the Weil group of F), so that ϕ is just a complex representation of W_F^{ab} which is continuous with respect to the discrete topology on $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$.
The Local Langlands Conjecture for GL_n concerns a very similar statement for every n, not just $n = 1$. In practice, local class field theory gives us the conjecture for $n = 1$.

Let's now recall the main ingredients.
The absolute Galois group of F surjects canonically onto the absolute Galois group of the residue field $k_F = k$:

$$v : \mathrm{Gal}_F = \mathrm{Gal}(\overline{F}|F) \longrightarrow \mathrm{Gal}(\overline{k}|k) = \mathrm{Gal}_k \longrightarrow 0$$

and one defines the *inertia subgroup* I_F to be the kernel of this map. As k is finite of cardinality q, we know very well its finite extensions and it's easy to check that $\mathrm{Gal}(\overline{k}|k) \cong \hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}$ embeds canonically into this group: in fact $\mathbb{Z} \subset \mathrm{Gal}_k$ is generated by the (arithmetic) Frobenius $x \mapsto x^{|k|}$.

Definition 3. We define the Weil group to be the preimage of $\mathbb{Z}$ under the above surjective map:

$$W_F = v^{-1}(\mathbb{Z})$$

and so we have an exact sequence

$$1 \longrightarrow I_F \longrightarrow W_F \xrightarrow{v} \mathbb{Z} \longrightarrow 0$$

However, we do not give W_F the induced topology, but we give it the topology such that I_F is open in W_F with its usual (profinite) topology and v is continuous once $\mathbb{Z}$ has the discrete topology.
Next, fix a *geometric Frobenius element* $\Phi \in W_F$ i.e. an element such that its image in Gal_k is the inverse of the arithmetic Frobenius. We can take the map v as sending Φ to 1. The second condition on the topology of W_F is equivalent to saying that multiplication by Φ is a homeomorphism.

We also get a map

$$\|\cdot\| : W_F \longrightarrow \mathbb{R}_+^\times \quad x \mapsto \|x\| = q^{-v(x)}$$

so that in particular $\|\Phi\| = q^{-1}$.

Theorem 10 (Local reciprocity map). *We have a topological isomorphism*

$$\theta_F : F^\times \longrightarrow W_F^{\mathrm{ab}}.$$

Moreover, this sends $\mathcal{O}^\times$ onto the inertia subgroup I_F and is compatible with the valuation on $F^\times$ and the map v defined above.

This lets us express $\|\cdot\|$ in a different way: it is the composition of

$$W_F \twoheadrightarrow W_F^{\mathrm{ab}} \xrightarrow{\theta_F^{-1}} F^\times \xrightarrow{|\cdot|_F} \mathbb{R}_+^\times.$$

Now we encounter a small technical problem. Due to the particular nature of local fields, representations of the Weil group are not exactly what will parametrize representations of $\mathrm{GL}_n(F)$. We need a slight twist to take into account, for instance, the existence of special representations i.e. of irreducible subquotients of a non-irreducible principal series.

Definition 4. Define the Weil-Deligne group

$$W_F' = W_F \ltimes \mathbb{C}$$

where W_F acts on $\mathbb{C}$ as $wxw^{-1} = \|w\|x$.
An *n-dimensional semisimple Weil-Deligne representation* (also called an *admissible homomorphism* or *Frobenius-semisimple representation*) of W_F' is a pair (ρ, N) such that

- $$\rho : W_F \longrightarrow \mathrm{GL}_n(\mathbb{C})$$
  is a continuous homomorphism (with respect to the discrete topology on $\mathrm{GL}_n(\mathbb{C})$) such that the image consists of semisimple matrices; in fact, it's enough to check that the image of any Frobenius element is semisimple.

- N is a nilpotent endomorphism of $\mathbb{C}^n$ such that
  $$\rho(w)N\rho(w)^{-1} = \|w\|N \quad \forall w \in W_F.$$

Notice that when $n = 1$, the nilpotent endomorphism is just 0 and thus we recover a continuous representation of the Weil group W_F, i.e. an admissibile character of $F^\times$ via the local reciprocity map.

Example 4. This example will come up later, so it's both non-trivial (because $N \neq 0$) and particularly important.
Fix n and define the following n-dimensional Weil-Deligne representation $\mathrm{Sp}(n) = (\rho, N)$. Setting $V = \mathbb{C}e_1 \oplus \ldots \oplus \mathbb{C}e_n$ we put

$$\rho(w)e_i = \|w\|^i e_i$$

and

$$Ne_i = e_{i+1}\, \forall i < n \quad Ne_n = 0.$$

Fact 11 (see [16], section 4.1.5)**.** *Every indecomposable representation of W_F' is of the form $\phi \otimes$ $\mathrm{Sp}(n)$ with ϕ irreducible.*

The tensor product of Weil-Deligne representations is defined as

$$(\rho_1, N_1) \otimes (\rho_2, N_2) = (\rho_1 \otimes \rho_2, N_1 \otimes 1 + 1 \otimes N_2).$$

This definition, at least the one for the nilpotent endomorphism, may seem unreasonable but it is a theorem of Grothendieck ([15], Appendix) that any l-adic representation ρ of G_F admits an open subgroup where ρ acts by unipotent matrices, and hence as the exponential of a nilpotent matrix

N. Taking the tensor product of two such representations ρ_i with respective nilpotent matrices N_i give us a representation where (on the appropriate open subgroup) the nilpotent endomorphism is $N_1 \otimes 1 + 1 \otimes N_2$.

We will state now the Local Langlands Conjecture for GL_n over a non-archimedean field, then we will proceed to define the various objects involved with it.
Denote by $\mathrm{Rep}_n(W'_F)$ the equivalence classes of n-dimensional semisimple Weil-Deligne representations, up to isomorphism, and similarly by $\mathcal{A}_n(F)$ the category of smooth irreducible complex representations of $\mathrm{GL}_n(F)$. Then we have

Theorem 12 (Local Langlands Conjecture for $\mathrm{GL}_n(F)$)**.** *With F fixed, there exists a collection of bijective maps $\mathcal{A}_n(F) \longrightarrow \mathrm{Rep}_n(W'_F)$ sending π to ρ_π having the following properties:*

1. *For $n = 1$ the bijection is the local class field theory case as above;*

2. *For any $\pi \in \mathcal{A}_n(F)$ and $\pi' \in \mathcal{A}_{n'}(F)$ we have a correspondence of L-functions and ε-factors: this means that to each Frobenius-semisimple representation of W'_F we can associate an L-function and an ε-factor (satisfying nice analytical properties), and similarly we associate to each smooth representation of $GL_n(F)$ an L-function and an ε-factor; then $\pi \in \mathcal{A}_n(F)$ corresponds to $\rho \in \mathrm{Rep}_n(W'_k)$ if and only if the respective L-functions coincide (and then the ε-factors will coincide as well).*
   Moreover, tensor products preserve this:
   $$L(s, \rho_\pi \otimes \rho_{\pi'}) = L(s, \pi \times \pi') \quad \varepsilon(s, \rho_\pi \otimes \rho_{\pi'}, \psi) = \varepsilon(s, \pi \times \pi', \psi),$$
   where ψ is a nontrivial additive character of F;

3. *For any $\pi \in \mathcal{A}_n(F)$, $\det \rho_\pi$ corresponds to the central character ω_π under the local class field theory bijection;*

4. *The bijection commutes with taking contragradients;*

5. *The bijection commutes with twisting by multiplicative characters of $F^\times$, i.e.*
   $$\rho_{\pi \otimes \chi} = \rho_\pi \otimes \chi$$
   for any character χ of $F^\times$.

6. *Supercuspidal representations of $\mathrm{GL}_n(F)$ correspond to irreducible semisimple Weil-Deligne representations, and in fact each such natural correspondence respecting the above five properties give a natural correspondence of $\mathcal{A}(F)$ with $\mathrm{Rep}(W'_F)$.*

Let's now define the objects involved in the theorem, starting with L-functions, that are easier to define.

Definition 5 (L-function for a Weil-Deligne representation)**.** Let $\phi = (\rho, N)$ be an n-dimensional semisimple Weil-Deligne representation. Then its associated L-function is

$$L(s, \phi) := \det \left(1 - \rho(\Phi)|_{V_N^{I_F}} q^{-s}\right)^{-1}$$

where $V_N^{I_F}$ is the subspace of $\ker N$ fixed by $\rho(I_F)$, which is a finite group as I_K is profinite and $\mathrm{GL}_n(\mathbb{C})$ contains no small subgroups.

Definition 6 (L-function for a smooth representation of $\mathrm{GL}_n(F)$)**.** We follow [11], section 1, for this definition.
Fix an admissibile representation $\pi \in \mathrm{Rep}(\mathrm{GL}_n(F))$. Let Φ be a Bruhat-Schwartz function on $\mathrm{Mat}_n(F)$ (i.e. locally constant and compactly supported), f a matrix coefficient of π. The *zeta integral* associated to such data is

$$Z(\Phi, s, f) = \int_G \Phi(g)|\det g|^s f(g) d\mu(g)$$

where s is a complex variable and $d\mu$ any fixed Haar measure on G.
If π is irreducible, the integrals above converge absolutely for $\Re s > s_0$ for some $s_0 \in \mathbb{C}$, and they are represented by rational functions in q^{-s}. As such rational functions, they admit a common denominator independent of Φ and f.
One proves that the subvector space

$$I(\pi) := \mathrm{span}_{\mathbb{C}} Z\left(\Phi, s + \frac{n-1}{2}, f\right) \subset \mathbb{C}\left(q^{-s}\right)$$

as Φ and f vary, is a fractional ideal of $\mathbb{C}\left[q^s, q^{-s}\right]$ and admits a generator of the form $P\left(q^{-s}\right)^{-1}$ for P a complex polynomial. Normalize it so that $P(0) = 1$ and set

$$L(s, \pi) := P(q^{-s})^{-1}.$$

This is a quite involved definition, but in many important cases the L-function associated to a representation π is in fact very simple.

Example 5 ([16], example 1.3.5)**.** If π is supercuspidal and $n > 2$, then $L(s, \pi) = 1$.

Example 6 ([9], chapter 3)**.** Let $\pi = \mathrm{Ind}_{G,T}(\chi)$ be the parabolic induction from a character χ of the diagonal torus T. If π is unramified with Satake parameter t_χ, then $L(s, \pi) = \det\left(1 - q^{-s} t_\chi\right)^{-1}$.

We are now ready to define the ε-factor for a representation $\pi \in \mathrm{Rep}(G)$.

Definition 7. Fix $\psi \neq 1$ a nontrivial additive character of F and define the Fourier transform of a Bruhat-Schwartz function Φ as

$$\widehat{\Phi}(x) := \int_{\mathrm{Mat}_n(F)} \Phi(y)\psi\left(\mathrm{Tr}(yx)\right) \mathrm{d}\, y$$

where $\mathrm{d}\, y$ is the self-dual Haar measure on $\mathrm{Mat}_n(F)$. Then there exists a rational function $\gamma = \gamma(s, \pi, \psi)$ such that

$$Z\left(\widehat{\Phi}, s + \frac{n-1}{2}, \check{f}\right) = \gamma(s, \pi, \psi) Z\left(\Phi, s, f\right)$$

for every f and every Φ, where we denote $\check{f}(g) := f(g^{-1})$. Therefore we set

$$\varepsilon(s, \pi, \psi) := \gamma(s, \pi, \psi)\frac{L(s, \pi)}{L(1 - s, \widetilde{\pi})}$$

and it turns out that $\varepsilon(s, \pi, \psi)$ is a monomial in q^{-s}.

Remark. We focus on ε rather than γ because it can be proven to have a simpler form (namely it is a monomial in q^{-s}) and because it satisfies some very nice functional relations mirroring the L-function; e.g. if α is the modulus character of F we have

$$L(s, \pi \otimes \alpha^t) = L(s+t, \pi) \quad \varepsilon(s, \pi \otimes \alpha^t, \psi) = \varepsilon(s+t, \pi, \psi).$$

Finally, it remains to define the ε-factor for a Weil-Deligne representation. This is not at all as easy as the above definitions, and in fact besides a few particular cases there is no explicit definition.
Let $\phi = (\rho, N)$ be an n-dimensional semisimple Weil-Deligne representation. We reduce the problem to defining an ε-factor for representation of the usual Weil group by setting

$$\varepsilon(s, \phi, \psi) := \varepsilon(s, \rho, \psi) \det \left(-\Phi|_{V^{I_F}/V_N^{I_F}} \right)$$

where Φ is again a geometric Frobenius element in W_F and $\varepsilon(s, \rho, \psi)$ is the ε-factor associated to a semisimple representation of W_F which we now go on to "define", following the approach in [12], where the following theorem is Theorem A of the Introduction. Another (simpler) proof of this theorem can be found in [8], chapter 4.

Theorem 13 (Existence of the Local Constant)**.** *Fix F and a nontrivial additive character $\psi \neq 1$ of F. Then there is a unique way to associate to each finite separable extension $E \supset F$ a complex number $\lambda(E|F, \psi)$ and to each (equivalence class of) complex semisimple representation ρ of W_E a complex number $\varepsilon(\rho, \psi_{E|F})$ (where $\psi_{E|F} = \psi \circ \mathrm{Tr}_{E|F}$) such that*

1. *if $\dim \rho = 1$ and then $\rho = \chi_E$ is a character of $E^\times$, then $\varepsilon(\rho, \psi_{E|F})$ corresponds to the usual ε-factor defined in the abelian case;*

2. *ε is multiplicative on short exact sequences, i.e.*

$$\varepsilon(\rho_1 \oplus \rho_2, \psi_{E|F}) = \varepsilon(\rho_1, \psi_{E|F}) \varepsilon(\rho_2, \psi_{E|F});$$

3. *if ν is a representation of W_F induced by a representation ρ of W_E (there's a canonical embedding $W_E \subset W_F$), then*

$$\varepsilon(\nu, \psi) = \lambda(E|F, \psi)^{\dim \rho} \varepsilon(\rho, \psi_{E|F}).$$

Given the theorem, denote as above by α the modulus of k, i.e. $\alpha(x) = |x|_k$. We set

$$\varepsilon(s, \rho, \psi) := \varepsilon(\alpha^{s-\frac{1}{2}} \otimes \rho, \psi)$$

and we finally managed to "define" our ε-factor for a representation of W_E.

Before giving a few examples, let's remark better what condition 6 of the LLC means. Suppose we have defined for every n a correspondence between irreducible supercuspidal representations of $\mathrm{GL}_n(F)$ and irreducible n-dimensional semisimple Weil-Deligne representations. The fantastic feature of the LLC is that it respects parabolic induction, in the sense that supercuspidal representations build up irreducible representations on the representation side, just with the same pattern as irreducible Weil-Deligne representation build up reducible ones on the Galois side. Let's see how (we roughly follow [9] here, section 4.2).

Let $\sigma \in \mathrm{Irrep}(\mathrm{GL}_m(F))$ be supercuspidal and set $n = mr$. Denote by $\sigma(s) = \sigma \otimes |\det|_F^s$ for every complex number s. We call a *segment* a representation

$$\Delta = \sigma \otimes \sigma(1) \otimes \ldots \otimes \sigma(r-1)$$

of $\mathrm{GL}_n(F)$, with the obvious embedding of $\times_{i=1}^r \mathrm{GL}_m(F) \subset \mathrm{GL}_n(F)$.
Two segments $\Delta_1 = \sigma_1 \otimes \ldots \otimes \sigma_1(r_1 - 1)$ and $\Delta_2 = \sigma_2 \otimes \ldots \otimes \sigma_2(r_2 - 1)$ are *linked* if none contains the other and the union is still a (necessarily larger) segment. We say Δ_1 *precedes* Δ_2 if they are linked and $\sigma_2 = \sigma_1(k)$ for some positive integer k.

Theorem 14 (Langlands quotient theorem)**.** *1. For every segment Δ the parabolically induced representation $\mathrm{Ind}_{G,P}(\Delta)$ has a unique irreducible quotient $Q(\Delta)$, called the* Langlands quotient*. Moreover, this is essentially square-integrable, i.e. up to twisting by a character of $F^\times$ every matrix coefficient is in $L^2(Z\backslash G)$.*

2. *Moreover, every square integrable representation π of G has the form $Q(\Delta)$ for some $\Delta = \sigma \otimes \ldots \otimes \sigma(r-1)$ with $\sigma\left(\frac{r-1}{2}\right)$ unitary.*

3. *In general, given segments $\Delta_1, \ldots, \Delta_r$ such that if $i < j$ then Δ_i does not precede Δ_j, then the induced representation*

$$I_P^G\left(Q(\Delta_1) \otimes \ldots Q(\Delta_r)\right)$$

admits a unique irreducible quotient $Q(\Delta_1, \ldots, \Delta_r)$ which we again call the Langlands quotient.

4. *Finally, every irreducible admissible representation of $\mathrm{GL}_n(F)$ is isomorphic to some $Q(\Delta_1, \ldots, \Delta_r)$.*

These results are basically due to Bernstein and Zelevinski.

Suppose then that the bijection between irreducible Weil-Deligne reps and irreducible supercuspidals has been built:

$$\rho \mapsto \pi_\rho$$

. Now if $\rho \otimes \mathrm{Sp}(r)$ is an indecomposible representation of W'_F, it is natural to consider the segment

$$\Delta = \pi_\rho \otimes \ldots \otimes \pi_\rho(r-1)$$

and thus thanks to the previous theorem we have a natural association

$$\rho \otimes \mathrm{Sp}(r) \mapsto Q(\Delta).$$

Finally, suppose we have a semisimple Weil-Deligne representation that we decomposed in indecomposable components

$$\rho = (\rho_1 \otimes \mathrm{Sp}(r_1)) \oplus \ldots \oplus (\rho_m \otimes \mathrm{Sp}(r_m))$$

Then if Δ_i is the i-th segment given by the indecomposable representation $\rho_i \otimes \mathrm{Sp}(r_i)$ as above, we have the natural association

$$\rho \mapsto Q(\Delta_1, \ldots, \Delta_m).$$

Example 7. This example is meant to (partially) answer a question by Iurie.
Consider the Weil-Deligne representation $\mathrm{Sp}(n)$ defined as in example 4, what's the correspondent admissibile representation? It is easy to see that $\mathrm{Sp}(n)$ is indecomposable but very far from being irreducible, e.g. using the notation of example 4 the span of $e_j, \ldots, e_n$ for each $1 < j \leqslant n$ is a proper subrepresentation.
Using the procedure mentioned above, we associate to $\mathrm{Sp}(n)$ the Langlands quotient of the segment

$$\Delta = 1 \otimes \ldots \otimes 1(n-1) = 1 \otimes |\cdot|_F^1 \otimes \ldots \otimes |\cdot|_F^{n-1},$$

(denoting by 1 the trivial representation) that is, $\pi_{\mathrm{Sp}(n)} = Q(\Delta)$ is the unique irreducible quotient of

$$I_{\mathrm{GL}(n),B}\left(1 \times \ldots \times |\cdot|_F^{n-1}\right)$$

where B is the usual Borel of upper triangular matrices and we are parabolically inducing the aforementioned character of the diagonal torus.
It is well-known that this parabolically induced representation is a non-irreducible principal series, and it turns out its unique irreducible quotient is a Steinberg representation.

What about the subrepresentations of $\mathrm{Sp}(n)$ we outlined above? If we fix $1 < j \leqslant n$, we'll get the subrepresentation $\rho_j \subset \mathrm{Sp}(n)$ of dimension $n-j+1$ spanned by $e_j, \ldots, e_n$.
By relabeling this basis as $f_i = e_{i-j+1}$ we obtain that the semisimple part acts like $\mathrm{Sp}(n-j+1)$ up to a twist of $\|\cdot\|^{j-1}$, while the nilpotent operator is the exactly the one defined for $\mathrm{Sp}(n-j+1)$. Hence, the correspondent admissible representation will be a twist of the Steinberg representation associated to $\mathrm{Sp}(n-j+1)$.

Let's try to be even more concrete and give one more example of how we construct the correspondence, starting from the Galois side. Suppose $\rho : W_F \longrightarrow T \subset \mathrm{GL}_n(\mathbb{C})$ is a continuous homomorphisms, so that it is determined by n characters χ_i of $F^\times$. Setting $N = 0$ we obtain a semisimple Weil-Deligne representation $\phi \in \mathrm{Rep}_n(W_F')$.
The data $\{\chi_i\}$ is all we have, so the natural way to associate to it a representation is taking the parabolically induced principal series

$$\pi = \mathrm{Ind}_{G,TN}\left(\chi_1, \ldots, \chi_n\right),$$

where N is the unipotent subgroup of upper triangular unipotent matrices.
If π is irreducible, this is exactly the representation we associate to ϕ in $\mathcal{A}_n(F)$, moreover we've seen before that π is reducible if and only if $\chi_i = \chi_j \cdot |\cdot|_F$ for some i, j. If this is the case, we associate to ϕ a subquotient of π, the Langlands subquotient, explicitly defined in the following way.
We can associate to each χ_i a complex number s_i such that

$$|\chi_i|_{\mathbb{C}} = |\cdot|_F^{s_i}$$

and the real part of the s_i's is uniquely determined. Suppose that the s_i's are ordered so that they have nonincreasing real parts, then the Langlands subquotient is the *unique* irreducible quotient of $\pi = \mathrm{Ind}_{G,TN}\left(\chi_1, \ldots, \chi_n\right)$. Note that the existence and uniqueness of such quotient is guaranteed by the theorem mentioned above!

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