The Jacquet-Langlands correspondence

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Contents

1. Introduction

This paper contains the notes of some lectures I gave during my visit to the Tata Institute of Fundamental Research in Mumbai, in November-December 2001. They present the proof by Deligne, Kazhdan and Vignéras ([DKV]) of the correspondence of Jacquet-Langlands in zero characteristic. The Jacquet-Langlands statement is that there exists a correspondence respecting characters (in a sense to be defined) between the set of equivalence classes of square integrable representations of GL_n over a local field and the set of equivalence classes of square integrable representations of an inner form of GL_n. The main tool is the simple trace formula of Deligne and Kazhdan. Even though all the ideas involved are from [DKV], the proof presented here is simpler. This is because the authors of [DKV] put in their article much more information than is necessary for the proof, intending to do also a survey of some techniques which had become classical in the theory of reductive p-adic groups at that time. The proof here is organized such that the largest possible part be true without any assumption regarding the characteristic of the base field. We have also developed some steps of the proof which may be found in the original papers of Jacquet and Langlands ([JL]) or Flath ([Fl2]) and have only been quoted in [Ro], [DKV] and all the other papers about this subject, wishing to bring them to the attention of the reader. We have tried to give the simplest complete proof, admitting only one result (the result of finitude of [BB], in order to avoid definitions of ϵ-factors and types).

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2. Notations, definitions, general results

Let F be a local non-archimedian field and n a positive integer. Set $G = GL_n(F)$. Let r and d be two positive integers such that $rd = n$. Let D be a central division algebra over F of dimension d^2. Central means that F is the center of D. Set $G' = GL_r(D)$. Then G' is an *inner form of* G that is G and G' are isomorphic over an algebraic closure of F.

If $g \in G$ we say that g is *regular semisimple* if the characteristic polynomial of g has distinct roots over an algebraic closure of F. We say that g is *elliptic* if g is regular semisimple and its characteristic polynomial is irreducible over F (the second condition implies the first only when F is of characteristic zero). Elliptic elements are exactly those elements which do not belong to any proper parabolic subgroup of G. The conjugacy class of a regular semisimple element g (call it a *regular semisimple conjugacy class*) is characterized by the characteristic polynomial of g. Just like for G, any element of G' has a characteristic polynomial which is monic of degree n ([Pi], 16.1). The same definitions and properties hold for elements of G'.

2.1. Admissible representations and their characters.

The groups G and G' inherit a locally compact topology from F. The unit has a basis of open compact neighborhood. A representation (π, V) of G or G' is said to be *smooth* if every $v \in V$ is fixed by an open neighborhood of the unit element under π. π is said to be *admissible* if it is smooth, and for every open compact subgroup K (of G or G') the space V^K of fixed vectors under K by π is finite dimensional. We know that an irreducible smooth representation is always admissible. Suppose (π, V) is an admissible representation of

G. As V is in general not finite dimensional, we cannot in general define $\mathrm{tr}\pi(g)$ for an element g.

Let O_F be the ring of integers of F and O_D be the ring of integers of D. Then $K_0 = GL_n(O_F)$ and $K_0' = GL_r(O_D)$ are maximal (open) compact subgroups of G and G' respectively. Fix Haar measures on G, resp. G', such that the volume of K_0, resp. K_0', is one.

Define the Hecke algebra under convolution $H(G)$ of locally constant and compactly supported functions on G. If π is an admissible representation of G set, for every $f \in H(G)$,

$$\pi(f) = \int_G f(g)\pi(g)dg.$$

Then $\pi(f)$ has finite rank (because of the admissibility of π) hence $\mathrm{tr}\pi(f)$ is well defined. This means that we can look at $\mathrm{tr}\pi$ as a linear functional on $H(G)$, that is as a distribution. Let G^s be the set of regular semisimple elements of G (we call it the *regular semisimple set*). G^s is an dense open subset of G, and its complement has measure zero. By the submersion theorem of Harish-Chandra ([H-C]), the trace distribution of π, when restricted to functions with support in G^s, may always be represented by a function χ_π defined on G^s, which is locally constant and class invariant. This means that for all $f \in H(G)$ with support in G^s, we have

$$\mathrm{tr}\pi(f) = \int_{G^s} \chi_\pi(g)f(g)dg. \tag{1}$$

The same is true for G'.

2.2. **Coefficients; square integrable representations.** Let (π, V) be an admissible representation of G. Pick an element v in V and a linear form v' over V. The formula

$$g \mapsto v'(\pi(g)(v))$$

gives a (locally constant) function on G. Such a function is called a *coefficient* of π. Now, say that π has central character ω if there exists a character ω of the center Z of G such that, for all $z \in Z$ and all $g \in G$, $\pi(zg) = \omega(z)\pi(g)$. A smooth irreducible representation always has a central character (by Schur's lemma). If a representation is unitary then its central character is unitary. We shall say that a non zero function $f : G \to \mathbf{C}$ has central character ω if there exists a character ω of Z such that, for all $z \in Z$ and all $g \in G$, $f(zg) = \omega(z)f(g)$. If π has central character ω then every non zero coefficient of π has central character ω.

Definition. *We say that a smooth irreducible representation π is* cuspidal *if there exists a coefficient f of π, not identically zero, such that the support of f is compact modulo Z.*

Definition. *We say that a smooth irreducible representation π is* square integrable *if π is unitary and if there exists a coefficient f of π, not identically zero, such that $|f|^2$*

is integrable over G/Z.

The same definition holds for G' too (and generally for all connected reductive groups on F).

2.3. **Corresponding elements.**

Definition. *If $g \in G^s$ and $g' \in G'^s$, we shall say that g* corresponds *to g' if they have the same characteristic polynomial. We also write in this case $g \leftrightarrow g'$.*

This correspondence induces an injection from the set of regular semisimple conjugacy classes on G' to the set of regular semisimple conjugacy classes on G (obvious). It may be proved that, when restricted to the elliptic classes, this injection becomes a bijection from the set of elliptic conjugacy classes on G' to the set of elliptic conjugacy classes on G. The last bijection is realized via the set of monic irreducible separable polynomials of degree n in $F[X]$.

2.4. **Statement of the Jacquet-Langlands correspondence.** We can now state the Jacquet-Langlands correspondence. Let $E^2(G)$ be the set of equivalence classes of square integrable representations of G and $E^2(G')$ the set of equivalence classes of square integrable representations of G'.

Jacquet-Langlands. *There exists a (unique) bijection*

$$\mathbf{C} : E^2(G) \to E^2(G')$$

such that for all $\pi \in E^2(G)$,

$$\chi_\pi(g) = (-1)^{n-r}\chi_{\mathbf{C}(\pi)}(g')$$

holds for all $g \leftrightarrow g'$.

Jacquet and Langlands prove this statement in the case $n = 2$ in [JL]. Flath proved it in the case $n = 3$ if $char(F) = 0$ in his thesis ([Fl]). Rogawski gave a proof in the case $r = 1$ for every n provided $char(F) = 0$ in [Ro] and finally Deligne, Kazhdan and Vignéras proved the general case if $char(F) = 0$ ([DKV]). Here I shall talk about a simplified version of the Deligne, Kazhdan and Vignéras proof.

3. Orthogonality relations for characters

Let Z be the center of G. Let dg and dz be Haar measures on G and Z. Let G_e be the set of elliptic elements of G (i.e. elements with irreducible separable characteristic polynomial). A maximal torus T in G is said to be *elliptic* if the quotient T/Z is compact. On every elliptic torus T of G fix a Haar measure dt such that volume of T/Z is one for the quotient measure $d\bar{t} = dt/dz$. Let T^{reg} be the set of elliptic elements of T, W_T the Weyl group of T and $|W_T|$ the cardinality of this group.

If g is an elliptic element of G it belongs to a maximal elliptic torus T and T is its centralizer. Let $D(g)$ be the absolute value of the determinant of the operator $\mathrm{Ad}(g^{-1}) - \mathrm{Id}$ acting on $\mathrm{Lie}(G)/\mathrm{Lie}(T)$.

For every unitary character ω of Z let $L_0(G_e;\omega)$ be the space of functions f defined on G_e with complex values which are locally constant, invariant under conjugation by elements of G and of central character ω.

Let $\mathcal{T}_e$ be a set of representatives of conjugacy classes of elliptic tori. Let $L^2(G_e;\omega)$ be the space of $f \in L_0(G_e;\omega)$ such that

$$\sum_{T\in\mathcal{T}_e} |W_T|^{-1} \int_{T^{reg}/Z} D(\bar{t})|f(\bar{t})|^2 d\bar{t}$$

is convergent (Z act on T^{reg} by multiplication, D and $|f|$ are invariant under this action). Define a scalar product on $L^2(G_e;\omega)$ by:

$$< f_1; f_2 >_e = \sum_T |W_T|^{-1} \int_{T^{reg}/Z} D(\bar{t}) f_1(\bar{t}) \overline{f_2(\bar{t})} d\bar{t}.$$

$(L^2(G_e;\omega), <>_e)$ is a pre-hilbert space.

Clozel showed ([Cl]) that if $char(F) = 0$ then, for every square integrable representation π of G of central character ω, the restriction of χ_π to G_e is in $L^2(G_e;\omega)$ and the set of these functions form an orthonormal family for $< ; >_e$ in $L^2(G_e;\omega)$. This property is usually called *orthogonality of characters on G*. Clozel showed it for every connected reductive group over a local field of zero characteristic. Thus this hold for G' too.

4. Orbital integrals

For every $g \in G^s$, the centralizer of g in G is a maximal torus T_g. Fix Haar measures on maximal tori of G such that if two such tori are conjugated then the measures correspond to each other via conjugation (is independent of choice of the conjugation). For each maximal torus T consider the quotient Haar measure on G/T. If $f \in H(G)$ we may define a map $\Phi : G^s \to \mathbb{C}$ by setting, for all $g \in G^s$,

$$\Phi(f;g) = \int_{G/T_g} f(xgx^{-1})dx.$$

The map Φ is called the *orbital integral* of f. Then Φ is locally constant on G^s and invariant under conjugation.

The submersion theorem of Harish-Chandra ([H-C]) implies a weak converse:

> **Proposition 4.1.** *If Φ is a function on G^s which is locally constant, class invariant and compactly supported mod conjugation then Φ is the orbital integral of a function $f \in H(G)$ supported in G^s.*

The same holds for G' too.

5. Simple trace formula

Let $\mathbb{F}$ be a global field, and G a reductive group defined over $\mathbb{F}$. Let Z be the center of G. For every place v of $\mathbb{F}$ let $\mathbb{F}_v$ be the completion of $\mathbb{F}$ at v, G_v the group $G(\mathbb{F}_v)$ and $Z_v = Z(\mathbb{F}_v)$ the center of G_v. Let $\mathbb{A}$ be the ring of adèles of $\mathbb{F}$. To simplify definitions for choice of maximal compacts and Haar measures, from now on we suppose G is GL_n. Things are trivially the same for its inner forms. For every finite place v let O_v be the ring of integers of $\mathbb{F}_v$, and put $K_v = G(O_v)$. From now on, *for almost every* will mean "for every but a finite number" and *a.e.* ("almost everywhere") will mean "for almost every place v of $\mathbb{F}$". We consider the adèle group $G(\mathbb{A})$ of G, which is the restricted product of G_v with respect to K_v. It is the group of elements $(g_v)_v$ of the direct product of all G_v such that $g_v \in K_v$ a.e..

Let $H(G(\mathbb{A}))$ be the set of functions $\tilde{f} : G(\mathbb{A}) \to \mathbb{C}$ of the form $\tilde{f} = \prod_v f_v$, where f_v is C^∞ with compact support for every infinite place v, $f_v \in H(G_v)$ for every finite place v, and f_v is the characteristic function of K_v a.e..

In the following we'll consider groups $G(\mathbb{F})$ and $Z(\mathbb{F})$ as subgroups of $G(\mathbb{A})$ and $Z(\mathbb{A})$ respectively (diagonal inclusion). Let $\tilde{\omega}$ be a unitary character of $Z(\mathbb{A})$ trivial on $Z(\mathbb{F})$. For every place v we choose a Haar measure on G_v such that, if v is finite, the volume of K_v is one. For every place v, there is an obvious isomorphism between $Z_v = Z(\mathbb{F}_v)$ and $\mathbb{F}_v^*$. Put Haar measures on Z_v such that, if v is finite, the volume of the image of $O_{\mathbb{F}_v}^*$ is one. The product measure gives measures on $G(\mathbb{A})$ and $Z(\mathbb{A})$.

Define $L^2(G(\mathbb{A}); \tilde{\omega})$ as the space of functions $\phi : G(\mathbb{A}) \to \mathbb{C}$ such that

- ϕ is left invariant under $G(\mathbb{F})$
- for all $z \in Z(\mathbb{A})$ and $g \in G(\mathbb{A})$, $\phi(zg) = \tilde{\omega}(z)\phi(g)$ and
- $\int_{G(\mathbb{F})Z(\mathbb{A})\backslash G(\mathbb{A})} |\phi(g)|^2 < \infty$.

It is a Hilbert space with respect to the inner product

$$< \phi \; ; \; \phi' >= \int_{G(\mathbb{F})Z(\mathbb{A})\backslash G(\mathbb{A})} \phi(g)\overline{\phi'(g)}dg.$$

Say that $\phi \in L^2(G(\mathbb{A}); \tilde{\omega})$ is a *cuspidal form* if, for all $g \in G(\mathbb{A})$ and all unipotent radicals N of proper parabolic subgroups of G, we have:

$$\int_{N(\mathbb{F})\backslash N(\mathbb{A})} \phi(ng)dn = 0.$$

Let $L^2_c(G(\mathbb{A}); \tilde{\omega})$ be the subspace of cuspidal forms in $L^2(G(\mathbb{A}); \tilde{\omega})$.

The group $G(\mathbb{A})$ acts on $L^2(G(\mathbb{A}); \tilde{\omega})$ by right translations. Call ρ this representation of $G(\mathbb{A})$. The space $L^2_c(G(\mathbb{A}); \tilde{\omega})$ is stable under ρ. Let ρ_c be the representation of $G(\mathbb{A})$ in $L^2_c(G(\mathbb{A}); \tilde{\omega})$ induced by ρ. Then ρ and ρ_c are unitary representations with central character $\tilde{\omega}$. The representation ρ_c is a reducible representation. We know that it decomposes discretely and every irreducible subrepresentation has finite multiplicity. An irreducible subrepresentation of ρ_c is called *automorphic cuspidal* (it is not necessary to define here what "automorphic representation" means). We know that every automorphic cuspidal representation $\tilde{\sigma}$ of $G(\mathbb{A})$ breaks into a restricted product over v of smooth irreducible representations $\tilde{\sigma}_v$ of G_v. Moreover σ_v is unramified (i.e. has a non zero fixed vector under the maximal compact K_v) a.e. (see [Fl1]).

If $\tilde{f} \in H(G(\mathbb{A}))$ define an operator on $L^2(G(\mathbb{A}); \tilde{\omega})$ by:

$$[\rho(\tilde{f})\phi](g) = \int_{G(\mathbb{A})} \tilde{f}(h)\phi(gh)dh$$

It is a unitary operator.

Let v be a place of $\mathbb{F}$. If π is a cuspidal representation of G_v, with central character ω, then we say that $f \in H(G_v)$ is a *coefficient with compact support* for π if $\mathrm{tr}\pi(f) = 1$ and $\mathrm{tr}\pi'(f) = 0$ for all smooth irreducible representations π' of G_v with central character ω. For every cuspidal representation there always exist coefficients with compact support.

Let X be the set of elliptic orbits in $Z(\mathbb{F})\backslash G(\mathbb{F})$. For every $O \in X$ choose an element $\gamma_O \in O$. Let G_{γ_O} denote the centralizer of γ_O. Put on $G_{\gamma_o}(\mathbb{A})$ the product measure with respect to the local fixed measures (at every place v, $G_{\gamma_o}(F_v)$ is a maximal torus, and for almost every v it is an elliptic torus). If $\tilde{f} \in H(G(\mathbb{A}))$ set

$$\Phi(\tilde{f}; \gamma_O) = \int_{G_{\gamma_O}(\mathbb{A})\backslash G(\mathbb{A})} \tilde{f}(x^{-1}\gamma_O x)\, dx$$

where dx is the quotient measure.

Theorem 5.1. (Simple trace formula) *Let $\tilde{f} \in H(G(\mathbb{A}))$. Suppose there are two finite places v_1 and v_2 such that f_{v_1} is a coefficient with compact support for a cuspidal representation and the orbital integral of f_{v_2} is supported in the elliptic set of G_{v_2}. Then $\rho_c(\tilde{f})$ and $\rho(\tilde{f})$ are operators of trace class and*

$$\mathrm{tr}(\rho_c(\tilde{f})) = \mathrm{tr}(\rho(\tilde{f})) = \sum_{O \in X} vol(G_{\gamma_O}(F)Z(\mathbb{A})\backslash G_{\gamma_O}(\mathbb{A})) \int_{Z(\mathbb{A})} \tilde{\omega}(z)\Phi(\tilde{f}; z\gamma_O)dz.$$

Proof. The proof may be found in [Ro] for example. The fact that the operator $\rho_c(\tilde{f})$ is an operator of trace class is a general result independent of the condition on $\tilde{f}$. The fact $\mathrm{tr}(\rho_c(\tilde{f})) = \mathrm{tr}(\rho(\tilde{f}))$ comes from $\rho(\tilde{f})\big(L^2(G(\mathbb{A}); \tilde{\omega})\big) \subset L^2_c(G(\mathbb{A}); \tilde{\omega})$ which is a consequence of the condition at v_1. I give here the proof of the formula since it is slighter different form the one we usually find (I consider here functions with compact support).

The operator $\rho(\tilde{f})$ in $L^2(G(\mathbb{A}); \tilde{\omega})$ is a kernel operator. One may see that:

$$K(x, y) = \sum_{\gamma \in Z(\mathbb{F})\backslash G(\mathbb{F})} \int_{Z(\mathbb{A})} \tilde{\omega}(z)\tilde{f}(x^{-1}z\gamma y)\, dz$$

is a kernel for $\rho(\tilde{f})$ on the space $G(\mathbb{F})Z(\mathbb{A})\backslash G(\mathbb{A})$. Usually $K(x;x)$ is not integrable, but the condition at v_2 implies

$$K(x;x) = \sum_{\gamma\in Z(\mathbb{F})\backslash G(\mathbb{F})_e} \int_{Z(\mathbb{A})} \tilde{\omega}(z)\tilde{f}(x^{-1}z\gamma x)\,dz,$$

where the index e means that we sum only over the elliptic elements. Then, our $K(x;x)$ IS integrable (see the calculus) and, as we have a trace operator with kernel and the kernel is integrable over the diagonal, we have

$$\mathrm{tr}\rho(\tilde{f}) = \int_{G(\mathbb{F})Z(\mathbb{A})\backslash G(\mathbb{A})} K(x;x)\,dx =$$

$$= \int_{G(\mathbb{F})Z(\mathbb{A})\backslash G(\mathbb{A})} \sum_{\gamma\in Z(\mathbb{F})\backslash G(\mathbb{F})_e} \int_{Z(\mathbb{A})} \tilde{\omega}(z)\tilde{f}(x^{-1}z\gamma x)\,dz\,dx =$$

$$= \int_{G(\mathbb{F})Z(\mathbb{A})\backslash G(\mathbb{A})} \sum_{O\in X}\sum_{\gamma\in O} \int_{Z(\mathbb{A})} \tilde{\omega}(z)\tilde{f}(x^{-1}z\gamma x)\,dz\,dx =$$

$$= \sum_{O\in X}\int_{G(\mathbb{F})Z(\mathbb{A})\backslash G(\mathbb{A})} \sum_{\gamma\in O} \int_{Z(\mathbb{A})} \tilde{\omega}(z)\tilde{f}(x^{-1}z\gamma x)\,dz\,dx =$$

$$= \sum_{O\in X}\int_{G(\mathbb{F})Z(\mathbb{A})\backslash G(\mathbb{A})} \sum_{g\in G_{\gamma_O}(\mathbb{F})\backslash G(\mathbb{F})} \int_{Z(\mathbb{A})} \tilde{\omega}(z)\tilde{f}(x^{-1}g^{-1}z\gamma_O gx)\,dz\,dx =$$

$$= \sum_{O\in X}\int_{G_{\gamma_O}(\mathbb{F})Z(\mathbb{A})\backslash G(\mathbb{A})} \int_{Z(\mathbb{A})} \tilde{\omega}(z)\tilde{f}(x^{-1}z\gamma_O x)\,dz\,dx =$$

$$= \sum_{O\in X} vol(G_{\gamma_O}(\mathbb{F})Z(\mathbb{A})\backslash G_{\gamma_O}(\mathbb{A})) \int_{Z(\mathbb{A})} \tilde{\omega}(z)\Phi(\tilde{f};z\gamma_O)\,dz.$$

□

6. Transfer

Let $G = GL_n(F)$ and $G' = GL_r(D)$ like before. Identify the centers of G and G' and call them both Z. There exists a coherent *bijection* between the set of conjugacy classes of maximal *elliptic* tori of G' and the set of conjugacy classes of maximal *elliptic* tori of G, and also an injection between the set of conjugacy classes of maximal tori of G' and the set of conjugacy classes of maximal tori of G. The first one is, of course, a restriction of the second one. They are defined as follows: let $g \leftrightarrow g'$. As g is regular semisimple, the centralizer T_g of g is a maximal torus of G, equal to the group of invertible elements of the subalgebra $F[g]$ of $M_n(F)$. Also, the centralizer $T'_{g'}$ of g' is a maximal torus of G', equal to the group of invertible elements of the subalgebra $F[g']$ of $M_n(D)$. As g and g' have the same characteristic (= minimal) polynomial, there exist a unique isomorphism from $F[g]$ onto $F[g']$ sending g to g', and this isomorphism induces an isomorphism $i_{g,g'} : T_g \to T'_{g'}$ which send g to g'. It is clear that for all $t \in T_g$, $t \leftrightarrow i_{g,g'}(t)$. Note that

$i_{g,g'}$ is a homeomorphism for the p-adic topologies. If T is a maximal torus in G and T' is a maximal torus in G', we say that T corresponds to T' if there exist $g \in T$ and $g' \in T'$ which correspond to each other.

Fix now Haar measures on the maximal tori of G' such that, if two tori are conjugated, their measures corresponds to each other via conjugation (it is independent of the choice of the conjugation) and suppose that, on elliptic tori T, measures are chosen as in section 3 ($vol(T/Z) = 1$).

Let T be a maximal torus of G. If T corresponds to a torus T' of G', choose $g \in T$ and $g' \in T'$ such that $g \leftrightarrow g'$, and put on T the inverse image by $i_{g,g'}$ of the measure fixed on T'. It is independent of the choice of g, g'. On tori of G' which do not correspond to any torus of G', put any Haar measure such that if two tori are conjugated, their measures correspond to each other via conjugation (it is independent of the choice of the conjugation). We fix from now on measures associated like this on tori of both groups.

Fix a character ω of Z. We obviously may define a map $i : L_0(G_e;\omega) \to L_0(G'_e;\omega)$ (see section 3 for definitions): for every $g' \in G'_e$ take a $g \in G_e$ which corresponds to g' and for every $f \in L_0(G_e;\omega)$ set $i(f)(g') = f(g)$. Using correspondence of elliptic tori classes just discussed, it is clear that the restriction of i to $L^2(G_e;\omega)$ is onto $L^2(G'_e;\omega)$ and is an isometry.

Definition. *If $f \in H(G)$ and $f' \in H(G')$ say that f* corresponds *to f' if f and f' are supported in the regular semisimple set, and if*

- for every $g \leftrightarrow g'$ we have $\Phi(f,g) = \Phi(f',g')$ and

- for all $g \in G^s$ which do not correspond to any $g' \in G'^s$ we have $\Phi(f,g) = 0$

(orbital integrals calculated with the choice of measures like before). Notation: $f \leftrightarrow f'$.

Proposition 4.1 implies now

> **Proposition 6.1.** *a) For every $f' \in H(G')$ supported in the regular semisimple set there exists $f \in H(G)$ such that $f \leftrightarrow f'$.*
>
> *b) For every $f \in H(G)$ supported in the regular semisimple set and whose orbital integral vanishes on every g which does not correspond to any $g' \in G'$ there exists $f' \in H(G')$ such that $f \leftrightarrow f'$.*

This is a weak variant of the definition and proof of transfer, but it is the only one which we may prove in non-zero characteristic. We'll see that this transfer of orbital integrals is sufficient for the proof of the Jacquet-Langlands correspondence.

7. The proof

Let's start the proof of the Jacquet-Langlands statement. I make no assumption on the characteristic of F yet. I suppose $G \neq G'$; if not, all is trivial.

Let π_0 be a square integrable representation of G and ω its (unitary) central character.

7.1. Finding a representation which corresponds to π_0.

7.1.1. *Application of the simple trace formula.* First recall some arithmetical facts. References are [Pi] and [We]. I shall follow [Pi] where the exposition is more clear. When a result I need is proven in [Pi] only in the zero characteristic base field case, I shall add a reference to [We], where the result is proven also if the characteristic of the base field is positive. Let $\mathbb{F}$ be a global field. For every finite place v of $\mathbb{F}$, there is a bijective map INV_v (notation of [Pi]) from the set of isomorphism classes of central simple algebras A_v of dimension n^2 over $\mathbb{F}_v$ onto the set $\{\frac{k}{n},\ k \in \{0,1,...,n-1\}\}$. Also, $INV_v(M_n(\mathbb{F}_v)) = 0$ and for every other (class of) central simple algebra $A_v = M_p(D)$, $q > 1$, with $dim_F D = q^2$ (hence $pq = n$), $INV_v(A_v) = \frac{k}{q}$, with $(k,q) = 1$. In particular, $INV_v^{-1}(\frac{1}{n})$ must always be the class of a division algebra. It is a consequence of a deep result ([Pi], th. 18.5 and [We], th. 4, sect. 6, XIII) that, if for every finite place v of $\mathbb{F}$ we fix an element $a_v \in \{\frac{k}{n},\ k \in \{0,1,...,n-1\}\}$ such that $a_v = 0$ for almost every v and the sum of positive a_v's is an integer, then there exists a central simple algebra A over $\mathbb{F}$ such that, for every infinite place v $A(\mathbb{F}_v) \simeq M_n(\mathbb{F}_v)$, and for every finite place v, $INV_v(A(\mathbb{F}_v)) = a_v$. For a place v of $\mathbb{F}$, we say that A is unramified at v if $A(\mathbb{F}_v) \simeq M_n(\mathbb{F}_v)$. If not, we say that A is ramified at v.

Consider now a global field $\mathbb{F}$ and a central division algebra $\mathbb{D}$ over $\mathbb{F}$ such that:
- there exists a place v_0 of $\mathbb{F}$ such that $\mathbb{F}_{v_0} \simeq F$ and $\mathbb{D}(\mathbb{F}_{v_0}) \simeq M_r(D)$;
- at infinite places $\mathbb{D}$ is nonramified;
- at every place $v \neq v_0$ where $\mathbb{D}$ is ramified, $\mathbb{D}(\mathbb{F}_v)$ is isomorphic to a division algebra over $\mathbb{F}_v$.

The existence of such a $\mathbb{D}$ is a simple consequence of the previous facts (exercice).

Set $\mathbb{D}_v = \mathbb{D}(\mathbb{F}_v)$ for every place v. Let $V = \{v_0, v_1...v_m\}$ be the set of places where $\mathbb{D}$ is ramified. Fix once and for all an isomorphism $\mathbb{D}_{v_0} \simeq M_r(D)$ and isomorphisms $\mathbb{D}_v \simeq M_n(\mathbb{F}_v)$ for all places $v \notin V$. For all v note $GL_n(\mathbb{F}_v)$ by G_v et $\mathbb{D}_v^*$ by G'_v. We shall put $G = GL_n$ and $G' = \mathbb{D}^*$ when no confusion may occurs. In fact, our local G and G' in the statement of the theorem have become, in our global situation, G_{v_0} and G'_{v_0}.

Identify centers of G and G' and call them Z. For every v, fix local measures on G_v, G'_v and $Z(\mathbb{F}_v)$ as in the section 5. The measures on $G(\mathbb{A})$ and $Z(\mathbb{A})$ will be taken to be the product measures. For every v, for every maximal torus of G_v or G'_v, fix a Haar measure as in the section 6 (such that measures correspond to each other). If γ is an elliptic element of $G(\mathbb{A})$ (resp. $G'(\mathbb{A})$), then the measure on $G_\gamma(\mathbb{A})$ (resp. $G'_\gamma(\mathbb{A})$) will be taken to be the product measure of local measures on tori.

Let $v_{m+1} \notin V$ be a finite place of $\mathbb{F}$. Set $S = V \cup \{v_{m+1}\} = \{v_0...v_{m+1}\}$. For all $v \in S$ choose π_v as follows:
- $\pi_{v_0} = \pi_0$;
- for all $v \in V \backslash \{v_0\}$, π_v is the Steinberg representation of G_v;
- $\pi_{v_{m+1}}$ is a cuspidal representation of $G_{v_{m+1}}$.

We know that there exists then a cuspidal automorphic representation $\tilde{\pi}$ such that $\tilde{\pi}_v \simeq \pi_v$ for all $v \in S$ ([AC], lemma 6.5 for example). Let $\tilde{\omega}$ be the central character of $\tilde{\pi}$.

Let $\tilde{f} \in H(G(\mathbb{A}))$ and $\tilde{f}' \in H(G'(\mathbb{A}))$. Say that $\tilde{f}$ and $\tilde{f}'$ *correspond to each other* and write $\tilde{f} \leftrightarrow \tilde{f}'$ if for all $v \in V$ we have $\tilde{f}_v \leftrightarrow \tilde{f'}_v$ (sect. 6) and for all $v \notin V$ we have $\tilde{f}_v = \tilde{f'}_v$ (recall we identified via some fixed isomorphisms G_v and G'_v for all these places).

Denote by ρ_c (resp. ρ'_c) the representations of $G(\mathbb{A})$ (resp. $G'(\mathbb{A})$) in the space of cuspidal forms with central character equal to $\tilde{\omega}$ (see sect. 5).

Proposition 7.1. *Let $\tilde{f} \in H(G(\mathbb{A}))$ and $\tilde{f}' \in H(G'(\mathbb{A}))$ be such that $\tilde{f} \leftrightarrow \tilde{f}'$. Suppose that $\tilde{f}_{v_{m+1}} = \tilde{f}'_{v_{m+1}}$ is a coefficient with compact support of a cuspidal representation of $G_{v_{m+1}}$. Then we have:*

$$\mathrm{tr}\rho_c(\tilde{f}) = \mathrm{tr}\rho'_c(\tilde{f}').$$

Proof. We may apply the simple trace formula to $G(\mathbb{A})$ and $G'(\mathbb{A})$ for $\tilde{f}$ and $\tilde{f}'$. Hypothesis are fulfilled : $\tilde{f}_{v_{m+1}}$ and $\tilde{f}'_{v_{m+1}}$ are coefficients of compact support of a cuspidal representation ; moreover, at every place in $v \in V\backslash\{v_0\}$, the orbital integral of $\tilde{f}'_v$ is supported in the set of elliptic elements (because $\tilde{f}'_v$ itself is supported in the elliptic set, as every regular semisimple element of G'_v is elliptic) and the orbital integral of $\tilde{f}_v$ is supported in the set of elliptic elements because it corresponds to $\tilde{f}'_v$.

After application of the trace formula, it remains to prove that, if X is a set of representatives in $G(\mathbb{F})$ of elliptic conjugacy classes of $Z(\mathbb{F})\backslash G(\mathbb{F})$ and X' is a set of representatives in $G'(\mathbb{F})$ of elliptic conjugacy classes of $Z(\mathbb{F})\backslash G'(\mathbb{F})$, then we have

$$(2) \qquad \sum_{\gamma\in X} vol(G_\gamma(F)Z(\mathbb{A})\backslash G_\gamma(\mathbb{A})) \int_{Z(\mathbb{A})} \tilde{\omega}(z)\Phi(\tilde{f}; z\gamma)dz =$$

$$\sum_{\gamma'\in X'} vol(G'_{\gamma'}(F)Z(\mathbb{A})\backslash G'_{\gamma'}(\mathbb{A})) \int_{Z(\mathbb{A})} \tilde{\omega}(z)\Phi(\tilde{f}'; z\gamma')dz.$$

For every $\gamma' \in X'$, there exists $\gamma \in G(\mathbb{F})$ which has the same characteristic polynomial (take the companion matrix). Then γ is elliptic. Without changing the sum, we may then clearly modify the choice of X such that there exists an injection $j : X' \to X$ such that, if $\gamma \in X'$, the characteristic polynomial of γ equals the characteristic polynomial of $j(\gamma)$. As we have chosen on all groups over $\mathbb{A}$ the product measure, the term concerning γ' on the right hand side of (2) is equal to the term concerning $j(\gamma')$ on the left hand side of (2) (both are the product of a common part, corresponding to a restricted product over places $v \notin V$, by the product over $v \in V$ of local terms which correspond to each other). We shall prove now that all the terms concerning elements $\gamma \in X$ which are not in the image of j vanish.

Let $\gamma \in X$. For every place v, denote by γ_v the image of γ under the inclusion $G(\mathbb{F}) \to G(\mathbb{F}_v)$. The element γ_v is regular semisimple, but need not be elliptic (the characteristic polynomial of γ may be reducible over $\mathbb{F}_v$).

Lemma 7.2. *Suppose that for every $v \in V$ there exists an element $x'_v \in G'_v$ such that $\gamma_v \leftrightarrow x'_v$. Then there exists an element $\gamma' \in \mathbb{D}$ such that γ'_v is a conjugate of γ_v for every place $v \notin V$ and $\gamma_v \leftrightarrow \gamma'_v$ for every $v \in V$.*

Proof. This follows from a more general fact but I shall give here the proof in our particular case. Let P be the characteristic polynomial of γ. P is irreducible over $\mathbb{F}$ because γ is elliptic. The subalgebra of $M_n(\mathbb{F})$ generated by γ is isomorphic to $\mathbb{F}[X]/(P)$. It is a sub-field L of $M_n(\mathbb{F})$ of dimension n over $\mathbb{F}$. For every place $v \in V \backslash \{v_0\}$, $\gamma_v \leftrightarrow x'_v$ which is elliptic, hence $L(\mathbb{F}_v)$ is a field (i.e. L/F is not ramified at v). As the characteristic polynomial of γ_{v_0} is equal to the characteristic polynomial of x'_{v_0} which is a regular semisimple element of $GL_r(D)$, every irreducible factor of P over $\mathbb{F}_{v_0}$ has degree divisible by d. Hence, L is either unramified at v_0, or splits into a product of extensions of $\mathbb{F}_{v_0}$ of degrees which are all divisible by d. As $d \times INV_{v_0}(M_r(D)) \in \mathbb{N}$, we may apply ii), Cor. b, sect. 18.4 in [Pi] (or the equivalence between (ii) and (iii) in prop. 5, sect. 3, XIII, [We] for the case of non zero characteristic) to obtain that L is isomorphic to a subalgebra (actually a sub-field) of $\mathbb{D}$. The image of γ under this isomorphism is an element γ' which has the same characteristic polynomial as γ. $\square$

The lemma implies that for a $\gamma \in X$ which doesn't have the same characteristic polynomial as some $\gamma' \in \mathbb{D}$, there exist $v \in V$ such that the orbital integral of $\tilde{f}_v$ is zero at $z\gamma_v$ for all $z \in Z(\mathbb{F}_v)$. Hence $\Phi(\tilde{f}; z\gamma) = 0$ for all $z \in Z(\mathbb{A})$ and all these terms vanish. $\square$

7.1.2. *First simplification of the equality.* Let T be the (finite) set of places v where G' is nonramified, but $\tilde{\pi}$ is ramified (see sect. 5). Let Y be the set of infinite places of $\mathbb{F}$ (void if the characteristic is non-zero). Set

$$W = S \cup T \cup Y.$$

Set $G_W = \Pi_{v \in W} G_v$, $G'_W = \Pi_{v \in W} G'_v$. Note $\tilde{\pi}_W$ the representation of G_W obtained by restricting $\tilde{\pi}$ at places in W. If $f_W = \prod_{v \in W} f_v \in \prod_{v \in W} H(G_v)$ and $f'_W = \prod_{v \in W} f_v \in \prod_{v \in W} H(G'_v)$, write $f_W \leftrightarrow f'_W$ if for all $v \in V$ we have $f_v \leftrightarrow f'_v$ and for all $v \in W \backslash V$ we have $f_v = f'_v$.

Proposition 7.3. *Let $f_W \leftrightarrow f'_W$. If $f_{v_{m+1}} = f'_{v_{m+1}}$ is a coefficient with compact support of a cuspidal representation of $G_{v_{m+1}} = G'_{v_{m+1}}$ we have:*

$$\mathrm{tr}\tilde{\pi}_W(f_W) = \sum_{\tilde{\pi}' \in U'} m(\tilde{\pi}')\mathrm{tr}\tilde{\pi}'_W(f'_W)$$

where

- U' is the set of automorphic cuspidal representations $\tilde{\pi}'$ of $G'(\mathbb{A})$ such that for all $v \notin W$ we have $\tilde{\pi}'_v \simeq \tilde{\pi}_v$,

- $m(\tilde{\pi}')$ is the multiplicity of $\tilde{\pi}'$ in ρ'_c.

Proof. The proof is based on the following lemma. Let $\{F_i\}_{i\in I}$ be a countable set of non-archimedian local fields. For every $i \in I$, let O_i be the ring of integers of F_i. Put $K_i = GL_n(O_i)$. Let H be the restricted product of $GL_n(F_i)$ over I with respect to compacts K_i. Let K be the product of all K_i's. Let M be the set of product functions $f = \prod_{i\in I} f_i$, $f_i \in H(GL_n(F_i))$ such that

- for all i, f_i is bi-invariant (i.e. right and left invariant) under K_i, and
- for all but a finite number of indices i, f_i is the characteristic function of K_i.

Let $\{\pi_j\}_{j\in J}$ be a countable set of irreducible, mutually non-equivalent representations of H such that every π_j is a restricted product (see [Fl1]) of unitary and unramified representations $\pi_{v,i}$ of the $GL_n(F_i)$'s.

Lemma 7.4. *Suppose given complex numbers c_j such that for every $f \in M$, the series*

$$\sum_{j\in J} c_j \mathrm{tr}\pi_j(f) \tag{3}$$

converges absolutely to zero. Then all the c_j are zero.

Proof. A proof of this lemma may be found in [Fl2]. I give here a sketch. Any unramified representation of GL_n is a sub-quotient of a parabolic induced representation from a character $|\cdot|^{z_1} \otimes |\cdot|^{z_2} \otimes ... \otimes |\cdot|^{z_n}$, ($z_i \in \mathbb{C}$) of the diagonal torus. Conversely, every such induced representation has only one unramified sub-quotient. Then the set of equivalence classes of unramified representations of GL_n is in bijection with $\mathbb{C}^n$ modulo relations $z_i \cong z'_i$ if $z_i - z'_i \in 2\pi i\mathbb{Z}$ and modulo permutation of components. It is also known that an unramified representation is unitary iff, for all i, $|Re(z_i)| \leq \frac{1}{2}$. This gives a structure of compact topological space to the set of equivalence classes of unramified unitary representations of GL_n. Using this, put a topology on the set of all representations of H which are locally unitary and unramified. Denote R this space. It is a compact space (product of compact spaces). With each f like in the lemma, we associate a function $F_f : R \to \mathbb{C}^n$ by the formula $F_f(\pi) = \mathrm{tr}\pi(f)$. The set of all these functions verifies the Stone-Weierstrass conditions:

- it contains the constant functions (consider f equal to a scalar multiple of the characteristic function of K);
- it is stable by complex conjugation because, if we put $f^*(g) = \overline{f(g^{-1})}$, then for every locally unitary representation π of H we have $\mathrm{tr}\pi(f^*) = \overline{\mathrm{tr}\pi(f)}$, hence $F_{f^*} = \overline{F_f}$.
- it separate points, as locally this is a classical result (particular case of [Be], cor. 3.9 for example).

Hence, the set of all functions F_f, $f \in M$, is a dense subset of the set $C(R)$ of continuous functions on R. Now, putting $f = 1$ in formula (3), the absolute convergence implies that $\sum_J |c_j|$ converges. Then choose u in J such that $|c_u|$ is maximal. Suppose $|c_u| \neq 0$. Choose a finite subset J_0 of J such that $\sum_{J\backslash J_0} |c_j| < \frac{|c_u|}{4}$. The density result implies that we may find a function f such that

- $|\mathrm{tr}\pi_j(f)| < 2$ for all $j \in J$,
- $|\mathrm{tr}\pi_u(f)| > 1$ and
- $|\mathrm{tr}\pi(f)| < \frac{|c_u|}{2|J_0|}$ for all $\pi \in J_0\backslash\{u\}$.

Then

$$| \sum_{j\in J\backslash\{u\}} c_j \mathrm{tr}\pi_j(f)| \leq \sum_{j\in J\backslash\{u\}} |c_j||\mathrm{tr}\pi_j(f)| <$$

$$< (|J_0|-1)\frac{|c_u|}{2|J_0|} + 2\frac{|c_u|}{4} < |c_u| < |c_u \mathrm{tr}\pi_u(f)|$$

which contradicts the hypothesis. Hence $|c_u|$ must be 0. □

Now we prove the proposition. Write

$$\mathrm{tr}\rho_c(\tilde{f}) - \mathrm{tr}\rho'_c(\tilde{f}') = 0$$

Decompose ρ_c and ρ'_c as a sum of irreducible representations. The index set I in the lemma will be the set of places $v \notin W$. The restricted product of $GL_n(\mathbb{F}_v)$'s, $v \notin W$ will play the role of the space H in the lemma. And the restriction to H of the irreducible sub-representations of ρ_c and ρ'_c which are unramified at every place $v \notin W$ will give the set $\{\pi_j\}$ in the lemma. Let us determine the coefficients c_j in the last relation. Take an irreducible sub-representation $\tilde{\pi}_i$ of ρ_c or ρ'_c. Let $\pi_{i,H}$ denote the restriction of $\tilde{\pi}_i$ to H. Let $U_{\tilde{\pi}_i}$ (resp. $U'_{\tilde{\pi}_i}$) be the set of automorphic cuspidal representations $\tilde{\pi}'$ of $G(\mathbb{A})$ (resp. of $G'(\mathbb{A})$) whose restriction to H is equivalent to $\pi_{i,H}$. By multiplicity one and the strong multiplicity one theorem for $G = GL_n$ ([Sh]), if $\tilde{\pi}_i$ is a sub-representation of ρ_c, then $U_{\tilde{\pi}_i}$ has a unique element (which must then be $\tilde{\pi}_i$) which appears with multiplicity one. The contribution of the terms in $U_{\tilde{\pi}_i} \cup U'_{\tilde{\pi}_i}$ to the sum may be written

$$c_{\tilde{\pi}_i} \mathrm{tr}\tilde{\pi}_{i,H}(\prod_{v\notin W} \tilde{f}_v)$$

($\tilde{f}_v = \tilde{f}'_v$ at these places), where $c_{\tilde{\pi}_i}$ is

$$c_{\tilde{\pi}_i} = \mathrm{tr}\tilde{\pi}_{i,W}(\tilde{f}_W) - \sum_{\tilde{\pi}'\in U'_{\tilde{\pi}_i}} m(\tilde{\pi}')\mathrm{tr}\tilde{\pi}'_W(\tilde{f}'_W)$$

where $m(\tilde{\pi}')$ is the multiplicity of $\tilde{\pi}'$ in ρ'_c. For every $v \notin W$, we let $\tilde{f}_v = \tilde{f}'_v$ vary in the set of bi-invariant under K_v functions in $H(G_v)$. For all such $\tilde{f}$, for every representation $\tilde{\sigma}$ which is ramified at at least one place $x \notin W$ we have $\mathrm{tr}\tilde{\sigma}(\tilde{f}) = 0$ because $\mathrm{tr}\tilde{\sigma}_x(\tilde{f}_x) = 0$. In the sum remains then only terms unramified at all the places $v \notin W$. For such representations, if we fix $\tilde{f}_W$ and $\tilde{f}'_W$, then the number of $c_{\tilde{\pi}_i}$ which do not vanish is finite (because it corresponds to cuspidal automorphic representations which have the same central character, and have a fixed non-zero vector under a given compact subgroup of the adèle group). We are then in the situation of the preceding lemma and we may conclude that all coefficients are 0. In particular $c_{\tilde{\pi}} = 0$, because $\tilde{\pi}$ is unramified at places $v \notin W$. This proves the proposition. $\square$

7.1.3. *Finitude on the G' side.*

Proposition 7.5. *U' is a finite set.*

Proof. Admitted. See [BB].

7.1.4. *New simplifications.* Recall $V = \{v_0, v_1 ... v_m\}$ is the set of places where G' is ramified and we have put $S = V \cup \{v_{m+1}\}$.

Proposition 7.6. *Let U'' be the set of those $\tilde{\pi}' \in U'$ such that $\tilde{\pi}'_{v_{m+1}} \cong \tilde{\pi}_{v_{m+1}}$. Then, for all $f_V = \prod_{v \in V} f_v \in \prod_{v \in V} H(G_v)$ and $f'_V = \prod_{v \in V} f_v \in \prod_{v \in V} H(G'_v)$ such that $f'_v \leftrightarrow f_v$ for all $v \in V$ we have:*

$$\mathrm{tr}\tilde{\pi}_V(f_V) = \sum_{\tilde{\pi}' \in U''} m(\tilde{\pi}')\mathrm{tr}\tilde{\pi}'_V(f'_V).$$

Proof. First put in the formula, at place v_{m+1}, a coefficient with compact support of $\tilde{\pi}_{v_{m+1}}$. Then the traces of all representations in $U' \backslash U''$ are zero. Now apply the same proof as for the first simplification (prop.7.3), replacing lemma 7.4 by the theorem of linear independence of a finite number of characters on the finite product $\prod_{v \in W \backslash S} GL_n$. $\square$

Proposition 7.7. *a) For all representations $\tilde{\pi'} \in U''$, for all $v \in V \backslash \{v_0\}$, $\tilde{\pi'}_v$ is the trivial representation of G'_v.*
b) We have:

$$\chi_{\pi_0}(g) = (-1)^{(n-1)m} \sum_{\tilde{\pi}' \in U''} m(\tilde{\pi}')\chi_{\tilde{\pi}'_{v_0}}(g') \qquad \forall g \leftrightarrow g'. \tag{4}$$

Proof. a) The local components of $\tilde{\pi}_V$ at places $v_1, v_2 ... v_m$ are Steinberg representations. The character of a Steinberg representation is equal to $(-1)^{n-1}$ on the elliptic set (general fact). Use then like before the independence of characters on G'_{v_i} (the character of the trivial representation is 1 on the elliptic set).

b) It is clear now from a) that we may simplify at places $v \in V \backslash \{v_0\}$. It is also clear that then the sign $(-1)^{(n-1)m}$ appears. We have

$$\mathrm{tr}\pi_0(f) = (-1)^{(n-1)m} \sum_{\tilde{\pi}' \in U''} m(\tilde{\pi}')\mathrm{tr}\tilde{\pi}'_{v_0}(f') \qquad \forall f \leftrightarrow f'. \tag{5}$$

Let us explain how we may pass from the distributions relation to characters relation. Choose any $g \leftrightarrow g'$. As we have a finite number of representations, there exist neighborhoods N and N' of g and g' such that χ_{π_0} is constant, equal to $\chi_{\pi_0}(g)$ in N, and every $\chi_{\tilde{\pi}'_{v_0}}$ is constant, equal to $\chi_{\tilde{\pi}'_{v_0}}(g')$ on N'. Using the submersion theorem of Harish-Chandra, we may suppose N is a crossed product of a neighborhood V_1 of g in the torus T_g containing no two conjugate elements and a neighborhood V_2 of 1 in G/T_g which acts by conjugation on V_1, and N' is the crossed product of the neighborhood $i_{g,g'}(V_1)$ (see sect. 6) of g' in the torus $T_{g'}$ and a neighborhood V_2 of 1 in $G'/T_{g'}$. If measures on tori are associated like in sect. 6, then one may see that $\frac{1}{vol\, V_2} 1_N \leftrightarrow \frac{1}{vol\, V'_2} 1_{N'}$. We may apply the relation 5 to those functions. But in the mean time we have $\mathrm{tr}\pi_0(1_N) = \chi_{\pi_0}(g)\, vol(N)$ and $\mathrm{tr}\tilde{\pi}'_{v_0}(1_{N'}) = \chi_{\tilde{\pi}'_{v_0}}(g')\, vol(N')$. By a well known calculus, $vol(N) = vol(V_2) \int_{V_1} D(t)\, dt$. We also have $vol(N') = vol(V'_2) \int_{i_{g,g'}(V_1)} D(i_{g,g'}(t))\, dt = vol(V'_2) \int_{V_1} D(t)\, dt$ (the function D has been defined at sect. 3).

We get then after simplifications:

$$\chi_{\pi_0}(g) = (-1)^{(n-1)m} \sum_{\tilde{\pi}' \in U''} m(\tilde{\pi}')\chi_{\tilde{\pi}'_{v_0}}(g').$$

which is the required relation. $\square$

7.1.5. *Representations on the G' side are square integrable.* Let F be a non-archimedian local field. Set $G = GL_n(F)$ and let $G' = GL_r(D)$ be an inner form of G, where D is a central division algebra of finite dimension d^2 over F. Let L' be a standard Levi subgroup of G', i.e. the product $GL_{n'_1}(D) \times GL_{n'_2}(D) \times ... \times GL_{n'_p}(D)$ "diagonally" embedded in G'. Let P' be the parabolic subgroup containing L' as a Levi subgroup and the upper triangular invertible matrix group (such a group will be called "standard parabolic subgroup"). Let L be the standard Levi subgroup of G corresponding to L', i.e. the product $GL_{dn'_1}(F) \times GL_{dn'_2}(F) \times ... \times GL_{dn'_p}(F)$ "diagonally" embedded in G. Let P be the (standard) parabolic subgroup of G which contains L as a Levi subgroup and the upper triangular invertible matrix group.

Let π be an irreducible smooth representation of G and π'_i, $i \in \{1, 2...k\}$ irreducible smooth representations of G'. Suppose there exists complex numbers a_i, $i \in \{1, 2...k\}$, such that

$$\chi_\pi(g) = \sum_{i=1}^{k} a_i \chi_{\pi'_i}(g')$$

for all $g \leftrightarrow g'$. We have the following proposition following [DKV].

Proposition 7.8. *We have*

$$\chi_{res_P^G \pi}(g) = \sum_{i=1}^{k} a_i \chi_{res_{P'}^{G'} \pi'_i}(g')$$

for all $g \in L^s$, $g' \in L'^s$ such that $g \leftrightarrow g'$.

Proof. ([DKV]) Let $g' \in L'$ be a regular semisimple element and let $g \in L$ such that $g \leftrightarrow g'$. Write $g = (g_1, g_2...g_p)$, $g_i \in GL_{dn'_i}(F)$, and $g' = (g'_1, g'_2...g'_p)$, $g'_i \in GL_{n'_i}(D)$.

Then we have $g_i \leftrightarrow g'_i$. Assume firs that g' is elliptic. We shall use the result of [Ca1] which we translate here in the simple case of our particular groups. Put

$$N(g) = \sup_{1 \leq i \leq p-1} \frac{|det(g_i)|_F^{n'_{i+1}}}{|det(g_{i+1})|_F^{n'_i}}.$$

This definition obviously extends to any element in L. Let $z = (z_1, z_2...z_p)$ in the center of L identified with the center of L' such that $N(z) < N(g)^{-1}$. Then we have $N(zg) < 1$. Take a finite extension F' of F which splits the characteristic polynomial of zg. For all i, the characteristic polynomial of $z_i g_i$ is irreducible over F, and its roots in F' have the same absolute value a_i. Moreover, if $i > j$, then $a_i > a_j$. Then zg is semisimple. For the same reasons, the parabolic group P_{zg} associated to an element zg in [Ca1] is nothing but P, and the th. 5.2 of [Ca1] implies $\chi_\pi(zg) = \chi_{res_P^G \pi}(zg)$. The same holds for g' and we have, for all $i \in \{1, 2...k\}$, $\chi_{\pi'_i}(zg) = \chi_{res_{P'}^{G'} \pi'_i}(zg')$. We then have

$$\chi_{res_P^G \pi}(zg) = \sum_{i=1}^{k} a_i \chi_{res_{P'}^{G'} \pi'_i}(zg').$$

Now write the character of each representation as the sum of characters of its irreducible sub-factors. After this operation, our relation becomes: a sum of characters of irreducible representations of L applied to zg equal a sum of characters of irreducible representations of L' applied to zg'. But every irreducible smooth representation has a central character, and then the z may be removed using the next lemma:

Lemma 7.9. *Let c be a real positive number. Let $\omega_1, \omega_2...\omega_v$ be distinct characters of the center Z_L of L and $b_1, b_2...b_v$ complex numbers such that:*

$$\forall z \in Z_L \text{ such that } N(z) < c, \quad \Sigma_{i=1}^{v} b_i \omega_i(z) = 0.$$

Then $b_i = 0$ for all i.

Proof. Suppose v is the smallest positive integer such that

$$\sum_{i=1}^{v} b_i \omega_i(z) = 0 \tag{6}$$

for all $z \in Z_L$ with $N(z) < c$ and at least one b_i doesn't vanish.

Then $v \geq 2$ and every b_i doesn't vanish. Let $z_0 \in Z_L$ such that $N(z_0) < 1$. Then, for every z such that $N(z) < c$ we have $N(z_0 z) < c$ hence $\Sigma_{i=1}^{v} b_i \omega_i(z_0 z) = 0$ which gives $\Sigma_{i=1}^{v} b_i \omega_i(z_0) \omega_i(z) = 0$. Multiplying (6) by $\omega_1(z_0)$ and subtracting with the last obtained relation, we find another (6)-type relation. The number of characters being inferior, all the coefficients vanish. Then, for every $i \in \{1, 2..v\}$ we must have $\omega_i(z_0) = \omega_1(z_0)$. In particular,

$$\omega_1(z_0) = \omega_2(z_0) \qquad \forall z_0 \text{ such that } N(z_0) < 1.$$

Then, since the ω_i's are characters, we also have

$$\omega_1(z_0^{-1}) = \omega_2(z_0^{-1}) \qquad \forall z_0 \text{ such that } N(z_0) < 1.$$

As every $h \in Z_L$ may be written $h = xy^{-1}$ where $N(x) < 1$ and $N(y) < 1$ (take y such that $N(y) < 1$ and $N(hy) < 1$, then set $x = hy$) we find $\omega_1 = \omega_2$ which is impossible. □

The proposition is proved if g' is elliptic. If g' is not elliptic, then there exists a proper standard Levi subgroup H' of L', such that g' is conjugated in L' with an elliptic element $h' \in H'$. If H is the standard Levi subgroup of L corresponding to H', there is an element h in H conjugated to g in L. It is clear that $h \leftrightarrow h'$. Since characters of representations are invariant by conjugation, it is sufficient to prove the proposition for h and h' instead of g and g'. But this is a consequence of the preceding proof (elliptic case) applied to G, G', H, H' and to L, L', H, H' and the transitivity of the Jacquet functor. □

Proposition 7.10. *Representations $\tilde{\pi}'_{v_0}$, $\tilde{\pi}' \in U''$, are all square integrable.*

Proof. ([DKV]) Let P' be a standard subgroup of G', and P the standard parabolic subgroup of G corresponding to P'. The restriction of π_0 to P is either zero or irreducible ([Ze]). We apply the last proposition to the relation (4) for P and P'. If $res_P^G \pi_0$ is zero, the restriction of every representation $\tilde{\pi}'_{v_0}$ must be also zero, by independence of characters and the fact that the coefficients in the sum are all positive. If $res_P^G \pi_0$ is an irreducible representation σ then for the same reasons (idependence of characters and positivity of coefficients), for every π'_i, for every irreducible sub-quotient σ' of $res_{P'}^{G'} \pi'_i$, the central character of σ' equals the one of σ. Then, applying the Casselman's criterion ([Ca2], 4.4.6), we find that π_0 square integrable implies all π'_i are square integrable. □

7.1.6. *Using the orthogonality to conclude.* The previous equality between characters may be written, when restricted to the elliptic set

$$i(\chi_{\pi_0}) = (-1)^{(n-1)m} \sum_{\tilde{\pi}' \in U''} m(\tilde{\pi}') \chi_{\tilde{\pi}'_{v_0}}(g')$$

(see sect. 6). Now *assume the orthogonality of characters is true for both groups.* It is always the case in zero characteristic. Then the norm of the left hand term is equal to the norm of the right hand teem. This gives

$$1 = \sum_{\tilde{\pi}' \in U''} m(\tilde{\pi}')^2.$$

It means that U'' has exactly one element $\tilde{\pi}'$, and $m(\tilde{\pi}') = 1$. Using (4), we have

$$\chi_{\pi_0}(g) = (-1)^{(n-1)m}\chi_{\tilde{\pi}'_{v_0}}(g')$$

for all $g \leftrightarrow g'$.

7.2. **Definition of C.** The construction we made works for any square integrable representation π_0. Set

$$\mathbf{C}(\pi_0) = \tilde{\pi}'_{v_0}.$$

We have then, for every $\pi_0 \in E^2(G)$,

$$\chi_{\pi_0}(g) = (-1)^{(n-1)m}\chi_{\mathbf{C}(\pi_0)}(g')$$

for all $g \leftrightarrow g'$. Now, if St_G is the Steinberg representation of G, then its character is constant $(-1)^{n-1}$ on the elliptic set. Then the character of $\mathbf{C}(St_G)$ must be constant, equal to $(-1)^{(n-1)(m-1)}$ on the elliptic set of G'. As the character of the Steinberg representation $St_{G'}$ of G' is constant $(-1)^{n-r}$ on the elliptic set of G', it is clear that $\mathbf{C}(St_G) = St_{G'}$, by the orthogonality of the characters of $\mathbf{C}(St_G)$ and $St_{G'}$. This proves that $(-1)^{(n-1)m} = (-1)^{n-r}$, hence

$$\chi_{\pi_0}(g) = (-1)^{n-r}\chi_{\mathbf{C}(\pi_0)}(g')$$

for all $g \leftrightarrow g'$, as required by the Jacquet-Langlands statement.

7.3. **Injectivity of C.** The assumption $\mathbf{C}(\pi_0) = \mathbf{C}(\pi_1)$ gives us equality of characters of π_0 and π_1 only on the set of regular semisimple elements of $G(F)$ which correspond to some $g' \in G'(F)$. But this set always contains the set of elliptic elements. And the orthogonality of characters on $G(F)$ implies that two *square integrable* representations could not have the same character on the elliptic set. (It is not true in general: a square integrable representation has the same character on the elliptic set as another representation on GL_n if $n > 3$.)

7.4. **Surjectivity of C.** We would like to do the same construction in the other sense: take a square integrable representation π' of G' and put it in a global situation and finally find something on G which corresponds to. But the problem is that I don't know how to prove that a square integrable representation of groups other than GL_n may always be a local component of an automorphic cuspidal one. I'll have to use once more the orthogonality of characters.

First, *suppose that G' is the group of invertible elements of a division algebra ($r = 1$).* Then every representation of G' is cuspidal, and we know that a cuspidal representation may always be a local component of an automorphic cuspidal one (true for any reductive group over a local field, [He] appendix 1). Then we may do the construction in the other sense and get surjectivity and complete correspondence with a division algebra (up to sign). Now, as in this case G' is compact mod Z, the set of restriction of all characters of representations of G' to G'_e form a *complete* orthonormal family in the prehilbert space $L^2(G'_e;\omega)$. The first corollary is that the same happens on G since we established a surjective correspondence here.

Now, *return to the general case.* Suppose $\mathbf{C}$ is not surjective. But, if $\pi' \in E^2(G',\omega)$ is not in the image, it means by orthogonality relations that it is orthogonal to the image. And as the image contains a complete orthonormal set in $L^2(G'_e;\omega)$ (because i is an isometry between $L^2(G_e;\omega)$ and $L^2(G'_e;\omega)$) it must be zero. Contradiction with the fact that it has norm 1.

The proof is done. □

8. Conclusion

This proof works in all characteristics except for the problem of orthogonality of characters. It has been used in three places: construction, injectivity, surjectivity.

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