Example: $G = SL(2,\mathbb{R}) \curvearrowright \mathbb{R}^2 - \{0\} \curvearrowleft \mathbb{R}^\times$

$\curvearrowright$ Functions on $\mathbb{R}^2 - \{0\} \curvearrowleft \mathbb{R}^\times$

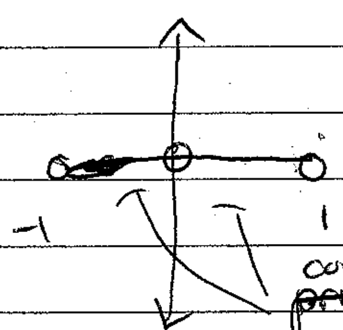
$\lambda \in \mathbb{C}$, $\varepsilon \in \{0,1\}$ $\quad V(\varepsilon,\lambda) = \{ f : \mathbb{R}^2 - \{0\} \to \mathbb{C} \mid f(av) = |a|^{-\lambda-1} (\operatorname{sign} a)^\varepsilon f(v) \}$

ie up to quasi character, invariant.

When unitary? $\rightsquigarrow$ $SL_2(\mathbb{R})$-rep $\sim$ $(\mathfrak{g}, K)$-module.

$\left(\begin{matrix} V(\varepsilon, -\bar{\lambda}) \\ \text{conjugate dual?} \end{matrix} \right)$

λ: where unitary?



$\lambda = 0$ should be easy.
principal series

complementary ~~principal~~ series (~~Induct from Borel~~)

~~Furthermore, for each integer get some: $\lambda = 1$ $\varepsilon = 0$ the $V(\varepsilon, \lambda)$~~

Contains $D^+_{\pm 2} \oplus D^-_{\pm 2}$ two unitary "discrete series"

$\lambda = 2$, $\varepsilon = 1$ so forth...

12/11 Akshay

Where we are: Defined automorphic representation

G reductive group / F: irreducible subrep of $L^2_\chi(G(\mathbb{A})/G(F))$

F: $\mathcal{O}$ integers; subrep: $G(\mathbb{A})$; χ central character, omit if G semisimple.

Such a rep is restricted completed tensor product $\bigotimes \pi_v$
where π_v unitary irreducible rep of $G(F_v)$
for almost all v, π_v has a $G(\mathcal{O}_v)$-invariant vector unique up to scaling.

$V \mapsto G(\mathcal{O}_v)$ - fixed vectors

Irreducible reps of $G(F_v)$ w/ $G(\mathcal{O}_v)$ fixed vect $\leftrightsquigarrow$ irr modules over $\mathcal{H}(G(F_v), G(\mathcal{O}_v))$

Rmk: This is bijection of irreducibles, not equivalence of categories.

Commutative b/c of Gelfand argument w/ Chevalley math Marks, showed using Satake.

For each v, defined class of "smooth rep" of $G(F_v)$

"purely algebraic" non-arch means smooth. For archimedean, interpret $(\mathfrak{g}, K)$ module.

$\left\{\begin{matrix}\text{irreducible unitary}\\ \text{reps}\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}\text{irr smooth}\\ \text{reps}\end{matrix}\right\}$ "enlargement".

For GL_n, we can explicitly parametrize $GL_n(F_v)$ v non-arch
{irreducible smooth reps with $GL_n \mathcal{O}_v$ - fixed vector}

$\updownarrow$

$\{(\alpha_1 \ldots \alpha_n) \in \mathbb{C}^{\times n} / S_n\}$

via: $\mathcal{H}(GL_n F_v, GL_n \mathcal{O}_v) \simeq \mathbb{C}[x_1^{\pm 1} \ldots x_n^{\pm 1}]^{S_n}$ (See Mark's talk)

$\begin{bmatrix} \pi^{\,j} & & 0 \\ & \pi & \\ 0 & & 1 \end{bmatrix} \to j^{\text{th}}$ symmetric function

Use tuple to produce character $x_1 \mapsto \alpha_1 \ldots x_n \mapsto \alpha_n$.
Call rep $\pi_v(\alpha_1 \ldots \alpha_n)$ (uses relationship w/ Hecke module)

Sample Conjecture. Suppose $\rho: Gal_F \to GL_n(\mathbb{C})$ is irreducible (continuous). Then $\exists!$ automorphic rep π for GL_n/F s.t. $\forall\forall v$

$\pi = \bigotimes' \pi_v$, $\pi_v \simeq \pi_v(\alpha_1 \ldots \alpha_n)$ where $\alpha_1 \ldots \alpha_n$ are eigenvalues of $\rho(\mathrm{Frob}_v)$ (See Frobenius).

Note: not quite correct

Example $G_{\mathbb{Q}} \to GL_2 \mathbb{C}$

Let $K = \mathbb{Q}(\sqrt{d})$, $\psi : G_K \to \mathbb{C}^{\times}$ sue character

$$\begin{array}{ccc} G_K & \xrightarrow{\psi} & \mathbb{C}^{\times} \\ \downarrow & \nearrow & \\ \text{class grp } K & & \end{array}$$

$\rho = \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \psi$ two rep of $G_{\mathbb{Q}}$.

If p ~~split~~ inert in K, $\rho(\mathrm{Frob}_p) \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

split $p = \mathfrak{p}_1 \mathfrak{p}_2$, $\rho(\mathrm{Frob}_p) \sim \begin{pmatrix} \psi(\mathfrak{p}_1) & 0 \\ 0 & \psi(\mathfrak{p}_2) \end{pmatrix}$.

Ex: $d = -23$, $|Cl_K| = 3$ Unique nontrivial character to conjugation.

automorphic rep comes from mod form level 23 weight 1

$$f = q \prod_{n \geq 1} (1 - q^n)(1 - q^{23n}) = \sum a_n q^n.$$

$$a_p = \mathrm{tr}(\mathrm{Frob}_p) = \begin{cases} 0 & p \text{ inert} \\ \psi(\mathfrak{p}_1) + \psi(\mathfrak{p}_2) & p \text{ split} \end{cases}$$

Rk: f is $\sum_{\mathfrak{a} \subset \mathcal{O}_K} q^{Nm(\mathfrak{a})} \psi(\mathfrak{a})$.

Problems: Too few Galois reps b/c of restricted possible images. No one is quite sure which enlargement of Galois group is correct.

"after enlargement of $Gal(F)$, the sure conjecture gives bijection between ρ's and π's

Conjecture There is an "enlarged Galois group" L_F for any global F or local K ~~local field~~ F satisfying (1) - (5) and map

$$\begin{Bmatrix} \text{irred reps} \\ \text{of } L_F \text{ in } GL_n\mathbb{C} \end{Bmatrix} \xleftrightarrow[\text{Langlands}]{\text{global}} \begin{Bmatrix} \text{aut reps} \\ \text{of } GL_n/F \end{Bmatrix}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\begin{pmatrix} \text{reps for } L_K \\ \text{in } GL_n\mathbb{C} \end{pmatrix} \xrightarrow[\text{Langlands}]{\text{local}} \begin{Bmatrix} \text{irr smooth reps} \\ \text{of } GL_n/K \end{Bmatrix}.$$

(1) natural embedding $L_{F_v} \to L_F$ (dist conj class)

(2) Diagram commutes; down right is take piece of restricted tensor product.

(3) Map $L_F \to G_F$

(4) for K non-arch $\exists f$ st.

$$\begin{array}{ccc} L_K & \to & Gal_K \\ f \downarrow & \circlearrowright & \downarrow \\ \mathbb{Z} & \to & \hat{\mathbb{Z}} \end{array}$$

(5) normalization of local Langlands

$$\left(L_{F_v} \to \mathbb{Z} \to \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix} \mathbb{Z} \right) \longleftrightarrow \pi_v(\alpha_1 \ldots \alpha_n)$$

$$m \mapsto \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}^m$$

(Plus continuity + semisimplicity)

Note: That of LCFT $F^\times \to G_F^{ab}$ need to de-$\hat{\mathbb{Z}}$-ify to set up: some problem here.

Simple prediction: without knowing L_F:

π automorphic rep on GL_n $\pi_v \cong \pi_v(\alpha_1 \ldots \alpha_{n_v})$ $\forall v$

π' another such rep GL_m ~~If~~ Conjecture $\Rightarrow$ $\exists$ rep π'' on $GL_{n \cdot m}$

st. $\pi'' \cong \pi_v(\{\alpha_i \beta_j\})$ for almost all v.

Proof: Tensor on Galois rep side.

Called "functoriality". Some of Langlands motivations. Some special cases known $n=m=2$ for example.

Rmk:

Rmk: Which reps for $\{L_{F_v} \to GL_n \mathbb{C}\}$ show up from automorphic reps is ~~...~~ interesting, conjectural.

~~What happens for $\mathbb{R}$ or $\mathbb{C}$?~~

Rmk: Direct sums? Under local Langlands, $\{L_{F_v} \to GL_n(\mathbb{C})\}$ w/ bounded image $\to$ $\left(\begin{array}{c}\text{tempered} \\ \text{irr smooth} \\ \text{reps of } GL_n(F_v)\end{array}\right)$

Note: bounded should be automatic for automorphic stuff

(occurs in L^2, possibly continuously).

$\rho_1 : L_F \to GL_n$ $\rho_2 : L_F \to GL_n$ in class

$\rho_1 \oplus \rho_2 \to GL_{n+m}$. ~~...~~

$\mathrm{Ind}\begin{pmatrix} \boxed{n} & * \\ 0 & \boxed{m} \end{pmatrix}$ is irreducible corresponds to direct sum.

Rmk: Local Langlands for GL_n is known.

What happens for $\mathbb{R}$ and $\mathbb{C}$: $L_{\mathbb{C}} = \mathbb{C}^\times$: why? analogy w/ local CFT,

$L_{\mathbb{R}} =$ "$\mathbb{C}^\times \times \{\pm 1\}$"

Rmk: "Think of Hodge theory as mainly Galois group of $\mathbb{C}$"

$< \mathbb{C}^\times, j \mid j z j^{-1} = \bar{z}, j^2 = -1 >$.

$L_{\mathbb{R}} =$ normalizer of $\mathbb{C}^\times$ in $\mathbb{H}$

Local Langlands for $\mathbb{R}$ or $\mathbb{C}$ $(= K)$

$\{L_K \to GL_n \mathbb{C}\} \longleftrightarrow \left(\begin{array}{c}\text{irr smooth} \\ GL_n K\text{-rep} \\ = (\mathfrak{g}, K) \text{ module}\end{array}\right)$

For bounded (unitary) reps, and ρ irreducible [$\oplus \to$ parabolic induct]

$K = \mathbb{C}$ $\mathbb{C}^\times$: all are dim $\longleftrightarrow \{\text{irr } \mathbb{C}^\times \cdot L_{\mathbb{C}} \text{ rep}\}$

Local Langlands for $\mathbb{R}$: "b/c $\mathbb{R}$ is fun"

irreducible reps of $L_{\mathbb{R}}$? one dim reps factoring through $L_{\mathbb{R}}^{ab} \simeq \mathbb{R}^{\times}$ just as for $\mathbb{C}$.

Note $L_{\mathbb{R}}$ often called Weil group of $\mathbb{R}$.

or

two dim'l: start with character of $\mathbb{C}^{\times}$, induce to $L_{\mathbb{R}}$.

Char: $re^{i\theta} \in \mathbb{C}^{\times} \mapsto r^{s} e^{in\theta}$

parametrized by $s \in \mathbb{C}$, $n \in \mathbb{Z}$. $\operatorname{Re} s = 0$ if want to be bounded.

For $n \neq 0$, induce to irr rep of $L_{\mathbb{R}}$. $n, -n$ give same.

The corresponding rep of $GL_2 \mathbb{R}$: $|\det g|^{s} \otimes \mathcal{D}_n$

So just understand $\mathcal{D}_n$:

What is D_n? Rep. of $GL_2 \mathbb{R}$

$n=1$: $GL_2(\mathbb{R}) \circlearrowright \mathbb{H}^{\pm} = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$

$\circlearrowright$ hol 1-forms on $\mathbb{H}^{\pm}$.

D_1: $\circlearrowright$ L^2 hol 1-forms $[L^2$ means $\int \omega \wedge \bar{\omega} < \infty.]$

provides irreducible unitary rep.

D_n called discrete series

for n odd; L^2 sections of bundle of 1 forms raised to ~~2nd~~ $\frac{n+1}{2}$ power.

Another description: $GL_2 \mathbb{R} \circlearrowright$ hom. funcs on $\mathbb{R}^2$ $f(\lambda x, \lambda y) = \lambda^{k} f(x,y)$.

not obviously unitary. Parabolically induced via unramified det.

If $k \in \mathbb{Z}$, natural subspace of polys of degree k.

Quotient in obvi' category of $(\mathfrak{g}, K)$ mods realizes $D_{k+1} \otimes |\det|$".

Relationship between two descriptions: ?

Third: start with hol modular form of weight $k+1$ $\rightsquigarrow$ automorphic rep the π_{∞}.

$\hookrightarrow$ think of a section of bundle to connect.

Conceptual Explanation: compact shouldn't be characterized by tori + characters

$\mathbb{R}^{\times} \times \mathbb{R}^{\times}$ $\rightsquigarrow$ the ones from parabolic induction.

two tori

$\mathbb{C}^{\times}$ $\rightsquigarrow$ discrete series

How to classify reps of $SL_2\mathbb{R}$ ($(\mathfrak{g},K)$ mods) or for $GL_2\mathbb{R}$.
$K = SO(2)$ are diag. span of h in $sl_2\mathbb{R}$. Use as h in $sl_2\mathbb{C}$ thing:
Pick basis for $sl_2\mathbb{R}$ e, f, h for $\mathfrak{g}\otimes\mathbb{C}$ as usual
If V is $(\mathfrak{g},K)$ mod, split into weight spaces for K: $V = \bigoplus_{n\in\mathbb{Z}} V(n)$

e, f act as raising + lowering operators.
If V irreducible, pick $v \in V(n)$. Then $v, ev, e^2v, \ldots fv, f^2v, \ldots$ span V. issue: some checks that things act as the notation suggests in terms of Lie alg action

So reps are characterized by parity, length, and whether ends; i.e. if eigenvalue is like integer then allow formula to map it to 0.

1/10 Zhiwei: continued
Infinitesimal Characters, Induction, SL_2

$\underline{G}$ connected reductive group $/\mathbb{R}$. $G = \underline{G}(\mathbb{R}) \supset K$ max compact,
$\mathfrak{g} = \mathfrak{g}_{\mathbb{C}} = (\mathrm{Lie}\, G)_{\mathbb{C}}$

$\{$representations of $G\}$ on nice enough inf. dim. space $\longleftrightarrow \{(\mathfrak{g}, K_{\mathbb{C}})\text{-modules}\}$

Σ V with compatible action of $U(\mathfrak{g})$ and $K_{\mathbb{C}}$.

$V \longmapsto \underbrace{V^{sm} \cap V^{(K)}}_{\subset K\text{ finite}}$ ie if K connected, same action as $\mathfrak{k} = \mathrm{Lie}\, K_{\mathbb{C}}$.

Infinitesimal Character: $\mathfrak{Z}(\mathfrak{g}) :=$ center of $U(\mathfrak{g}) = \{z \in U(\mathfrak{g}) \mid [x,z]=0\ \forall x\in\mathfrak{g}\}$
if connected $= U(\mathfrak{g})^{G_{\mathbb{C}}}$