INTRODUCTION TO REAL GROUP REPRESENTATIONS

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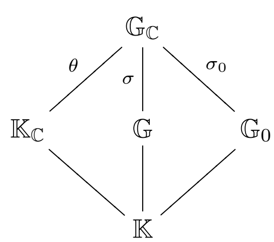
Contents

1. Some structure theory

General reference: Knapp's lecture in [3].

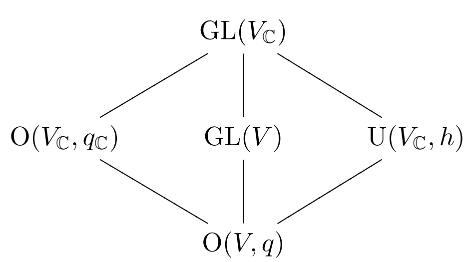
1.1. **The groups.** Let $\mathbb{G}$ be a connected reductive algebraic group over $\mathbb{R}$, and $G = \mathbb{G}(\mathbb{R})$. Starting with $\mathbb{G}$ one can construct the following diagram

(1.1)



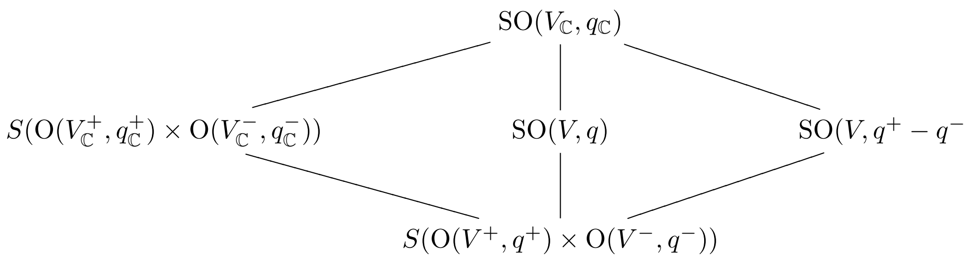
We first explain the groups. Here $\mathbb{G}_{\mathbb{C}} = \mathbb{G} \otimes_{\mathbb{R}} \mathbb{C}$. The $\mathbb{R}$-group $\mathbb{K} \subset \mathbb{G}$ is a (possibly disconnected) reductive subgroup such that $K = \mathbb{K}(\mathbb{R})$ is Zariski dense in $\mathbb{K}$ and K is a maximal compact subgroup of G. We set $\mathbb{K}_{\mathbb{C}} = \mathbb{K} \otimes_{\mathbb{R}} \mathbb{C}$. Finally $\mathbb{G}_0$ is the unique compact real form of $\mathbb{G}_{\mathbb{C}}$ containing $\mathbb{K}$. The lines in (1.1) are inclusions of real algebraic groups (viewing $\mathbb{G}_{\mathbb{C}}$, $\mathbb{K}_{\mathbb{C}}$ as real groups via Weil's restriction of scalars).

1.2. **Example.** Let $\mathbb{G} = \mathrm{GL}_n = \mathrm{GL}(V)$ for a n-dimensional $\mathbb{R}$-vector space V, the diagram (1.1) looks like



Here $q : V \to \mathbb{R}$ is a *positive definite* quadratic form; $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$; $q_{\mathbb{C}}$ is the complexification of q; $h : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ is the unique Hermitian form extending q, i.e., $h(av, av) = |a|^2 q(v)$ for all $v \in V_{\mathbb{R}}, a \in \mathbb{C}$.

1.3. **Example.** Let (V, q) be a nondegenerate quadratic space over $\mathbb{R}$ and let $\mathbb{G} = \mathrm{SO}(V, q)$. Choose an orthogonal decomposition $V = V^+ \oplus V^-$ such that $q^+ := q|_{V^+}$ (resp. $q^- := q|_{V^-}$) is positive (resp. negative) definite. Then the diagram (1.1) looks like



1.4. **Involutions.** We then explain the involutions θ, σ and σ_0. The real form $\mathbb{G}$ of $\mathbb{G}_{\mathbb{C}}$ corresponds to an involution $\sigma : \mathbb{G}_{\mathbb{C}} \to \mathbb{G}_{\mathbb{C}} \otimes_{\mathbb{C},c} \mathbb{C}$ where c denotes complex conjugation (we call such an involution anti-holomorphic). We have $\mathbb{G} = (\mathrm{Res}^{\mathbb{C}}_{\mathbb{R}} \mathbb{G}_{\mathbb{C}})^{\sigma}$, where we view σ as an involution of $\mathrm{Res}^{\mathbb{C}}_{\mathbb{R}} \mathbb{G}_{\mathbb{C}}$. Similarly, the compact real form $\mathbb{G}_0$ corresponds to an anti-holomorphic involution σ_0 of $\mathbb{G}_{\mathbb{C}}$. The involutions σ and σ_0 commute with each other and $\theta = \sigma\sigma_0 = \sigma_0\sigma \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{G}_{\mathbb{C}})$ is an involution of $\mathbb{G}_{\mathbb{C}}$ over $\mathbb{C}$. We have $\mathbb{K}_{\mathbb{C}} = \mathbb{G}^{\theta}_{\mathbb{C}}$. Moreover, $\mathbb{K} = (\mathrm{Res}^{\mathbb{C}}_{\mathbb{R}} \mathbb{G}_{\mathbb{C}})^{\sigma_0, \sigma}$.

1.5. **Theorem** (Cartan)**.** *The $\mathbb{G}(\mathbb{C})$-conjugacy class of θ is uniquely determined by the real form $\mathbb{G}$ of $\mathbb{G}_{\mathbb{C}}$. The correspondence $\mathbb{G} \mapsto \theta$ gives a bijection*

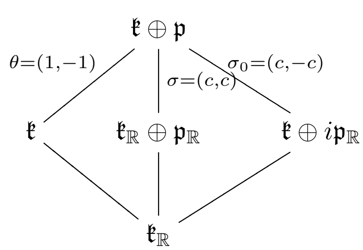
$$\{\textit{real forms of } \mathbb{G}_{\mathbb{C}}\}/\textit{isom} \leftrightarrow \{\textit{involutions of } \mathbb{G}_{\mathbb{C}} \textit{ over } \mathbb{C}\}/\mathbb{G}_{\mathbb{C}}(\mathbb{C})$$

Special cases: compact real form $\leftrightarrow \theta = 1$; split real form $\leftrightarrow$ Chevalley involutions.

1.6. **Lie algebras.** Let $\mathfrak{g} = \mathrm{Lie}\ \mathbb{G}_{\mathbb{C}}$. We have a decomposition of $\mathfrak{g}$ into eigenspaces of θ

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where $\mathfrak{k} = \mathfrak{g}^{\theta}$ and $\mathfrak{p}$ is the (-1)-eigenspace of θ on $\mathfrak{g}$. Similarly, the real Lie algebra $\mathfrak{g}_{\mathbb{R}} = \mathrm{Lie}\ \mathbb{G}$ has a decompostion $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$ into eigenspaces of $\theta|_{\mathfrak{g}_{\mathbb{R}}}$. On the level of Lie algebras, diagram (1.1) becomes



Here c denotes the complex conjugation on $\mathfrak{k}$ or $\mathfrak{p}$ with respect to the real structure $\mathfrak{k}_{\mathbb{R}}$ or $\mathfrak{p}_{\mathbb{R}}$.

1.7. **Polar decomposition.** The map

$$\begin{array}{rcl} K \times \mathfrak{p}_{\mathbb{R}} & \to & G \\ (k, X) & \mapsto & k \exp(X) \end{array}$$

is a diffeomorphism. In particular the symmetric space G/K is diffeomorphic to the vector space $\mathfrak{p}_{\mathbb{R}}$.

1.8. **Iwasawa decomposition.** Let $\mathfrak{a}_{\mathbb{R}} \subset \mathfrak{p}_{\mathbb{R}}$ be a maximal subalgebra (automatically commutative). Then $A = \exp(\mathfrak{a}_{\mathbb{R}})$ is a subgroup of G, and there is a unique split $\mathbb{R}$-torus $\mathbb{A} \subset \mathbb{G}$ such that $A = \mathbb{A}(\mathbb{R})^{\circ}$ (neutral component). Let $\mathbb{M} = C_{\mathbb{K}}(\mathbb{A}) \subset \mathbb{K}$ and let $\mathbb{N}$ be a maximal unipotent subgroup of $\mathbb{G}$ normalized by $\mathbb{A}$. Let $M = \mathbb{M}(\mathbb{R})$ and $N = \mathbb{N}(\mathbb{R})$. Then

(1) There is a unique minimal parabolic subgroup $\mathbb{P} \subset \mathbb{G}$ with Levi factor $\mathbb{L} = \mathbb{M}\mathbb{A}$ and unipotent radical $\mathbb{N}$. Multiplication gives a diffeomorphism $M \times A \times N \cong P = \mathbb{P}(\mathbb{R})$. The decomposition $P = MAN$ is called the *Langlands decomposition* for P. In particular, $A = \mathbb{A}(\mathbb{R})^{\circ}$ where $\mathbb{A}$ is the split center of $\mathbb{L}$ (i.e., the maximal $\mathbb{R}$-torus in the center of $\mathbb{L}$).

(2) Multiplication gives a diffeomorphism (Iwasawa decomposition)

$$K \times A \times N \cong G.$$

1.9. **Cartan decomposition.** $G = KAK$.

1.10. **Example.** Polar decomposition for $\mathbb{G} = \mathrm{GL}_n$. Choose $K = O(q)$ with $q = \sum_i x_i^2$. Then $\mathfrak{p}_{\mathbb{R}}$ consists symmetric real matrices and $\exp(\mathfrak{p}_{\mathbb{R}})$ consists of positive definite symmetric matrices. Every matrix A can be written uniquely as $A = OS$ where O is orthongal with respect to q and S is symmetric and positive definite. In fact, $A^t \cdot A$ is positive definite, hence admits a square root S which is again positive definite. Then let $O = AS^{-1}$.

Iwasawa decomposition for $\mathbb{G} = \mathrm{GL}_n$. Take A to be the group of diagonal matrices with positive entries and N upper triangular unipotent real matrices. Every matrix A can be written uniquely as $A = ODU$ with $O \in K, D \in A$ and $U \in N$. This follows from the Gram-Schmidt orthogonalization procedure for the positive definite matrix $A^t \cdot A$.

1.11. **Example.** Let us take a non-split example $\mathbb{G} = \mathrm{U}(V, h)$ for a non-degenerate Hermitian form h on a complex vector space V of signature (p, q) and $p \leq q$. Choose an orthogonal decomposition $V = V^+ \oplus V^-$ such that $h|_{V^+_{\mathbb{C}}} > 0, h|_{V^-_{\mathbb{C}}} < 0$. Then $K = \mathrm{U}(V^+, h) \times \mathrm{U}(V^-, h)$ is a maximal compact of G, and $\mathfrak{p}_{\mathbb{R}} \cong \mathrm{Hom}_{\mathbb{C}}(V^+, V^-)$ (really should be thought of as a pair of maps $V^+ \to V^-$ and $V^- \to V^+$ adjoint to each other).

Choose an orthonormal basis $e_1^+, \cdots, e_p^+$ of V^+ and an orthonormal (norms are -1) basis $e_1^-, \cdots, e_q^-$ of V^-. Let $\mathfrak{a}_{\mathbb{R}} \subset \mathfrak{p}_{\mathbb{R}}$ consist of maps $V^+ \to V^-$ sending e_j^+ to $\mathbb{R}e_j^-$. Then $\mathfrak{a}_{\mathbb{R}}$ is a maximal subalgebra of $\mathfrak{p}_{\mathbb{R}}$, and it corresponds to a maximally split torus in $\mathbb{G} = \mathrm{U}(V, h)$.

For each j, $f_j = e_j^+ + e_j^-$ is isotropic. Let $F_j = \mathrm{Span}_{\mathbb{C}}\{f_1, \cdots, f_j\}$ for $j = 1, \cdots, p$. The parabolic $\mathbb{P}$ adapted to $\mathfrak{a}_{\mathbb{R}}$ in this situation is the stabilizer of the flag

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_p \subset F_p^{\perp} \subset \cdots \subset F_1^{\perp} \subset V. \tag{1.2}$$

For a description of M, A and N, see Example 3.2.

2. Notions of representations

General reference: Baldoni's lecture in [3]. Partially follow [5].

2.1. **Definition.** (1) A *representation* of G is a complete locally convex topological [1] $\mathbb{C}$-vector space V with a continuous action $G \times V \to V$. We denote the homomorphism $G \to \mathrm{GL}(V)$ by π.

(2) A *unitary representation* of G is a Hilbert space V with a continuous unitary action of G.

(3) A representation (π, V) of G is called *irreducible* if it does not contain nonzero proper *closed* subspace $V' \subset V$ stable under G.

2.2. **Smooth vectors.** A vector $v \in V$ is C^1 if for any $X \in \mathfrak{g}_{\mathbb{R}}$ the derivative

$$X \cdot v := \lim_{t \to 0} \frac{\pi(\exp(tX))v - v}{t}$$

exists. Similarly we may define C^k vectors. A *smooth vector* $v \in V$ is one which is C^k for all $k \geq 1$. Let $V^\infty \subset V$ be the subspace of smooth vectors. This is stable under G.

2.3. **Theorem** (Garding)**.** *Let (π, V) be a representation of G. Then*

(1) *The subspace V^∞ is dense in V;*

(2) *The subspace V^∞ carries a natural action of $\mathfrak{g}$ (hence $U(\mathfrak{g})$).*

Sketch of proof of (1). For any smooth compactly supported measure ϕ on G and $v \in V$, the vector

$$\pi(\phi)v = \int_G \phi(g)\pi(g)v$$

belongs to V^∞. The space spanned by such $\pi(\phi)v$ is already dense in V. □

[1] It means a vector space whose topology is induced from a family of seminorms.

For example, consider the right regular representation of G on $V = L^2(G)$. Then V^∞ consists of smooth L^2-functions on G with all derivatives (of arbitrary order) still L^2.

2.4. **K-finite vectors.** Let V_1 be a continuous K-module. A vector $v \in V_1$ is K-finite if $\pi(k)v$ ($\forall k \in K$) span a finite-dimensional subspace. Let $V_1^{(K)}$ denote the K-finite vectors in V_1; V_1 is called locally finite (under the action of K) if $V_1^{(K)} = V_1$. Representation theory of the compact group K gives

$$V_1^{(K)} \cong \bigoplus_{\mu \in \mathrm{Irr}(K)} E_\mu \otimes \mathrm{Hom}_K(E_\mu, V_1).$$

Here E_μ is the finite-dimensional $\mathbb{C}$-vector space affording the irreducible representation μ of K. In particular, the action of K on $V_1^{(K)}$ is analytic, because the action of K on each E_μ is.

When V is a representation of G, $V^{(K)}$ is dense in V. Warning: $V^{(K)}$ is not stable under G! (It depends on the choice of K).

2.5. **Definition.** A $(\mathfrak{g}, K)$-*module* is a $\mathbb{C}$-vector space V equipped with

- A representation of $\mathfrak{g}$ on V; ($\Leftrightarrow V$ is a $U(\mathfrak{g})$-module)
- A locally finite and continuous action of K (hence analytic);

subject to the conditions

(1) The differential of the K-action on V is equal to the $\mathfrak{k}$-action restricted from the $\mathfrak{g}$-action on V.
(2) For $k \in K, X \in \mathfrak{g}, v \in V$ we have

$$k \cdot (X \cdot v) = (\mathrm{Ad}(k)X) \cdot (k \cdot v).$$

Note that the second condition is only needed when K is disconnected. One can similarly define the notation of $(\mathfrak{g}, \mathbb{K}_\mathbb{C})$-modules by requiring V to be a union of finite-dimensional algebraic representations of $\mathbb{K}_\mathbb{C}$. This notion is equivalent to the notion of $(\mathfrak{g}, K)$-modules. Therefore, $(\mathfrak{g}, K)$-module is a purely algebraic notion.

We have a functor

$$\begin{array}{rcl} \{\text{representations of } G\} & \to & (\mathfrak{g}, K)\text{-mod} \\ V & \mapsto & V^{\infty,(K)} := V^\infty \cap V^{(K)}. \end{array}$$

The subspace $V^{\infty,(K)}$ is also dense in V. Two representations of G are called *infinitesmially equivalent* if they give the same $(\mathfrak{g}, K)$-module by the above functor.

2.6. **Definition.** A $(\mathfrak{g}, K)$-module V is called *admissible* if each irreducible representation of K appears in V with finite multiplicity. Likewise, a representation (π, V) of G is *admissible* if each irreducible representation of K appears in $V^{(K)}$ with finite multiplicity.

2.7. **Theorem** (Harish-Chandra)**.** *If (π, V) is an admissible representation of G, then there is a one-to-one bijection between* closed *G-invariant subspaces of V and sub-$(\mathfrak{g}, K)$-modules of $V^{\infty,(K)}$. In particular, an admissible representation (π, V) is irreducible if and only the $(\mathfrak{g}, K)$-module $V^{\infty,(K)}$ is irreducible.*

Key ingredient: If (π, V) is admissible, then $V^{(K)} \subset V^\infty$ (even contained in analytic vectors). Proof uses regularity of elliptic operators.

2.8. **Corollary.** *Schur's lemma holds for admissible representations of G.*

Proof. Let (π, V) be an irreducible admissible representation of G. Let $\mathrm{End}_G(V)$ be the continuous G-endomorphisms of V. The map $\mathrm{End}_G(V) \to \mathrm{End}_{(\mathfrak{g},K)}(V^{(K)})$ is injective since $V^{(K)}$ is dense in V. However, since $V^{(K)}$ is irreducible as a $(\mathfrak{g}, K)$-module by Theoerem 2.7, $\mathrm{End}_{(\mathfrak{g},K)}(V^{(K)})$ is a division algebra over $\mathbb{C}$. Moreover, admissibility of $V^{(K)}$ implies that $\mathrm{End}_{(\mathfrak{g},K)}(V^{(K)})$ has countable dimension. Therefore $\mathrm{End}_{(\mathfrak{g},K)}(V^{(K)}) = \mathbb{C}$ (same as Jacquet's argument for p-adic groups). $\square$

Discrete series will be elaborated in the next lecture by Akshay. We only give definition here.

2.9. **Definition-Lemma** (Godement). *The following are equivalent for a unitary representation (π, V) of G*

(1) *(π, V) unitarily embeds into the left regular representation of G on $L^2(G)$;*
(2) *Every matrix coefficient of V is square-integrable;*
(3) *There exists a nonzero matrix coefficient of V which is square-integrable.*

If (π, V) satisfies the above conditions, it is called a discrete series representation *of G.*

2.10. **Infinitesimal characters.** Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Let $\mathfrak{Z}(\mathfrak{g})$ denote the center of $U(\mathfrak{g})$. To get a sense of how large $\mathfrak{Z}(\mathfrak{g})$ is, consider the filtration of $U(\mathfrak{g})$ by degree. Since $\mathfrak{Z}(\mathfrak{g}) = U(\mathfrak{g})^{\mathbb{G}}$, $\mathrm{Gr}\mathfrak{Z}(\mathfrak{g}) \cong (\mathrm{Gr}U(\mathfrak{g}))^{\mathbb{G}}$ (exactness of $(-)^{\mathbb{G}}$) $\cong \mathrm{Sym}(\mathfrak{g})^{\mathbb{G}} \cong \mathrm{Sym}(\mathfrak{h})^W$ (last isom: Chevalley).

2.11. **Definition** (Harish-Chandra). A representation (π, V) of G is *quasi-simple* if the center $\mathfrak{Z}(\mathfrak{g})$ acts as a scalar on V^∞.

Note: Irreducible *unitary* representations of G are quasi-simple.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ with Weyl group W. Note that the quotient $\mathfrak{h}^* /\!\!/ W$ is independent of the choice of $\mathfrak{h}$: it is identified with $\mathfrak{g}^* /\!\!/ G$ by Chevalley's theorem.

2.12. **Theorem** (Harish-Chandra). *There is a canonical isomorphism* $\mathrm{Spec}\ \mathfrak{Z}(\mathfrak{g}) \cong \mathfrak{h}^* /\!\!/ W$. *For an irreducible $\mathfrak{g}$-module V_λ with highest weight (with respect to the choice of a Borel $\mathfrak{b}$ containing $\mathfrak{h}$) $\lambda \in \mathfrak{h}^*$, $\mathfrak{Z}(\mathfrak{g})$ acts on V_λ via the $\xi \in$* $\mathrm{Spec}\ \mathfrak{Z}(\mathfrak{g})$ *which corresponds to the W-orbit of $\lambda + \rho$ (2ρ is the sum of roots appearing in $\mathfrak{b}$).*

Sketch of proof. Choose a triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. We have $U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes \mathrm{Sym}(\mathfrak{h}) \otimes U(\mathfrak{n}^+)$ as vector spaces. Every $Z \in \mathfrak{Z}(\mathfrak{g})$ can be written as $Z = Z_0 + Z_+$ where $Z_+ \in U(\mathfrak{g})\mathfrak{n}^+$ and $Z_0 \subset U(\mathfrak{n}^-) \otimes \mathrm{Sym}(\mathfrak{h})$. The fact that Z commutes with $\mathfrak{h}$ implies $Z_0 \in \mathrm{Sym}(\mathfrak{h})$.

One checks that Z_0 is invariant under the dot-action of W on $\mathfrak{h}^*$

$$w \cdot \lambda = w(\lambda + \rho) - \rho, \forall w \in W, \lambda \in \mathfrak{h}^*.$$

The assignment $Z \mapsto Z_0 \in \mathrm{Sym}(\mathfrak{h})^{(W,\cdot)}$ gives an algebra isomorphism $\mathfrak{Z}(\mathfrak{g}) \cong \mathrm{Sym}(\mathfrak{h})^{(W,\cdot)}$. Shifting by ρ gives an isomorphism $\mathfrak{h}^* /\!\!/ (W, \cdot) \cong \mathfrak{h}^* /\!\!/ W$. □

For a quasi-simple representation (π, V), the action of $\mathfrak{Z}(\mathfrak{g})$ is via a character of $\xi \in \mathrm{Spec}\ \mathfrak{Z}(\mathfrak{g})$, which correspond to a W-orbit in $\mathfrak{h}^*$ by the above theorem. The character ξ is called the *infinitesimal character* of (π, V).

2.13. **Theorem** (Harish-Chandra; Lepowsky). *Every finitely-generated quasi-simple $(\mathfrak{g}, K)$-module is admissible. The multiplicity of $\lambda \in \mathrm{Irr}(K)$ in any irreducible $(\mathfrak{g}, K)$-module V is bounded by a constant which only depends on λ.*

The proof of the Theorem involves a detailed study of the algebra $(U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} R(K))^K$ (Lepowsky). Here $R(K) = \oplus_{\mu \in \mathrm{Irr}(K)} \mathrm{End}(E_\mu)$ is the space of matrix coefficients of K. The key point is to show an algebra embedding

$$(U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} R(K))^K \hookrightarrow \mathrm{Sym}(\mathfrak{a}) \otimes R(K)^{M,\mathrm{op}}$$

and the target has a large center. The argument is similar to that of Theorem 2.12.

Using this theorem, Theorem 2.7 can be extended to all quasi-simple G-representations.

2.14. **Summary.** We have

$$\mathrm{Irr}(G)_{\mathrm{disc}} \hookrightarrow \mathrm{Irr}(G)_{\mathrm{unitary}} \hookrightarrow \mathrm{Irr}(G)_{\text{quasi-simple}} \twoheadrightarrow \mathrm{Irr}(\mathfrak{g}, K).$$

All the above are admissible.

The composition $\mathrm{Irr}(G)_{\mathrm{unitary}} \to \mathrm{Irr}(\mathfrak{g}, K)$ is injective, with image consisting of those irreducible $(\mathfrak{g}, K)$-modules admitting a positive definite $(\mathfrak{g}, K)$-invariant Hermitian form. (Reason for injectivity: suppose V_1 and V_2 are unitary reps with $T : V_1^{(K)} \cong V_2^{(K)}$ as $(\mathfrak{g}, K)$-modules. First try to extend T to a K-equivariant isometry $\widetilde{T} : V_1 \xrightarrow{\sim} V_2$. Then using the fact that the K-finite matrix coefficients are analytic functions on G, one checks that $\widetilde{T}$ sends matrix coefficients of V_1 to the corresponding matrix coefficients of V_2 (enough to check $U(\mathfrak{g})$-action by analyticity). Therefore $\widetilde{T}$ is also G-equivariant.)

3. Parabolic induction

3.1. **Standard parabolics.** Let $P = \mathbb{P}(\mathbb{R}) = MAN$ be a minimal parabolic. A standard parabolic of G is the $\mathbb{R}$-points of a parabolic of $\mathbb{G}$ containing $\mathbb{P}$. These are in bijection with subsets of $\Delta \subset \mathbb{X}^*(\mathbb{A})$ (simple roots of A with respect to N).

Let Q_J be a standard parabolic corresponding to $J \subset \Delta$, then it has a Langlands decomposition $Q_J = M_J A_J N_J$. Here $A_J = \cap_{\alpha \in J} \ker(\alpha) \subset A$, which is the neutral component of the $\mathbb{R}$-points of a subtorus $\mathbb{A}_J \subset \mathbb{A}$. Let $\mathbb{L}_J = C_{\mathbb{G}}(\mathbb{A}_J)$ be the Levi subgroup of $\mathbb{Q}_J$ with $\mathbb{R}$-points L_J. There is unique subgroup $M_J \subset L_J$ with compact center which is complementary to A_J (M_J may not be connected). We also write the Langlands decomposition as $Q = M_Q A_Q N_Q$.

3.2. **Example.** Let $G = U(V, h)$ as in Example 1.11. Standard parabolic subgroups of G are stabilizers of a partial flag (a self-dual subset of (1.2)):

$$0 \subset F_{i_1} \subset \cdots \subset F_{i_s} \subset F_{i_s}^{\perp} \subset \cdots \subset F_{i_1}^{\perp} \subset V$$

with $1 \le i_1 < i_2 < \cdots < i_s \le p$. For Q equal to the stabilizer of this partial flag, we have

$$\begin{aligned} L_Q &= \prod_{j=1}^{s} \mathrm{GL}_{\mathbb{C}}(F_{i_j}/F_{i_{j-1}}) \times U(F_{i_s}^{\perp}/F_{i_s}, \overline{h}). \\ A_Q &= \prod_{j=1}^{s} \mathbb{R}_{>0} \cdot \mathrm{id} \times \{1\}. \\ M_Q &= \ker(L_Q \xrightarrow{|\det|} \prod_{j=1}^{s} \mathbb{R}_{>0}). \\ N_Q &= \ker(Q \to L_Q). \end{aligned}$$

3.3. **Induction.** Let $Q = M_Q A_Q N_Q \subset G$ be a standard parabolic. Let (σ, V_σ) be a representation of M_Q and $\lambda \in \mathfrak{a}_Q^*$ (where $\mathfrak{a}_Q = (\mathrm{Lie}\, A_Q)_{\mathbb{C}}$). Let $2\rho_Q$ be the weight of the action of $\mathbb{A}_Q$ on $\mathbb{N}_Q$. Then the induced representation $\mathrm{Ind}_Q^G(\sigma, \lambda) := \mathrm{Ind}_Q^G(\sigma \otimes (\lambda + \rho_Q) \otimes 1)$ is the completion of the space of continuous functions $f : G \to V_\sigma$ such that

$$f(gman) = e^{-\langle \lambda + \rho_Q, \log(a) \rangle} \sigma(m)^{-1} f(g).$$

(with G acting by left translation). Alternatively, this can be viewed as a space of sections of a homogeneous bundle over G/Q with fibers V_σ.

3.4. **Special case: induction from a minimal parabolic.** When $Q = P$ is minimal, M is compact. In this case we take a finite-dimensional irreducible representation $\sigma \in \mathrm{Irr}(M)$.

Since $G = KAN$, we may alternative describe $\mathrm{Ind}_P^G(\sigma, \lambda)$ as (completion of) the space of functions $f : K \to V_\sigma$ satisfying $f(km) = \sigma(m^{-1}) f(k)$. Hence, as K-module we have

$$(\mathrm{Res}_K^G \mathrm{Ind}_P^G(\sigma, \lambda))^{(K)} \cong (\mathrm{Ind}_M^K V_\sigma)^{(K)}.$$

In particular, the multiplicity of $\mu \in \mathrm{Irr}(K)$ in $\mathrm{Ind}_P^G(\sigma, \lambda))$ is

$$\mathrm{Hom}_K(E_\mu, \mathrm{Ind}_P^G(\sigma, \lambda))) = \mathrm{Hom}_K(E_\mu, \mathrm{Ind}_M^K V_\sigma) = \mathrm{Hom}_M(\mathrm{Res}_M^K E_\mu, V_\sigma). \tag{3.1}$$

3.5. **Theorem** (Casselman's submodule theorem)**.** *Any irreducible $(\mathfrak{g}, K)$-module appears as a sub-$(\mathfrak{g}, K)$-module of some induced representation $\mathrm{Ind}_P^G(\sigma, \lambda)$ for some $\sigma \in \mathrm{Irr}(M)$ and $\lambda \in \mathfrak{a}^*$.*

Outline of proof. By Frobenius reciprocity, it suffices to show that $V_{\mathfrak{n}} := V/\mathfrak{n}V \neq 0$, where $\mathfrak{n} = (\mathrm{Lie}\, \mathbb{N})_{\mathbb{C}} \subset \mathfrak{g}$. There are two proofs of this fact.

Casselman's original proof uses estimates of matrix coefficents. For $v \in V^{(K)}$ and $v^* \in (V^*)^{(K)}$, the matrix coefficient $a \mapsto \langle v^*, \pi(a)v \rangle$ ($a \in A^+$, dominant part of A) can be expanded as an ansolutely convergent series $\sum_{\lambda,\mu} c_{\lambda,\mu}(v^*, v) a^\lambda (\log a)^\mu$ for a discrete bounded above subset $\lambda \in \mathfrak{a}^*$ and μ in the positive root semigroup. Fix v^*, take a maximal λ (under the order induced from simple roots Δ^+) such that $f_\lambda(v, a) = \sum_\mu c_{\lambda,\mu} a^\lambda (\log a)^\mu \neq 0$, then f_λ gives a nonzero map $V_{\mathfrak{n}} \to C^\infty(A^+)$; in particular $V_{\mathfrak{n}} \neq 0$.

Beilinson and Bernstein [1] gave an algebraic proof of the fact $V_{\mathfrak{n}} \neq 0$. They reduce to showing that for any finitely generated $U(\mathfrak{g})$-module V, $V_{\mathfrak{n}'} \neq 0$ for a Zariski dense choice of $\mathfrak{n}'$ (parametrized by the flag variety X of $\mathbb{G}_{\mathbb{C}}$). Then when V is a $(\mathfrak{g}, K)$-module, the action of $\mathbb{K}_{\mathbb{C}}$ on X allows one to conclude that $V_{\mathfrak{n}} \neq 0$ because $\mathfrak{n}$ lies in the open $\mathbb{K}(\mathbb{C})$-orbit of X. The strategy for showing $V_{\mathfrak{n}'} \neq 0$ is by relating V to (twisted) D-modules over X. Suppose the infinitesimal character ξ corresponds to the W-orbit of $\lambda + \rho \in \mathfrak{h}^*$ under Theorem 2.12, there is a localization functor

$$\Delta_\lambda : U(\mathfrak{g})_\xi\text{-mod} \to D_\lambda\text{-mod}(X)$$

When χ is regular and dominant, this is an equivalence of categories. The stalk of $\Delta_\lambda(V)$ at $\mathfrak{n}' \in X$ is $V_{\mathfrak{n}'}(\lambda)$ (weight space for $\mathfrak{h}$), and we reduce to show that the support of $\Delta_\lambda(V)$ is Zariski dense. The dominant $\lambda + \rho$ may not give this right away, and one uses intertwining operators to switch between different $\lambda + \rho$'s in the W-orbit, to eventually find one λ such that $\Delta_\lambda(V)$ has full support.

□

Combining this theorem with the calculation (3.1), we see that the multiplicity of $\mu \in \mathrm{Irr}(K)$ in any irreducible $(\mathfrak{g}, K)$-module is bounded by the maximum of the mutiplicities of irreducible representations of M appearing in $\mathrm{Res}^K_M E_\mu$. This is a number which only depends on μ and not on the $(\mathfrak{g}, K)$-module. This gives a proof of the second of part of Theorem 2.13.

4. GL_2 and SL_2

4.1. **The maximal compact.** Let W be a two-dimensional vector space over $\mathbb{R}$. Let $\mathbb{G} = \mathrm{SL}(W)$ and $\mathbb{G}' = \mathrm{GL}(W)$. Choose a volume form $\omega \in \wedge^2(W)$ and a positive definite quadratic form $q : W \to \mathbb{R}$. Let $\mathbb{K} = \mathrm{SO}(W, q) < \mathbb{G}$ and $\mathbb{K}' = \mathrm{O}(V, q) < \mathbb{G}'$. Note that (ω, q) uniquely determines a complex structure $J : W \to W$ such that $b_q(Jx, y) = (x \wedge y)/\omega$ (b_q is the symmetric bilinear form associated with q), so that W becomes a 1-dimensional $\mathbb{C}$-vector space. Elements in $\mathbb{K} = \mathrm{SO}(W, q)$ preserve both q and ω, hence commutes with J. This gives a canonical embedding $\iota : \mathbb{K} \hookrightarrow \mathrm{Res}^{\mathbb{C}}_{\mathbb{R}} \mathbb{G}_m = \mathrm{Aut}_J(W)$ and identifies K with the unit circle in $\mathbb{C}^\times$.

4.2. **Center of $U(\mathfrak{g})$.** Let $z = \mathrm{diag}(1, 1) \in \mathfrak{g}' = \mathfrak{gl}(W_{\mathbb{C}})$. Since $K = \mathrm{SO}(W, q)$ is a maximal torus in G, we may choose a basis $\{e, h, f\}$ for $\mathfrak{g} = \mathfrak{sl}(W_{\mathbb{C}})$ such that $\mathfrak{k} = \mathrm{Span} h$, $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. Then Theorem 2.12 specializes to an isomorphism

$$\mathfrak{Z}(\mathfrak{sl}_2) \cong \mathbb{C}[\Delta]; \qquad \mathfrak{Z}(\mathfrak{gl}_2) \cong \mathbb{C}[z, \Delta].$$

where $\Delta = \frac{h^2}{2} + fe + ef = \frac{h^2}{2} + h + 2fe = \frac{h^2}{2} - h + 2ef$.

4.3. **Principal series.** First consider $\mathbb{G} = \mathrm{SL}(W)$. A line $W_1 \subset W$ gives a Borel subgroup $\mathbb{B} \subset \mathbb{G}$. Using the quadratic form we get a decomposition $W = W_1 \oplus W_1^\perp$. We have $B = MAN$ where $M = \{\pm 1\}$, $A = \mathbb{R}_{>0}$ acting as diagonal matrices with respect to the decomposition above, and N acts as the identity on W_1. For any $\lambda \in \mathbb{C}$ and $\epsilon \in \{0, 1\}$ we may define a character $(\epsilon, \lambda) : M \times A \to \mathbb{C}^\times$ such that $(m, a) \mapsto m^\epsilon a^\lambda$. The induction $\mathrm{Ind}^G_B(\epsilon, \lambda)$ has the following more concrete realization

$$\mathrm{Ind}^G_B(\epsilon, \lambda) = \{\text{continuous } f : W - \{0\} \to \mathbb{C} | f(aw) = |a|^{-\lambda-1} \mathrm{sgn}(a)^\epsilon f(w), \forall w \in W - \{0\}, a \in \mathbb{R}^\times\}.$$

Recall that we may view W as a 1-dimensional $\mathbb{C}$-vector space and K acts on W by multiplication via $\iota : K \hookrightarrow \mathbb{C}^\times$. For $i \in \mathbb{Z}$ let $f_i : W - \{0\} \to \mathbb{C}$ be a function satisfying $f(a\iota(k)w) = \iota(k)^{-i} a^{-\lambda-1} f(w)$ for all $a \in \mathbb{R}^{>0}, w \in W - \{0\}$ and $k \in K$. Such functions are unique up to a scalar. Let $V(\epsilon, \lambda)$ be the $(\mathfrak{g}, K)$-module of $\mathrm{Ind}^G_B(\epsilon, \lambda)$. Then

$$V(\epsilon, \lambda) = \mathrm{Span}\{f_i\}_{i \equiv \epsilon (2)}.$$

The infinitesimal character of $\mathrm{Ind}^G_B(\epsilon, \lambda)$ is $\Delta \mapsto \frac{\lambda^2 - 1}{2}$.

For $\mathbb{G}' = \mathrm{GL}(W)$, let $B' = M'A'N$ be the Borel in $\mathbb{G}'$ containing B fixed above. We have $M' \cong \{\pm 1\} \times \{\pm 1\}$ and $A' \cong \mathbb{R}_{>0} \times \mathbb{R}_{>0}$. We similarly define $\mathrm{Ind}^{G'}_{B'}(\epsilon_1, \epsilon_2, \lambda_1, \lambda_2)$, with $\epsilon_1, \epsilon_2 \in \{0, 1\}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, as the space of functions $f : G \to \mathbb{C}$ such that $f(gman) = m_1^{\epsilon_1} m_2^{\epsilon_2} a_1^{-\lambda-1/2} a_2^{-\lambda_2+1/2} f(g)$.

The center $\mathbb{R}^\times \subset G'$ acts on it via the character $a \mapsto \mathrm{sgn}(a)^{\epsilon_1+\epsilon_2}|a|^{-\lambda_1-\lambda_2}$. Restricting to G we get an isomorphism

$$\mathrm{Res}^{G'}_G \mathrm{Ind}^{G'}_{B'}(\epsilon_1, \epsilon_2, \lambda_1, \lambda_2) \cong \mathrm{Ind}^G_B(\epsilon_1 - \epsilon_2, \lambda_1 - \lambda_2).$$

Denote the $(\mathfrak{g}', K')$-module of $\mathrm{Ind}^G_{B'}(\epsilon_1, \epsilon_2, \lambda_1, \lambda_2)$ by $V(\epsilon_1, \epsilon_2, \lambda_1, \lambda_2)$, then

$$V(\epsilon_1, \epsilon_2, \lambda_1, \lambda_2) = \bigoplus_{i\geq 0, i\equiv \epsilon_1-\epsilon_2(2)} \mathrm{Span}\{f_i, f_{-i}\}.$$

with each summand an irreducible representation of K'.

4.4. **Irreducible $(\mathfrak{g}, K)$-modules.** First consider the case $\mathbb{G} = \mathrm{SL}(V)$. Let V be a $(\mathfrak{g}, K)$-module. Since K is a compact torus, we have a decomposition into weight spaces of K

$$V = \bigoplus_{n\in\mathbb{Z}} V(n),$$

with K acting on $V(n)$ via the character $k \mapsto \iota(k)^n$. We also have $e : V(n) \to V(n+2)$ and $f : V(n) \to V(n-2)$, satisfying that $[e, f] = n$ on $V(n)$.

Now assume V is irreducible with infinitesimal character $\Delta \mapsto \xi$. Then V has a parity $\epsilon(V) \in \{0, 1\}$: $V(n) = 0$ unless $n \equiv \epsilon(V) \mod 2$. This can be read from the action of the center $\{\pm 1\} \subset K \subset G$. Starting from some nonzero vector $v \in V(\ell)$, then $U(\mathfrak{g})v$ is spanned by $\{v, e^n v, f^n v\}_{n=1,2,\cdots}$ (e.g., to compute $fe^n v$, we only need to note that $fe = \frac{1}{4}(\Delta - (h+1)^2) = \frac{1}{4}(\xi - (h+1)^2))$. Therefore, for irreducible V, $\dim V(n) \leq 1$ and n's such that $V(n) \neq 0$ form a chain with step 2. There are three cases:

1. ξ cannot be written as $\frac{1}{2}\ell(\ell+2)$ for some integer $\ell \equiv \epsilon(V) \mod 2$. Then there is up to isomorphism a unique irreducible $(\mathfrak{g}, K)$-module with infinitesimal character $\Delta \mapsto \xi$ and parity $\epsilon(V)$. It is isomorphic to $V(\epsilon, \lambda)$ for $(\lambda + 1)^2 = \xi$.
2. $\xi = \frac{1}{2}\ell(\ell + 2)$ for some integer $\ell \geq 0$ and $\ell \equiv \epsilon(V) \mod 2$. Then either $V \cong \mathrm{Sym}^\ell(W_\mathbb{C})$ (if $\ell \geq 0$); or $V \cong V^+_{\ell+2} := \bigoplus_{n>\ell,n\equiv\ell(2)} V(n)$; or $V \cong V^-_{\ell+2} := \bigoplus_{n<-\ell,n\equiv\ell(2)} V(n)$. The last two are the *holomorphic and anti-holomorphic* discrete series representations of G respectively. We have exact sequences of $(\mathfrak{g}, K)$-modules
$$0 \to V^+_{\ell+2} \oplus V^-_{\ell+2} \to V(\ell \bmod 2, \ell + 1) \to \mathrm{Sym}^\ell(W_\mathbb{C}) \to 0;$$
$$0 \to \mathrm{Sym}^\ell(W_\mathbb{C}) \to V(\ell \bmod 2, -\ell - 1) \to V^+_{\ell+2} \oplus V^-_{\ell+2} \to 0.$$
Realization of $V^\pm_{\ell+2}$: holomorphic sections of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-\ell-2)$ over the two components of $\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$.
3. $\xi = -\frac{1}{2}$ and $\epsilon(V) = 1$. In this case either $V = V^+_1 := \bigoplus_{n\geq 1,n\equiv 1(2)} V(n)$; or $V = V^-_1 := \bigoplus_{n\leq -1,n\equiv 1(2)} V(n)$. These are called the *limits of discrete series representations*, and
$$V(1, 0) = V^+_1 \oplus V^-_1.$$

For $\mathbb{G}' = \mathrm{GL}(W)$, we have an extra freedom of a central character of $\mathbb{R}^{>0} \ni a \mapsto a^{-\lambda_0}$ for some $\lambda_0 \in \mathbb{C}$. We again have three cases as above. The only difference is that in cases (2) and (3), $V^+_{\ell+2} \oplus V^-_{\ell+2}$ is an irreducible $(\mathfrak{g}', K')$-module.

4.5. **Unitary representations.** Reference: [4]. For $G = \mathrm{SL}(W)$, the following is a complete list of irreducible unitary $(\mathfrak{g}, K)$-modules without repetition (Bargmann's theorem)

- The trivial representation $\mathbb{C}$;
- The principal series $V(\epsilon, \lambda)$ for $\lambda \in i\mathbb{R}^{>0}$ (note $V(\epsilon, \lambda) \cong V(\epsilon, -\lambda)$);
- The complementary series $V(\epsilon, \lambda)$ for $0 < \lambda < 1$ (note $V(\epsilon, \lambda) \cong V(\epsilon, -\lambda)$);
- The discrete series V^+_n and V^-_n for $n \geq 2$;
- The limits of discrete series V^+_1 and V^-_1.

Complete list of irreducible unitary $(\mathfrak{g}', K')$-modules:

- The 1-dimensional unitary representations $g \mapsto \mathrm{sgn}\det(g)^\epsilon|\det(g)|^\lambda$ with $\epsilon \in \{0, 1\}$ and $\lambda \in i\mathbb{R}$;

- The principal series $V(0,0,\lambda_1,\lambda_2), V(1,1,\lambda_1,\lambda_2)$ (they differ by $\otimes$sgn(det)) with $\lambda_1,\lambda_2 \in i\mathbb{R}$ and $\lambda_1/i < \lambda_2/i$; $V(0,1,\lambda_1,\lambda_2)$ for $\lambda_1,\lambda_2 \in i\mathbb{R}$ and $\lambda_1 \neq \lambda_2$;
- The complementary series $V(0,0,\lambda_1,\lambda_2), V(1,1,\lambda_1,\lambda_2)$ (they differ by $\otimes$sgn(det)) with $0 < \lambda_1 - \lambda_2 < 1$ and $\lambda_1+\lambda_2 \in i\mathbb{R}$; $V(0,1,\lambda_1,\lambda_2)$ with $\lambda_1 \neq \lambda_2, -1 < \lambda_1-\lambda_2 < 1$ and $\lambda_1+\lambda_2 \in i\mathbb{R}$;
- The discrete series $V_n^+ \oplus V_n^-$ for $n \geq 2$;
- The limit of discrete series $V_1^+ \oplus V_1^-$.

4.6. **Classification in general: D-modules.** Reference: [2]. Now $\mathbb{G}$ is a general connected real reductive group. Fix a character $\xi \in \mathrm{Spec}\ \mathfrak{Z}(\mathfrak{g})$. Let $(\mathfrak{g},K)$-mod$_\xi$ be the abelian category of finitely generated $(\mathfrak{g},K)$-modules on which $\mathfrak{Z}(\mathfrak{g})$ acts by scalars via ξ. Suppose ξ corresponds to the W-orbit of $\lambda+\rho \in \mathfrak{h}^*$ by Theorem 2.12 (under the usual W-action), and that $\lambda+\rho$ is dominant and regular, then the localization functor gives an equivalence of categories

$$(\mathfrak{g},K)\text{-mod}_\xi \cong D_\lambda\text{-mod}(X)^{\mathbb{K}_\mathbb{C}}$$

where the superscript $\mathbb{K}_\mathbb{C}$ stands for $\mathbb{K}_\mathbb{C}$-equivariant twiste D-modules. In particular, irreducible $(\mathfrak{g},K)$-modules with infinitesimal character ξ are parametrized by irreducible D_λ-modules over the $\mathbb{K}_\mathbb{C}$-orbit closures on X. When $\lambda+\rho$ is integral, regular and dominant (which means $\langle\lambda,\alpha\rangle \in \mathbb{Z}_{\geq 0}$ for all positive roots α), then we have a bijection

$$\begin{aligned} & \{\text{irreducible } (\mathfrak{g},K)\text{-modules with infinitesimal character } \xi\} \\ \leftrightarrow\ & \{(O,\rho) | O \subset X \text{ is a } \mathbb{K}_\mathbb{C}\text{-orbit, } \rho \text{ is an irreducible representation of } \pi_0(\mathbb{K}_{\mathbb{C},x}) \text{ for some } x \in O\}. \end{aligned}$$

When $\mathbb{G} = \mathrm{SL}(W)$, there are three $\mathbb{K}_\mathbb{C}$-orbits on $X = \mathbb{P}^1_\mathbb{C}$: two points which we call $\{0\}$ and $\{\infty\}$ and $U = \mathbb{P}^1 - \{0,\infty\}$. The stabilizer of $\mathbb{K}_\mathbb{C}$ on U is $\{\pm 1\}$. When $\xi = \frac{1}{2}\ell(\ell+2)$ for some integer $\ell \geq 0$, the corresponding $\lambda+\rho$ can be whosen to be $(\ell+1)\rho$, which is integral, regular and dominant. In this case we have four pairs (O,ρ) (when $O = U$ we have two choices of ρ). The discrete series $D^\pm_{\ell+2}$ correspond to the two point orbits. Let $\epsilon = \ell \mod 2$, which also denotes the trivial or sign representation of $\{\pm 1\}$ (stabilizer of $\mathbb{K}_\mathbb{C}$ on U). The pair (U,ϵ) then corresponds to the finite-dimensional representation $\mathrm{Sym}^\ell(W_\mathbb{C})$; the rest corresponds to the irreducible principal series $V(1-\epsilon,\ell+1)$.

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