I have had the good fortune to learn most of the material in these notes from lectures given by Robert Kottwitz at The University of Chicago. I am making use of some unpublished notes of Casselman [4] and Bernstein [3, 2]. The notes you are reading come with no warranty – there are hundreds of millions of mistakes in them. Indeed, no one (including the author) has taken time to read what occurs between pages 79 and 115 in a serious way. By reading these notes, you agree to send me a list of all the mistakes you find. I thank Brian Conrad, Florian Herzig, Christopher Malon, and Joseph Rabinoff for helping to identify many mistakes. I thank Brian Conrad for suggesting many substantive improvements.

1. Nonarchimedean local fields

We fix some notation and spend a little time recalling basic facts.

We let k denote a nonarchimedean local field with finite residue field $\mathfrak{f}$, ring of integers[1] R, maximal ideal $\wp$, and residue field $\mathfrak{f} = R/\wp$. We fix a uniformizer $\varpi \in k$ (that is, an element of R such that $\wp = \varpi R$). Note that $\wp^n = \varpi^n R$ for all $n \in \mathbb{Z}$. We let v denote a valuation on k, normalized such that $v(k^\times) = \mathbb{Z}$. We suppose that the cardinality of $\mathfrak{f}$ is q.

Example 1.0.1. Up to isomorphism, the field k is either a finite extension of $\mathbb{Q}_p$ (the p-adic completion of $\mathbb{Q}$), or k is the field of Laurent series in an indeterminate t over the finite field $\mathfrak{f}$.

In the former case, k has characteristic zero. We present a way to construct the field $\mathbb{Q}_p$. Let p be any prime. If r is a nonzero rational number, then there exists a unique integer ℓ such that $r = p^\ell \cdot a/b$ with $p \nmid a$ and $p \nmid b$. The *p-adic absolute value* $|\cdot|_p$ on $\mathbb{Q}$ is defined by $|r|_p = 0$ if $r = 0$ and $|r|_p = p^{-\ell}$ otherwise. The p-adic absolute value has the following properties.

Exercise 1.0.2. If r_1 and r_2 are rational numbers, then

(1) $|r_1|_p \geq 0$, and $|r_1|_p = 0$ if and only if $r_1 = 0$,
(2) $|r_1 \cdot r_2|_p = |r_1|_p \cdot |r_2|_p$, and
(3) $|r_1 + r_2|_p \leq \max(|r_1|_p, |r_2|_p)$.

Exercise 1.0.3. In the last item of the previous exercise, show that if $|r_1|_p \neq |r_2|_p$ then the inequality is an equality. Is the converse of this statement true?

From Exercise 1.0.2, it follows that we can define a metric on $\mathbb{Q}$ with respect to the p-adic absolute value. We define $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$. The p-adic absolute value on $\mathbb{Q}$ extends continuously (and uniquely) to an absolute value $|\cdot|_p : \mathbb{Q}_p \to \{0, p^k \mid k \in \mathbb{Z}\}$. We define the valuation v on $\mathbb{Q}_p$ by $|x|_p = p^{-v(x)}$ for $x \in \mathbb{Q}_p^\times$ and $v(0) = \infty$. Thus $v(p^m) = m$ for $m \in \mathbb{Z}$.

Exercise 1.0.4. Fix $\alpha \in \mathbb{R}_{>0}$. Show that $|\cdot|_p^\alpha$ satisfies all three parts of Exercise 1.0.2, and that the resulting metric recovers $\mathbb{Q}_p$ upon completion. For which $\alpha \in \mathbb{R}_{>0}$ does $|\cdot|^\alpha$ satisfy the triangle inequality, where $|\cdot|$ is the usual absolute value on $\mathbb{R}$?

[1] It is far more common to denote the ring of integers by the symbol $\mathcal{O}$. However, this is the notation we shall use for nilpotent orbits.

The *ring of integers* of $\mathbb{Q}_p$ consists of those elements of $\mathbb{Q}_p$ with nonnegative or infinite valuation. It can also be identified with the completion of $\mathbb{Z}$ with respect to $|\cdot|_p$, and so is usually denoted by $\mathbb{Z}_p$. The maximal ideal of $\mathbb{Z}_p$ is the principal ideal (p), so we can take p to be the uniformizer for $\mathbb{Q}_p$. The residue field is $\mathbb{Z}/p\mathbb{Z}$, the field with p elements.

In the latter case of Example 1.0.1, k has positive characteristic (equal to the characteristic of $\mathfrak{f}$), R is the ring of power series in the indeterminate t having coefficients in $\mathfrak{f}$, and we can take $\varpi = t$.

For $x \in k^\times$ there exists a unique $n = v(x) \in \mathbb{Z}$ and unit $u \in R^\times$ so that $x = u\varpi^n$. (Consequently, $k^\times$ is isomorphic to $\mathbb{Z} \times R^\times$ as a topological group.) For $x \in k$ we define

$$|x| := \begin{cases} q^{-v(x)} & \text{if } x \in k^\times \\ 0 & \text{otherwise.} \end{cases}$$

As in Exercise 1.0.2, this defines a norm. With respect to the metric defined by this norm, we have that k is a complete Hausdorff topological ring and that the map from $k^\times$ to itself which sends x to x^{-1} is continuous. For each $n \in \mathbb{Z}$, the sets $\varpi^n R = \wp^n$ can be written as either

$$\{x \in k \mid |x| \leq q^{-n}\} \quad \text{or} \quad \{x \in k \mid |x| < q^{(1-n)}\}$$

and so are simultaneously open and closed. In fact, because $\mathfrak{f}$ is finite, these sets are also compact. Thus the ideals $\wp, \wp^2, \wp^3, \ldots$ form a neighborhood-basis of the identity consisting of compact open subgroups. (So, in particular, k is a totally disconnected topological space.)

Exercise 1.0.5. Show that $k/\wp^n$ is countable for any $n \in \mathbb{Z}$.

We have just shown that the additive topological group k has the most important algebraic-topological properties of nearly all of the groups that we will study. We give a name to these properties:

Definition 1.0.6. If G is a Hausdorff topological group such that

(1) G has a countable neighborhood-basis of the identity consisting of compact open subgroups, and
(2) for any open subgroup K of G, the quotient space G/K is countable,

then we say that G is a *t.d.-group.*

Note that an open subgroup of a t.d.-group is a t.d.-group, and a closed subgroup of a t.d.-group is a t.d.-group.

Remark 1.0.7. The t.d.-group terminology is not standard. It is a specialization of the standard concept of an ℓ-group, which is a Hausdorff topological group with a neighborhood-basis of the identity consisting of compact open subgroups.

Exercise 1.0.8. Prove that an ℓ-group is totally disconnected, meaning the only connected subsets are the points. For a challenge, try proving the following converse: a Hausdorff, locally compact, and totally disconnected group is an ℓ-group.

2. Representation theory of $\mathrm{GL}_1(k)$

We now look at the representation theory of $\mathrm{GL}_1(k) = k^\times$.

The group $k^\times$ is a Hausdorff topological group with a filtration by compact open subgroups:

$$k^\times \supset R^\times \supset (1+\wp) \supset (1+\wp^2) \supset (1+\wp^3) \supset \cdots \supset \{1\}.$$

For notational ease, for $n \geq 1$ we define $K_n := 1 + \wp^n$. The collection $\{K_n \,|\, n \geq 1\}$ is a neighborhood basis of the identity. Note that every compact open subgroup of $k^\times$ is contained in $R^\times = R \setminus \wp$, making it the unique maximal compact open subgroup of $k^\times$.

The filtration $\{K_n\}$ and the following exercise show that $k^\times$ is also a t.d.-group.

Exercise 2.0.9. (1) Let G be a topological group with a countable base of open sets. Show that for every open subgroup $K \subset G$, the set G/K is countable.
(2) Show that $k^\times$ has a countable base of open sets.

2.1. Some basic definitions.

Let G denote any t.d.-group. Keep $k^\times$ in mind as a model for G.

A *representation* of G is a pair (π, V) where

(1) V is a complex vector space, and
(2) π is a homomorphism from G to $\mathrm{Aut}_{\mathbb{C}}(V)$.

For two representations (π, V) and (σ, W) of G, the set of morphisms of (π, V) into (σ, W), denoted $\mathrm{Hom}_G(V, W)$, is the space of linear maps $f\colon V \to W$ for which

$$\sigma(g)f(v) = f(\pi(g)v)$$

for all $v \in V$ and $g \in G$. Thus representations of G form a category, denoted by Rep(G).

We will restrict our attention to the following class of representations:

Definition 2.1.1. A representation (π, V) of G is said to be *smooth*, or *algebraic*, provided that for all $v \in V$, the stabilizer Stab$_G(v)$ of v in G is open.

If we place the discrete topology on V and the product topology on $G\times V$, then the requirement that the stabilizer of each $v \in V$ be open is equivalent to requiring that the map from $G \times V$ to V which sends (g, v) to $\pi(g)v$ be continuous.

Remark 2.1.2. We offer a few remarks on smooth representations.

(1) For $v \in V$, we note that the stabilizer of v in G is open if and only if there exists a compact open subgroup $K \subset G$ such that

$$v \in V^K := \{v \in V \,|\, \pi(x)v = v \text{ for all } x \in K\}.$$

(2) In the literature $\mathfrak{R}(G)$, $\mathfrak{S}(G)$, or $\mathrm{Alg}(G)$ denotes the full subcategory of Rep(G) consisting of smooth G-representations, meaning the morphisms are the same as in Rep(G) but we consider only smooth representations as objects. We will generally use the notation $\mathfrak{R}(G)$. Since subrepresentations, quotients, and direct sums of smooth representations are again smooth representations, $\mathfrak{R}(G)$ is an abelian category.

(3) Suppose that (σ, W) is any representation of G. We define the set W^∞ of smooth vectors for σ by

$$\begin{aligned} W^\infty :=& \{w \in W \mid w \in W^K \text{ for some compact open subgroup } K \subset G\} \\ =& \{w \in W \mid \mathrm{Stab}_G(w) \text{ is open}\}. \end{aligned}$$

Then (σ, W^∞) is the largest smooth subrepresentation of (σ, W). For all smooth representations (π, V) we have

$$\mathrm{Hom}_G(V, W) = \mathrm{Hom}_G(V, W^\infty).$$

We are most interested in those smooth representations which have no nontrivial proper G-invariant subspaces.

Definition 2.1.3. A representation (π, V) of G is called *irreducible* provided that it is nonzero and the only G-subrepresentations of V are the trivial G-representation (that is, $\{0\}$) and V.

Lemma 2.1.4. *If (π, V) is an irreducible smooth representation of G, then the dimension of V is countable.*

Proof. Let $v \in V$ be nonzero. Since (π, V) is smooth, there exists a compact open subgroup K of G such that $v \in V^K$. Because (π, V) is irreducible, $G \cdot v = G/K \cdot v$ spans V, and G/K is countable, so the lemma follows. ■

We have the following fundamental result. The proof in our setting is due to Jacquet [8, *Lemme* 1].

Lemma 2.1.5 (Schur's lemma)**.** *If (π, V) is an irreducible smooth representation of G, then the natural map*

$$\mathbb{C} \to \mathrm{End}_G(V) := \mathrm{Hom}_G(V, V)$$

is an isomorphism.

Proof. Since (π, V) is irreducible, we have that $\mathrm{End}_G(V)$ is a division algebra.

Choose a nonzero $v \in V$, and let $A \in \mathrm{End}_G(V)$. Since $\{\pi(g)v \mid g \in G\}$ generates V as a complex vector space and $A(\pi(g)v) = \pi(g)(Av)$, we have that A is uniquely determined by Av. Consequently, the map $A \mapsto Av$ from $\mathrm{End}_G(V)$ to V is injective. Thus Lemma 2.1.4 implies that the dimension of $\mathrm{End}_G(V)$ is countable.

If $A \in \mathrm{End}_G(V)$, then $\mathbb{C}(A) \subset \mathrm{End}_G(V)$ is a field of countable dimension over $\mathbb{C}$. If $\mathbb{C} \neq \mathbb{C}(A)$ then $(A - \alpha)$ is invertible for all $\alpha \in \mathbb{C}$, and the subset

$$\{(A - \alpha)^{-1} \mid \alpha \in \mathbb{C}\}$$

of $\mathbb{C}(A)$ consists of uncountably many $\mathbb{C}$-linearly independent elements of $\mathbb{C}(A)$, a contradiction. ■

Remark 2.1.6. Lemma 2.1.5 holds (with the same proof) for any irreducible representation (π, V) of an arbitrary group, provided that the dimension of V is countable.

Corollary 2.1.7. *Let* (π, V) *and* (σ, W) *be two irreducible smooth representations of* G*. If* (π, V) *and* (σ, W) *are isomorphic then* $\mathrm{Hom}_G(V, W) \cong \mathbb{C}$*, but otherwise* $\mathrm{Hom}_G(V, W) = 0$.

As always, isomorphism is an equivalence relation. We denote by $\mathrm{Irr}(G)$ the set of isomorphism classes of smooth irreducible representations of G.

Exercise 2.1.8. Let (π, V) be a finite-dimensional complex representation of an arbitrary group G. Show directly that if (π, V) is irreducible then the natural map $\mathbb{C} \to \mathrm{End}_G(V)$ is surjective.

2.2. **The irreducible representations of** $k^\times$**.** Let (π, V) be an irreducible smooth representation of $k^\times$. Since $k^\times$ is abelian, for every $x \in k^\times$ we have $\pi(x) \in \mathrm{End}_{k^\times}(V)$. Consequently, from Schur's lemma, we have $\pi(x) = z_x \cdot \mathrm{Id}_V$ for a unique $z_x \in \mathbb{C}^\times$. Thus, V is one-dimensional. When V is one-dimensional, π is called a *smooth character* of $k^\times$, and we write $(\pi, \mathbb{C}_\pi)$ for (π, V).

Since the map $x \mapsto (v(x), x \cdot \varpi^{-v(x)})$ is a group isomorphism of $k^\times$ with $\mathbb{Z} \times R^\times$, any smooth character $(\widetilde{\psi}, \mathbb{C}_{\widetilde{\psi}})$ of $k^\times$ can be written as

$$\widetilde{\psi}(x) = z_{\widetilde{\psi}}^{v(x)} \cdot \psi(x \cdot \varpi^{-v(x)}) \tag{1}$$

where $z_{\widetilde{\psi}} \in \mathbb{C}^\times$ and ψ lies in the group $\widehat{R^\times}$ of smooth characters of $R^\times$. Since ψ is a smooth character of $R^\times$, there exists an $m \in \mathbb{Z}_{>0}$ for which $\mathrm{res}_{(1+\wp^m)} \psi$ is trivial. (Here, $\mathrm{res}_{(1+\wp^m)} \psi$ denotes the restriction of ψ to $(1 + \wp^m)$.) That is, we can think of the character ψ as a character of the abelian group $R^\times/(1 + \wp^m)$, which is finite because $1 + \wp^m$ is an open subgroup of the compact group $R^\times$. Consequently, it must be the case that $\psi(R^\times) \subset S^1 = \{z \in \mathbb{C}^\times \mid z\overline{z} = 1\}$. We call such a character *unitary*.

Exercise 2.2.1. Show that if G is any t.d.-group and $\psi : G \to \mathbb{C}^\times$ is a continuous character, then ψ is smooth. Give an example of a smooth character of $k^\times$ which is not unitary.

Exercise 2.2.2. Prove that the cardinality of $R^\times/(1 + \wp^m)$ is $(q - 1)q^{(m-1)}$ by showing that $R^\times/(1 + \wp) \cong \mathfrak{f}^\times$ and $(1 + \wp^k)/(1 + \wp^{k+1}) \cong \mathfrak{f}$ (here $k \geq 1$) as abelian groups.

Definition 2.2.3. A smooth character of $k^\times$ is called an *unramified character* provided that its restriction to $R^\times$ is trivial. The set of unramified characters of $k^\times$ is denoted by $\mathbf{X}(k^\times)$.

Exercise 2.2.4. Show that the map $\psi \mapsto \psi(\varpi)$ induces an isomorphism of $\mathbf{X}(k^\times)$ with $\mathbb{C}^\times$.

2.3. **The category** $\mathfrak{R}(k^\times)$**.** We would like to say something reasonable about the category $\mathfrak{R}(k^\times)$. In the representation theory of compact or finite groups, every representation[2] is *completely decomposable* (or *semisimple*). That is, if V is a representation of this type of group, then

$$V = \bigoplus_{(\pi, V_\pi)} V_\pi$$

with each of the (π, V_π) irreducible. This does **not** happen here.

[2]Recall that our vector spaces are complex vector spaces. If we remove this assumption, this sentence can be false.

Exercise 2.3.1. Consider the representation (π, V) defined as follows. Let $V = \mathbb{C}^2$ and $\pi(x) = \left(\begin{smallmatrix} 1 & v(x) \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Aut}(V)$ for $x \in k^\times$. Show that this representation is an object in $\mathfrak{R}(k^\times)$, but it is not completely decomposable.

2.4. **A different approach.** Fix $(\pi, V) \in \mathfrak{R}(k^\times)$.

Let $\psi \in \widehat{R^\times}$. Consider the projection operator $e_\psi \in \mathrm{End}_{\mathbb{C}}(V)$ defined by

$$e_\psi(v) = \int_{R^\times} \overline{\psi}(x) \cdot \pi(x) v \, dx$$

for $v \in V$, where dx is the normalized Haar measure[3] on $R^\times$. (Since π and ψ are smooth, this integral is really just a finite sum.) Here $\overline{\psi}(x)$ is the complex conjugate of $\psi(x)$.

Exercise 2.4.1. Let G be a compact topological group, e.g. a finite group, or more generally a profinite group like $R^\times$. Fix a nontrivial continuous character $\psi : G \to \mathbb{C}^\times$ and a Haar measure dg on G. There is a unique Haar measure such that G has measure 1, but any Haar measure will do for this exercise. Prove that

$$\int_G \psi(g) \; dg = 0.$$

Exercise 2.4.2. Verify the following standard facts.

(1) For all $y \in R^\times$ we have $\pi(y) e_\psi = \psi(y) e_\psi$.
(2) If $\psi' \in \widehat{R^\times}$ and $\psi \neq \psi'$, then $e_{\psi'} \cdot e_\psi = 0$.
(3) We have $e_\psi \cdot e_\psi = e_\psi$.

Definition 2.4.3. For $\psi \in \widehat{R^\times}$, we define $V_\psi := e_\psi V$.

Since $R^\times$ is a normal subgroup of $k^\times$, V_ψ is a smooth representation of $k^\times$. Moreover, V_ψ is ψ-isotypic. That is, as a representation of $R^\times$, V_ψ is a direct sum of copies of $(\psi, \mathbb{C}_\psi)$.

Lemma 2.4.4. *As a representation of $k^\times$, we have*

$$V = \bigoplus_{\psi \in \widehat{R^\times}} V_\psi.$$

Proof. From Exercise 2.4.2, it is enough to show that if $v \in V$, then

$$v = \sum_{\psi \in S} e_\psi v$$

where S is a finite subset of $\widehat{R^\times}$. Fix $v \in V$. Since (π, V) is smooth, there exists a compact open subgroup $K \subset G$ for which $v \in V^K$. If $\psi \in \widehat{R^\times}$, then

$$e_\psi(v) = \int_{R^\times} \overline{\psi}(x) \cdot \pi(x) v \, dx = \sum_{\bar{y} \in R^\times / K} \overline{\psi}(y) \cdot \pi(y) v \int_K \overline{\psi}(x) \, dx.$$

Consequently, $e_\psi(v)$ will be zero unless the restriction of ψ to K is trivial. Let S be the (finite) subset of $\widehat{R^\times}$ consisting of all ψ which restrict trivially to K. We can think of S as the set of all irreducible representations of the finite abelian group $R^\times / K$. Let W be the $R^\times$-submodule of V

[3] That is, $\int_{R^\times} dx = 1$.

generated by v. Then W is a finite-dimensional complex representation of a finite group, so it is completely decomposable as

$$W = \bigoplus_{\psi \in S} e_\psi W$$

which shows that

$$v = \sum_{\psi \in S} e_\psi(v).$$

■

Remark 2.4.5. Note that if $\psi, \psi' \in \widehat{R^\times}$ such that $\psi \neq \psi'$, then for $(\pi, V) \in \mathfrak{R}(k^\times)$ we have $\mathrm{Hom}_{k^\times}(V_\psi, V_{\psi'}) = 0$. Consequently, the category $\mathfrak{R}(k^\times)$ decomposes into a product of full subcategories:

$$\mathfrak{R}(k^\times) = \prod_{\psi \in \widehat{R^\times}} \mathfrak{R}^\psi(k^\times).$$

Here $\mathfrak{R}^\psi(k^\times)$ is the full subcategory of $\mathfrak{R}(k^\times)$ consisting of the ψ-isotypic representations of $\mathfrak{R}(k^\times)$.

2.5. **Toward an understanding of $\mathfrak{R}^\psi(k^\times)$.**

2.5.1. *Natural transfomations and equivalences of categories.* We will often try to understand certain categories of representations by finding equivalent categories to study. We therefore present a brief review of natural transformations and categorical equivalences.

Definition 2.5.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two categories, and let F and G be covariant functors $\mathcal{A} \to \mathcal{B}$. A *natural transformation* $\rho : F \to G$ from F to G is a rule which associates to each object X of $\mathcal{A}$ a morphism $\rho_X : F(X) \to G(X)$ such that for any morphism $f : X \to Y$ between two objects X and Y of $\mathcal{A}$, the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\rho_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\rho_Y"'] & G(Y) \end{tikzcd}$$

commutes. We let $\mathrm{Hom}(F, G)$ denote the collection[4] of all natural transformations $F \to G$. A *natural isomorphism* of F and G is thus a natural transformation $\rho : F \to G$ that has a two-sided inverse.

Definition 2.5.2. If $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are covariant functors such that $F \circ G$ and $G \circ F$ are naturally isomorphic to the identity functors $\mathrm{Id}_\mathcal{B}$ and $\mathrm{Id}_\mathcal{A}$, respectively, then we say that F and G define an *equivalence of categories.*

Unwrapping the definitions, this means that for all $X \in \mathcal{A}$ and $Y \in \mathcal{B}$ we have natural isomorphisms $\rho_X : G(F(X)) \to X$ and $\sigma_Y : F(G(Y)) \to Y$, so F and G define bijections of

[4]In general, $\mathrm{Hom}(F, G)$ is not a set.

isomorphism classes of objects of $\mathcal{A}$ and $\mathcal{B}$. The naturality property of ρ and σ guarantee that for all $X, X' \in \mathcal{A}$ and $Y, Y' \in \mathcal{B}$,

$$F: \operatorname{Hom}_{\mathcal{A}}(X, X') \longrightarrow \operatorname{Hom}_{\mathcal{B}}(F(X), F(X'))$$

and

$$G: \operatorname{Hom}_{\mathcal{B}}(Y, Y') \longrightarrow \operatorname{Hom}_{\mathcal{A}}(G(Y), G(Y'))$$

are bijections.

2.5.2. *A categorical equivalence.* Fix $\psi \in \widehat{R^\times}$. As we saw in Exercise 2.3.1, we cannot expect to decompose objects of $\mathfrak{R}^\psi(k^\times)$ into a direct sum of irreducible $k^\times$-representations. We will try to find a category that is equivalent to $\mathfrak{R}^\psi(k^\times)$ which we hope will be easier to understand.

Fix a smooth character $\widetilde{\psi}$ of $k^\times$ whose restriction to $R^\times$ is ψ. Then every irreducible object in $\mathfrak{R}^\psi(k^\times)$ is equivalent to $\widetilde{\psi} \otimes \chi$, where χ is some element of $\mathbf{X}(k^\times)$ (this is just Equation (1) on page 5).

Let B denote the set of regular functions on $\mathbf{X}(k^\times) \cong \mathbb{C}^\times$ (viewed as an algebraic variety over $\mathbb{C}$). Thus, we have $B \cong \mathbb{C}[t, t^{-1}]$. Define an action χ_{unr} of the group $k^\times$ on B as follows. For $x \in k^\times$, let $\mathrm{ev}_x : \mathbf{X}(k^\times) \to \mathbb{C}$ be evaluation at x: that is, $\mathrm{ev}_x(\chi) = \chi(x)$. Then $\mathrm{ev}_x \in B$: indeed, if we identify B with $\mathbb{C}[t, t^{-1}]$, then $\mathrm{ev}_x = t^{\nu(x)}$. Finally, for $b \in B$ set $\chi_{\mathrm{unr}}(x)b = \mathrm{ev}_x \cdot b$. We let B act on $\mathbb{C}_{\widetilde{\psi}} \otimes_{\mathbb{C}} B$ by $b \cdot (v \otimes b') := v \otimes b \cdot b'$. The $(k^\times, B)$-module $(\widetilde{\psi} \otimes \chi_{\mathrm{unr}}, \mathbb{C}_{\widetilde{\psi}} \otimes_{\mathbb{C}} B)$ is called a $(k^\times, B)$-representation.

Let $\chi \in \mathbf{X}(k^\times)$. If $\mathfrak{m}_\chi$ denotes the maximal ideal $\{b \in B \mid b(\chi) = 0\}$ in B, then

$$\mathbb{C}_{\widetilde{\psi}} \otimes_{\mathbb{C}} (B/\mathfrak{m}_\chi) \cong (\mathbb{C}_{\widetilde{\psi}} \otimes_{\mathbb{C}} B)/\mathfrak{m}_\chi(\mathbb{C}_{\widetilde{\psi}} \otimes_{\mathbb{C}} B) \cong \mathbb{C}_{\widetilde{\psi}\otimes\chi}$$

where the second map is given by $v \otimes b \mapsto b(\chi)v$. That is, every irreducible object in $\mathfrak{R}^\psi(k^\times)$ is equivalent to a quotient of $(\widetilde{\psi} \otimes \chi_{\mathrm{unr}}, \mathbb{C}_{\widetilde{\psi}} \otimes_{\mathbb{C}} B)$.

Remark 2.5.3. The object $\mathbb{C}_{\widetilde{\psi}} \otimes_{\mathbb{C}} (B/\mathfrak{m}_\chi)$ is usually called the *specialization* of $\mathbb{C}_{\widetilde{\psi}} \otimes_{\mathbb{C}} B$ at χ, and it is denoted by $\mathrm{sp}_\chi(\mathbb{C}_{\widetilde{\psi}} \otimes_{\mathbb{C}} B)$.

Lemma 2.5.4. *Let (π, V) be an object in $\mathfrak{R}^(k^\times)$. We have a natural isomorphism*

$$\operatorname{Hom}_{k^\times}(\mathbb{C}_{\widetilde{\psi}} \otimes_{\mathbb{C}} B, V) \cong \operatorname{Hom}_{R^\times}(\mathbb{C}_\psi, V)$$

as $R^\times$-modules.

Proof. The isomorphism is just precomposition with the natural inclusion $\mathbb{C}_\psi \to \mathbb{C}_{\widetilde{\psi}} \otimes_{\mathbb{C}} B$, which sends $z \mapsto z \otimes 1$. Inversely, given an $R^\times$-morphism $\varphi : \mathbb{C}_\psi \to V$, we can extend this to a $k^\times$-morphism $\mathbb{C}_{\widetilde{\psi}} \otimes_{\mathbb{C}} B \to V$ by letting $1 \otimes t^n \mapsto \widetilde{\psi}(\varpi^n) \cdot \varphi(1)$. ■

Corollary 2.5.5. *The functor from $\mathfrak{R}^\psi(k^\times)$ to the category of B-modules given by*

$$(\pi, V) \mapsto \operatorname{Hom}_{k^\times}(\mathbb{C}_{\widetilde{\psi}} \otimes_{\mathbb{C}} B, V)$$

is exact and faithful. Moreover, it defines an equivalence of categories.

Exercise 2.5.6. Prove Corollary 2.5.5. Hint: note that the right side of the isomorphism in Lemma 2.5.4 can be identified with V_ψ.

Exercise 2.5.7. Show that the category of B-modules is equivalent to $\mathfrak{R}(\mathbb{Z})$ (note that since $\mathbb{Z}$ is discrete, smoothness is automatic and $\mathfrak{R}(\mathbb{Z}) = \mathrm{Rep}(\mathbb{Z})$). Moreover, both of these are equivalent to the following very concrete category: the objects consist of pairs (V, T) where V is a $\mathbb{C}$-vector space and $T \in \mathrm{GL}(V)$, and a morphism $\varphi : (V, T) \to (W, S)$ is a $\mathbb{C}$-linear map such that $\varphi(T(v)) = S(\varphi(v))$ for all $v \in V$.

Remark 2.5.8. The center of the category $\mathfrak{R}^{\psi}(k^{\times})$ (as defined in Section 13) is isomorphic to $B = \mathbb{C}[t, t^{-1}]$.

3. THE REPRESENTATION THEORY OF HEISENBERG GROUPS

The Heisenberg group can be found in many places in representation theory (see, for example, [5, 10, 15]) and, more broadly, in a great many branches of mathematics (see, for example, [6])

Here we consider the simplest incarnation of a Heisenberg group, namely the Heisenberg group in three variables, which we shall denote by the letter H. The group H can be identified with the subgroup of $\mathrm{GL}_3(k)$ consisting of the matrices

$$\left\{ [s, t, z] := \left(\begin{smallmatrix} 1 & s & z \\ 0 & 1 & t \\ 0 & 0 & 1 \end{smallmatrix} \right) \mid s, t, z \in k \right\}.$$

The group multiplication law is given by

$$[s, t, z] \cdot [s', t', z'] = [s + s', t + t', z + z' + t's],$$

so the center $Z = Z(H)$ of H is $\{[0, 0, z] \mid z \in k\}$. The quotient $P = H/Z$ is isomorphic to $k \oplus k$ as a topological group. For all $r \in \mathbb{R}$, we can define the compact open subgroup

$$K_r := \left\{ [s, t, z] \mid v(s), v(t) \geq \frac{r}{2} \text{ and } v(z) \geq r \right\} = \{[s, t, z] \mid s, t \in \wp^{\lceil \frac{r}{2} \rceil} \text{ and } z \in \wp^{\lceil r \rceil}\}.$$

Since $K_r = K_{2\lceil r/2 \rceil}$, the set $\{K_r \mid r > 1\}$ is a countable neighborhood-basis of the identity consisting of compact open subgroups. For each K in this basis, the coset space H/K is countable, so H is a t.d.-group.

3.0.3. *The representation theory of* k. For the Heisenberg group, the central character controls nearly everything; thus, before we can begin to understand the representation theory of H, we must first consider the representation theory of Z (which is isomorphic to k).

Since k is abelian, it follows from Schur's lemma that every smooth irreducible representation of k is a smooth character. The group of smooth characters of k (or more generally, any abelian topological group) is called the *Pontrjagin dual* of k and is denoted $\widehat{k}$.

Exercise 3.0.9. Prove that there exists a nontrivial smooth character $\Lambda : k \to \mathbb{C}^{\times}$. (Or see, for example, [11, Exercise 3 on p. 297].)

Fix one nontrivial smooth character $\Lambda : k \to \mathbb{C}^{\times}$. For all $m \in \mathbb{Z}$, we have that $\mathrm{res}_{\wp^m} \Lambda$ is a character of the compact open subgroup $\wp^m$. Thus, the image of $\mathrm{res}_{\wp^m} \Lambda$ in $\mathbb{C}^{\times}$ lies in S^1. Since $k = \cup_{m \geq 1} \wp^m$ and this is true for arbitrary m, we conclude that Λ is a unitary character of k.

Choose $x \in k$. The function $\Lambda_x \colon k \to S^1$ which maps $y \in k$ to $\Lambda(yx)$ is a smooth additive character of k, so $x \mapsto \Lambda_x$ is a map from k to $\widehat{k}$.

Exercise 3.0.10. Show that this map is a topological isomorphism of k and $\widehat{k}$. (The topology on $\widehat{k}$ is the compact open topology.)

Exercise 3.0.11. More generally, for a finite-dimensional k-vector space W equipped with the obvious topology, show that the characters on W are all unitary, and that $W^* \cong \widehat{W}$ via the map $\lambda \mapsto (x \mapsto \Lambda(\lambda(x)))$. (Here $W^* = \mathrm{Hom}_k(W, k)$.)

Exercise 3.0.12. Let Γ be a finite group, and let A be any Γ-module, i.e. an abelian group with an action of Γ by group automorphisms. Define

$$A^\Gamma = \{a \in A \mid \gamma \cdot a = a \text{ for all } \gamma \in \Gamma\}.$$

Let $A(\Gamma) \subset A$ be the submodule generated by $\{\gamma a - a \mid a \in A,\ \gamma \in \Gamma\}$, and define $A_\Gamma = A/A(\Gamma)$; thus A_Γ is the largest quotient module of A on which Γ acts trivially.

Let $\widehat{A} = \mathrm{Hom}(A, S^1)$ be the Pontrjagin dual of A (without topological considerations). Then Γ acts on $\widehat{A}$ by $(\gamma \cdot \lambda)(a) = \lambda(\gamma^{-1} \cdot a)$. Suppose that the natural injection $A \to \widehat{\widehat{A}}$ is surjective (which is true when, for instance, A is finitely generated as an abelian group). Show that there is a natural isomorphism $A^\Gamma \to ((\widehat{A})_\Gamma)^\wedge$.

3.1. **Some basic definitions.** In this subsection, let G be any t.d.-group.

Definition 3.1.1. A smooth representation (π, V) of G is called *admissible* provided that, for each compact open subgroup K of G, the dimension of V^K is finite.

Let (π, V) be a smooth representation of G. We define the representation (π^*, V^*) of G on V^*, the linear dual of V, via

$$(\pi^*(g)\lambda)(v) = \lambda(\pi(g^{-1})v).$$

Generally, this representation will not be smooth, so we define the *contragredient* representation of (π, V), denoted $(\widetilde{\pi}, \widetilde{V})$, to be the restriction of π^* to the smooth vectors in V^*.

Let K be a compact subgroup of G, and let dx denote a normalized Haar measure on K. Define a projection operator e_K on V by

$$e_K v = \int_K \pi(x) v \, dx$$

for $v \in V$. Note that since (π, V) is smooth, the integral defining e_K is a finite sum.

Exercise 3.1.2. Verify that e_K is a projection operator. Show that $e_K V = V^K$ and that, as a representation of K, we have $(1 - e_K)V \oplus V^K = V$. Show that for $\lambda \in \widetilde{V}$ and $v \in V$, $(e_K\lambda)v = \lambda(e_K v)$, and conclude that $\widetilde{V}^K = \mathrm{Hom}_{\mathbb{C}}(V^K, \mathbb{C})$.

For $\lambda \in \widetilde{V}$ and $v \in V$, we call the function $m_{\lambda,v} : G \to \mathbb{C}$ sending g to $\lambda(\pi(g)v)$ the *matrix coefficient* of G corresponding to v and λ. Since (π, V) and $(\widetilde{\pi}, \widetilde{V})$ are both smooth, there exists a compact open subgroup K of G such that $m_{\lambda,v}(g) = m_{\lambda,v}(k_1 g k_2)$ for all $k_1, k_2 \in K$ and all $g \in G$.

Definition 3.1.3. A smooth representation (π, V) of G is called *supercuspidal* provided that its matrix coefficients are compactly supported modulo the center Z of G. That is, the image of the support of any matrix coefficient in the topological space G/Z is compact.

It follows from Schur's lemma that for an irreducible smooth representation (π, V) of G, the center of G will act on V by a character. We call this the *central character* associated to (π, V).

I believe that it was Jacquet [8] who first observed the following fact.

Lemma 3.1.4. *If (π, V) is a smooth, irreducible, supercuspidal representation of G, then (π, V) is admissible.*

Proof. Let K be a compact open subgroup of G, so we want to show that V^K is finite-dimensional. Fix a nonzero $v \in V^K$, so by the irreducibility assumption $G \cdot v = G/K \cdot v$ spans V and hence $e_K \cdot G/K \cdot v$ spans V^K. Let $S \subset G$ be a subset such that $e_K \cdot S \cdot v = e_K \cdot SK/K \cdot v$ is a basis for V^K. Letting $(\widetilde{\pi}, \widetilde{V})$ denote the smooth contragredient, we have $\widetilde{V}^K = (V^K)^*$ and consequently we can find $\lambda \in \widetilde{V}^K$ such that

$$\lambda(\pi(g)v) = \lambda(e_K\pi(g)v) = 1$$

for all $g \in S$. Since (π, V) is supercuspidal, $g \mapsto \lambda(\pi(g)v)$ has compact support modulo Z, whence the open cover $\{sKZ/Z \mid s \in S\}$ of SZ/Z has a finite subcover. But by Schur's lemma Z acts on V by scalars, so it follows that the $\{sKZ/Z\}$ are all disjoint, whence SZ/Z is finite. Again, the $\{sZ/Z \mid s \in S\}$ are all distinct by Schur's lemma, so S is finite as desired. ■

3.2. **Basic properties of the representations of the Heisenberg group.** We are now in a position to investigate whether or not smooth irreducible representations of H are admissible.

Lemma 3.2.1. *If (π, V) is an irreducible smooth representation of H with trivial central character, then (π, V) is a smooth character. In particular, (π, V) is admissible but not supercuspidal.*

Proof. Since the central character of (π, V) is trivial, the representation (π, V) descends to a smooth irreducible representation of $P = H/Z \cong k \oplus k$. As in subsection 3.0.3, any such representation of P is a unitary character. ■

We therefore have a complete understanding of those irreducible representations of H with trivial central character. We now turn our attention to understanding the remaining irreducible representations.

Lemma 3.2.2. *If (π, V) is an irreducible smooth representation of H with nontrivial central character χ, then (π, V) is admissible and supercuspidal.*

We follow the proof of [13].

Proof. From Lemma 3.1.4, it will be enough to show that (π, V) is supercuspidal.

Fix $\lambda \in \widetilde{V}$ and $v \in V$. Choose $m \in \mathbb{Z}_{\geq 1}$ such that $v \in V^{K_m}$ and $\lambda \in \widetilde{V}^{K_m}$, so the matrix coefficient $m_{\lambda,v}$ satisfies $m_{\lambda,v}(k_1hk_2) = m_{\lambda,v}(h)$ for all $k_1, k_2 \in K_m$ and $h \in H$. Let $[s, t, 0] \in H$, and choose $[s', t', 0] \in K_m$ with s', t' nonzero. Since

$$m_{\lambda,v}([s, t, 0]) = m_{\lambda,v}([0, t', 0] \cdot [s, t, 0] \cdot [s', 0, 0]) = m_{\lambda,v}([s + s', t + t', 0]),$$

it follows that

$$m_{\lambda,v}([s,t,0]) = m_{\lambda,v}([s,t,0]\cdot[s',t',0]) = \chi(t's)\cdot m_{\lambda,v}([s,t,0])$$

and that

$$m_{\lambda,v}([s,t,0]) = m_{\lambda,v}([s',t',0]\cdot[s,t,0]) = \chi(ts')\cdot m_{\lambda,v}([s,t,0]).$$

Since χ is a nontrivial character, we must have that $m_{\lambda,v}$ is zero if either of s or t is too far from the origin. Consequently, $m_{\lambda,v}$ is compactly supported modulo Z. ■

Remark 3.2.3. Note that the bound for the support of $m_{\lambda,v}$ obtained in the proof is independent of v and λ, in the sense that it depends only on the central character χ and the integer m. Thus for any $m \in \mathbb{Z}$, the functions in the set

$$\{m_{\lambda,v} \mid v \in V^{K_m} \text{ and } \lambda \in \widetilde{V}^{K_m}\}$$

are uniformly supported (that is, they all have support within some fixed subset of G which is compact modulo the center).

3.3. **Some more definitions and results in a general context.** In order to continue investigating the irreducible representations of H with nontrivial central character, we first must introduce some general notation and results. As before, G will denote any t.d.-group in this subsection.

3.3.1. *Unitary representations.*

Definition 3.3.1. A smooth representation (π, V) of G is *unitary* provided that there exists a positive-definite G-invariant Hermitian form $(\ ,\ )$ on V. That is:

(1) the form $(\ ,\ )$ is linear in the first variable and conjugate-linear in the second variable, and $\overline{(v,w)} = (w,v)$ for all $v, w \in V$;
(2) for all $v \in V$ we have $(v,v) \geq 0$, and $(v,v) = 0$ if and only if $v = 0$; and
(3) for all $v, w \in V$ and for all $g \in G$, we have $(\pi(g)v, \pi(g)w) = (v,w)$.

Lemma 3.3.2. *Let (π, V) be an admissible unitary representation of G with Hermitian form $(\ ,\ )$. If V_1 is any G-submodule of V, then*

$$V_1^{\perp} := \{v \in V \mid (v, v_1) = 0 \textit{ for all } v_1 \in V_1\}$$

is also a G-submodule, and $V = V_1 \oplus V_1^{\perp}$.

Proof. The only nonobvious part of the lemma is the claim that if $v \in V$, then $v \in V_1 \oplus V_1^{\perp}$. Let $v \in V$. Choose a compact open subgroup K of G such that $v \in V^K$. Note that $V_1^K = V^K \cap V_1$, and that $(\ ,\ )$ descends to a positive-definite K-invariant Hermitian form $(\ ,\ )_K$ on V^K. Let $V_2 \subset V^K$ denote the perpendicular to V_1^K with respect to $(\ ,\ )_K$. Since V^K is finite-dimensional, we have $V^K = V_1^K \oplus V_2$, and one checks that $V_1^{\perp} \cap V^K = V_2$. ■

Corollary 3.3.3. *Let (π, V) be an admissible unitary representation of G. Then $\mathrm{End}_G(V) = \mathbb{C}$ if and only if (π, V) is irreducible.*

Corollary 3.3.4. *Any admissible unitary representation is semisimple.*

3.3.2. *Quotients, induction, and coinvariants.* We begin with a consequence of Zorn's lemma.

Lemma 3.3.5. *Let (π, V) be a smooth representation of G. If V is nonzero, then V has an irreducible subquotient. That is, there exist smooth subrepresentations (π_1, V_1) and (π_2, V_2) of G such that $V_2 \subset V_1 \subset V$ and V_1/V_2 is an irreducible G-representation.*

Proof. We can replace V with any nonzero finitely generated subrepresentation: for example, pick some nonzero $v \in V$ and consider the subrepresentation generated by v. Now consider the partially ordered set S of proper G-representations of V. This is a nonempty set, since $\{0\} \in S$. Suppose that $\mathcal{C}$ is a chain of proper G-representations. If $v_1, \ldots, v_n$ is a set of generators of V as a G-representation, then no $W \in \mathcal{C}$ contains v_j for all $1 \leq j \leq n$, whence $\cup_{W \in \mathcal{C}} W \in S$. Thus $S \neq \varnothing$ is closed with respect to chain unions, so by Zorn's lemma, it has a maximal element. ∎

Exercise 3.3.6. Let (π, V) be a nonzero finite-dimensional complex representation of an arbitrary group G. Show that (π, V) has an irreducible subrepresentation.

Next we describe a canonical way of inducing a smooth representation of a closed subgroup F of G to a smooth representation of G. Fix such an F.

Definition 3.3.7. Let (σ, W) be a smooth representation of F. We define the *induced representation* $(R, \mathrm{Ind}_F^G(W))$ as follows. We let $\mathrm{Ind}_F^G(W)$ denote the space of functions $h\colon G \to W$ such that

(1) for all $f \in F$ and for all $g \in G$, we have $h(fg) = \sigma(f)h(g)$; and
(2) there exists a compact open subgroup K in G (depending only on h) such that for all $g \in G$ and all $x \in K$, we have $h(gx) = h(g)$.

The action of G is given by the *right regular action*: $(R(g)h)(x) = h(xg)$ for all $x, g \in G$. Property (2) guarantees that this action is smooth.

There are other types of induction one might wish to consider, but we shall take up that discussion at a later time.

We need a definition before the statement of the next result: given a representation (π, V) of G, we write $\mathrm{res}_F V = (\pi|_F, V)$ for the *restriction* to F, which has the same underlying vector space V thought of as a representation of $F \subset G$.

Lemma 3.3.8 (Frobenius Reciprocity)**.** *Let (σ, W) be a smooth representation of F and let (π, V) be a smooth representation of G. We have*

$$\mathrm{Hom}_G(V, \mathrm{Ind}_F^G W) \cong \mathrm{Hom}_F(\mathrm{res}_F V, W).$$

In other words, induction is the right adjoint of the restriction functor.

Proof. Given $\alpha \in \mathrm{Hom}_G(V, \mathrm{Ind}_F^G W)$, define $\beta_\alpha \in \mathrm{Hom}_F(\mathrm{res}_F V, W)$ by $\beta_\alpha(v) = \alpha(v)(e)$, where e denotes the identity element of F. Conversely, given $\beta \in \mathrm{Hom}_F(\mathrm{res}_F V, W)$, define $\alpha_\beta \in \mathrm{Hom}_G(V, \mathrm{Ind}_F^G W)$ via $\alpha_\beta(v) = (g \mapsto \beta(\pi(g)v))$. One easily checks that these define inverse maps between $\mathrm{Hom}_G(V, \mathrm{Ind}_F^G W)$ and $\mathrm{Hom}_F(\mathrm{res}_F V, W)$. ∎

Let (π, V) be a representation of G and $S \subset V$ any subset. The *subrepresentation generated by* S, denoted by $\langle S \rangle$, is defined to be the intersection of all subrepresentations of (π, V) which contain S. Concretely, $\langle S \rangle$ is the $\mathbb{C}$-span of $G \cdot S = \{\pi(g)v \mid g \in G, v \in S\}$.

Let ψ be a smooth character of F and (π, V) a smooth representation of G. We would like to construct the largest quotient of (π, V) on which F acts by ψ. First define an F-subrepresentation of V by

$$V(F, \psi) := \langle \pi(f)v - \psi(f)v \,|\, v \in V \text{ and } f \in F \rangle,$$

and then set

$$V_{(F,\psi)} := V/V(F, \psi).$$

Clearly F acts by ψ on $V_{(F,\psi)}$, so we call $V_{(F,\psi)}$ the space of ψ*-coinvariants*. Note that if F is normal, then $V(F, \psi)$ is a G-subrepresentation of V.

Exercise 3.3.9. Prove that $V(F, \psi) = \cap \ker f$ where the intersection ranges over all $f \in \operatorname{Hom}_F(V, \psi)$. Deduce that the natural map

$$\operatorname{Hom}_F(V_{(F,\psi)}, \psi) \to \operatorname{Hom}_F(V, \psi)$$

is an isomorphism.

Exercise 3.3.10. Suppose that we can write $F = \bigcup K_n$ where $K_1 \subset K_2 \subset K_3 \subset \cdots$ are compact open subgroups. In this case, show that we can characterize $V(F, \psi)$ as the set of $v \in V$ for which

$$\int_K \psi^{-1}(x) \cdot \pi(x)v \, dx = 0$$

for some compact open subgroup $K \subset F$. Show that in this case, the above integral is zero for every compact open subgroup K' containing K as well. Also, deduce that if $V_{(F,\psi)} = 0$, then $W_{(F,\psi)} = 0$ for every F-subrepresentation $W \subset V$.

3.4. **The irreducible representations of the Heisenberg group with nontrivial central character.** We now prove the p-adic version of the Stone–von Neumann theorem. Our approach will be very concrete: for a more conceptual approach which applies to all p-adic unipotent groups, see Rodier's paper (*insert citation*).

We define two closed subgroups

$$S = \{[s, 0, 0] \mid s \in k\} \cong k \quad \text{ and } \quad \widehat{S} = \{[0, \widehat{s}, 0] \mid \widehat{s} \in k\} \cong k$$

of H; in addition, let $Z = \{[0, 0, z] \mid z \in k\} \cong k$ be the center of H (which is closed). We will use the following properties of these subgroups:

(1) The multiplication map $S \times \widehat{S} \times Z \to H$ is a homeomorphism.
(2) The subgroup $SZ = \{[s, 0, z] \mid s, z \in k\} \cong k \oplus k$ is normal in H.

Fix a nontrivial smooth character χ of Z, and let $\widetilde{\chi}$ be any smooth character of SZ which restricts to χ. For $\widehat{s} \in \widehat{S}$, define another smooth character $\widetilde{\chi}_{\widehat{s}}$ of SZ by

$$\widetilde{\chi}_{\widehat{s}}([s, 0, z]) := \widetilde{\chi}(\widehat{s}[s, 0, z]\widehat{s}^{-1}) = \chi([0, 0, -s\widehat{s}])\widetilde{\chi}([s, 0, z]). \tag{2}$$

Exercise 3.4.1. Show that $\widetilde{\chi} \mapsto \widetilde{\chi}_{\widehat{s}}$ defines a simply transitive action of $\widehat{S}$ on the set of characters of SZ which agree with χ on Z (use Exercise 3.0.10).

Fix a smooth character $\widetilde{\chi}$ of SZ which restricts to χ on Z (for example, set $\widetilde{\chi}(sz) = \chi(z)$).

Lemma 3.4.2. *The representation* $\mathrm{Ind}_{SZ}^{H}(\widetilde{\chi})$ *is admissible.*

Proof. Fix $m \in \mathbb{N}$, and let $f \in \mathrm{Ind}_{SZ}^{H}(\widetilde{\chi})^{K_m}$ and $[s, \widehat{t}, z] \in K_m$. Then for $\widehat{s} \in k$, we have

$$[0, \widehat{s}, 0][s, \widehat{t}, z] = [s, 0, z][0, 0, -s(\widehat{s} + \widehat{t})][0, \widehat{s} + \widehat{t}, 0]$$

which implies that

$$f([0, \widehat{s}, 0]) = \widetilde{\chi}([s, 0, z]) \cdot \chi([0, 0, -s(\widehat{s} + \widehat{t})]) \cdot f([0, \widehat{s} + \widehat{t}, 0]) \tag{3}$$

for all $[0, \widehat{s}, 0] \in \widehat{S}$. By setting $\widehat{t} = 0$ in Equation (3) and choosing s to be nonzero, we see that this implies that $f([0, \widehat{s}, 0]) = 0$ for all $\widehat{s}$ outside of some compact set in k, and this compact set is completely determined by m. By setting $s = z = 0$, we see (again from Equation (3)) that the restriction of f to $\widehat{S}$ is locally constant with respect to $K_m \cap \widehat{S}$. Consequently, the dimension of $\mathrm{Ind}_{SZ}^{H}(\widetilde{\chi})^{K_m}$ is finite, so $\mathrm{Ind}_{SZ}^{H}(\widetilde{\chi})$ is admissible. ∎

The proof of the above lemma also shows the following.

Remark 3.4.3.
(1) Since any $f \in \mathrm{Ind}_{SZ}^{H}(\widetilde{\chi})$ is determined by its values on $\widehat{S} = SZ\backslash H$, we have that the space $C_c^{\infty}(\widehat{S})$ of locally constant compactly supported functions on $\widehat{S}$ is isomorphic as a complex vector space to $\mathrm{Ind}_{SZ}^{H}(\widetilde{\chi})$ under the map

$$f \mapsto ([s, \widehat{s}, z] \mapsto \widetilde{\chi}([s, 0, z - s\widehat{s}]) \cdot f([0, \widehat{s}, 0])). \tag{4}$$

If we let $[s, \widehat{s}, z] \in H$ act on $f \in C_c^{\infty}(\widehat{S})$ by the formula

$$([s, \widehat{s}, z] \cdot f)(\widehat{t}) = \widetilde{\chi}([s, 0, z - s(\widehat{s} + \widehat{t})]) \cdot f(\widehat{s} + \widehat{t}),$$

then formula 4 is an H-intertwiner, i.e. defines an isomorphism of H-representations.

(2) This will imply that $\mathrm{Ind}_{SZ}^{H}(\widetilde{\chi})$ is not a semisimple S-module. Indeed, suppose that $\mathbb{C}f$ is stable under S for some $f \in \mathrm{Ind}_{SZ}^{H}(\widetilde{\chi})$. For any $[0, \widehat{s}, 0] \in \widehat{S}$ and $[s, 0, 0] \in S$, we have that

$$(R([s, 0, 0])f)([0, \widehat{s}, 0]) = f([0, \widehat{s}, 0][s, 0, 0][0, \widehat{s}, 0]^{-1}[0, \widehat{s}, 0]) = \chi([0, 0, -s\widehat{s}])f([0, \widehat{s}, 0]).$$

Since S acts by scalars, we have $R([s, 0, 0])f = c \cdot f$ for some $c \in \mathbb{C}^{\times}$. If $f([0, \widehat{s}, 0]) \neq 0$, then, the above equation shows that $\chi([0, 0, -s\widehat{s}]) = c$ for all $s \in k$, so since χ is nontrivial, $\widehat{s} = 0$. Thus $f|_{\widehat{S}}$ can only be nonzero at $[0, 0, 0]$, so since $f|_{\widehat{S}}$ is locally constant, we must have $f|_{\widehat{S}} = 0$, so $f = 0$.

Lemma 3.4.4. *The representation* $\mathrm{Ind}_{SZ}^{H}(\widetilde{\chi})$ *is unitary.*

Proof. For $f, f' \in \mathrm{Ind}_{SZ}^{H}(\widetilde{\chi})$, set

$$(f, f') := \int_{\widehat{S}} f(\widehat{s}) \cdot \overline{f}'(\widehat{s})\, d\widehat{s}$$

where $\overline{f}'(\widehat{s})$ denotes the complex conjugate of $f'(\widehat{s})$ and $d\widehat{s}$ is a Haar measure on $\widehat{S} \cong k$. Since f and f' are locally constant compactly supported functions on $\widehat{S}$ (Remark 3.4.3(1)), the above integral makes sense. This pairing is clearly a positive-definite Hermitian form; we need to check that it is H-invariant. For $g = [s, \widehat{t}, z] \in H$, we have

$$\begin{aligned}(R(g)f, R(g)f') &= \int_{\widehat{S}} f([0, \widehat{s}, 0][s, \widehat{t}, z]) \cdot \overline{f}'([0, \widehat{s}, 0][s, \widehat{t}, z])\, d\widehat{s} \\ &= \int_{\widehat{S}} \left|\widetilde{\chi}([s, 0, z - s\widehat{t} - s\widehat{s}])\right|^2 \cdot f([0, \widehat{s} + \widehat{t}, 0]) \cdot \overline{f}'([0, \widehat{s} + \widehat{t}, 0])\, d\widehat{s} \\ &= (f, f').\end{aligned}$$

The last in this string of equalities follows from the facts that $\widetilde{\chi}$ is a unitary character, and that a Haar measure is translation-invariant. ■

Theorem 3.4.5 (Stone-von Neumann)**.** *The representation* $\mathrm{Ind}_{SZ}^{H}(\widetilde{\chi})$ *is, up to isomorphism, the unique irreducible smooth representation of* H *with central character* χ.

We follow the proof in [14].

Remark 3.4.6. Before we begin the proof, we note that the theorem implies that $\mathrm{Ind}_{SZ}^{H}(\widetilde{\chi})$ depends only on χ and not on the choice of $\widetilde{\chi}$.

Proof. Let (π, V) be an irreducible representation of H with central character χ, so V restricts to a smooth representation of SZ. Therefore, by Lemma 3.3.5 and Exercise 3.4.1, there is an irreducible subquotient of V on which SZ acts by the character $\widetilde{\chi}_{\widehat{s}}$ for some $\widehat{s} \in \widehat{S}$. By the last statement in Exercise 3.3.10, the largest quotient $V_{(SZ, \widetilde{\chi}_{\widehat{s}})} = V/V(SZ, \widetilde{\chi}_{\widehat{s}})$ of V on which SZ acts by $\widetilde{\chi}_{\widehat{s}}$ is nonzero. Since SZ is stable under conjugation in H, we have that $\pi(\widehat{s}) \cdot V(SZ, \widetilde{\chi}_{\widehat{s}}) = V(SZ, \widetilde{\chi})$, so by Exercise 3.3.9 $\mathrm{Hom}_{SZ}(V, \widetilde{\chi})$ is nonzero. Thus, by Frobenius reciprocity, V embeds into $\mathrm{Ind}_{SZ}^{H}(\widetilde{\chi})$.

To complete the proof, it will suffice to show that $\mathrm{Ind}_{SZ}^{H}(\widetilde{\chi})$ is irreducible. By Frobenius reciprocity, we have $\mathrm{End}_H(\mathrm{Ind}_{SZ}^{H}(\widetilde{\chi})) = \mathrm{Hom}_{SZ}(\mathrm{res}_{SZ} \mathrm{Ind}_{SZ}^{H} \widetilde{\chi}, \widetilde{\chi})$. Note that for any $\varphi \in \mathrm{Hom}_{SZ}(\mathrm{res}_{SZ} \mathrm{Ind}_{SZ}^{H} \widetilde{\chi}, \widetilde{\chi})$, $f \in \mathrm{Ind}_{SZ}^{H} \widetilde{\chi}$, and $g \in H$, we have $\varphi(R(g)f) = \widetilde{\chi}(g)\varphi(f) = \varphi(\widetilde{\chi}(g)f)$, so φ reduces to a homomorphism $(\mathrm{Ind}_{SZ}^{H}(\widetilde{\chi}))_{(SZ, \widetilde{\chi})} \to \widetilde{\chi}$. Therefore, by Corollary 3.3.3, 3.4.2, and Lemma 3.4.4, it suffices to show that $(\mathrm{Ind}_{SZ}^{H}(\widetilde{\chi}))_{(SZ, \widetilde{\chi})}$ is one-dimensional.

Let $F : C_c^{\infty}(\widehat{S}) \to \mathrm{Ind}_{SZ}^{H}(\widetilde{\chi})$ be the isomorphism given in Remark 3.4.3(1), and let $C_0 \subset C_c^{\infty}(\widehat{S})$ be the subspace of functions which are zero at $[0, 0, 0]$. Since C_0 has codimension one, it is enough to show that $F(C_0) \subset (\mathrm{Ind}_{SZ}^{H}(\widetilde{\chi}))(SZ, \widetilde{\chi})$. We use Exercise 3.3.10 again. Let $f \in C_0$ and $[0, \widehat{s}, 0] \in \widehat{S}$, and let $U \subset SZ$ be any compact open subgroup. For $u \in SZ$ and $x \in \widehat{S}$, we have $F(f)(xu) = F(f)(xux^{-1}x) = \widetilde{\chi}(xux^{-1})f(x)$, so

$$\int_U \widetilde{\chi}^{-1}(u) \cdot F(f)([0, \widehat{s}, 0] \cdot u)\, du = \int_U \widetilde{\chi}^{-1}(u) \cdot \widetilde{\chi}_{\widehat{s}}(u) \cdot f([0, \widehat{s}, 0])\, du.$$

When $\widehat{s} = 0$, $f([0, \widehat{s}, 0]) = 0$ by the definition of C_0. When $\widehat{s} \neq 0$, the character $\widetilde{\chi}^{-1} \cdot \widetilde{\chi}_{\widehat{s}}$ is nontrivial since $\widehat{S}$ acts simply transitively on the characters of SZ which restrict to χ. In either case, the integral is zero for large enough U (when $\widehat{s} \neq 0$ apply Exercise 2.4.1). We can choose a single U that will work for each $[0, \widehat{s}, 0]$ since f is compactly supported. ■

3.5. **A more general setting.** The proof of the Stone–von Neumann theorem presented above holds in the following more general context.

Suppose that H is a t.d.-group which is a central extension of an abelian group P by the center Z of H (we assume $Z \neg \{1\}$). That is, we have an exact sequence (of topological groups)

$$1 \to Z \to H \to P \to 1$$

and the commutator subgroup (H, H) of H is a subgroup of Z.

Given a nontrivial smooth character χ of Z, we define a form $\langle\ ,\ \rangle$ on $P = H/Z$ by

$$\langle p_1, p_2 \rangle = \chi(h_1 h_2 h_1^{-1} h_2^{-1}),$$

where $p_1, p_2 \in P$, and h_i is any lift of p_i (note $langle\ ,\ \rangle$ is well-defined). We have that $\langle \cdot, \cdot \rangle$ is

(1) *alternating*: i.e., $\langle p_1, p_2 \rangle = \langle p_2, p_1 \rangle^{-1}$ for all $p_1, p_2 \in P$, and
(2) *bimultiplicative*: i.e., $\langle pp_1, p_2 \rangle = \langle p, p_2 \rangle \langle p_1, p_2 \rangle$ and $\langle p_1, pp_2 \rangle = \langle p_1, p \rangle \langle p_1, p_2 \rangle$ for all $p, p_1, p_2 \in P$ (see Exercise 3.5.1 below).

By the Pontrjagin duality there exists a nontrivial smooth character χ of Z, and we will assume that the associated form $\langle \cdot, \cdot \rangle$ is *nondegenerate*: i.e., if $\langle p, p' \rangle = 1$ for all $p' \in P$, then $p = 1$.

Exercise 3.5.1. Show that $\langle \cdot, \cdot \rangle$ is bimultiplicative.

If H is a t.d.-group, we say that $H = S\widehat{S}Z$ is a *complete polarization* of H with respect to $\langle \cdot, \cdot \rangle$ if:

(1) S, $\widehat{S}$, and Z are closed abelian subgroups of H;
(2) the multiplication map $S \times \widehat{S} \times Z \to H$ is a homeomorphism; and
(3) the image of S (resp. $\widehat{S}$) in P is a maximal isotropic subgroup with respect to $\langle\ ,\ \rangle$ — that is, the image of S (resp. $\widehat{S}$) in P is a maximal subgroup having the property that $\langle S, S \rangle = 1$ (resp. $\langle \widehat{S}, \widehat{S} \rangle = 1$); and
(4) for any compact open subgroup K of P, the group

$$K^\perp = \{p \in P \mid \langle p, p' \rangle = 1 \text{ for all } p' \in K\}$$

is also compact and open.

If a complete polarization exists, one can show the following.

Remark 3.5.2. (1) Let $K \subset S$ and $\widehat{K} \subset \widehat{S}$ be any compact open subgroups. Writing $P = H/Z = S \oplus \widehat{S}$, we have that $K^\perp \cap \widehat{S} = (K \oplus \widehat{K})^\perp \cap \widehat{S}$ (resp. $\widehat{K}^\perp \cap S = (K \oplus \widehat{K})^\perp \cap S$) is a compact open subgroup of $\widehat{S}$ (resp. S). In other words, the subgroups

$$\{\widehat{s} \in \widehat{S} \mid \langle \widehat{s}, s \rangle = 1 \text{ for all } s \in K\} \quad \text{and} \quad \{s \in S \mid \langle s, \widehat{s} \rangle = 1 \text{ for all } \widehat{s} \in \widehat{K}\}$$

are compact and open.

(2) Since the image of S in P is a maximal isotropic subspace, a calculation shows that SZ is a normal closed subgroup of H.
(3) There is a natural injective homomorphism of $\widehat{S}$ into the Pontrjagin dual of S via the map $\widehat{s} \mapsto \langle \widehat{s}, \cdot \rangle$.

(4) This homomorphism gives an action of the group $\widehat{S}$ on the (nonempty) set of smooth characters of SZ which restrict to χ on Z. To wit,

$$\widetilde{\chi}_{\widehat{s}}(s) := \widetilde{\chi}(\widehat{s}s\widehat{s}^{-1}) = \langle \widehat{s}, s\rangle \cdot \widetilde{\chi}(s)$$

where $\widetilde{\chi}$ is such a character of SZ, $\widehat{s} \in \widehat{S}$, and $s \in SZ$.

Exercise 3.5.3. Let H, S, $\widehat{S}$, and χ be as in Subsection 3.4. Show that

(1) H is a central extension of $H/Z \cong k \oplus k$ by $Z \cong k$,
(2) $\langle \cdot, \cdot \rangle$ is nondegenerate, and
(3) $H = S\widehat{S}Z$ is a complete polarization of H.

Remark 3.5.4. Let H be as in Subsection 3.4. If the characteristic of k is not two, set $S' = \{[s, s, s^2/2] \mid s \in k\}$ and $\widehat{S}' = \{[\widehat{s}, -\widehat{s}, -\widehat{s}/2] \mid \widehat{s} \in k\}$. Then $H = S'\widehat{S}'Z$ is another complete polarization of H.

The proof of the following theorem is similar to the proof of Theorem 3.4.5 (see Exercise 3.5.6).

Theorem 3.5.5. *Suppose that H is a t.d.-group which is an increasing union of compact open subsets (so χ is unitary), and that it has a complete polarization $H = S\widehat{S}Z$. Suppose also that the abelian group $\widehat{S}$ acts simply transitively on the set of continuous characters of SZ that restrict to χ on Z. Then the representation $\mathrm{Ind}_{SZ}^{H}(\widetilde{\chi})$ is the unique irreducible representation (up to equivalence) with central character χ. (Here $\widetilde{\chi}$ is any continuous character of SZ whose restriction to Z is χ.)*

Exercise 3.5.6. Modify the proof of Lemma 3.4.2 to work under the hypotheses of Theorem 3.5.5 (hint: use Remark 3.5.2(1)). Conclude that Remark 3.4.3(1) holds as well.

3.5.1. *A case of interest.* We consider what happens when P is finite (with the discrete topology).

Exercise 3.5.7. Prove that in this case, any irreducible representation of H on which the center acts by a character is finite-dimensional. (Hint: use Frobenius reciprocity.)

Remark 3.5.8. Since H is homeomorphic to a finite disjoint union of copies of Z, all topological considerations reduce to the topology of Z and the smoothness of the central character of an irreducible representation.

It would be nice if the hypotheses of Theorem 3.5.5 were valid in this context. However, this need not be the case — we may not be able to find subgroups S and $\widehat{S}$ with the required properties. We therefore follow the treatment of Jeff Adler given in Séminair Paul Sally, 1994.

Let S' be a maximal isotropic subgroup of P (it is easy to see that such a subgroup exists). Define a map Φ from P to the character group $\widehat{P}$ of P by sending p to the character $\chi_p : P \to \mathbb{C}^\times$ given by $p' \mapsto \langle p, p'\rangle$. Since $\langle\ ,\ \rangle$ is assumed to be nondegenerate, Φ is an isomorphism. Since S' is a maximal isotropic subgroup of P, for $p \in P$ we have that

$$\mathrm{res}_{S'} \Phi(p) = 1 \text{ if and only if } p \in S'.$$

Therefore, Φ descends to an isomorphism from P/S' to the character group of S' (so $|P| = |S'|^2$).

Let SZ denote the preimage of S' in H. (This is a notational convenience; there may be no subgroup S.) Note that SZ is normal. We now define a character of SZ whose restriction to Z is χ. By the fundamental theorem for finitely generated abelian groups, we can write S' as the direct sum of n cyclic subgroups $\langle s_i' \rangle$ for $1 \leq i \leq n$. For each $1 \leq i \leq n$, let d_i denote the order of s_i and let $s_i \in SZ$ be any element that maps to s_i', so $s_i^{d_i} \in Z$. Let α_i denote a d_ith root of $\chi(s_i^{d_i})$. Noting that any element of SZ can be written as $s_1^{r_1} s_2^{r_2} \cdots s_n^{r_n} z$ for integers r_i and $z \in Z$, we define $\widetilde{\chi}$ by

$$\widetilde{\chi}(s_1^{r_1} s_2^{r_2} \cdots s_n^{r_n} z) := \alpha_1^{r_1} \alpha_2^{r_2} \cdots \alpha_n^{r_n} \chi(z).$$

Exercise 3.5.9. Check that $\widetilde{\chi}$ is a well-defined character of SZ that restricts to χ on Z.

Remark 3.5.10. Alternatively, one can prove the existence of an extension $\widetilde{\chi}$ of χ to SZ using the following basic lemma from the field of homological algebra [**?**]:

Lemma 3.5.11. *A module M over a principal ideal domain A is injective (that is, if $N' \subset N$ are two A-modules then any homomorphism $N' \to M$ extends to N) if and only if it is divisible (that is, the map $x \mapsto ax : M \to M$ is surjective for every nonzero $a \in A$).*

Since any $x \in \mathbb{C}^\times$ has an nth root for any $n \in \mathbb{Z} \setminus \{0\}$, $\mathbb{C}^\times$ is injective in the category of abelian groups. Since the character χ is trivial on the commutator subgroup (SZ, SZ) of SZ, it reduces to a homomorphism $Z/(SZ, SZ) \to \mathbb{C}^\times$ and thus extends to a homomorphism $SZ/(SZ, SZ) \to \mathbb{C}^\times$, which gives an extension $\widetilde{\chi} : SZ \to \mathbb{C}^\times$ of χ.

In this context, we define the induced representation $\operatorname{Ind}_{SZ}^H \widetilde{\chi}$ to be the set of all functions $f : H \to \mathbb{C}$ such that $f(sh) = \widetilde{\chi}(s) f(h)$ for all $s \in SZ$ and $h \in H$. As before, we give this space the right regular action; since H is homeomorphic to a finite disjoint union of copies of Z, this defines a continuous representation. One shows that Frobenius reciprocity holds for $\operatorname{Ind}_{SZ}^H \widetilde{\chi}$, with the same proof.

With these definitions, a version of the Stone–von Neumann theorem holds. Note that we do not assume that χ is unitary.

Theorem 3.5.12. *The representation $\operatorname{Ind}_{SZ}^H \widetilde{\chi}$ is the unique (up to equivalence) irreducible representation of H with central character χ.*

Proof. For ease of notation, set $\pi_{\widetilde{\chi}} = \operatorname{Ind}_{SZ}^H \widetilde{\chi}$.

First we claim that any irreducible representation of SZ with central character χ is one-dimensional. Let (π, V) be such a representation. Since S' is isotropic,

$$\pi(sts^{-1}t^{-1}) = \chi(sts^{-1}t^{-1}) = \langle s, t \rangle = 1$$

for any $s, t \in SZ$. Thus $\pi \in \operatorname{Aut}_G(V)$, so the claim follows by Schur's lemma, which holds in this context since (π, V) is finite-dimensional in any case by Exercise **??** (cf. Exercise 2.1.8).

Now let (π, V) be any irreducible representation of H with central character χ. Since V is finite-dimensional, $\operatorname{res}_{SZ} V$ has an irreducible quotient, so there is some character $\widetilde{\chi}'$ of SZ

which restricts to χ on Z such that the space $V(SZ, \widetilde{\chi}')$ of coinvariants is not all of V. Since any character of $S' = SZ/Z$ is of the form $\langle p, \cdot \rangle$ for some $p \in P/S' = H/SZ$, there exists an $h \in H$ such that $\widetilde{\chi}'(s) = \langle h, s \rangle \widetilde{\chi}(s)$ for all $s \in SZ$. Therefore, as in the proof of Theorem 3.4.5, we have that $\pi(h)V(SZ, \widetilde{\chi}') = V(SZ, \widetilde{\chi})$, so by Frobenius reciprocity, V embeds into $\pi_{\widetilde{\chi}}$.

It remains to show that $\pi_{\widetilde{\chi}}$ is irreducible. Since any $f \in \pi_{\widetilde{\chi}}$ is determined by its values on a set $\{h_1, \dots, h_n\}$ of coset representatives of $SZ \backslash H$, a calculation shows that

$$f \mapsto \sum_{i=1}^{n} f(h_i) : \operatorname{res}_{SZ} \pi_{\widetilde{\chi}} \longrightarrow \bigoplus_{i=1}^{n} \widetilde{\chi}_{h_i}$$

is an isomorphism of SZ-modules, where $\widetilde{\chi}_{h_i}(s) = \langle h_i, s \rangle \widetilde{\chi}(s)$. Roughly, $\pi_{\widetilde{\chi}}$ will be irreducible because H acts transitively on the $\widetilde{\chi}_{h_i}$. More precisely, if we define $f_h \in \pi_{\widetilde{\chi}}$ for $h \in H$ by

$$f_h(x) = \begin{cases} \widetilde{\chi}(xh^{-1}) & \text{if } x \in SZh \\ 0 & \text{otherwise,} \end{cases}$$

then the above isomorphism shows that the elements $f_{h_1}, \dots, f_{h_n}$ are a basis for $\pi_{\widetilde{\chi}}$. We have $h_j^{-1} \cdot f_{h_i} = f_{h_i h_j}$, so if h_k represents the coset SZh_ih_j, then

$$f_{h_k} = \chi(h_i h_j h_k^{-1}) f_{h_i h_j} = \chi(h_i h_j h_k^{-1})(h_j^{-1} \cdot f_{h_i}).$$

Thus each f_{h_i} generates $\pi_{\widetilde{\chi}}$.

Let $g \in \pi_{\widetilde{\chi}}$. We would like to show that g generates $\pi_{\widetilde{\chi}}$, so it is enough to show that f_{h_1} is contained in the subrepresentation W of $\pi_{\widetilde{\chi}}$ generated by g. Assume that $h_1 = 1$, and assume without loss of generality that $g(1) \neq 0$. Set $g_1 = g$, and find $s_2 \in SZ$ such that $\langle h_2, s_2 \rangle \neq 1$. Let $g_2 = (\langle h_2, s_2 \rangle \widetilde{\chi}(s) - s) g_1 \in W$, so

$$g_2(h_i) = ((\langle h_2, s_2 \rangle \widetilde{\chi}(s) - s) g_1)(h_i) = \widetilde{\chi}(s)(\langle h_2, s_2 \rangle - \langle h_i, s_2 \rangle) g_1(h_i);$$

in particular, $g_2(1) \neq 0$ and $g_2(h_2) = 0$. Continuing in this fashion, we can inductively find a $g_n \in W$ that is a nonzero multiple of f_{h_1}. This completes the proof. ∎

Exercise 3.5.13. Prove that if χ is unitary, so are $\widetilde{\chi}$ and $\operatorname{Ind}_{SZ}^{H} \widetilde{\chi}$.

3.6. **Another look at representations of the Heisenberg group.** Let χ be a nontrivial smooth character of the center Z of H. Here we present a different way to construct the unique irreducible representation of H having central character χ. The method presented in Subsection 3.4 is akin to parabolic induction for reductive groups (see § 6.2). The approach we outline here is closer to the way in which supercuspidal representations of p-adic reductive groups are constructed (see Exercise 7.3.7 (2)).

Recall that for $r \in \mathbb{R}$, we have defined the compact open subgroup

$$K_r := \left\{ [s, t, z] \mid v(s), v(t) \geq \frac{r}{2} \text{ and } v(z) \geq r \right\} = \{ [s, t, z] \mid s, t \in \wp^{\lceil \frac{r}{2} \rceil} \text{ and } z \in \wp^{\lceil r \rceil} \},$$

so $K_s \subset K_r$ when $s \geq r$, and $K_r = K_{2\lceil r/2 \rceil}$. We set

$$K_{r^+} := \bigcup_{s > r} K_s = \left\{ [s, t, z] \mid v(s), v(t) > \frac{r}{2} \text{ and } v(z) > r \right\}.$$

Note that $K_{r^+} = K_r$ if and only if $r/2 \notin \mathbb{Z}$, and that $K_{r^+} \subset K_r$ in any case.

Let $\rho(\chi)$ be the unique integer for which

$$\operatorname{res}_{Z\cap K_{\rho(\chi)^+}} \chi = 1 \quad \text{and} \quad \operatorname{res}_{Z\cap K_{\rho(\chi)}} \chi \neq 1.$$

We extend χ to a character of $K_{\rho(\chi)^+}Z$ by setting

$$\chi(kz) = \chi(z)$$

for $k \in K_{\rho(\chi)^+}$ and $z \in Z$. Note that for $[a,b,c],\ [s,t,z] \in H$, we have

$$[s,t,z][a,b,c][s,t,z]^{-1} = [a,b,c+(bs-ta)].$$

Thus we see that $K_{\rho(\chi)^+}Z$ is a normal subgroup of H, and that the stabilizer in H of the character χ of $K_{\rho(\chi)^+}Z$ is

$$\left\{[s,t,z] \mid v(bs-ta) > \rho(\chi) \text{ for all } a,b \in k \text{ such that } v(a),v(b) > \frac{\rho(\chi)}{2}\right\} = K_{\rho(\chi)}Z.$$

Our goal is eventually to construct an irreducible representation of H, for which we first must extend χ to an irreducible representation $\widetilde{\chi}$ of $K_{\rho(\chi)}Z$. When $\rho(\chi)$ is odd, $K_{\rho(\chi)}Z = K_{\rho(\chi)^+}Z$, so we set $\widetilde{\chi} := \chi$. However, when $\rho(\chi)$ is even, these subgroups are not the same. Fortunately, in this case, $K_{\rho(\chi)}Z/K_{\rho(\chi)^+}$ is a central extension of the abelian group $\mathfrak{f} \oplus \mathfrak{f}$ by its center $K_{\rho(\chi)^+}Z/K_{\rho(\chi)^+}$. Hence, by Theorem 3.5.12, there is a unique unitary irreducible representation $\widetilde{\chi}$ of $K_{\rho(\chi)}Z$ with central character χ.

Lemma 3.6.1. *The representation $\pi_\chi := \operatorname{Ind}_{K_{\rho(\chi)}Z}^{H} \widetilde{\chi}$ is admissible.*

Proof. It will be enough to show that if $m > \rho(\chi)$ then

$$\dim_{\mathbb{C}}(\pi_\chi)^{K_m} < \infty.$$

Let $f \in (\pi_\chi)^{K_m}$, $h = [a,b,c] \in H$, and $x = [s,t,z] \in K_m$. We have $f(hx^{-1}) = f(h)$, and since $x \in K_{\rho(\chi)^+} \subset K_{\rho(\chi)^+}Z$ we have $\widetilde{\chi}(x) = \chi(x) = 1$ nomatter what $\rho(\chi)$ is. Therefore,

$$\begin{aligned} f(h) &= \widetilde{\chi}(x)f(hx^{-1}) = f(xhx^{-1}) \\ &= f([0,0,bs-ta]h) \\ &= \chi([0,0,bs-ta])f(h). \end{aligned}$$

Choosing s and t wisely, we see that when $a,b \in k$ are outside of some compact set depending only on m, we must have $f([a,b,c]) = 0$. Therefore, all functions in $(\pi_\chi)^{K_m}$ are supported in some set C (depending only on m) which is compact modulo the center Z. Any $f \in (\pi_\chi)^{K_m}$ is determined by its values on a (finite) set of representatives of the cosets $K_{\rho(\chi)}Z\backslash CK_{\rho(\chi)}Z$, so we have

$$\dim_{\mathbb{C}}(\pi_\chi)^{K_m} \leq \left|K_{\rho(\chi)}Z\backslash CK_{\rho(\chi)}Z\right| < \infty.$$

∎

Lemma 3.6.2. *The representation π_χ is unitary.*

Proof. Let $(\cdot,\cdot)_{\widetilde{\chi}}$ denote a positive-definite $K_{\rho(\chi)}Z$-invariant Hermitian form for the unitary representation $\widetilde{\chi}$ (cf. Exercise 3.5.13). For $f, g \in \pi_\chi$, then, we have that the map $h \mapsto (f(h), g(h)) : H \to \mathbb{C}$ descends to a map on the quotient group $K_{\rho(\chi)}Z\backslash H$. This and the fact that any $f \in \pi_\chi$ is compactly supported modulo Z (which we showed in the proof of Lemma 3.6.1) allows us to define a positive-definite H-invariant Hermitian form on π_χ by

$$(f, g) = \int_{K_{\rho(\chi)}Z\backslash H} (f(h), g(h))_{\widetilde{\chi}}\, dh^*$$

for $f, g \in \pi_\chi$, where dh^* denotes a right Haar measure on $K_{\rho(\chi)}Z\backslash H$. ■

Lemma 3.6.3. *The representation π_χ is irreducible.*

Proof. Since π_χ is unitary and admissible, by Corollary 3.3.3 it is enough to show that $\mathrm{End}_H(\pi_\chi) = \mathrm{Hom}_{K_{\rho(\chi)}Z}(\pi_\chi, \widetilde{\chi})$ is one-dimensional.

For $h \in H$, let V_h denote the subspace of π_χ consisting of the functions which are supported on the coset $K_{\rho(\chi)}Zh$. So, as complex vector spaces, we have $\pi_\chi = \bigoplus V_h$, where the sum runs over some set of coset representatives of $K_{\rho(\chi)}Z\backslash H$. Let $h = [a, b, c] \in H$ and $f \in V_h$. For $x = [s, t, z] \in K_{\rho(\chi)^+}$ and $y \in K_{\rho(\chi)}Z$, we have

$$\begin{aligned}
(R(x)f)(yh) &= \widetilde{\chi}(yhxh^{-1})f(h) \\
&= \widetilde{\chi}(x)\widetilde{\chi}(yhxh^{-1})f(h) \\
&= \widetilde{\chi}(y)\widetilde{\chi}(xhxh^{-1})f(h) \\
&= \chi([0, 0, bs - ta])f(yh),
\end{aligned}$$

i.e., $R(x)f = \chi([0, 0, bs - ta])f$. Therefore, for $\varphi \in \mathrm{Hom}_{K_{\rho(\chi)}Z}(\pi_\chi, \widetilde{\chi})$, we have

$$\varphi(f) = \widetilde{\chi}(x)\varphi(f) = \varphi(R(x)f) = \varphi(\chi([0, 0, bs - ta])f) = \chi([0, 0, bs - ta])\varphi(f).$$

If $h \notin K_{\rho(\chi)}Z$ then we may assume that $v(a) < \rho(\chi)/2$, so we can find a $t \in k$ with $v(t) > \rho(\chi)/2$ such that $\chi([0, 0, -ta]) \neq 1$ (since χ is nontrivial on $K_{\rho(\chi)} \cap Z$); choosing $x = [0, t, 0]$, the above equation tells us that $\varphi(f) = 0$. Thus φ reduces to a homomorphism $V_{[0,0,0]} \to \widetilde{\chi}$. But $V_{[0,0,0]}$ is isomorphic to $\widetilde{\chi}$ as a representation of $K_{\rho(\chi)Z}$, so $\mathrm{Hom}_{K_{\rho(\chi)}Z}(\pi_\chi, \widetilde{\chi})$ is one-dimensional as claimed. ■

3.6.1. *A glance at the Weil representation.* Although we will not attempt a full development of this topic, we note that the *Weil* (or *Shale-Weil*, or *oscillator*) representations play a crucial role in representation theory. For example, they can be used to realize the supercuspidal representations of $\mathrm{SL}_2(k)$ (see, for example, [12]), they are used to construct more general supercuspidal representations (see, for example, [15]), and they are used to define the θ-correspondence (see, for example, [**?**]).

For the remainder of this section, we assume k does not have characteristic 2. Here it is convenient to use a different realization of the Heisenberg group: let $W = k^2$ with the symplectic form $\omega : W \to k$ given by $\omega((s, t), (s', t')) = st' - s't$, and put $Z = k$, thought of as the additive group. As a set $H = W \times k$, with the group operation given by

$$(v, a) \cdot (w, b) = (v + w, a + b + \frac{1}{2}\omega(v, w)).$$

More explicitly, we can write $H = k^3$ and then the multiplication is

$$(s, t, z) \cdot (s', t', z') = (s + s', t + t', z + z' + \frac{1}{2}(st' - s't)).$$

This is isomorphic to our previous construction of H as a matrix group via

$$(s, t, z) \mapsto \left\{ \left(\begin{smallmatrix} 1 & s & z+\frac{1}{2}st \\ 0 & 1 & t \\ 0 & 0 & 1 \end{smallmatrix} \right) \right\}.$$

Suppose that H_1 is a locally compact Hausdorff topological group that has a continuous action μ on the Heisenberg group H via automorphisms that fix Z (pointwise). Fix a nontrivial central character χ of Z, and let (π, V) be the unique (up to equivalence) irreducible smooth representation of H with central character χ. For $g \in H_1$, we define another representation (π_g, V) of H by

$$\pi_g(h)v = \pi(\mu(g)h)v$$

for $v \in V$ and $h \in H$. Since π_g and π have the same central character, the Stone–von Neumann theorem tells us that π_g and π are equivalent. Thus, there exists $\rho(g) \in \mathrm{Aut}_{\mathbb{C}}(V)$ such that for all $h \in H$ we have

$$\rho(g)\pi(h) = \pi_g(h)\rho(g).$$

Choose one such $\rho(g)$ for each $g \in G$ — note that by Schur's lemma, $\rho(g)$ is unique up to scalar multiplication. Let $g_1, g_2 \in H_1$. Since $\rho(g_1g_2)$ and $\rho(g_1)\rho(g_2)$ both define equivalences between (π, V) and $(\pi_{g_1g_2}, V)$, we have that

$$\rho(g_1)\rho(g_2) = \beta(g_1, g_2) \cdot \rho(g_1g_2)$$

for an element $\beta(g_1, g_2) \in \mathbb{C}^\times$. We even have $\beta(g_1, g_2) \in S^1$ because (π, V) is unitary. One can verify that $\beta\colon H_1 \times H_1 \to S^1$ satisfies

(1) $\beta(g_1, g_2) \cdot \beta(g_1 \cdot g_2, g_3) = \beta(g_2, g_3) \cdot \beta(g_1, g_2 \cdot g_3)$ for all $g_1, g_2, g_3 \in H_1$ and
(2) $\beta(1, g) = \beta(g, 1) = 1$ for all $g \in H_1$.

That is, β defines a two-cochain on H_1. The homomorphism $\rho\colon H_1 \to \mathrm{PGL}(V)$ is called a *projective representation of* H_1 *which extends* χ (or, sometimes, a *projective* β*-representation*). It sometimes happens that the cohomology class of β in $\mathrm{H}^2(H_1, S^1)$ is trivial. In this case, one can extend χ to a representation of H_1.

When β represents a nontrivial class in $\mathrm{H}^2(H_1, S^1)$, one can extend χ to a representation by enlarging the group H_1. We offer an example.

Example 3.6.4. Let $H_1 = \mathrm{Sp}_2(k) = \mathrm{SL}_2(k)$ denote the isometry group for the symplectic form ω on $W = k^2$. The group $\mathrm{SL}_2(k)$ acts on H via

$$g \cdot (w, z) = (g(w), z),$$

and it is clear that these are group automorphisms of H which fix Z pointwise. As above, we get a projective representation ρ of $\mathrm{Sp}_2(k)$ on the space V which extends χ. The two-cycle, β, associated to this projective representation defines a central extension $\widetilde{Sp}_2(\beta)$ of $\mathrm{Sp}_2(k)$ by S^1:

$$1 \to S^1 \to \widetilde{Sp}_2(\beta) \to \mathrm{Sp}_2(k) \to 1$$

where the group law on $\widetilde{Sp}_2(\beta)$ is given by

$$(g, z) \cdot (g', z') = (gg', \beta(g, g')zz')$$

for $g, g' \in \mathrm{Sp}_2(k)$ and $z, z' \in S^1$. We have a representation ρ of $\widetilde{Sp}_2(\beta)$ defined by $\rho(g, z) = z\rho(g)$. This is the "universal" solution to the problem of linearizing ρ, but there is a more efficient one: write $\mathrm{Mp}_2(\beta) = [\widetilde{Sp}_2(\beta), \widetilde{Sp}_2(\beta)]$ for the commutator subgroup of $\widetilde{Sp}_2(\beta)$, called the *metaplectic group* associated with β. Then the short exact sequence above reduces to

$$1 \to \{\pm 1\} \to \mathrm{Mp}_2(\beta) \to \mathrm{Sp}_2(k) \to 1,$$

and ρ lifts to a (linear) representation of $\mathrm{Mp}_2(\beta)$.

4. Reductive p-adic groups: basic facts

In this section, we review some fundamental facts about reductive p-adic groups. We state many general theorems but provide proofs only for the general linear group. For a more general treatment, see, for example, [**?**] and [**?**, **?**]. Readers who are not comfortable with the theory of algebraic groups are advised to think of the general linear group everywhere below unless we say otherwise (e.g. we sometimes use Sp_4 as an example).

Let G be the group of k-rational points of a connected reductive group $\mathbf{G}$ defined over k. Let T denote the group of k-rational points of a maximal k-split torus in $\mathbf{G}$. Let $P_\emptyset$ denote a minimal parabolic subgroup of G which contains T. (That is, $P_\emptyset$ is the group of k-rational points of a parabolic subgroup $\mathbf{P}_\emptyset$ of $\mathbf{G}$, and $\mathbf{P}_\emptyset$ is a minimal element in the set of parabolic subgroups of $\mathbf{G}$ which are defined over k). We let K_0 denote a special (with respect to T) parahoric subgroup of G. From [**?**, Theorem 20.9], the set of maximal k-split tori in $\mathbf{G}$ form a single conjugacy class under the action of G and similarly for the set of minimal parabolic subgroups. However, in general it is not true that the special parahoric subgroups are all conjugate. (This fails already for SL_2.)

Example 4.0.5. Fix $n \in \mathbb{Z}_{\geq 1}$, and let $\mathbf{G} = \mathrm{GL}_n$, so $G = \mathrm{GL}_n(k)$. We shall always realize $\mathrm{GL}_n(k)$ as the set of invertible elements in $\mathrm{M}_n(k)$, the vector space of $n \times n$ matrices with entries in k. We shall take T to be the subgroup of $\mathrm{GL}_n(k)$ consisting of diagonal matrices. For $P_\emptyset$ we take the Borel subgroup consisting of upper triangular matrices in $\mathrm{GL}_n(k)$. For K_0 we can take $\mathrm{GL}_n(R)$; this is a maximal compact open subgroup of $\mathrm{GL}_n(k)$.

Theorem 4.0.6 (Iwasawa decomposition)**.** *We have $G = P_\emptyset K_0$.*

Remark 4.0.7. By taking inverses, we see that we also have $G = K_0 P_\emptyset$.

Proof. As mentioned at the start of this section, we will assume that $G = \mathrm{GL}_n(k)$, and that $P_\emptyset, K_0$, and T are all as in Example 4.0.5.

Note that the permutation matrices belong to K_0. Also, for $1 \leq i \neq j \leq n$ and $r \in R$, the matrix $g_{ij}(r) \in \mathrm{GL}_n(k)$ defined by

$$g_{ij}(r)_{kl} = \begin{cases} 1 & \text{if } k = l, \\ r & \text{if } k = i \text{ and } l = j, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

for $1 \leq k, l \leq n$ belongs to K_0. Multiplication on the right by the permutation matrices and by the matrices $g_{ij}(r)$ allows us to permute columns of matrices in G and add r-multiples of one column to another. We proceed as follows:

Fix $a \in G$. Permute the columns of a so that the entry a_{nn} has $v(a_{nn}) \leq v(a_{nj})$ for all $1 \leq j \leq n$; in particular, $a_{nn} \neq 0$. For $1 \leq j < n$, we have $a_{nj}/a_{nn} \in R$; consequently, we can add R-multiples of the last column to the other columns to clear the last row of all entries except the a_{nn} entry.

By applying the same reasoning to the first $(n-1)$ rows and $(n-1)$ columns of a, we can find an element g of K_0 such that the $n-1$st row of ag is zero in all entries except $(n-1, n-1)$ and $(n-1, n)$. Continuing in this fashion, we arrive at the result. ∎

Remark 4.0.8. Since K_0 is compact, $G/P_\emptyset$ is compact. Let $M_\emptyset := C_G(T)$, the centralizer of T in G. Let $\overline{P}_\emptyset$ denote the minimal parabolic opposite $P_\emptyset$ with respect to $M_\emptyset$. That is, $\overline{P}_\emptyset$ is the unique minimal parabolic for which $\overline{P}_\emptyset \cap P_\emptyset = M_\emptyset$. We let $N_\emptyset$ (resp. $\overline{N}_\emptyset$) denote the unipotent radical of $P_\emptyset$ (resp. $\overline{P}_\emptyset$). We have the Levi decomposition $P_\emptyset = M_\emptyset N_\emptyset$, and similarly for $\overline{P}_\emptyset$.

Example 4.0.9. For $\mathrm{GL}_n(k)$, we have $M_\emptyset = T$, and $\overline{P}_\emptyset$ is the set of lower triangular matrices in $\mathrm{GL}_n(k)$. The group $N_\emptyset$ is the group of upper triangular matrices in $\mathrm{GL}_n(k)$ with ones on the diagonal, and $\overline{N}_\emptyset$ is the group of lower triangular matrices in $\mathrm{GL}_n(k)$ with ones on the diagonal. For example, in $\mathrm{GL}_3(k)$, we have

$$\begin{aligned} P_\emptyset &= \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \\ \overline{P}_\emptyset &= \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix}. \end{aligned}$$

Definition 4.0.10. A compact open subgroup K of G is said to have an *Iwahori decomposition* with respect to $P_\emptyset = M_\emptyset N_\emptyset$ provided that

$$K = K^+ \cdot K^0 \cdot K^-,$$

where the product can be taken in any order, and $K^+ = K \cap N_\emptyset$, $K^- = K \cap \overline{N}_\emptyset$, and $K^0 = M_\emptyset \cap K$.

We have a filtration of G by compact open subgroups

$$G \supset K_0 \supset K_1 \supset K_2 \supset K_3 \supset \cdots \supset \{1\}$$

where K_i is normal in K_j for $0 \leq j \leq i$, and K_m has an Iwahori decomposition with respect to $P_\emptyset = M_\emptyset N_\emptyset$ for each $m \geq 1$. Moreover, the elements of this filtration form a neighborhood basis of the identity, and G/K_m is countable for each m, so G is a t.d.-group.

Example 4.0.11. For $m \in \mathbb{Z}_{\geq 1}$, define $K_m = 1 + \varpi^m \cdot \mathrm{M}_n(R)$. The group K_m is certainly compact and open in $G = \mathrm{GL}_n(k)$, and K_0/K_1 is canonically isomorphic to the finite group $\mathrm{GL}_n(\mathfrak{f})$. Left multiplication by K_m^+ corresponds to adding a $\wp^m$ multiple of a row to any row strictly above it, we can left multiply any element of K_m by elements of K_m^+ into a lower triangular matrix in K_m.

Since right multiplication by K_m^- corresponds to adding a $\varpi^m \cdot R$ multiple of a column to any column strictly to the left of it, we can right multiply any lower triangular matrix in K_m into a diagonal element of K_m.

Consequently, each K_m has an Iwahori decomposition.

Remark 4.0.12. In general, K_0 does not have an Iwahori decomposition, but we can always write

$$K_0 = (\overline{N}_\emptyset \cap K_0) \cdot (N_\emptyset \cap K_0) \cdot (\overline{N}_\emptyset \cap K_0) \cdot (M_\emptyset \cap K_0).$$

Let $\Phi = \Phi(G, T)$ denote the set of roots of G with respect to T and G. We let Φ^+ denote the set of positive roots with respect to $P_\emptyset$, and we let $\Delta \subset \Phi^+$ denote the set of simple roots with respect to $P_\emptyset$. We let T' denote the set of $t \in T$ for which $|\alpha(t)| \leq 1$ for all $\alpha \in \Delta$. We can and do choose a set of representatives for $T'/(T \cap K_0)$ which is closed with respect to products. We call this set of representatives T^+. Note that there is a natural monoid isomorphism between T^+ and $\mathbb{Z}^n_{\geq 0} \times \mathbb{Z}^m$ where n is the semisimple rank of G and m is the k-rank of the center of G.

Example 4.0.13. The elements of Φ are the nontrivial eigencharacters of T for its adjoint action on $\mathrm{M}_n(k)$ (that is, $t \cdot X = \mathrm{Ad}(t)X = tXt^{-1}$. Thus, for each pair (i, j) with $1 \leq i \neq j \leq n$, we have a root $\alpha_{ij} \in \Phi$ defined by $\alpha_{i,j}(\mathrm{diag}(t_1, t_2, \ldots, t_n)) = t_i/t_j$. The root α_{ij} is positive with respect to $P_\emptyset$ provided that $i < j$, and it is simple with respect to $P_\emptyset$ provided that $j = i + 1$. We have that $T \cap K_0$ is the group of elements in T for whom each entry belongs to $R^\times$. We will take T^+ to be the set

$$\{\mathrm{diag}(\varpi^{k_1}, \varpi^{k_2}, \ldots, \varpi^{k_n}) \,|\, k_1 \geq k_2 \geq \cdots \geq k_n \in \mathbb{Z}\}.$$

The set T^+ has some very important properties. For example, we can and will assume that our filtration of G by compact open subgroups K_m has the following property. For all $m > 1$, not only does K_m have an Iwahori decomposition with respect to $P_\emptyset = M_\emptyset N_\emptyset$, but for each $t \in T^+$ and $K = K_m$ we have

$${}^tK^+ \subset K^+,\ {}^tK^- \supset K^-, \text{ and } {}^tK^0 = K^0.$$

and

$${}^{t^{-1}}K^+ \supset K^+,\ {}^{t^{-1}}K^- \subset K^-, \text{ and } {}^{t^{-1}}K^0 = K^0.$$

(Here ${}^tK_m^+ = tK^+t^{-1}$, etc.) In other words, the action of T^+ preserves the $M_\emptyset$ part of K_m, shrinks the $N_\emptyset$ part, and enlarges the $\overline{N}_\emptyset$ part. Moreover, we have

$$N_\emptyset = \bigcup_{t \in T^+} {}^{t^{-1}}K^+ \text{ and } \overline{N}_\emptyset = \bigcup_{t \in T^+} {}^tK^-.$$

That is, both $N_\emptyset$ and $\overline{N}_\emptyset$ can be written as the union of compact open subgroups. Finally, we have that the elements of

$$\{{}^tK^+ \,|\, t \in T^+\}$$

form a neighborhood basis of the identity in $N_\emptyset$ and, similarly, the elements of

$$\{{}^{t^{-1}}K^+ \,|\, t \in T^+\}$$

form a neighborhood basis of the identity in $\overline{N}_\emptyset$.

Example 4.0.14. Consider the element $t = \mathrm{diag}(\varpi^{j_1}, \varpi^{j_2}, \varpi^{j_3}) \in \mathrm{GL}_3(k)$ with $j_1 \geq j_2 \geq j_3$. If

$$n = \begin{pmatrix} 1 & n_{12} & n_{13} \\ 0 & 1 & n_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

then ${}^tn$ is

$$\begin{pmatrix} 1 & \varpi^{(j_2-j_1)}n_{12} & \varpi^{(j_3-j_1)}n_{13} \\ 0 & 1 & \varpi^{(j_3-j_2)}n_{23} \\ 0 & 0 & 1 \end{pmatrix}.$$

In general, the element $t = \mathrm{diag}(\varpi^{\ell_1}, \varpi^{\ell_2}, \cdots, \varpi^{\ell_n})$ acts on $x \in \mathrm{GL}_n(k)$ by $({}^tx)_{ij} = \varpi^{(\ell_i - \ell_j)} \cdot x_{ij}$.

Exercise 4.0.15. Prove that any finite-dimensional smooth irreducible representation (π, V) of $\mathrm{GL}_n(k)$ has the form $\chi \circ \det$, where $\chi : k^\times \to \mathbb{C}^\times$ is a smooth character. (Hint: use the above calculation of the action of T on $N_\emptyset$ to deduce that π is trivial on $N_\emptyset$. Then prove that $\mathrm{SL}_n(k)$ is generated by the conjugates of $N_\emptyset$, so that π factors through $\det$.)

Theorem 4.0.16 (Cartan decomposition)**.** *There exists a finite subset ω of $M_\emptyset$ such that*

$$G = \coprod_{w \in \omega,\, t \in T^+} K_0 w t K_0$$

Remark 4.0.17. The set ω compensates for the fact that, in general, $M_\emptyset \neq T$. It is also true that the elements of ω stabilize K_0 and the K_m can be chosen so that they too are stabilized by the elements of ω.

Proof. Since $M_\emptyset = T$ for $\mathrm{GL}_n(k)$, we have $\omega = \{1\}$. We first show that we can write:

$$G = \bigcup_{t \in T^+} K_0 t K_0.$$

Indeed, since K_0 is now on the left and right, by multiplying by elements of K_0 on the left and right, we can permute rows and columns. Therefore, we can move the matrix entry with smallest valuation into the a_{nn} position. As in the proof of the Iwasawa decomposition, we can then clear out all entries in the bottom row except for a_{nn} and all entries in the final column except for a_{nn}. Note that a_{nn} remains the matrix entry with the smallest valuation. We continue in this way until we produce a diagonal matrix $\mathrm{diag}(a_{11}, a_{22}, \ldots, a_{nn})$ with $v(a_{11}) \geq v(a_{22}) \geq \cdots \geq v(a_{nn})$. By multiplying by an appropriate element of $T \cap K_0$, we arrive at an element of T^+.

We now show that if $K_0 \cdot \mathrm{diag}(\varpi^{k_1}, \varpi^{k_2}, \ldots, \varpi^{k_n}) \cdot K_0 = K_0 \cdot \mathrm{diag}(\varpi^{j_1}, \varpi^{j_2}, \ldots, \varpi^{j_n}) \cdot K_0$ with $k_1 \geq k_2 \geq \cdots \geq k_n$ and $j_1 \geq j_2 \geq \cdots \geq j_n$, then $k_i = j_i$ for $1 \leq i \leq n$.

If $a \in G$, then define $|a| = \max_{1 \le i,j \le n} |a_{ij}|$. Note that for $g \in G$ and $k_1, k_2 \in K$ we have $|k_1 x| \le |x| = \left|k_1^{-1} k_1 x\right| \le |k_1 x|$ and similarly for right multiplication by k_2. Therefore, $|k_1 g k_2| = |g|$.

Thinking of $g \in \mathrm{GL}_n(k)$ as a map from k^n to k^n we can then define $\wedge^\ell g$ from $\wedge^\ell k^n$ to $\wedge^\ell k^n$ for $1 \le \ell \le n$. Here $\wedge^\ell g$ is the $\binom{n}{\ell} \times \binom{n}{\ell}$ matrix whose entries are the determinants of $\ell \times \ell$ minors of g. We have $\wedge^\ell(k_1 g k_2) = \wedge^\ell k_1 \cdot \wedge^\ell g \cdot \wedge^\ell k_2$ and since $\wedge^\ell k_i \in \mathrm{GL}_{\binom{n}{\ell}}(R)$, we have that $\left|\wedge^\ell g\right|$ depends only on the the K_0 double coset of g. Thus, if $g \in K_0 \cdot \mathrm{diag}(\varpi^{k_1}, \varpi^{k_2}, \ldots, \varpi^{k_n}) \cdot K_0$, then $\left|\wedge^\ell g\right| = \left|\varpi^{k_n} \cdot \varpi^{k_{(n-1)}} \cdot \varpi^{k_{n-\ell+1}}\right|$. We conclude that $k_\ell = j_\ell$ for $1 \le \ell \le n$. ■

Corollary 4.0.18. *For all $m \ge 0$, G/K_m is countable.*

Proof. From the Cartan decomposition, the double coset space $K_0 \setminus G/K_0$ is countable. Since for all $g \in G$, the double coset $K_0 g K_0$ can be written as a finite union of left K_m-cosets, the corollary follows. ■

4.1. **Parabolic subgroups.** A parabolic subgroup P of G is called *standard* (or, more precisely, *standard with respect to $P_\emptyset$*) if $P_\emptyset \subset P \subset G$. Since every minimal parabolic is conjugate to $P_\emptyset$, every parabolic subgroup of G is conjugate to a standard parabolic. If H is a closed subgroup of G such that $P_\emptyset \le H \le G$, then H is a (standard) parabolic subgroup of G.

Remark 4.1.1. As we shall see, two standard parabolic subgroups of G can be conjugate.

There is a very nice description of the standard parabolic subgroups of G. We follow the presentation of [4]. For every subset θ of Δ, let $T_\theta \le T$ denote the connected component of

$$\bigcap_{\alpha \in \theta} \ker(\alpha).$$

Define $M_\theta = C_G(T_\theta)$ and $P_\theta = M_\theta P_\emptyset$. Note that, by definition, T_θ is the "split part" of the center of M_θ.

Remark 4.1.2. The minimal parabolic $P_\emptyset$ is what it should be, and $G = M_\Delta = P_\Delta$.

Example 4.1.3. For $\mathrm{GL}_3(k)$, there are four standard parabolic subgroups. Besides G and $P_\emptyset$, we have $P_{\alpha_{12}}$ consisting of matrices in $\mathrm{GL}_3(k)$ of the form

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$$

and $P_{\alpha_{23}}$ consisting of matrices in $\mathrm{GL}_3(k)$ of the form

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

For completeness, we note that

$$T_{\alpha_{12}} = \{\mathrm{diag}(a, a, b) \mid a, b \in k^\times\}$$

and

$$T_{\alpha_{23}} = \{\mathrm{diag}(a, b, b) \mid a, b \in k^\times\}$$

while $M_{\alpha_{12}}$ consists of those matrices in $P_{\alpha_{12}}$ of the form

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

and $M_{\alpha_{23}}$ consists of those matrices in $P_{\alpha_{23}}$ of the form

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

We denote by N_θ the unipotent radical[5] of P_θ. The parabolic P_θ has a Levi decomposition $P_\theta = M_\theta N_\theta$.

Example 4.1.4. Continuing the example above, we have that $N_{\alpha_{12}}$ consists of those matrices in $P_{\alpha_{12}}$ of the form

$$\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

and $N_{\alpha_{23}}$ consists of those matrices $P_{\alpha_{23}}$ of the form

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 4.1.5. For $\mathrm{GL}_2(k)$ and $\mathrm{GL}_4(k)$ describe, $\Phi, \Phi^+, \Delta, T^+$ and all possible P_θ, M_θ, N_θ, and T_θ.

Exercise 4.1.6. Do the same for $\mathrm{Sp}_4(k)$. In these notes, we shall always realize $\mathrm{Sp}_4(k)$ as the subgroup of $\mathrm{GL}_4(k)$ which is the isometry group for the form $J = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}$ where $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We take T, $P_\emptyset$, and K_m in $\mathrm{Sp}_4(k)$ to be the intersection of $\mathrm{Sp}_4(k)$ with the analogous objects in $\mathrm{GL}_4(k)$.

Definition 4.1.7. Suppose that P is a parabolic subgroup of G with Levi decomposition $P = MN$. A compact open subgroup K of G is said to have an Iwahori factorization with respect to $P = MN$ if we can write, in any order,

$$K = (K \cap \overline{N}) \cdot (K \cap M) \cdot (K \cap N)$$

where $\overline{N}$ is the unipotent radical of the parabolic opposite $P = MN$.

We can and will assume that each of our compact open subgroups K_m with $m > 1$ has an Iwahori decomposition with respect to every standard parabolic $P_\theta = M_\theta N_\theta$. Note that the action of T^+ on $K_m \cap M_\theta$ will no longer be trivial. To compensate for this, we fix a subset $T_\theta^+ \subset T_\theta$ of coset representatives for $T'_\theta/(T_\theta \cap K_0)$ where

$$T'_\theta := \{t \in T_\theta \mid |\alpha(t)| \leq 1 \text{ for all } \alpha \in \Delta \smallsetminus \theta\}.$$

Moreover, we can and do assume that $T_\theta^+ = T^+ \cap T_\theta$. Just as above, the elements of the set

$$\{{}^t(K_m \cap N_\theta) \mid t \in T_\theta^+\}$$

[5]The unipotent radical is the unique minimal normal subgroup of P_θ for which P_θ/N_θ is reductive. (This quotient is isomorphic to the Levi component M_θ, which is not unique.)

form a neighborhood basis of the identity element in N_θ and

$$N_\theta = \bigcup_{t \in T_\theta^+} {}^{t^{-1}}(K_m \cap N_\theta).$$

Remark 4.1.8. $P_\emptyset \cap M_\theta$ is a minimal parabolic subgroup of M_θ, T is the group of k-rational points of a maximal k-split torus in M_θ, and $K_0 \cap M_\theta$ is a special (with respect to T) compact open subgroup of M_θ. With respect to this data, we have the Iwasawa decomposition, Iwahori decomposition, and Cartan decomposition (of M_θ). Moreover, every standard parabolic subgroup of M_θ looks like $P_\delta \cap M_\theta$ for some standard parabolic subgroup P_δ of G. Moreover, $P_\delta \cap M_\theta$ has a Levi decomposition $P_\delta \cap M_\theta = (M_\theta \cap M_\delta)(M_\theta \cap N_\delta)$. Note that $M_\theta \cap M_\delta = M_{\theta\cap\delta}$.

Exercise 4.1.9. Show that $K_m \cap P_\theta$ forms a neighborhood basis of the identity in P_θ. Show that $P_\theta/(K_m \cap P_\theta)$ is countable.

Remark 4.1.10. I hope that this will be enough notation to keep us going for some time.

5. Some general basic facts

In this section we assume only that G is a t.d.-group.

5.1. **A second look at admissibility.** Recall that a smooth representation (π, V) is admissible provided that for all compact open subgroups K of G, the space of K-fixed vectors is finite-dimensional. We will give an alternative characterization of admissibility; first we need a lemma.

Lemma 5.1.1. *Let K be a compact t.d.-group.*

(1) *Any irreducible smooth representation of K is finite-dimensional.*
(2) *For every finite-dimensional smooth representation (π, V) of K, there is a normal compact open subgroup N of K that acts trivially on V.*
(3) *Every smooth representation of K is semisimple.*
(4) *Every smooth representation of K is unitary.*

Proof. (1) Let (π, V) be an irreducible smooth representation of K. Choose a nonzero $v \in V$, and let $U \subset K$ be a compact open subgroup that fixes v. Since K/U is finite, the set $Kv = \{\pi(x)v \mid x \in K\}$ is finite. Since Kv generates V as a vector space, $\dim_{\mathbb{C}} V < \infty$.

(2) Let (π, V) be a finite-dimensional smooth representation of K, and let $N = \ker \pi$ (recall that π is a homomorphism $K \to \mathrm{Aut}_{\mathbb{C}}(V)$), so N is a normal subgroup of K. Let $v_1, \dots, v_n$ be a $\mathbb{C}$-basis for V, so

$$N = \bigcap_{i=1}^{n} \mathrm{stab}_K(v_i).$$

Since (π, V) is smooth, $\mathrm{stab}_K(v) \subset K$ is open for any $v \in V$, so, since the above intersection is finite, N is open. Since K is compact, any open subgroup of K is compact.

(3) Let (π, V) be a smooth representation of K. It suffices to show that V is a (not necessarily direct) sum of irreducible representations (see, for example, [9, XVII, §2]). Let $v \in V$ be any nonzero vector, so we must show that v belongs to a sum of irreducible

subrepresentations of V. Let $W = Kv$ be the subrepresentation generated by V. As in the proof of (1), W is finite-dimensional, so by (2), there is an open normal subgroup N of K which acts trivially on W. Since K/N is finite, W is completely decomposable into a direct sum of irreducible representations of K. Since $v \in W$, this completes the proof.

(4) Let (π, V) be a smooth representation of K. Choose a positive-definite Hermetian form $\langle \cdot, \cdot \rangle$ on V. For $v, w \in V$, set

$$(v, w) = \int_K \langle \pi(x)v, \pi(x)w \rangle \, dx,$$

where dx is a Haar measure on K. Clearly $(\cdot, \cdot)$ defines a positive-definite K-invariant Hermetian form on V.

■

Thus any smooth representation of a compact t.d.-group K can be decomposed into a direct sum of irreducible representations. This decomposition is not necessarily canonical — e.g., there are many ways to decompose n copies of the trivial representation. We do have the following result, however:

Corollary 5.1.2. *Let (π, V) be a smooth representation of a compact t.d.-group K. Then (π, V) has a canonical decomposition (as K-representations) as*

$$V = \bigoplus_{\sigma} V(\sigma),$$

where the sum runs over a set of representatives for the isomorphism classes of irreducible smooth representations (σ, W_σ) of K, and $V(\sigma)$ is the image of the canonical map $W_\sigma \otimes_{\mathbb{C}} \operatorname{Hom}_K(W_\sigma, V) \to V$. The subrepresentation $V(\sigma)$ is isomorphic to a direct sum of copies of W_σ.

Proof. Let $V = \bigoplus_I V_i$ be a decomposition of V into irreducible K-modules, where I is some index set. If $V_i \cong W_\sigma$ for some σ, then there is an injection $\varphi : W_\sigma \to V$ whose image is V_i; thus $V_i \subset V(\sigma)$, so $V = \sum_\sigma V(\sigma)$. To show that the sum is direct, let $I_\sigma = \{i \in I \mid V_i \cong W_\sigma\}$, so $\bigoplus_{i \in I_\sigma} V_i \subset V(\sigma)$. Any homomorphism $\varphi : W_\sigma \to V$ is zero when composed with each projection map $V \to V_i$ for $i \notin I_\sigma$, so $V(\sigma) \subset \bigoplus_{i \in I_\sigma} V_i$.

It is clear from the proof that $V(\sigma)$ is isomorphic to a direct sum of copies of W_σ. ■

The submodule $V(\sigma)$ has a name:

Definition 5.1.3. Let G be any t.d.-group, let $(\sigma, W) \in \mathfrak{R}(G)$ be irreducible, and let $(\pi, V) \in \mathfrak{R}(G)$. We define the *$\sigma$-isotypic submodule* of (π, V) to be the image of the canonical homomorphism $W \otimes_{\mathbb{C}} \operatorname{Hom}_G(W, V) \to V$, and we denote it by $V(\sigma)$. As above, one can show that $V(\sigma)$ is the unique largest submodule of W which is isomorphic to a direct sum of copies of V. If we can write $V = V(\sigma) \oplus V'$ for a unique submodule $V' \subset V$, then we will also call $V(\sigma)$ the *σ-isotypic component* of V.

Corollary 5.1.2 allows us to make the following definition.

Definition 5.1.4. Let K be a compact t.d.-group, let (σ, W_σ) be any irreducible smooth representation of K, and let (π, V) be any smooth representation of K. We define the *multiplicity* $m(\sigma)$ *of* σ *in* (π, V) to be

$$m(\sigma) := \frac{\dim_{\mathbb{C}}(V(\sigma))}{\dim_{\mathbb{C}}(W_\sigma)}$$

where $V(\sigma)$ is the σ-isotypic component of V.

Exercise 5.1.5. Show that $m(\sigma) = \dim_{\mathbb{C}}(\mathrm{Hom}_K(W_\sigma, V))$.

Lemma 5.1.6. *A smooth representation (π, V) of G is admissible if and only if for every compact open subgroup K of G, each irreducible representation of K occurs with finite multiplicity in $(\mathrm{res}_K\, \pi, V)$.*

Proof. "$\Rightarrow$" Choose a compact open subgroup K of G and an irreducible representation (σ, W) of K. Let N be a normal compact open subgroup of K such that σ factors through K/N. Then $V(\sigma) \subset V^N$ so $V(\sigma)$ is finite-dimensional.

"$\Leftarrow$" Let K be a compact open subgroup of G. Since the trivial one-dimensional representation of K is irreducible, the dimension of the space of K-fixed vectors is finite by hypothesis. ∎

If (π, V) is an admissible representation, then from the previous two lemmas we can write

$$\mathrm{res}_K\, \pi = \bigoplus_\sigma \sigma^{\oplus m(\sigma)}$$

where σ runs over a set of representatives for the isomorphism classes of irreducible representations of K, and $m(\sigma) < \infty$.

Lemma 5.1.7. *A smooth representation (π, V) is admissible if and only if $(\widetilde{\pi}, \widetilde{V})$ is admissible. In this case, the natural map $V \to \widetilde{\widetilde{V}}$ is an isomorphism of G-representations.*

Proof. Let K be a compact open subgroup of G. By Exercise 3.1.2, there is a natural isomorphism $\widetilde{V}^K \cong \mathrm{Hom}_{\mathbb{C}}(V^K, \mathbb{C})$, so V^K is finite-dimensional if and only if $\widetilde{V}^K$ is. This proves the first statement.

For all compact open subgroups K, we have $\widetilde{\widetilde{V}}^K \cong \mathrm{Hom}_{\mathbb{C}}(\widetilde{V}^K, \mathbb{C})$, so since V^K and $\widetilde{\widetilde{V}}^K$ have the same dimension, the natural injection $\varphi : v \mapsto (\lambda \mapsto \lambda(v))$ is an isomorphism. Since this is true for all compact open subgroups, we conclude that φ is an isomorphism of V with $\widetilde{\widetilde{V}}$. It is easy to check that φ is a G-homomorphism. ∎

Formally, the previous lemma says that $V \mapsto \widetilde{V}$ is a duality on the full subcategory $\mathcal{A} \subset \mathcal{R}(G)$ consisting of admissible representations, meaning a contravariant equivalence $\mathcal{A} \to \mathcal{A}$ whose square is isomorphic to the identity functor. When G is the trivial group, this specializes to the duality of finite-dimensional vector spaces.

Remark 5.1.8. If (π, V) is smooth but not admissible, then the natural map $V \to \widetilde{\widetilde{V}}$ is an injection, but it is never an isomorphism.

We close this section with a definition which is closely related to semisimplicity.

Definition 5.1.9. Let V be a smooth G-representation. Define the *length* of V to be the maximum $n \in \mathbb{Z}_{\geq 0}$ such that there exists a filtration

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$$

of G-representations, assuming that such an n exists. If there is no maximum n, we say that V has *infinite length*.

Alternatively, suppose that the filtration

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$$

has the property that each V_i/V_{i-1} is simple (such a filtration exists for any finite length module). Then n is the length of V (and is thus independent of the filtration by the Jordan-Hölder theorem).

If V has a decomposition $V = \bigoplus_I V_i$ where each V_i is simple, then the length of V is the cardinality of the indexing set I.

Exercise 5.1.10. Show that any finite length module is finitely generated.

5.2. **Exactness properties.**

Lemma 5.2.1. *Suppose that* $V_i \in \mathfrak{R}(G)$, $i = 1, 2, 3$. *The sequence*

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

is exact in $\mathfrak{R}(G)$ *if and only if for each compact open subgroup* K *of* G*, the sequence of complex vector spaces*

$$0 \to V_1^K \to V_2^K \to V_3^K \to 0$$

is exact.

Proof. "$\Rightarrow$" Since taking invariants is always left exact, it is enough to show that if $\beta\colon W \to V$ is a surjective G-module homomorphism between two smooth representations, then for each compact open subgroup K of G, $\beta\colon W^K \to V^K$ is surjective. Let $v \in V^K$. Since β is surjective, there exists $w \in W$ such that $\beta(w) = v$. We have

$$v = e_K v = e_K \beta(w) = \beta(e_K w).$$

(Since the projection operator e_K on V and W is a finite sum, we may move e_K through β.)

"$\Leftarrow$" This is clear.

$\blacksquare$

Exercise 5.2.2. Let G be a t.d.-group with the property that all irreducible smooth representations of G are admissible (cf. Corollary 7.3.5). If V is a smooth representation of G which has finite length, show that V is admissible.

Corollary 5.2.3. *The functor* $V \mapsto \widetilde{V}$ *from* $\mathfrak{R}(G)$ *to itself is exact.*

Proof. Suppose that

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

is an exact sequence of smooth representations. We want to show that

$$0 \to \widetilde{V}_3 \to \widetilde{V}_2 \to \widetilde{V}_1 \to 0$$

is exact. For this, it will suffice to show that for all compact open subgroups K of G, we have that

$$0 \to \widetilde{V}_3^K \to \widetilde{V}_2^K \to \widetilde{V}_1^K \to 0$$

is exact. This follows immediately from the fact that for any smooth representation V of G, we have $\widetilde{V}^K = \mathrm{Hom}(V^K, \mathbb{C})$, so the functor $V \mapsto \widetilde{V}^K$ is the composition of two exact functors. ■

Corollary 5.2.4. *A smooth representation (π, V) is irreducible and admissible if and only if its contragredient is.*

Proof. By Lemma 5.1.7, we only have to show that if V is irreducible and admissible, then $\widetilde{V}$ is irreducible. Assume that V is irreducible and admissible, and suppose that

$$0 \to V_1 \to \widetilde{V} \to V_3 \to 0$$

is an exact sequence of smooth G-modules. By Lemma 5.2.1, the sequence

$$0 \to \widetilde{V}_3 \to \widetilde{\widetilde{V}} \to \widetilde{V}_1 \to 0$$

is exact. Since $V = \widetilde{\widetilde{V}}$ is irreducible, either $\widetilde{V}_1 = 0$ or $\widetilde{V}_3 = 0$, so either $V_1 = 0$ or $V_3 = 0$. Thus $\widetilde{V}$ is irreducible. ■

5.3. **Some comments on integration.** We first consider the type of functions in which we will be most interested.

For a complex vector space V, we let $C_c^\infty(G, V)$ denote the space of locally constant, V-valued, compactly supported functions on G. That is, $C_c^\infty(G, V)$ is the set of $f\colon G \to V$ such that f is compactly supported, and for each $g \in G$ there exists an open subgroup K_g in G such that $f(gx) = f(g)$ for all $x \in K_g$. When $V = \mathbb{C}$, we will write $C_c^\infty(G)$ rather than $C_c^\infty(G, \mathbb{C})$. Note that there is a left action of G on $C_c^\infty(G, V)$ given by the right regular action $(g \cdot f)(x) = f(xg)$. For a compact open subgroup K of G, we let $C_c(G/K, V)$ denote the space $C_c^\infty(G, V)^K$ of K-invariants of $C_c^\infty(G, V)$; that is, $C_c(G/K, V)$ is the space of all elements $f \in C_c^\infty(G, V)$ such that $f(xg) = f(x)$ for all $g \in K$.

Exercise 5.3.1. Show that $C_c^\infty(G, V)$ (with the right regular action) is a smooth representation of G — i.e., we can write

$$C_c^\infty(G, V) = \bigcup_{\text{compact open } K \leq G} C_c(G/K, V).$$

Conclude that the natural map

$$C_c^\infty(G) \otimes V \to C_c^\infty(G, V)$$

is an isomorphism of G-modules.

Exercise 5.3.2. There is also a right action of G on $C_c^\infty(G,V)$ given by the following: for $f \in C_c^\infty(G,V)$ and $x, g \in G$, we define $(f \cdot g)(x) = f(gx)$. Show that this action is smooth as well — i.e., for each $f \in C_c^\infty(G,V)$, there is a compact open subgroup $K_f \subset G$ such that $f(gx) = f(x)$ for all $g \in K_f$ and all $x \in G$.

We now consider a left Haar measure $d_\ell g$ on G. This is the (unique up to a positive real number) σ-regular nonzero Borel measure for which $\operatorname{meas}_{d_\ell g}(C) = \operatorname{meas}_{d_\ell g}(gC)$ for all Borel subsets C of G and all $g \in G$.

Let K and K' be compact open subgroups of G. In this case, $K' \cap K$ is an open subgroup of both K and K'. Consequently, both $[K\colon K' \cap K]$ and $[K'\colon K' \cap K]$ are finite. Since $d_\ell g$ is a left Haar measure, we have

$$\operatorname{meas}_{d_\ell g}(K') = [K'\colon K' \cap K] \cdot \operatorname{meas}_{d_\ell g}(K' \cap K) = \frac{[K'\colon K' \cap K]}{[K\colon K' \cap K]} \operatorname{meas}_{d_\ell g}(K).$$

Exercise 5.3.3. Let $f \in C_c^\infty(G,V)$. Show that

$$\int_G f(g)\, d_\ell g = \sum_{\overline{g} \in G/K} f(g) \cdot \operatorname{meas}_{d_\ell g}(K)$$

for any compact open subgroup K of G such that $f \in C_c(G/K, V)$.

Note that $f \mapsto \int_G f\, d_\ell g$ is a left invariant distribution on G. That is, it is a linear map $C_c^\infty(G) \to \mathbb{C}$ such that

$$\int_G f(hg)\, d_\ell g = \int_G f(g)\, d_\ell g$$

for all $f \in C_c^\infty(G)$ and all $h \in G$.

5.4. **The modulus character.** Let $x \in G$. Define ${}^x g = xgx^{-1}$ for $g \in G$ and ${}^x S := \{{}^x s \mid s \in S\}$ for $S \subset G$ any subset. If $C \subset G$ is a compact set then so is ${}^x C$, and similarly if $S \subset G$ is a Borel set then ${}^x S$ is also a Borel set. Thus if we define $d^x g$ by $\operatorname{meas}_{d^x g}(S) = \operatorname{meas}_{d_\ell g}({}^{x^{-1}}S)$, then $d^x g$ is again a left Haar measure. Since Haar measures are unique up to a positive real number, we can define a function δ_G on G by

$$d^x g = \delta_G(x) \cdot d_\ell g.$$

Exercise 5.4.1. (1) The definition of δ_G is independent of the choice of $d_\ell g$.

(2) The function $\delta_G\colon G \to \mathbb{R}_{>0}$ is a character whose kernel contains every compact open subgroup of G. In particular, δ_G is a smooth character; we call it the *modulus character* of G.

(3) The measure $\delta_G(g) \cdot d_\ell g$ is a right Haar measure on G with modulus character δ_G^{-1}. (One defines the modulus character δ'_G of a right Haar measure d_r by requiring that $\operatorname{meas}_{d_r g}({}^x S) = \delta'_G(x) \operatorname{meas}_{d_r} S$ for all Borel sets $S \subset G$).

Definition 5.4.2. A t.d.-group G is *unimodular* provided that $\delta_G = 1$, or equivalently, any left Haar measure on G is also a right Haar measure.

If G is unimodular, then as there is no question as to whether a Haar measure on G is left- or right-translation invariant, so we usually denote it by dg.

Example 5.4.3. Here are some examples of (obviously) unimodular groups we will encounter (cf. Exercise 5.4.1(2)):

(1) any abelian group,
(2) any compact group,
(3) any group which can be written as a union of compact open subgroups (for example, the Heisenberg group of Section 3).

We will soon discuss the most important example (for us) of a unimodular group.

Exercise 5.4.4. Let $G = k$, and let dg be a Haar measure on (the additive group) k. For $x \in k^\times$, define a measure $d(xg)$ on k by $\mathrm{meas}_{d(xg)}(S) = \mathrm{meas}_{dg}(xS)$, where $S \subset k$ is a Borel set. Prove that $d(xg) = |x| \cdot dg$.

Example 5.4.5. If P is a proper parabolic subgroup of a reductive p-adic group, then P is not unimodular. For example, consider the parabolic subgroup $P_\emptyset$ of $\mathrm{GL}_n(k)$ (for $n \in \mathbb{Z}_{\geq 2}$) with Levi decomposition $TN_\emptyset$. For $t \in T$ and $n \in N_\emptyset$ we have $\delta_{P_\emptyset}(tn) = \delta_{P_\emptyset}(t) \cdot \delta_{P_\emptyset}(n) = \delta_{P_\emptyset}(t)$ (see part (1) of Exercise 5.4.6 below). To keep this example simple, let us suppose that $t = \mathrm{diag}(\varpi^{j_1}, \varpi^{j_2}, \ldots, \varpi^{j_n})$ with $j_1 \geq j_2 \geq \cdots \geq j_n$. By definition, we have

$$\delta_{P_\emptyset}(t^{-1}) = \frac{\mathrm{meas}({}^t S)}{\mathrm{meas}(S)}$$

for all Borel subsets S of $P_\emptyset$ with finite measure. Let us take S to be the compact open subgroup K_1 obtained by intersecting $1 + \mathrm{M}_n(\wp)$ with $P_\emptyset$. A calculation shows that ${}^t K_1 \leq K_1$, so

$$\delta_{P_\emptyset}(t^{-1}) = \frac{\mathrm{meas}(K_1)}{[K_1 \colon {}^t K_1] \cdot \mathrm{meas}(K_1)} = \prod_{\alpha \in \Phi^+} |\alpha(t)| \,.$$

Exercise 5.4.6. Let P be a proper parabolic subgroup of $G = \mathrm{GL}_n(k)$ for some $n \geq 2$, and let $P = MN$ be a Levi decomposition of P.

(1) Show that there exist no nontrivial smooth characters $N \to \mathbb{R}_{>0}$. Conclude that if $p = mn$ with $m \in M$ and $n \in N$, then $\delta_P(mn) = \delta_P(m)$.
(2) Let $\theta \subset \Delta$, as in Section 4.1. Show that if $t \in T_\theta$ then $\delta_{P_\theta}(t) = \delta_{P_\emptyset}(t)$.
(3) By using the Cartan decomposition and a well chosen compact open subgroup with Iwahori decomposition, show that G is unimodular.
(4) Show that $\delta_P(m) = |\det(\mathrm{Ad}(m^{-1})|_{\mathfrak{n}})|$ for $m \in M$. Here Ad denotes the action of P on its Lie algebra, and $\mathfrak{n}$ is the Lie algebra of N in the Lie algebra of P.

All of the results in this exercise are true as stated when G is the group of k-points of any reductive algebraic group over k.

5.5. **A second look at induction.** Let $H \leq G$ be a closed subgroup of G and (σ, W) a smooth representation of H. The induced representation $(R, \mathrm{Ind}_H^G(\sigma))$ has a natural subrepresentation $(R, \text{c-}\mathrm{Ind}_H^G(\sigma))$, called the *compact induction* of σ from H to G, consisting of those functions $f \in \mathrm{Ind}_H^G(\sigma)$ whose support has compact image in $H \backslash G$.

Example 5.5.1. In Section 3 we showed that any f in the induced representation $\operatorname{Ind}_{SZ}^{H} \widetilde{\chi}$ (resp. $\operatorname{Ind}_{K_{\rho(\chi)}Z}^{H} \widetilde{\chi}$) has compact support modulo the inducing subgroup. In these cases, then, the inclusion $\text{c-Ind}_{SZ}^{H} \widetilde{\chi} \to \operatorname{Ind}_{SZ}^{H} \widetilde{\chi}$ (resp. $\text{c-Ind}_{K_{\rho(\chi)}Z}^{H} \widetilde{\chi} \to \operatorname{Ind}_{K_{\rho(\chi)}Z}^{H} \widetilde{\chi}$) is an isomorphism.

Lemma 5.5.2. *Let K be a compact open subgroup of G. Fix a set of representatives $\{g\}$ for $H\backslash G/K$. The $\mathbb{C}$-linear map*

$$(\operatorname{Ind}_H^G(\sigma))^K \to \prod_{\overline{g}\in H\backslash G/K} W^{(H\cap gKg^{-1})} \tag{5}$$

defined by $f \mapsto (f(g))_{\overline{g}}$ is an isomorphism.

Proof. We leave it to the reader to check that the map is well-defined. Bijectivity is then clear. ■

Corollary 5.5.3. *Let the notation be as in Lemma 5.5.2. The map in Equation (5) restricts to an isomorphism*

$$(\text{c-Ind}_H^G(\sigma))^K \to \bigoplus_{\overline{g}\in H\backslash G/K} W^{(H\cap gKg^{-1})}.$$

Corollary 5.5.4. *If $H\backslash G$ is compact and (σ, W) is admissible then $\operatorname{Ind}_H^G(\sigma) = \text{c-Ind}_H^G(\sigma)$ is admissible.*

Proof. For a compact open subgroup K of G, the set $H\backslash G/K$ is finite. ■

Corollary 5.5.5. *The functors $\text{c-Ind}_H^G, \operatorname{Ind}_H^G : \mathfrak{R}(H) \to \mathfrak{R}(G)$ are exact.*

Proof. Let

$$0 \to W_1 \to W_2 \to W_3 \to 0$$

be an exact sequence of smooth representations of H. We want to show that

$$0 \to \operatorname{Ind}_H^G W_1 \to \operatorname{Ind}_H^G W_2 \to \operatorname{Ind}_H^G W_3 \to 0$$

is exact. For this, it will be enough to show that for all compact open subgroups K of G the sequence

$$0 \to (\operatorname{Ind}_H^G W_1)^K \to (\operatorname{Ind}_H^G W_2)^K \to (\operatorname{Ind}_H^G W_3)^K \to 0$$

is exact. Fix a compact open subgroup $K \subset G$. Lemma 5.5.2 identifies the above sequence with the sequence

$$0 \longrightarrow \prod_{H\backslash G/K} W_1^{(H\cap gKg^{-1})} \longrightarrow \prod_{H\backslash G/K} W_2^{(H\cap gKg^{-1})} \longrightarrow \prod_{H\backslash G/K} W_3^{(H\cap gKg^{-1})} \longrightarrow 0,$$

which is a product of sequences

$$0 \to W_1^{K'} \to W_2^{K'} \to W_3^{K'} \to 0$$

which are exact by Lemma 5.2.1.

The same argument applies to c-Ind_H^G. ■

6. The Jacquet functor and Jacquet's lemma

We suppose that we are in the setting of Section 4. That is, G is the group of k-rational points of a connected reductive group defined over k, etc. Let P be a parabolic subgroup of G with Levi decomposition $P = MN$.

6.1. **A technical result.**

Lemma 6.1.1. *Let K_1 be any compact t.d.-group, and let $K_2, K_3 \subset K_1$ be compact subgroups such that $K_1 = K_2K_3$. If (π, V) is a smooth representation of K_1, then*

$$e_{K_1} = e_{K_2}e_{K_3}.$$

Proof. Fix $v \in V$. We need to show that $e_{K_1}v = e_{K_2}e_{K_3}v$. By Lemma 5.1.1, there is a normal compact open subgroup N of K_1 which acts trivially on the (finite-dimensional) representation K_1v generated by v. Set $\overline{K}_i = K_i/(K_i \cap N)$ for $i = 1, 2, 3$, so each $\overline{K}_i$ is a finite group, and π descends to a representation of $\overline{K}_i$. We then have

$$e_{K_i}v = \int_{K_i} \pi(k_i)v\, dk_i = \frac{1}{|\overline{K}_i|} \sum_{\overline{k}_i \in \overline{K}_i} \pi(\overline{k}_i)v$$

(where dk_i is the normalized Haar measure on K_i).

Since $e_{K_i}v \in K_1v$ for each $i = 1, 2, 3$, we have $\pi(n)e_{K_i}v = e_{K_i}v$ for all $n \in N$. Therefore,

$$\begin{aligned}
e_{K_2}e_{K_3}v &= \frac{1}{|\overline{K}_2|} \sum_{\overline{k}_2 \in \overline{K}_2} \pi(\overline{k}_2)e_{K_3}v \\
&= \frac{1}{|\overline{K}_2| \cdot |\overline{K}_3|} \sum_{\overline{k}_2 \in \overline{K}_2} \sum_{\overline{k}_3 \in \overline{K}_3} \pi(\overline{k}_2)\pi(\overline{k}_3)v \\
&= \frac{|\overline{K}_2 \cap \overline{K}_3|}{|\overline{K}_2| \cdot |\overline{K}_3|} \sum_{\overline{k}_1 \in \overline{K}_1} \pi(\overline{k}_1)v \\
&= e_{K_1}v.
\end{aligned}$$

■

Remark 6.1.2. Suppose that $K \subset G$ has an Iwahori decomposition with respect to some parabolic subgroup $P = MN$, i.e., $K = K^- \cdot K^0 \cdot K^+$, where

$$K^- = K \cap \overline{N}, \quad K^0 = K \cap M, \quad \text{and} \quad K^+ = K \cap N.$$

Then $K^-K^0 = K^0K^-$ and $K^+K^0 = K^0K^+$ are groups, so Lemma 6.1.1 shows that $e_K = e_{K^+} \cdot e_{K^0} \cdot e_{K^-}$, where the product can be taken in any order.

6.2. **The Jacquet functor.** Let $(\pi, V) \in \mathfrak{R}(G)$ and $(\sigma, W) \in \mathfrak{R}(M)$.

Since N is a normal subgroup of P with quotient $P/N \cong M$, we may extend (σ, W) to a smooth representation of P by defining $\sigma(mn) = \sigma(m)$. Consider the representation $\mathrm{Ind}_P^G(\sigma) \in \mathfrak{R}(G)$. From Corollary 5.5.4, this latter representation is an admissible representation of G if σ is an admissible representation of M.

We now recall the definition of coinvariants from Subsection 3.3.2. We define

$$V(N) := V(N,1) = \langle \pi(n)v - v \mid v \in V, n \in N \rangle.$$

Since N can be written as an increasing union of compact open subgroups, Exercise 3.3.10 shows that

$$V(N) = \{v \in V \mid e_K v = 0 \text{ for some compact open subgroup } K \text{ of } N\}.$$

We let $V_N := V_{N,1} = V/V(N)$ denote the coinvariants of V; it is the maximal quotient of V on which N acts trivially.

Since N is a normal subgroup of P, $V(N)$ is a P-subrepresentation of V, so V_N is a P-representation on which N acts trivially. Consequently, for all $W' \in \mathfrak{R}(P)$ on which N acts trivially, we have

$$\mathrm{Hom}_P(V, W') = \mathrm{Hom}_P(V_N, W') = \mathrm{Hom}_M(V_N, W').$$

Therefore, we can recast Lemma 3.3.8 as follows.

Lemma 6.2.1 (Frobenius Reciprocity)**.** *For $(\pi, V) \in \mathfrak{R}(G)$ and $(\sigma, W) \in \mathfrak{R}(M)$ we have*

$$\mathrm{Hom}_G(V, \mathrm{Ind}_P^G W) = \mathrm{Hom}_M(V_N, W).$$

It follows that the functor $\mathrm{Ind}_{P=MN}^G : \mathfrak{R}(M) \to \mathfrak{R}(G)$ is the right adjoint of the functor $V \mapsto V_N : \mathfrak{R}(G) \to \mathfrak{R}(M)$. We call the functor $V \mapsto V_N$ the *Jacquet functor*.

Definition 6.2.2. For $(\pi, V) \in \mathfrak{R}(G)$ and P a parabolic subgroup of G with Levi decomposition $P = MN$, the M-representation (π_N, V_N) is called the *Jacquet module* of V with respect to $P = MN$.

Note that the Jacquet module depends on the choice of a Levi decomposition of P, which corresponds to the choice of maximal torus. There are many such choices: one can show that N acts simply transitively on the set of Levi subgroups $M \subset P$ which give a Levi decomposition $P = MN$.

6.3. **Properties of the Jacquet functor.**

Lemma 6.3.1. *Let $(\pi, V) \in \mathfrak{R}(G)$, and let P be a parabolic subgroup of G with a Levi decomposition $P = MN$.*

(1) *If (π, V) is a finitely generated G-module, then (π_N, V_N) is a finitely generated M-module.*
(2) *The functor $V \mapsto V_N$ from $\mathfrak{R}(G)$ to $\mathfrak{R}(M)$ is exact.*

Proof. (1) Suppose that $v_1, v_2, \ldots, v_\ell$ is a set of generators for V. Choose a compact open subgroup K of G so that $v_i \in V^K$ for $1 \le i \le \ell$. From the Iwasawa decomposition (or, more directly, since $\mathbf{P}\backslash\mathbf{G}$ is projective), $P\backslash G$ is compact. Thus we can choose a finite set $\{g_1, g_2, \ldots, g_m\}$ of coset representatives for $P\backslash G/K$. As a P-module, V is therefore generated by the finite set $\{\pi(g_j)v_i \mid 1 \le i \le \ell\,, 1 \le j \le m\}$. Hence, since N acts trivially on V_N, V_N is generated as an M-module by the images of the $\pi(g_j)v_i$ in V_N.

(2) Since the functor of coinvariants is tautologically right exact, it suffices to show that if $\rho\colon W \to V$ is an injective map between two objects in $\mathfrak{R}(G)$, then $\rho_N\colon W_N \to V_N$ is injective. Let $\overline{w} \in \ker \rho_N$, and let $w \in W$ be a lift of $\overline{w}$. Since $\rho_N(\overline{w}) = 0$, there exists a compact open subgroup K of N such that $e_K\rho(w) = 0$. Since ρ is a G-homomorphism, we have $0 = e_K\rho(w) = \rho(e_K w)$. Since ρ is injective, we conclude that $e_K w = 0$, so $\overline{w} = 0$.

■

Theorem 6.3.2 (Jacquet's lemma). *If $(\pi, V) \in \mathfrak{R}(G)$ is admissible, P is a parabolic subgroup of G with Levi decomposition $P = MN$, and K is a compact open subgroup of G admitting an Iwahori factorization with respect to $P = MN$, then the projection map $V \to V_N$ maps V^K surjectively onto $(V_N)^{K\cap M}$.*

Remark 6.3.3. In fact, this lemma is true for an arbitrary smooth representation of G; however, the proof is difficult. See [1].

Before proving Theorem 6.3.2, we note an important corollary: just as parabolic induction carries an admissible representation of a Levi to an admissible representation of G, the Jacquet functor preserves admissibility. More precisely:

Corollary 6.3.4. *Let P be a parabolic subgroup of G with Levi decomposition $P = MN$. If (π, V) is an admissible representation of G, then (π_N, V_N) is an admissible representation of M.*

Proof. Since there is a neighborhood basis of the identity in G consisting of compact open subgroups possessing Iwahori factorizations with respect to $P = MN$, this follows immediately from Theorem 6.3.2. ■

Proof of Theorem 6.3.2. Let $j\colon V \to V_N$ denote the P-representation quotient map. If $v \in V^K$, then for all $k \in K \cap M$, we have $\pi_N(k)j(v) = j(\pi(k)v) = j(v)$. Consequently, $j(V^K) \subset V_N^{K\cap M}$. We now show that the other inclusion holds.

Without loss of generality, suppose that $P = P_\theta$ is a standard parabolic subgroup for some $\theta \subset \Delta$ (and let $M = M_\theta$, etc). Choose $t \in T_\theta$ such that $|\alpha(t)| < 1$ for all $\alpha \in \Delta \smallsetminus \theta$. For example, if $G = \mathrm{GL}_n(k)$ and $P = MN$ is the parabolic subgroup corresponding to the partition $(k_1, k_2, \dots, k_\ell)$ of n, we can take t to be the element

$$\mathrm{diag}(\underbrace{\varpi^{k_\ell}, \varpi^{k_\ell}, \dots, \varpi^{k_\ell}}_{k_1}, \underbrace{\varpi^{k(\ell-1)}, \varpi^{k(\ell-1)}, \dots, \varpi^{k(\ell-1)}}_{k_2}, \dots, \underbrace{\varpi, \varpi, \dots, \varpi}_{k_\ell}).$$

Let $\overline{N}$ denote the unipotent radical of the parabolic opposite of $P = MN$ with respect to M. Define

$$K^+ = K \cap N, \quad K^0 = K \cap M, \quad K^- = K \cap \overline{N}.$$

The set

$$\{t^{-m}K^-t^m \,|m \in \mathbb{Z}_{\geq 0}\}$$

forms a neighborhood basis of the identity in $\overline{N}$.

First we claim that $\pi_N(t)j(V^K) = j(V^K)$. Let $v \in V^K$. Since $t^{-1}K^-t \subset K^-$ and $t^{-1}K^0t = K^0$, we have that $e_{K^-} \cdot \pi(t)v = e_{K^0} \cdot \pi(t)v = \pi(t)v$, so by Lemma 6.1.1,

$$\begin{aligned} j(e_K \cdot \pi(t)v) &= j(e_{K^+} \cdot e_{K^0} \cdot e_{K^-} \cdot \pi(t)v) \\ &= j(e_{K^+} \cdot \pi(t)v) = e_{K^+} \cdot \pi_N(t)j(v) \\ &= \pi_N(t)j(v), \end{aligned}$$

where the final equality holds since $K^+ \subset N$ acts trivially on V_N. Since $j(v)$ was an arbitrary element of $j(v^K)$, we have shown that $\pi_N(t)j(V^K) \subset j(V^K)$. Since $\pi_N(t)$ is invertible on V_N and $j(V^K)$ is finite-dimensional, we have that $\pi_N(t)j(V^K) = j(V^K)$. Consequently, $\pi_N(t^m)j(V^K) = j(V^K)$ for any $m \in \mathbb{Z}$.

Let $\overline{v} \in V_N^{K^0}$, let $v' \in V$ be any lift of $\overline{v}$, and let $v = e_{K^0}(v')$. Thus $j(v) = \overline{v}$ and $v \in V^{K^0}$. Fix $m > 0$ such that $t^{-m}K^-t^m \leq \mathrm{stab}_G(v)$, so K^- fixes $\pi(t^m)v$. Since t^m is in the center of M, it follows that K^0 fixes $\pi(t^m)v$. As before, we have that

$$j(e_K \cdot \pi(t^m)v) = e_{K^+}\pi_N(t^m)j(v) = \pi_N(t^m)\overline{v},$$

so $\overline{v} \in \pi_N(t^{-m})j(V^K) = j(V^K)$. ■

7. Basic properties of supercuspidal representations

In this section we discuss various properties of supercuspidal representations.

7.1. **Finite representations.** In this subsection, G denotes any t.d.-group.

Definition 7.1.1. A smooth representation (π, V) of G is *finite* provided that, for all $v \in V$ and all $\lambda \in \widetilde{V}$, the matrix coefficient $m_{\lambda,v}$ is compactly supported.

Example 7.1.2. If the center of G is compact, then every supercuspidal representation of G is finite.

Exercise 7.1.3. Show that if there exists a nonzero, finite representation $(\pi, V) \in \mathfrak{R}(G)$, then the center of G is compact.

Lemma 7.1.4. *If π is finitely generated and finite, then π is admissible.*

Proof. This proof is nearly identical to that of Lemma 3.1.4. In the proof of that lemma, the representation was assumed to be irreducible (a) so that it would have one generator, and (b) so that the center would act through a central character. Since our matrix coefficients are now assumed to be compactly supported, we can replace the condition of irreducibility with that of being finitely generated. ■

Lemma 7.1.5. *A smooth representation (π, V) of G is finite if and only if for all compact open subgroups K of G and for all $v \in V$, the function $(g \mapsto e_K\pi(g)v) \in C^\infty(G, V)$ is compactly supported.*

Proof. "$\Leftarrow$" This is clear.

"$\Rightarrow$" Fix $v \in V$ and a compact open subgroup K of G. Let V_1 denote the subrepresentation of (π, V) generated by v. By Corollary 5.2.3, the natural map from $\widetilde{V}$ to $\widetilde{V_1}$ is surjective, so the representation (π, V_1) is finite. By Lemma 7.1.4, V_1 is admissible, so the dimension of V_1^K is finite. Choose a basis $\lambda_1, \lambda_2, \dots \lambda_k$ for $\widetilde{V_1}^K = \mathrm{Hom}(V_1^K, \mathbb{C})$. The support of the map $g \mapsto e_K \pi(g) v$ is contained in the union

$$\bigcup_{1 \le i \le k} \mathrm{supp}(m_{\lambda_i, v}),$$

which is compact. (Here $\mathrm{supp}(m_{\lambda_i,v})$ denotes the support of the function $m_{\lambda_i,v}$.) ∎

7.2. **The subgroup G^1.** Suppose again that G is the group of k-rational points of a connected reductive group $\mathbf{G}$ defined over k.

Definition 7.2.1. We let $\mathrm{Rat}(G)$ denote the group of algebraic characters from $\mathbf{G}$ to GL_1 that are defined over k.

Example 7.2.2. We have

$$\mathrm{Rat}(\mathrm{GL}_n(k)) = \{\det^n : n \in \mathbb{Z}\} \quad \text{and} \quad \mathrm{Rat}(\mathrm{Sp}_4(k)) = \{1\}.$$

Definition 7.2.3. We define

$$G^1 := \bigcap_{\chi \in \mathrm{Rat}(G)} \ker |\chi| .$$

Example 7.2.4. If $G = k^\times$ then $G^1 = R^\times$. More generally, if $G = \mathrm{GL}_n(k)$ then

$$G^1 = \{y \in G \mid \det(y) \in R^\times\}.$$

One can prove that G^1 has the following properties (see Exercise 7.2.6):

Remark 7.2.5.
(1) Every compact subgroup of G belongs to G^1. In particular, for all parabolics $P \le G$ with a Levi decomposition $P = MN$, we have $N \le G^1$.
(2) The group G^1 is an open, closed, normal, unimodular subgroup of G.
(3) The quotient G/G^1 is isomorphic to $\mathbb{Z}^m$, where m denotes the rank of the k-split part of the center of $\mathbf{G}$. In particular, if the center of G is compact, then $G = G^1$.
(4) The quotient $G/(Z(G)G^1)$ is finite. In fact, $Z(G)/(Z(G) \cap G^1)$ is a full rank sublattice of G/G^1.
(5) The intersection $Z(G) \cap G^1$ is compact.

Exercise 7.2.6. Prove the facts given in Remark 7.2.5. Assume that $G = \mathrm{GL}_n(k)$ for parts (3)–(5). In particular, show that $G/(Z(G)G^1) \cong \mathbb{Z}/n\mathbb{Z}$ in part (4).

It follows from Remark 7.2.5 that a representation (π, V) of G is smooth if and only if its restriction to G^1 is smooth as a representation of G^1. We also have a Cartan decomposition for G^1:

$$G^1 = \coprod_{\substack{w \in \omega \cap G^1 \\ t \in T^+ \cap G^1}} K_0 w t K_0.$$

Example 7.2.7. When $G = \mathrm{GL}_n(k)$, the element $\mathrm{diag}(\varpi^{j_1}, \varpi^{j_2}, \dots, \varpi^{j_n})$ belongs to $T^+ \cap G^1$ if and only if $j_1 \geq j_2 \geq \cdots \geq j_n$ and $j_1 + j_2 + \cdots + j_n = 0$.

Definition 7.2.8. We define the group of *unramified characters* of G to be

$$\mathbf{X}(G) := \mathrm{Hom}(G/G^1, \mathbb{C}^\times).$$

7.3. **Various ways to think about supercuspidal representations.** We have the following theorem, due to Jacquet and Harish-Chandra.

Theorem 7.3.1. *Let $(\pi, V) \in \mathfrak{R}(G)$. The following statements are equivalent.*

(1) *(π, V) is a supercuspidal representation of G.*
(2) *$\mathrm{res}_{G^1} \pi$ is finite.*
(3) *For all proper parabolics P of G with a Levi decomposition $P = MN$, we have $V_N = \{0\}$.*

Proof. We will show (1) ⇒ (2) ⇒ (3) ⇒ (1).

"(1) ⇒ (2)" Let $v \in V$ and $\lambda \in \widetilde{V}$. We must show that the function

$$g \mapsto m_{\lambda,v}(g) : G^1 \to \mathbb{C}$$

is compactly supported.

Let $C \subset G$ be the support of $m_{\lambda,v}$, let $C^1 = C \cap G^1$, and let $\overline{C^1}$ be the image of $C \cap G^1$ in $G^1/(G^1 \cap Z(G))$. If we think of $G^1/(G^1 \cap Z(G))$ as a (closed) subgroup of $G/Z(G)$, then since the image $\overline{C}$ of C in $G/Z(G)$ is compact, we have that $\overline{C^1} = \overline{C} \cap (G^1/(G^1 \cap Z(G)))$ is compact as well. Since $G^1 \cap Z(G)$ is compact, C^1 is thus compact.

"(2) ⇒ (3)" Let P be a proper parabolic subgroup of G with a Levi decomposition $P = MN$. Without loss of generality, we may assume that P is a standard parabolic subgroup corresponding to some $\theta \subset \Delta$. As in the proof of Jacquet's lemma (Theorem 6.3.2), we choose $t \in T_\theta^+ \cap G^1$ such that $|\alpha(t)| < 1$ for all $\alpha \in \Delta \smallsetminus \theta$. Note that $\{t^m \mid m \in \mathbb{Z}\}$ is not contained in any compact set since $\{|\alpha(t^m)| \mid m \in \mathbb{Z}\}$ is not contained in any compact subset of $\mathbb{R}$.

For example, if $G = \mathrm{GL}_n(k)$ and $P = MN$ corresponds to the partition $(k_1, k_2, \dots, k_\ell)$ of n, we can take t to be the element

$$\mathrm{diag}(\underbrace{\varpi^{k_\ell}, \dots, \varpi^{k_\ell}}_{k_1}, \underbrace{\varpi^{k_{(\ell-1)}}, \dots, \varpi^{k_{(\ell-1)}}}_{k_2}, \dots, \underbrace{\varpi^{-k_2}, \dots, \varpi^{-k_2}}_{k_{\ell-1}}, \underbrace{\varpi^{-k_1}, \dots, \varpi^{-k_1}}_{k_\ell}).$$

Let $v \in V$. We want to show that $v \in V(N)$. Choose a compact open subgroup K of G with an Iwahori decomposition with respect to $P = MN$, such that $v \in V^K$. Since $\mathrm{res}_{G^1} \pi$ is finite, from Lemma 7.1.5 we have

$$e_K \pi(t^m) v = 0$$

for all m sufficiently large. This implies that for all sufficiently large m we have

$$\begin{aligned} 0 &= e_{t^{-m}Kt^m}v \\ &= e_{t^{-m}(K^+)t^m}e_{t^{-m}(K^0)t^m}e_{t^{-m}(K^-)t^m}v \end{aligned}$$

where $K^+ = N\cap K$, $K^0 = \mathrm{M}\cap K$, and $K^- = \overline{N}\cap K$ (as usual, $\overline{N}$ denotes the unipotent radical of the parabolic opposite $P = MN$). Since t^m belongs to the center of M and $t^{-m}K^-t^m \subset K^-$, we conclude that

$$0 = e_{t^{-m}(K^+)t^m}e_{t^{-m}(K^0)t^m}e_{t^{-m}(K^-)t^m}v = e_{t^{-m}(K^+)t^m}v.$$

Hence, $v \in V(N)$.

"(3) $\Rightarrow$ (1)" We will show that if (π, V) is not supercuspidal, then there exists a proper parabolic P with Levi decomposition $P = MN$ such that $V_N \neq \{0\}$.

Suppose that there exist $v \in V$ and $\lambda \in \widetilde{V}$ such that the matrix coefficient $m_{\lambda,v}$ does not have compact support modulo the center of G. From the Cartan decomposition of G, this implies that $\mathrm{supp}(m_{\lambda,v}) \cap K_0wtK_0 \neq \emptyset$ for infinitely many choices of $w \in \omega$ and $t \in T^+$. Since ω and Δ are finite sets, there exist $w \in \omega$, $\alpha \in \Delta$, and $\{t_m\} \subset T^+$ such that

$$\mathrm{supp}(m_{\lambda,v}) \cap K_0wt_mK_0 \neq \emptyset \quad \text{and} \quad |\alpha(t_m)| \to 0 \text{ as } m \to \infty.$$

Choose a compact open normal subgroup K of K_0 such that

(1) K has an Iwahori decomposition with respect to the maximal standard parabolic $P = P_{\Delta\smallsetminus\{\alpha\}}$ with (standard) Levi decomposition $P = MN$, and

(2) $v \in V^K$ and $\lambda \in \widetilde{V}^K$.

Since the group K_0/K is finite, we can choose $k_1, k_2 \in K_0$ such that

$$\mathrm{supp}(m_{\lambda,v}) \cap k_1Kwt_mKk_2 \neq \emptyset$$

for each m. By replacing v with $\pi(k_2)v \in V^K$ and λ with $\widetilde{\pi}(k_1^{-1})\lambda \in \widetilde{V}^K$, we may assume that $m_{\lambda,v}(wt_m) \neq 0$ for all $m \in \mathbb{Z}_{\geq 0}$. Replacing v with $\pi(w)v$, then, we have $m_{\lambda,v}(t_m) \neq 0$ for all m. We therefore have

$$\begin{aligned} 0 \neq m_{\lambda,v}(t_m) &= \lambda(\pi(t_m)v) = (e_{K\cap N}\lambda)(\pi(t_m)v) \\ &= \lambda(e_{K\cap N}\pi(t_m)v) = \lambda(\pi(t_m)e_{t_m^{-1}(K\cap N)t_m}v). \end{aligned}$$

Consequently, for all $m \in \mathbb{Z}_{\geq 0}$ we have $e_{t_m^{-1}(K\cap N)t_m}v \neq 0$. Since $|\alpha(t_m)| \to 0$, we conclude that the compact open subgroups $t_m^{-1}(K\cap N)t_m$ fill out N, so $e_Uv \neq 0$ for all compact open subgroups U of N. Consequently, $v \notin V(N)$.

■

Remark 7.3.2. Let $P = MN$ be any parabolic subgroup of G, and let $g \in G$. Then $V(gNg^{-1}) = gV(N)$, so $V_N = \{0\}$ if and only if $V_{gNg^{-1}} = \{0\}$. Since any parabolic subgroup of G is conjugate to a standard parabolic, we may replace "all proper parabolics P" with "all proper standard parabolics P" in Theorem 7.3.1(3).

Corollary 7.3.3. *If $(\pi, V) \in \mathfrak{R}(G)$ is irreducible, then there exist a parabolic $P = MN$ and an irreducible supercuspidal $\sigma \in \mathfrak{R}(M)$ such that π is a subrepresentation of $\operatorname{Ind}_P^G \sigma$.*

Proof. Let $P' \leq P$ be standard parabolic subgroups of G and choose Levi decompositions $P = MN$ and $P' = M'N'$ so that $M' \leq M$ ($N \leq N'$ automatically). Moreover, $P' \cap M$ is a standard parabolic subgroup of M with Levi decomposition $M'(N' \cap M)$, and $N' = N(N' \cap M)$. It follows that $V(N) \subset V(N')$ and $V_{N'} = (V_N)_{(N' \cap M)}$. In addition, every standard parabolic of M arises in this way.

Let P be a standard parabolic subgroup that is minimal with respect to the property $V_N \neq \{0\}$. The previous paragraph and Remark 7.3.2 make it clear that (π_N, V_N) is a supercuspidal object in $\mathfrak{R}(M)$.

By Lemma 6.3.1, V_N is a finitely generated M-module. Thus, by Lemma 3.3.5, V_N has an irreducible quotient σ. The M-module σ is supercuspidal and we have, via Frobenius reciprocity,

$$0 \neq \operatorname{Hom}_M(V_N, \sigma) = \operatorname{Hom}_G(V, \operatorname{Ind}_P^G \sigma).$$

■

From the above results, it is clear that supercuspidal representations play a distinguished role in the representation theory of reductive p-adic groups. Unfortunately, outside of the general linear group and related groups, we do not have a complete understanding of the supercuspidal representations. We have the following very old conjecture.

Conjecture 7.3.4. *If $(\pi, V) \in \mathfrak{R}(G)$ is irreducible and supercuspidal, then there exist an open subgroup K of G which is compact modulo the center of G, and an irreducible smooth representation σ of K, such that $\pi = \text{c-}\operatorname{Ind}_K^G \sigma$.*

We collect a few more consequences of Theorem 7.3.1.

Corollary 7.3.5. *If $(\pi, V) \in \mathfrak{R}(G)$ is irreducible, then it is admissible.*

Proof. By Corollary 7.3.3, there exist a parabolic subgroup P of G with Levi decomposition $P = MN$ and an irreducible supercuspidal representation $\sigma \in \mathfrak{R}(M)$ such that V embeds in $\operatorname{Ind}_P^G \sigma$. Therefore, it is enough to show that $\operatorname{Ind}_P^G \sigma$ is admissible. Lemma 3.1.4 guarantees that σ is an admissible representation of M, so by Corollary 5.5.4, the representation $\operatorname{Ind}_P^G \sigma$ is admissible. ■

Corollary 7.3.6. *Let $(\pi, V) \in \mathfrak{R}(G)$. Then*

(1) *π is irreducible if and only if its contragredient is.*
(2) *π is supercuspidal and irreducible if and only if its contragredient is.*

Proof. By Corollary 7.3.5, an irreducible smooth representation is admissible, so the first statement follows immediately from Corollary 5.2.4.

Assume that V (resp. $\widetilde{V}$) is irreducible, and let $F : V \to \widetilde{\widetilde{V}}$ be the natural isomorphism. For $v \in V$, $\lambda \in \widetilde{V}$, and $g \in G$, we have

$$m_{F(v),\lambda}(g) = F(v)(\widetilde{\pi}(g)\lambda) = (\widetilde{\pi}(g)\lambda)(v) = \lambda(\pi(g^{-1})v) = m_{\lambda,v}(g^{-1}),$$

so $m_{F(v),\lambda}$ is compactly supported modulo the center of G if and only if $m_{\lambda,v}$ is. Thus V is irreducible and supercuspidal if and only if $\widetilde{V}$ is. ■

Exercise 7.3.7. (1) Let $(\pi, V) \in \mathfrak{R}(G)$ be irreducible. Show that if one nonzero matrix coefficient of π has compact support modulo the center of G, then π is supercuspidal.

(2) (Mautner) Let $K \subset G$ be an open subgroup which is compact modulo the center of G, and let (σ, W) be an irreducible smooth representation of K. Show that if $\text{c-Ind}_K^G \sigma$ is irreducible, then it is supercuspidal.

8. Square integrable representations

8.1. **Some generalities.**

We again suppose that G is a t.d.-group.

Lemma 8.1.1. *Let $(\pi, V) \in \mathfrak{R}(G)$ be finitely generated and let W be a complex vector space (endowed with the trivial G-action). Then*

(1) *the natural map $\mathrm{End}_G(V) \otimes_{\mathbb{C}} W \to \mathrm{Hom}_G(V, V \otimes_{\mathbb{C}} W)$ is an isomorphism, and*
(2) *if (π, V) is irreducible then $W \to \mathrm{Hom}_G(V, V \otimes_{\mathbb{C}} W)$ is an isomorphism and the map $U \mapsto V \otimes_{\mathbb{C}} U$ is a bijective correspondence between the sets*

$$\{\mathbb{C}\text{-vector subspaces of } W\} \text{ and } \{\, G\text{-submodules in } V \otimes_{\mathbb{C}} W\}.$$

Proof. (1) Choose a basis $W \cong \oplus_I \mathbb{C}$, so that our map is identified with

$$\bigoplus_I \mathrm{End}_G(V) \to \mathrm{Hom}_G(V, \bigoplus_I V).$$

Since V is finitely generated, it has the property that for any morphism $V \to \oplus_I V$ there exists a finite subset $J \subset I$ such that this morphism factors through $\oplus_J V \to \oplus_I V$: if we choose a finite generating set $v_1, \cdots, v_n \in V$ then the image of each v_j has only finitely many nonzero summands in $\oplus_I V$. This provides an inverse to the map above.

(2) The isomorphism $W \to \mathrm{Hom}_G(V, V \otimes_{\mathbb{C}} W)$ follows from (1) and Schur's lemma. We must show that if X is a G-submodule of $V \otimes_{\mathbb{C}} W$ then $X = V \otimes_{\mathbb{C}} U$ for some $U \subset W$. The natural G-map $\varphi : V \otimes_{\mathbb{C}} \mathrm{Hom}_G(V, X) \to X$ is compatible with the embedding $\mathrm{Hom}_G(V, X) \subset \mathrm{Hom}_G(V, V \otimes_{\mathbb{C}} W) \cong W$, and by part (1), it is injective. Choosing a basis $W \cong \oplus_I \mathbb{C}$, we see that $X \subset V \otimes W \cong \oplus_I V$ is a direct sum of copies of V; thus φ is surjective.

■

Lemma 8.1.2. *Let G_1 and G_2 be two t.d.-groups. If the representations $(\pi_1, V_1) \in \mathfrak{R}(G_1)$ and $(\pi_2, V_2) \in \mathfrak{R}(G_2)$ are irreducible, then $V_1 \otimes_{\mathbb{C}} V_2$ is an irreducible representation of $G_1 \times G_2$.*

Proof. Let X be a $(G_1 \times G_2)$-submodule of $V_1 \otimes_{\mathbb{C}} V_2$. Regarding X as a G_1-module, Lemma 8.1.1 tells us that $X = V_1 \otimes_{\mathbb{C}} U$ for some complex vector subspace $U \subset V_2$. If $U \neq \{0\}$ then since any nonzero vector in U generates V_2 as a G_2-module, we must have $X = V_1 \otimes V_2$. ■

Lemma 8.1.3. *Suppose $(\pi, V) \in \mathfrak{R}(G)$ is irreducible and admissible. If $f \colon V \times \widetilde{V} \to \mathbb{C}$ is bilinear and G-invariant, then there exists $c_f \in \mathbb{C}$ such that $f(v, \lambda) = c_f \cdot \lambda(v)$ for all $v \in V$ and $\lambda \in \widetilde{V}$. In particular, f is degenerate if and only if $f = 0$.*

Proof. If f is degenerate then there exists some $\lambda \in \widetilde{V}$ (or some $v \in V$) such that $f(v, \lambda) = 0$ for all $v \in V$ (resp. for all $\lambda \in \widetilde{V}$). Thus the kernel of the G-map $\lambda \mapsto f(\cdot, \lambda) : \widetilde{V} \to \widetilde{V}$ (resp. $v \mapsto f(v, \cdot) : V \to \widetilde{\widetilde{V}}$) is nontrivial, so since $\widetilde{V}$ (resp. V) is irreducible, $f = 0$.

If f is nondegenerate, then we have two G-module isomorphisms of V with $\widetilde{\widetilde{V}}$: the canonical map φ_c which maps $v \in V$ to $(\lambda \mapsto \lambda(v))$ and the map $\varphi_f : v \mapsto f(v, \cdot)$. By Schur's lemma, there is some $c_f \in \mathbb{C}^\times$ such that $\varphi_f = c_f \cdot \varphi_c$ ■

Lemma 8.1.4. *Let G_1 and G_2 be two t.d.-groups, and let $(\pi_1, V_1) \in \mathfrak{R}(G_1)$ and $(\pi_2, V_2) \in \mathfrak{R}(G_2)$ be irreducible admissible representations. Then the natural $(G_1 \times G_2)$-map $\widetilde{V}_1 \otimes_{\mathbb{C}} \widetilde{V}_2 \to (V_1 \otimes_{\mathbb{C}} V_2)^{\sim}$ is an isomorphism.*

Proof. By Lemma 8.1.2, $V_1 \otimes_{\mathbb{C}} V_2$ is an irreducible admissible $(G_1 \times G_2)$-module; thus by Corollary 5.2.4, the representations $\widetilde{V}_1$, $\widetilde{V}_2$, $\widetilde{V}_1 \otimes_{\mathbb{C}} \widetilde{V}_2$, and $(V_1 \otimes_{\mathbb{C}} V_2)^{\sim}$ are all irreducible. Thus since the map $\widetilde{V}_1 \otimes_{\mathbb{C}} \widetilde{V}_2 \to (V_1 \otimes_{\mathbb{C}} V_2)^{\sim}$ is nonzero, it must be an isomorphism. ■

8.2. **Square integrable representations.** Let $Z(G)$ denote the center of the t.d.-group G. We suppose that $G/Z(G)$ is unimodular, and we let dg^* denote a Haar measure on $G/Z(G)$.

Remark 8.2.1. In this subsection and the next one, all we actually assume is that $Z(G)$ is a closed subgroup of the center of G such that the quotient $G/Z(G)$ is unimodular.

Example 8.2.2. When G is the group of k-points of a connected reductive group over k, then by Exercise 5.4.6, G is unimodular. Since $Z(G)$ is unimodular, $G/Z(G)$ is unimodular.

Definition 8.2.3. Let χ be a unitary character of $Z(G)$. We denote by $L^2(G/Z(G), dg^*)_\chi$ the space of functions in $C^\infty(G)$ for which

(1) $f(gz) = \chi(z) f(g)$ for all $g \in G$ and $z \in Z(G)$, and
(2)
$$\int_{G/Z(G)} |f(g)|^2 \, dg^* < \infty.$$

With respect to the right regular action and the inner product

$$(f_1, f_2)_{L^2,\chi} := \int_{G/Z(G)} f_1(g) \cdot \overline{f}_2(g) \, dg^*,$$

$L^2(G/Z(G), dg^*)_\chi$ is a unitary representation of G.

Definition 8.2.4. A representation $(\pi, V) \in \mathfrak{R}(G)$ is *square-integrable modulo the center* or *discrete series* provided that

(1) the center of G acts on V by a unitary character χ, and
(2) for all $v \in V$ and $\lambda \in \widetilde{V}$, the matrix coefficient $m_{\lambda,v}$ belongs to $L^2(G/Z(G), dg^*)_\chi$.

A representation $(\pi, V) \in \mathfrak{R}(G)$ is *essentially square-integrable modulo the center* provided that there is a smooth character ω of G such that $\pi \otimes \omega$ is square-integrable modulo the center.

Exercise 8.2.5. If $(\pi, V) \in \mathfrak{R}(G)$ is an irreducible representation that is essentially square-integrable modulo the center, then it is admissible.

Definition 8.2.6. For a character χ of $Z(G)$, we let $\mathfrak{R}(G)_\chi$ denote the full subcategory of $\mathfrak{R}(G)$ whose objects transform with respect to χ under the action of $Z(G)$.

Lemma 8.2.7. *If χ is a unitary character of Z and $(\pi, V) \in \mathfrak{R}(G)_\chi$ is irreducible and square-integrable modulo the center, then (π, V) is unitary.*

Proof. Fix $0 \neq \lambda \in \widetilde{V}$. We have a natural G-embedding $m\colon V \to L^2(G/Z(G), dg^*)_\chi$ given by $m(v) = m_{\lambda,v}$ for $v \in V$. For $v, w \in V$, define

$$(v, w) := (m(v), m(w))_{L^2,\chi}.$$

This defines a positive-definite, G-invariant, Hermitian form on V. ■

8.3. **Schur orthogonality.** The next two lemmas constitute *Schur orthogonality*.

For a function f on a group G, we define $\check{f}(g) := f(g^{-1})$.

Lemma 8.3.1. *If $(\pi, V) \in \mathfrak{R}(G)$ is irreducible and essentially square-integrable modulo the center, then there exists a unique $\deg(\pi) \in \mathbb{R}_{>0}$ such that for all $v_1, v_2 \in V$ and $\lambda_1, \lambda_2 \in \widetilde{V}$, we have*

$$\int_{G/Z(G)} m_{\lambda_1,v_2}(g) \cdot \check{m}_{\lambda_2,v_1}(g)\, dg^* = \frac{\lambda_1(v_1) \cdot \lambda_2(v_2)}{\deg(\pi)}.$$

Remark 8.3.2. The number $\deg(\pi)$ is called the *formal degree* of π; it depends only on π and the measure dg^*. When (π, V) is an irreducible representation of a finite group G, one has

$$\frac{1}{|G|} \sum_{g \in G} \lambda_1(\pi(g)v_2) \cdot \lambda_2(\pi(g)v_1) = \frac{\lambda_1(v_1) \cdot \lambda_2(v_2)}{\deg(\pi)},$$

where $\deg(\pi) = \dim_{\mathbb{C}} V$ is the ordinary degree of the (finite-dimensional) representation (π, V).

Proof. If $\deg(\pi)$ exists, then it is unique.

Without loss of generality, we may assume that (π, V) is square-integrable modulo the center. Therefore, by Lemma 8.2.7, there is a positive-definite G-invariant Hermetian form $(\cdot,\cdot)$ on V.

Consider the $G\times G$-module $V \otimes_{\mathbb{C}} \widetilde{V}$. This is an irreducible admissible smooth representation of $G \times G$ by Exercise 8.2.5, Corollary 5.2.4, and Lemma 8.1.2. By Lemma 8.1.4, the contragredient of $V \otimes_{\mathbb{C}} \widetilde{V}$ is naturally isomorphic as a $(G \times G)$-module to $\widetilde{V} \otimes_{\mathbb{C}} V$.

Define a bilinear form on $(V \otimes_{\mathbb{C}} \widetilde{V}) \times (\widetilde{V} \otimes_{\mathbb{C}} V)$ by

$$\begin{aligned}(v_1 \otimes \lambda_1, \lambda_2 \otimes v_2)_1 &:= \int_{G/Z(G)} m_{\lambda_1,v_1}(g) \cdot \check{m}_{\lambda_2,v_2}(g)\, dg^* \\ &= \int_{G/Z(G)} \lambda_1(\pi(g)v_1) \cdot \lambda_2(\pi(g^{-1})v_2)\, dg^*.\end{aligned}$$

The form $(\cdot,\cdot)_1$ is $(G \times G)$-invariant since dg^* is unimodular. By Lemma 8.1.3, in order to show that $(\cdot,\cdot)_1$ is nondegenerate, it suffices to show that it is nonzero. Choose a nonzero $v \in V$, and define $\lambda_v \in \widetilde{V}$ by $\lambda_v(w) = (w, v)$. Note that

$$\lambda_v(\pi(g^{-1})v) = (\pi(g^{-1})v, v) = (v, \pi(g)v) = \overline{(\pi(g)v, v)} = \overline{\lambda_v(\pi(g)v)},$$

so

$$(v \otimes \lambda_v, \lambda_v \otimes v)_1 = \int_{G/Z(G)} \lambda_v(\pi(g)v) \cdot \lambda_v(\pi(g^{-1})v)\, dg^* = \int_{G/Z(G)} |\lambda_v(\pi(g)v)|^2 \, dg^* \neq 0.$$

Define another bilinear form on $(V \otimes_{\mathbb{C}} \widetilde{V}) \times (\widetilde{V} \otimes_{\mathbb{C}} V)$ by

$$(v_1 \otimes \lambda_1, \lambda_2 \otimes v_2)_2 := \lambda_1(v_2) \cdot \lambda_2(v_1)$$

(identifying $\widetilde{V} \otimes_{\mathbb{C}} V$ with the contragredient of $V \otimes_{\mathbb{C}} \widetilde{V}$, this is just the natural pairing $(v, \lambda) \mapsto \lambda(v)$). The form $(\cdot, \cdot)_2$ is clearly $(G \times G)$-invariant, and

$$(v \otimes \lambda_v, \lambda_v \otimes v)_2 = |v|^2 \cdot |v|^2 \neq 0,$$

where v and λ_v are defined as before.

Lemma 8.1.3 tells us that the two $(G \times G)$-invariant nondegenerate bilinear forms $(\cdot, \cdot)_1$ and $(\cdot, \cdot)_2$ differ by a constant $c \in \mathbb{C}^\times$; since $(v \otimes \lambda_v, \lambda_v \otimes v)_1$ and $(v \otimes \lambda_v, \lambda_v \otimes v)_2$ are both positive real numbers, $c \in \mathbb{R}_{>0}$. ■

Lemma 8.3.3. *If $(\pi_1, V_1), (\pi_2, V_2) \in \mathfrak{R}(G)$ are inequivalent irreducible representations with the same central character that are square-integrable modulo the center, then for all $v_1 \in V_1,\ v_2 \in V_2,\ \lambda_1 \in \widetilde{V}_1$, and $\lambda_2 \in \widetilde{V}_2$, we have*

$$\int_{G/Z(G)} m_{\lambda_1, v_1}(g) \cdot \check{m}_{\lambda_2, v_2}(g)\, dg^* = 0.$$

Proof. Fix $\lambda_1 \in \widetilde{V}_1$ and $v_2 \in V_2$. We define a G-homomorphism from V_1 to $\widetilde{\widetilde{V}}_2 \cong V_2$ by sending $v_1 \in V_1$ to the linear map

$$\lambda_2 \mapsto \int_{G/Z(G)} m_{\lambda_2, v_2}(g) \cdot \check{m}_{\lambda_1, v_1}(g)\, dg^*$$

for $\lambda_2 \in \widetilde{V}_2$. As V_1 and V_2 are assumed to be nonequivalent, this map must be zero. ■

Corollary 8.3.4. *Let $(\pi_i, V_i) \in \mathfrak{R}(G)$ be a set of inequivalent irreducible finite representations of G. For each i, choose nonzero $v_i \in V_i$ and $\lambda_i \in \widetilde{V}_i$. Then the functions $m_{\lambda_i, v_i} : G \to \mathbb{C}$ are linearly independent.*

Proof. Since all of the matrix coefficients m_{λ_i, v_i} are compactly supported, they are all square-integrable, so we may ignore the center of G altogether (cf. Remark 8.2.1 and Exercise 7.1.3).

Suppose that there exist constants $c_i \in \mathbb{C}$ (with $c_i = 0$ except for finitely many i) such that $\sum_i c_i \cdot m_{\lambda_i, v_i} = 0$. Fix an index j, and choose $\lambda'_j \in \widetilde{V}_i$ such that $\lambda'_j(v_j) = 1$ and $v'_j \in V_j$ such that $\lambda_j(v'_j) = 1$. Then the Schur orthogonality relations (Lemma 8.3.1 and Lemma 8.3.3) give

$$\begin{aligned} 0 &= \int_G \left(\sum_i c_i \cdot m_{\lambda_i, v_i}(g) \right) \cdot \check{m}_{\lambda'_j, v'_j}(g)\, dg^* \\ &= c_j \cdot \int_G m_{\lambda_j, v_j}(g) \cdot \check{m}_{\lambda'_j, v'_j}(g)\, dg^* \\ &= c_j \cdot \frac{1}{\deg(\pi_j)}, \end{aligned}$$

so $c_j = 0$. ■

8.4. **Application to supercuspidal representations.** We continue to assume that $G/Z(G)$ is unimodular.

There is an obvious class of representations that are likely to be square-integrable modulo the center: those whose matrix coefficients are compactly supported modulo the center, i.e., the supercuspidal representations. All that we need to require is that $Z(G)$ acts by a unitary character:

Lemma 8.4.1. *A supercuspidal representation $(\pi, V) \in \mathfrak{R}(G)$ is essentially square-integrable modulo the center if and only if there exists a smooth character ω of $Z(G)$ such that $Z(G)$ acts by a unitary character under $\pi \otimes \omega$.*

Example 8.4.2. Suppose that G is the group of k-rational points of a connected, reductive, algebraic group defined over k. Then any irreducible supercuspidal representation of G is automatically essentially square-integrable modulo the center:

Lemma 8.4.3. *If χ is a smooth character of the center $Z(G)$ of G, then there exists a unique real-valued character $\omega\colon G \to \mathbb{R}_{>0}$ which is trivial on G^1 such that $(\mathrm{res}_Z\,\omega) \cdot \chi$ is a unitary character of $Z(G)$.*

Proof. We must show that the smooth character $|\chi|^{-1}$ extends uniquely to a homomorphism $G/G^1 \to \mathbb{C}^\times$ (note that $|\chi|^{-1}$ is trivial on the compact group $G^1 \cap Z(G)$). This follows immediately from the fact that $Z(G)/(Z(G) \cap G^1)$ is a full rank sublattice of G/G^1. ■

Let $\mathcal{C}$ be an abelian category. An object P in $\mathcal{C}$ is said to be *projective* provided that the functor $X \mapsto \mathrm{Hom}(P, X)$ is exact. Equivalently, P is projective if and only if for any object X in $\mathcal{C}$ and every surjective map $\varphi\colon X \to P$, there exists a map $\psi\colon P \to X$ such that $\varphi \circ \psi = 1_P$ (cf. [9, III, §4]).

Similarly, an object I in $\mathcal{C}$ is *injective* provided that the functor $X \mapsto \mathrm{Hom}(X, I)$ is exact. Equivalently, I is injective if and only if for every injective map $i : X \to Y$ of objects of $\mathcal{C}$, every map $\varphi : X \to I$ extends to a map $Y \to I$ (cf. [9, XX, §4]).

Lemma 8.4.4. *Let $(\pi, V) \in \mathfrak{R}(G)$ be an irreducible, supercuspidal representation of G that is essentially square-integrable modulo the center. Let χ be the central character of G. Then (π, V) is projective in $\mathfrak{R}(G)_\chi$.*

Proof. Choose a nonzero $\lambda_0 \in \widetilde{V}$ and fix $v_0 \in V$ such that $\lambda_0(v_0) = \deg(\pi)$. Define a map $m\colon V \to \text{c-Ind}_Z^G\,\chi$ by setting $m(v) = m_{\lambda_0, v}$. Since V is irreducible, the map m is an injective G-homomorphism.

Let $(\sigma, U) \in \mathfrak{R}(G)_\chi$, and let $\varphi\colon U \to V$ be a surjective G-map. Choose $u_0 \in U$ such that $\varphi(u_0) = v_0$. Define $\tau\colon$ c-Ind$_Z^G\,\chi \to U$ by

$$\tau(f) := \int_{G/Z(G)} f(g^{-1}) \cdot \sigma(g) u_0 \, dg^*.$$

Let $\psi = \tau \circ m \colon V \to U$. For all $\lambda \in \widetilde{V}$ and $v \in V$, we have

$$\begin{aligned}\lambda(\varphi \circ \psi(v)) &= \lambda\left(\varphi\left(\int_{G/Z(G)} m_{\lambda_0,v}(g^{-1})\sigma(g)u_0\, dg^*\right)\right) \\ &= \int_{G/Z(G)} \check{m}_{\lambda_0,v}(g)\cdot m_{\lambda,v_0}(g)\, dg^* \\ &= \frac{\lambda(v)\cdot\lambda_0(v_0)}{\deg(\pi)} \\ &= \lambda(v).\end{aligned}$$

We conclude that $\varphi \circ \psi = 1_V$. Thus (π, V) is projective in the category $\mathfrak{R}(G)_\chi$. ∎

Remark 8.4.5. If G is the group of k-points of a connected reductive group defined over k, then the converse is true as well: if π is a simple projective object in $\mathfrak{R}(G)_\chi$, then it is supercuspidal.

Lemma 8.4.6. *Let $(\pi, V) \in \mathfrak{R}(G)$ be an irreducible, supercuspidal representation of G that is essentially square-integrable modulo the center. Let χ be the central character of G. Then (π, V) is injective in $\mathfrak{R}(G)_\chi$.*

Proof. We must show that the functor $X \mapsto \mathrm{Hom}(X, V)$ is exact when $X \in \mathfrak{R}(G)_\chi$. Since $\widetilde{V}$ is also supercuspidal and irreducible, by Lemmas 8.4.4 and 5.2.1, the functor

$$X \mapsto \widetilde{X} \mapsto \mathrm{Hom}_G(\widetilde{V}, \widetilde{X})$$

is exact (note that $\widetilde{X}$ and $\widetilde{V}$ are in the category $\mathfrak{R}(G)_{\chi^{-1}}$). We have natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_G(\widetilde{V}, \widetilde{X}) &= \mathrm{Hom}_G(\widetilde{V}, X^*) = \mathrm{Hom}_G(\widetilde{V} \otimes_{\mathbb{C}} X, \mathbb{C}) \\ &= \mathrm{Hom}_G(X, \widetilde{V}^*) = \mathrm{Hom}_G(X, \widetilde{\widetilde{V}}) \\ &= \mathrm{Hom}_G(X, V).\end{aligned}$$

∎

9. The Hecke algebra $\mathcal{H}(G)$

In this section, G denotes any unimodular t.d.-group.

9.1. **Idempotented rings.** Let $\mathcal{H}$ be an associative ring (we do not assume that $\mathcal{H}$ is commutive or has an identity element). Let I denote the set of idempotents in $\mathcal{H}$. That is,

$$I = \{h \in \mathcal{H} \mid h^2 = h\}.$$

Lemma 9.1.1. *If $e, f \in I$, then the following are equivalent.*

(1) $e\mathcal{H}e \subset f\mathcal{H}f$
(2) $e \in f\mathcal{H}f$
(3) $e = fef$

Proof. "(1) $\implies$ (2)": If $e\mathcal{H}e \subset f\mathcal{H}f$ then $e = eee \in e\mathcal{H}e \subset f\mathcal{H}f$.

"(2) $\implies$ (3)": If $e \in f\mathcal{H}f$ then there exists $h \in \mathcal{H}$ such that $e = fhf$, so $fef = f(fhf)f = fhf = e$.

"(3) $\implies$ (1)": This is clear. ∎

Definition 9.1.2. Let $e, f \in I$. If the equivalent statements in Lemma 9.1.1 are satisfied by e and f, then we write $e \le f$.

Exercise 9.1.3. The relation $\le$ on I is a partial order on I, with unique minimal element 0. If $1 \in \mathcal{H}$, then 1 is the unique maximal element.

Definition 9.1.4. We call $\mathcal{H}$ an *idempotented ring* provided that for any finite subset S of $\mathcal{H}$ there exists an $e \in I$ such that $ese = s$ for all $s \in S$.

Exercise 9.1.5. Show that $\mathcal{H}$ is an idempotented ring if and only if

$$\mathcal{H} = \bigcup_{e \in I} e\mathcal{H}e$$

and I is filtered with respect to $\le$ (that is, any finite collection of elements in I is dominated by an element of I). Can you find an example of a ring $\mathcal{H}$ such that $\mathcal{H} = \bigcup_{e \in I} e\mathcal{H}e$ but $\mathcal{H}$ is not idempotented? (I can't.)

An idempotented ring $\mathcal{H}$ has a natural topology: a neighborhood basis of zero is

$$\{\mathcal{H}(1-e) \mid e \in I\},$$

where we define $\mathcal{H}(1-e) = \{h - he \mid h \in \mathcal{H}\}$. We obtain a basis at any other point via left translation. (Note that if $e \le f$, then $e = fef = ffef = fe$, so if $h \in \mathcal{H}$, then

$$(h - hf)(1-e) = (h - hf) - (he - hfe) = (h - hf) - (he - he) = h - hf;$$

thus $\mathcal{H}(1-f) \subset \mathcal{H}(1-e)$.)

Exercise 9.1.6. Show that a net $h_\alpha \in \mathcal{H}$ converges to the zero element if and only if for all $e \in I$ there exists an A such that for all $\alpha \ge A$ we have $h_\alpha e = 0$. (Hint: for all $e \in I$ we have $\mathcal{H} = \mathcal{H}e \oplus \mathcal{H}(1-e)$.)

9.2. **The Hecke algebra $\mathcal{H}(G)$.**

Definition 9.2.1. A *distribution* on G is any $\mathbb{C}$-linear function $C_c^\infty(G) \to \mathbb{C}$.

As in Exercises 5.3.1 and 5.3.2, there is a left action of $(G \times G)$ on $C_c^\infty(G)$ given by

$$((g_1, g_2) \cdot f)(x) = (g_2 \cdot f \cdot g_1^{-1})(x) = f(g_1^{-1} x g_2).$$

This induces an action of $(G \times G)$ on the space of distributions on G: if T is a distribution, then we set $((g_1, g_2) \cdot T)(f) = T((g_1^{-1}, g_2^{-1}) \cdot f)$ for $(g_1, g_2) \in G \times G$ and $f \in C_c^\infty(G)$.

Remark 9.2.2. If $\mathcal{K} = \{K\}$ is a neighborhood basis of the identity of G consisting of compact open subgroups, then $\{K \times K \mid K \in \mathcal{K}\}$ is a neighborhood basis of the identity in $G \times G$.

For any open set $U \subset G$, we can regard $C_c^\infty(U)$ as a subspace of $C_c^\infty(G)$ via extension by 0; therefore, one can define the restriction $\operatorname{res}_U T = \operatorname{res}_{C_c^\infty(U)} T$ of a distribution T to $C_c^\infty(U)$. We define the *support* $\operatorname{supp}(T)$ of a distribution T to be the smallest closed set S of G such that $\operatorname{res}_{G\setminus S} T = 0$; that is, the support of T is the complement of

$$\bigcup\{U \subset G \text{ open} \mid \operatorname{res}_U T = 0\}.$$

We let $\mathcal{H} = \mathcal{H}(G)$ denote the set of of locally constant, compactly supported distributions on G. (A locally constant distribution is a distribution T such that for all $f \in C_c^\infty(G)$, there is some compact open $K_f \subset G \times G$ such that $T((g_1, g_2) \cdot f) = T(f)$ for all $(g_1, g_2) \in K_f$.) Equivalently, $\mathcal{H}$ can be characterized as the set of locally constant, compactly supported, complex-valued measures on G (or in other words $C_c^\infty(G) \cdot \mu$ where μ is a Haar measure on G).

The complex vector space $\mathcal{H}$ is an algebra with respect to convolution. If μ_1 and μ_2 are two measures in $\mathcal{H}$, then convolution is defined via the product measure:

$$\int_G f(g)\, d(\mu_1 * \mu_2)(g) := \int_{G\times G} f(g_1 g_2)\, d(\mu_1 \otimes \mu_2)(g_1, g_2)$$

for $f \in C_c^\infty(G)$. Note that the $(G \times G)$-action does not respect convolution.

Exercise 9.2.3. Let T_1 and T_2 be distributions in $\mathcal{H}$, and let $f \in C_c^\infty(G)$. Show that

$$(T_1 * T_2)(f) = T_2(g \mapsto T_1(R(g)f)).$$

If we fix a Haar measure dg on G, then the map $f \mapsto f \cdot dg$ defines a vector space isomorphism from $C_c^\infty(G)$ to $\mathcal{H}(G)$.

Exercise 9.2.4. Show that the isomorphism above is an algebra isomorphism, where convolution on $C_c^\infty(G)$ is defined by

$$(f_1 * f_2)(h) = \int_G f_1(hg^{-1}) \cdot f_2(g)\, dg$$

for $f_1, f_2 \in C_c^\infty(G)$ and $h \in G$. Show that it is also an isomorphism of $(G \times G)$-modules.

Because it is often more convenient to deal with functions rather than distributions or measures, we will, unless specifically stated to the contrary, realize $\mathcal{H}(G)$ as the algebra $C_c^\infty(G)$ with respect to convolution.

Definition 9.2.5. For a compact open subgroup K of G, we let $C_c(G//K)$ denote the space of K-bi-invariant functions in $C_c^\infty(G)$; that is, $C_c(G//K) = C_c^\infty(G)^{K\times K}$.

Definition 9.2.6. If K is a compact open subgroup of G, then we define $e_K \in \mathcal{H}$ by

$$e_K = \frac{[K]}{\operatorname{meas}_{dg}(K)} \in C_c(G//K).$$

Here, $[K]$ denotes the characteristic function of K. Note that this definition depends on the choice of measure dg.

Alternatively, we can define $e_K \in \mathcal{H}$ to be the distribution

$$e_K(f) = \int_K f(x)\, dx,$$

where dx is the normalized Haar measure on K.

Remark 9.2.7. If $f \in C_c(G/\!/K)$, for some compact open subgroup K of G, then

$$e_K * f * e_K = f = f * e_K = e_K * f.$$

In particular, e_K is an idempotent element of $\mathcal{H}$. Since we have a neighborhood basis of the identity consisting of compact open subgroups, it follows that $\mathcal{H}$ is an idempotented ring (cf. Exercise 9.1.5).

9.3. **Modules over $\mathcal{H}(G)$.** First we define a functor from $\mathfrak{R}(G)$ to the category of $\mathcal{H}$-modules. If $(\pi, V) \in \mathfrak{R}(G)$, then we give V the structure of an $\mathcal{H}$-module via

$$f \cdot v = \pi(f)v := \int_G f(g)\pi(g)v\, dg$$

for $f \in C_c^\infty(G)$ and $v \in V$. It is easy to check that for $f, h \in \mathcal{H}$, $\pi(f * h) = \pi(f)\pi(h)$. In particular, if $f \in C_c(G/\!/K)$, then

$$\pi(f) = \pi(e_K * f * e_K) = \pi(e_K)\pi(f)\pi(e_K) = e_K\pi(f)e_K.$$

(We are using e_K in two roles – it will always be clear from context if we are to interpret e_K as an element of $\mathcal{H}$ or as an element of $\operatorname{End}_{\mathbb{C}}(V)$ for some representation space V. This overload of notation is justified by the fact that $\pi(e_K) = e_K$.) Thus, if (π, V) is admissible, then $\pi(f)$ is a finite rank operator for every $f \in C_c^\infty(G)$.

Remark 9.3.1. When (π, V) is admissible, we can define the *character distribution*

$$\Theta_\pi \colon C_c^\infty(G) \to \mathbb{C}$$

by setting $\Theta_\pi(f) := \operatorname{tr} \pi(f)$. We will discuss the character distributions of admissible representations at great length in some future course.

Definition 9.3.2. Let V be complex vector space. If V is a module for the $\mathbb{C}$-algebra $\mathcal{H} = \mathcal{H}(G)$, then we say that V is *nondegenerate* provided that $V = \mathcal{H}V$.

Let (π, V) be a smooth representation of G. For any $v \in V$, there exists a compact open subgroup K of G with $v \in V^K$. Thus $\pi(e_K)v = v$, so (π, V) defines a nondegenerate $\mathcal{H}$-module.

Lemma 9.3.3. *The functor from $\mathfrak{R}(G)$ to the category of nondegenerate $\mathcal{H}$-modules defines an equivalence of categories.*

Proof. Let V be a nondegenerate $\mathcal{H}$-module. We must give V the structure of a smooth G-module. Fix $v \in V$. By hypothesis, there exists $h \in \mathcal{H}$ and $w \in V$ such that $v = hw$. Since $h \in C_c(G/\!/K)$ for some compact open subgroup K of G, we have

$$e_K v = e_K(hw) = (e_K * h)w = hw = v.$$

It follows that the definition

$$\pi(g)v := \lim_K \frac{[gK]}{\mathrm{meas}_{dg}(K)} v$$

makes sense. It is left to the reader to verify that π is a group homomorphism that defines a smooth representation of G.

Let $(\pi, V) \in \mathfrak{R}(G)$, let $v \in V$, and let $K \subset G$ be a compact open subgroup with $v \in V^K$. Then for $h \in G$,

$$\pi(h)v = \pi(h)e_K v = \int_K \pi(hx)v\, dx = \int_G \frac{[hK](g)}{\mathrm{meas}_{dg}(K)} \pi(g)v\, dg,$$

so giving (π, V) the structure of a nondegenerate $\mathcal{H}$-module recovers its G-module structure.

Conversely, let V be a nondegenerate $\mathcal{H}$-module, and let v and K be as before. For $f \in C_c^\infty(G)$ and $h \in G$, we have

$$(f * e_K)(h) = \int_G f(g^{-1}) \frac{[K](gh)}{\mathrm{meas}_{dg}(K)}\, dg = \int_G f(g) \frac{[gK](h)}{\mathrm{meas}_{dg}(K)}\, dg$$

since $d(g^{-1}) = dg$. Thus

$$f \cdot v = (f * e_K) \cdot v = \int_G f(g) \frac{[gK](h)}{\mathrm{meas}_{dg}(K)} \cdot v\, dg,$$

which shows that giving V the structure of a G-representation recovers its original $\mathcal{H}$-module structure. ■

There is one more compatibility result that we need. Let L denote the left regular action of G on $\mathcal{H}$: that is,

$$(L(g)f)(x) := ((g,1)f)(x) = f(g^{-1}x).$$

Lemma 9.3.4. *Consider $\mathcal{H}$ as a nondegenerate $\mathcal{H}$-module by left multiplication. This action is the same as that induced by the left regular action L of G on $\mathcal{H}$: that is, for all $f, h \in \mathcal{H}$, we have $L(f)h = f * h$. Conversely, left multiplication in $\mathcal{H}$ induces the left regular action of G on $\mathcal{H}$.*

Proof. By Lemma 9.3.3, it suffices to prove the first statement. Let $f, h \in C_c^\infty(G)$ and $x \in G$. We have

$$\begin{aligned}(L(f)h)(x) &= \int_G f(g) \cdot (L(g)h)(x)\, dg \\ &= \int_G f(g^{-1}) \cdot (L(g^{-1})h)(x)\, dg \\ &= \int_G f(g^{-1}) \cdot h(gx)\, dg \\ &= \int_G f(xg^{-1}) \cdot h(g)\, dg \\ &= (f * h)(x).\end{aligned}$$

■

9.4. **Some equivalences.** Fix a compact open subgroup K of G, and let $e = e_K$. As $\mathbb{C}$-algebras, we have $C_c(G//K) = e\mathcal{H}e$, where $\mathcal{H} = C_c^\infty(G)$. The algebra $e\mathcal{H}e$ is actually a ring with unit: the element e is a two-sided identity. We say that an $e\mathcal{H}e$-module W is *unital* provided that $e \cdot w = w$ for all $w \in W$.

We treat $\mathcal{H}$ as a right $e\mathcal{H}e$-module. Since $\mathcal{H} = \mathcal{H}e \oplus \mathcal{H}(1-e)$, we have that, as (right) $e\mathcal{H}e$-modules, $\mathcal{H}e$ is a direct summand of $\mathcal{H}$. It follows that for any unital $e\mathcal{H}e$-module W, we have

$$\mathcal{H} \otimes_{e\mathcal{H}e} W = \mathcal{H}e \otimes_{e\mathcal{H}e} W \tag{6}$$

as a left $\mathcal{H}$-module.

We define $\mathrm{Irr}(e\mathcal{H}e)$ to be the set of equivalence classes of simple unital $e\mathcal{H}e$-modules, $\mathrm{Irr}(\mathcal{H})$ to be the set of equivalence classes of irreducible nondegenerate $\mathcal{H}$-modules, and $\mathrm{Irr}(\mathcal{H})^e$ to be the subset of $\mathrm{Irr}(\mathcal{H})$ consisting of V such that $eV \neq 0$.

Definition 9.4.1. If V is a nondegenerate $\mathcal{H}$-module, then we define the nondegenerate $\mathcal{H}$-module

$$V_e := V/\{v \in V | e\mathcal{H}v = 0\}.$$

Remark 9.4.2. V_e is the largest quotient of V having the property that every nonzero vector in V_e generates an $\mathcal{H}$-module not annihilated by e.

Exercise 9.4.3. Let V be a nondegenerate $\mathcal{H}$-module.

(1) Show that $(V_e)_e = V_e$.
(2) Define a map $\varphi : \mathcal{H} \otimes_{e\mathcal{H}e} eV \to V$ by $\varphi(h \otimes v) = h \cdot v$ for $v \in eV$. Show that $e\mathcal{H}(\ker \varphi) = 0$.
(3) Let V be a nondegenerate $\mathcal{H}$-module, and let W be a unital $e\mathcal{H}e$-module. Show that $V \mapsto eV$ and $W \mapsto (\mathcal{H} \otimes_{e\mathcal{H}e} W)$ are adjoint functors. Explicitly,

$$\mathrm{Hom}_{\mathcal{H}}(\mathcal{H} \otimes_{e\mathcal{H}e} W, V) = \mathrm{Hom}_{e\mathcal{H}e}(W, V) = \mathrm{Hom}_{e\mathcal{H}e}(W, eV).$$

Lemma 9.4.4. *The map $V \mapsto eV$ gives a bijective correspondence between $\mathrm{Irr}(\mathcal{H})^e$ and $\mathrm{Irr}(e\mathcal{H}e)$.*

In the opposite direction, we map $W \in \mathrm{Irr}(e\mathcal{H}e)$ to $(\mathcal{H} \otimes_{e\mathcal{H}e} W)_e$.

Proof. First we show that if W is a simple unital $e\mathcal{H}e$-module, then there is a canonical isomorphism $W \cong e(\mathcal{H} \otimes_{e\mathcal{H}e} W)_e$. Indeed, from Equation (6) we have

$$e(\mathcal{H} \otimes_{e\mathcal{H}e} W)_e = e(\mathcal{H}e \otimes_{e\mathcal{H}e} W)_e.$$

One checks that the maps

$$w \mapsto e \otimes w \quad \text{and} \quad e(he \otimes w) \mapsto ehew$$

are well-defined inverse isomorphisms between W and $e(\mathcal{H}e \otimes_{e\mathcal{H}e} W)_e$.

We now show that if V is a simple nondegenerate $\mathcal{H}$-module for which $eV \neq 0$, then eV is a simple $e\mathcal{H}e$-module. Choose $0 \neq ev \in eV$. Since V is a simple $\mathcal{H}$-module, we have

$$(e\mathcal{H}e)ev = (e\mathcal{H})ev = e(\mathcal{H}ev) = eV.$$

Since $0 \neq ev$ was arbitrary, we conclude that eV is a simple $e\mathcal{H}e$-module.

We now show that if W is a simple unital $e\mathcal{H}e$-module, then $(\mathcal{H} \otimes_{e\mathcal{H}e} W)_e$ is a simple $\mathcal{H}$-module. From above, we know that there is an element $0 \neq w' = ew' \in (\mathcal{H} \otimes_{e\mathcal{H}e} W)_e$. Since $((\mathcal{H} \otimes_{e\mathcal{H}e} W)_e)_e = (\mathcal{H} \otimes_{e\mathcal{H}e} W)_e$, we have

$$0 \neq e\mathcal{H}w' \subset e(\mathcal{H} \otimes_{e\mathcal{H}e} W)_e.$$

But from the first paragraph of the proof, this latter object is isomorphic to W. Since W is a simple $e\mathcal{H}e$-module, we conclude that

$$e\mathcal{H}w' = (e\mathcal{H}e\mathcal{H})w' = (e\mathcal{H}e)e\mathcal{H}w' = e(\mathcal{H} \otimes_{e\mathcal{H}e} W)_e.$$

Thus,

$$\mathcal{H}w' = \mathcal{H}(e\mathcal{H}w') = \mathcal{H}e(\mathcal{H} \otimes_{e\mathcal{H}e} W)_e = (\mathcal{H} \otimes_{e\mathcal{H}e} W)_e.$$

Since w' was arbitrary, we conclude that $(\mathcal{H} \otimes_{e\mathcal{H}e} W)_e$ is irreducible.

Finally, we need to show that if V is a simple nondegenerate $\mathcal{H}$-module for which $eV \neq 0$, then the natural map

$$h \otimes ev \mapsto he \cdot v : (\mathcal{H} \otimes_{e\mathcal{H}e} eV)_e \to V$$

is an isomorphism (it is well-defined by Exercise 9.4.3(2)). From the above paragraphs, we know that $(\mathcal{H} \otimes_{e\mathcal{H}e} eV)_e$ is a simple $\mathcal{H}$-module. Since the above map is nonzero, we conclude that it is an isomorphism. ■

Exercise 9.4.5. Generalize the previous lemma by proving that the functor from nondegenerate $\mathcal{H}$-modules to $e\mathcal{H}e$-modules which sends $V \mapsto eV$ becomes an equivalence if we restrict to modules with the property that $\mathcal{H}eV = V$.

We let $\mathrm{Irr}(G)^K$ denote the subset of $\mathrm{Irr}(G)$ consisting of those elements for which the space of K-fixed vectors is nontrivial.

Corollary 9.4.6. *The map $V \mapsto V^K$ is a bijective correspondence between $\mathrm{Irr}(G)^K$ and $\mathrm{Irr}(e\mathcal{H}e)$.*

Proof. The equivalence between $\mathfrak{R}(G)$ and the category of nondegenerate $\mathcal{H}$-modules gives us a natural correspondence between $\mathrm{Irr}(G)$ and $\mathrm{Irr}(\mathcal{H})$. Thus, the map $V \mapsto V^K$ induces a bijective correspondence between $\mathrm{Irr}(G)^K$ and $\mathrm{Irr}(\mathcal{H})^e$. The corollary now follows from Lemma 9.4.4. ■

9.5. **The separation lemma.** The goal of this subsection is to prove the following lemma.

Lemma 9.5.1 (Separation Lemma)**.** *Let $h \in \mathcal{H}(G)$ be nonzero. There exists $(\pi, V) \in \mathrm{Irr}(G)$ such that $\pi(h) \neq 0$.*

We will require the following result, which I believe is due to Jacobson [7].

Lemma 9.5.2. *Let $\mathcal{A}$ be a $\mathbb{C}$-algebra with unit of countable dimension. If $a \in \mathcal{A}$ is not nilpotent, then there exists a simple $\mathcal{A}$-module M such that $a \cdot M \neq 0$.*

Proof. Let 1 denote the unit of $\mathcal{A}$. Suppose that $a \notin \mathbb{C}\cdot 1$. We will show that there is a $\lambda \in \mathbb{C}^\times$ for which $a-\lambda$ is not invertible. Then $\mathcal{A}/\mathcal{A}(a-\lambda)$ is a nontrivial $\mathcal{A}$-module; since it is generated by the image of 1, Zorn's lemma guarantees us a simple quotient M (cf. the proof of Lemma 3.3.5). We see that a does not kill M since any $\mathcal{A}$-submodule of $\mathcal{A}$ that contains a and $\mathcal{A}(a-\lambda)$ is all of $\mathcal{A}$.

Suppose that $a-\lambda$ is invertible for all $\lambda \in \mathbb{C}^\times$. Since $\mathcal{A}$ has countable dimension, the elements of $\{(a-\mu)^{-1} \mid \mu \in \mathbb{C}^\times\}$ must be linearly dependent. Consequently, we can find a finite collection of $c_i \in \mathbb{C}^\times$ and $\mu_i \in \mathbb{C}^\times$ such that

$$\sum c_i(a-\mu_i)^{-1} = 0.$$

By multiplying through by the polynomial $\prod(a-\mu_i)$ and factoring, we find that there exist $\lambda_j \in \mathbb{C}^\times$, $n_j \in \mathbb{Z}_{\geq 1}$, and $n \in \mathbb{Z}_{\geq 0}$ such that

$$a^n \prod (a-\lambda_j)^{n_j} = 0.$$

Since $a^n \neq 0$, we conclude that some $a-\lambda_j$ is a zero divisor of $\mathcal{A}$, contradicting the hypothesis that it is invertible.

If $a \in \mathbb{C}\cdot 1$, then we only need to show that there exists a simple $\mathcal{A}$-module. As above, we can construct a simple $\mathcal{A}$-module by using Zorn's Lemma to find a maximal proper $\mathcal{A}$-submodule of $\mathcal{A}$ and taking the quotient. ∎

We can now prove the separation lemma.

Proof of Lemma 9.5.1. For $f \in \mathcal{H}(G)$, define $f^* \in \mathcal{H}(G)$ by $f^*(g) = \overline{f}(g^{-1})$. A calculation shows that $(f * f^*)^* = f * f^*$.

Let $h \in \mathcal{H}(G)$ be nonzero. Fix a compact open subgroup $K \subset G$ such that $h \in C_c(G/\!/K)$. Note that $h^* \in C_c(G/\!/K)$. Define $a = h * h^*$, so $a \in C_c(G/\!/K)$ and $a(e) = \int |h(g)|^2 \, dg$, where $e \in G$ is the identity. Since $h \neq 0$, we have $a \neq 0$. Therefore, since $a = a^*$,

$$a^2 := a * a = a * a^* \in C_c^\infty(G/\!/K)$$

has $a^2(e) = \int_G |a(g)|^2 \, dg \neq 0$. By induction, we have $a^{2^n} \neq 0$, $(a^{2^n})^* = a^{2^n}$, and $a^{2^n} \in C_c(G/\!/K)$ for all $n \in \mathbb{Z}_{\geq 1}$. Hence a is not nilpotent.

By Lemma 9.5.2, there exists a simple $C_c(G/\!/K)$-module V' upon which a, and thus h, acts nontrivially. It follows that h acts nontrivially on the $\mathcal{H}$-module $V = (\mathcal{H} \otimes_{e_K \mathcal{H} e_K} V')_{e_K}$, which is simple by Lemma 9.4.4. ∎

10. Splitting representations

10.1. **Splitting off finite representations.** Most of the representation theory that follows can be generalized in the following manner: replace "finite" with "square integrable" and replace $\mathfrak{R}(G)$ with $\mathfrak{R}(G)_\chi$ for a smooth unitary central character χ. We choose not to present the material in this level of generality because it greatly increases the typesetting demands and will not be used in the sequel.

Fix a unimodular t.d.-group G and a finite, irreducible representation $(\pi, V) \in \mathfrak{R}(G)$.

We have already defined an action of $G \times G$ on $\mathcal{H}$; we will also need:

Definition 10.1.1. Let $(\sigma_1, W_1),\ (\sigma_2, W_2) \in \mathfrak{R}(G)$. Define an action of $G \times G$ on $\operatorname{Hom}_{\mathbb{C}}(W_1, W_2)$ by

$$((g_1, g_2) \cdot A)(w) = \sigma(g_1)A(\sigma(g_2^{-1})w).$$

In particular, $G \times G$ acts on $\operatorname{End}(W)$ for $(\sigma, W) \in \mathfrak{R}(G)$.

Remark 10.1.2. Let $(\sigma, W) \in \mathfrak{R}(G)$. A calculation shows that the algebra homomorphism $\sigma : \mathcal{H} \to \operatorname{End}_{\mathbb{C}}(W)$ respects the above action of $G \times G$ on $\operatorname{End}_{\mathbb{C}}(W)$.

Lemma 10.1.3. *If $(\sigma_1, W_1),\ (\sigma_2, W_2) \in \mathfrak{R}(G)$ are two admissible representations, then the map*

$$\tau \colon W_2 \otimes_{\mathbb{C}} \widetilde{W}_1 \to \operatorname{Hom}_{\mathbb{C}}(W_1, W_2)^{\infty} \quad \textit{given by} \quad \tau(w \otimes \lambda)(w') = \lambda(w')w$$

is a $(G \times G)$-module isomorphism.

Proof. One checks that τ is a $(G \times G)$-module homomorphism.

First we show that τ is injective. Let

$$\sum_i w_i \otimes \lambda_i \in \ker \tau, \quad \text{i.e.,} \quad \sum_i \lambda_i(w') \cdot w_i = 0 \text{ for all } w' \in W_1.$$

Rewrite the sum $\sum_i w_i \otimes \lambda_i$ so that the λ_i are linearly independent, and for each i, find $w'_i \in W_1$ such that $\lambda_i(w'_j) = \delta_{ij}$. Then for each j,

$$0 = \sum_i \lambda_i(w'_j) \cdot w_i = w_j.$$

By Remark 9.2.2, we have

$$\operatorname{Hom}_{\mathbb{C}}(W_1, W_2)^{\infty} = \bigcup_{K \leq G \text{ compact open}} \operatorname{Hom}_{\mathbb{C}}(W_1, W_2)^{K \times K}.$$

Thus, for a fixed compact open subgroup K of G, it is enough to show that τ restricts to a surjection

$$W_2^K \otimes \widetilde{W}_1^K \to \operatorname{Hom}_{\mathbb{C}}(W_1, W_2)^{K \times K}.$$

For brevity, denote the above map by τ_K; note that $\tau(W_2^K \otimes \widetilde{W}_1^K)$ is indeed contained in $\operatorname{Hom}_{\mathbb{C}}(W_1, W_2)^{K \times K}$. One easily shows that $\operatorname{Hom}_{\mathbb{C}}(W_1, W_2)^{K \times K} = \operatorname{Hom}_{\mathbb{C}}(W_1^K, W_2^K)$, so since $W_2^K \otimes \widetilde{W}_1^K$ and $\operatorname{Hom}_{\mathbb{C}}(W_1^K, W_2^K)$ have the same (finite) dimension and τ is injective, τ_K is surjective. ■

Lemma 10.1.4. *Let $(\pi, V) \in \mathfrak{R}(G)$ be finite and irreducible. The map*

$$m \colon V \otimes_{\mathbb{C}} \widetilde{V} \to \mathcal{H} \quad \textit{given by} \quad m(v \otimes \lambda) = \deg(\pi) \cdot \check{m}_{\lambda, v}$$

$(G \times G)$-module injection.

Proof. It is immediate that m is a $(G \times G)$-homomorphism. By Lemma 7.1.4, Corollary 5.2.4, and Lemma 8.1.2, $V \otimes_{\mathbb{C}} \widetilde{V}$ is an irreducible $(G \times G)$-module. Hence, since m is not the zero map, it is injective. ■

Lemma 10.1.5. *The following diagram of $(G \times G)$-modules commutes.*

$$\begin{tikzcd} & \mathcal{H} \arrow[dr, "\pi"] & \\ V \otimes_{\mathbb{C}} \widetilde{V} \arrow[ur, hook, "m"] \arrow[rr, "\sim", "\tau"'] & & \mathrm{End}_{\mathbb{C}}(V)^{\infty} \end{tikzcd}$$

Remark 10.1.6. This shows that $\mathcal{H}$ surjects onto $\mathrm{End}_{\mathbb{C}}(V)^{\infty}$. In general, $\mathcal{H}$ has no unit, so Id_V is not a smooth vector of $\mathrm{End}_{\mathbb{C}}(V)$. Indeed, Id_V is smooth if and only if $V = V^K$ for some compact open subgroup $K \subset G$.

Proof. We have already verified that all of the maps are $(G \times G)$-module homomorphisms. We need to show that $\tau = \pi \circ m$. For $v, w \in V$ and $\lambda \in \widetilde{V}$, we have

$$\begin{aligned}(\pi \circ m(v \otimes \lambda))(w) &= \deg(\pi) \cdot \pi(\check{m}_{\lambda,v})w \\ &= \deg(\pi) \cdot \int_G \check{m}_{\lambda,v}(g)\pi(g)w \, dg.\end{aligned}$$

Thus for all $\lambda' \in \widetilde{V}$, Schur orthogonality gives

$$\lambda'((\pi \circ m(v \otimes \lambda))(w)) = \lambda'(\lambda(w)v).$$

It follows that $(\pi \circ m(v \otimes \lambda))(w) = \lambda(w)v = \tau(v \otimes \lambda)(w)$. ■

Lemma 10.1.7. *Both* $\ker(\pi)$ *and* $m(V \otimes_{\mathbb{C}} \widetilde{V})$ *are two-sided ideals in* $\mathcal{H}$*, so there is a ring isomorphism*

$$\mathcal{H} \cong \ker(\pi) \times m(V \otimes_{\mathbb{C}} \widetilde{V}).$$

Moreover, the composition $m \circ \tau^{-1} : \mathrm{End}_{\mathbb{C}}(V)^{\infty} \to \mathcal{H}$ *is a homomorphism of* $\mathbb{C}$*-algebras which respects the action of* $G \times G$*.*

Remark 10.1.8. Note that m, π, and τ are algebra homomorphisms, where we give $V \otimes_{\mathbb{C}} \widetilde{V}$ a ring structure using the composition

$$V \otimes_{\mathbb{C}} \widetilde{V} \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} \widetilde{V} \to V \otimes_{\mathbb{C}} \mathbb{C} \otimes_{\mathbb{C}} \widetilde{V} \cong V \otimes_{\mathbb{C}} \widetilde{V}.$$

Proof. It is clear that the kernel of the ring homomorphism π is a two-sided ideal. Two calculations show that for any $f \in C_c^{\infty}(G)$, $v \in V$, $\lambda \in \widetilde{V}$, and $x \in G$, we have

$$(f * \check{m}_{\lambda,v})(x) = \check{m}_{\lambda,\pi(f)v}(x) \quad \text{and} \quad (\check{m}_{\lambda,v}(x) * f)(x) = \check{m}_{\lambda\circ\pi(f),v}(x),$$

which shows that $m(V \otimes_{\mathbb{C}} \widetilde{V})$ is a two-sided ideal in $\mathcal{H}$ as well.

Since τ is an isomorphism, we have that $\mathcal{H} = \ker(\pi) \oplus m(V \otimes_{\mathbb{C}} \widetilde{V})$ as vector spaces. Since $\ker(\pi)$ and $m(V \otimes_{\mathbb{C}} \widetilde{V})$ are two-sided ideals with trivial intersection, $\mathcal{H} = \ker(\pi) \times m(V \otimes_{\mathbb{C}} \widetilde{V})$ as rings (note $IJ \subset I \cap J = 0$).

The composition $m \circ \tau^{-1}$ is the algebra isomorphism $\mathrm{End}_{\mathbb{C}}(V)^{\infty} \to m(V \otimes_{\mathbb{C}} \widetilde{V}) \times \{0\}$. It respects the action of $G \times G$ since τ^{-1} and m do. ■

Exercise 10.1.9. Prove directly that the composition $m \circ \tau^{-1} : \mathrm{End}_{\mathbb{C}}(V)^{\infty} \to \mathcal{H}$ is a homomorphism of $\mathbb{C}$-algebras (hint: use Schur orthogonality).

Definition 10.1.10. Let $K \subset G$ be a compact open subgroup. Let $e_K = \pi(e_K) \in \mathrm{End}_{\mathbb{C}}(V)^\infty$ be the canonical projection onto V^K, as defined in Subsection 3.1. Define $e_K^\pi \in \mathcal{H}$ by

$$e_K^\pi := (m \circ \tau^{-1})(e_K).$$

Remark 10.1.11. Sometimes it is helpful to have an explicit realization of e_K^π. We have $\mathrm{res}_K V = V^K \oplus (1 - e_K)V$ and $\widetilde{V}^K = \mathrm{Hom}(V^K, \mathbb{C})$. If $v_1, v_2, \dots, v_m \in V^K$ is a basis of V^K with dual basis $\lambda_1, \lambda_2, \dots, \lambda_m \in \widetilde{V}^K$, then

$$e_K^\pi = \deg(\pi) \cdot \sum_{j=1}^{m} \check{m}_{\lambda_j, v_j}.$$

Lemma 10.1.12. *Let $(\pi, V) \in \mathfrak{R}(G)$ be a finite irreducible representation, and let $K \leq G$ be a compact open subgroup. Then the following hold:*

(1) e_K^π *is idempotent.*
(2) *If K' is a compact open subgroup of K, then $e_{K'}^\pi * e_K = e_K^\pi$ and $e_K * e_{K'}^\pi = e_K^\pi$.*
(3) *For $g \in G$ we have $e_{gKg^{-1}}^\pi = (g, g) \cdot e_K^\pi$.*

Proof. (1) This is immediate since e_K^π is the image of e_K under the algebra homomorphism $m \circ \tau^{-1} \circ \pi$.

(2) Let $K' \leq K$ be a compact open subgroup. As above, it suffices to show that $\pi(e_{K'}^\pi * e_K) = \pi(e_K * e_{K'}^\pi) = \pi(e_K^\pi)$ Recalling Lemma 6.1.1, we calculate

$$\pi(e_{K'}^\pi * e_K) = \pi(e_{K'}^\pi)\pi(e_K) = e_{K'} \circ e_K = e_K = \pi(e_K^\pi)$$
$$\pi(e_K * e_{K'}^\pi) = \pi(e_K)\pi(e_{K'}^\pi) = e_K \circ e_{K'} = e_K = \pi(e_K^\pi).$$

(3) As above, it is enough to check that $\pi(e_{gKg^{-1}}^\pi) = (g, g) \cdot \pi(e_K^\pi)$. This is left to the reader. ■

Definition 10.1.13. For $(\sigma, W) \in \mathfrak{R}(G)$, define $e^\pi \colon W \to W$ as follows. Let $w \in W$, and choose a compact open subgroup K such that $w \in W^K$. Define

$$e^\pi w := \sigma(e_K^\pi) w;$$

this is independent of the choice of K by Lemma 10.1.12 (2).

Note that e^π is the identity when $W = V$ and $\sigma = \tau$.

Lemma 10.1.14. *Let $(\pi, V) \in \mathfrak{R}(G)$ be a finite irreducible representation, and let $(\sigma, W) \in \mathfrak{R}(G)$. Then*

(1) $e^\pi \in \mathrm{End}_{\mathbb{C}}(W)$ *is idempotent, and*
(2) $e^\pi \in \mathrm{End}_G(W)$.

Proof. (1) Fix $w \in W$ and a compact open subgroup K of G such that $w \in W^K$. By Lemma 10.1.12 (2), $e^\pi w \in W^K$. We have

$$(e^\pi \circ e^\pi) w = (\sigma(e_K^\pi) \circ \sigma(e_K^\pi)) w = \sigma(e_K^\pi * e_K^\pi) w = \sigma(e_K^\pi) w = e^\pi w.$$

Thus e^π is idempotent.

(2) Fix $g \in G$. Without loss of generality, assume that $w \in V^K \cap V^{gKg^{-1}}$. We have

$$\big((g,g)\cdot e^\pi\big)w = \sigma((g,g)\cdot e_K^\pi)w = \sigma(e_{gKg^{-1}}^\pi)w = e^\pi w.$$

Consequently, for all $g \in G$, we have $\sigma(g)e^\pi = e^\pi\sigma(g)$.

■

Lemma 10.1.15. *Let* $(\pi, V) \in \mathfrak{R}(G)$ *be a finite irreducible representation. Let* $(\sigma_1, W_1),\ (\sigma_2, W_2) \in \mathfrak{R}(G)$ *and* $\beta \in \mathrm{Hom}_G(W_1, W_2)$. *Then* $e^\pi \circ \beta = \beta \circ e^\pi$.

In fancy language, Lemma 10.1.15 asserts that e^π is an element of the center of the category $\mathfrak{R}(G)$.

Proof. Fix $w \in W_1$ and a compact open subgroup K such that $w \in W_1^K$. Then $\beta(w) \in W_2^K$, so

$$\beta(e^\pi w) = \beta(\sigma_1(e_K^\pi)w) = \sigma_2(e_K^\pi)\beta(w) = e^\pi\beta(w).$$

■

Lemma 10.1.16. *Let* $(\pi, V) \in \mathfrak{R}(G)$ *be a finite irreducible representation, and let* $(\sigma, W) \in \mathfrak{R}(G)$. *As G-modules, we have*

$$W = e^\pi W \oplus (e^\pi - 1)W.$$

Moreover, there exists an indexing set I *such that*

$$e^\pi W = \bigoplus_I (\pi, V)$$

as G*-modules.*

Proof. Only the final statement requires proof. It suffices to show that $e^\pi W$ is spanned by submodules isomorphic to V (cf. [9, XVII, §2]).

Let $w \in W$, and find a compact open subgroup K of G such that $w \in W^K$, so $\sigma(e_K^\pi)w \in e^\pi W$. Since $\mathcal{H}e_K^\pi \subset m(V \otimes \widetilde{V})$, by Lemma 9.3.4 we can think of $\mathcal{H}e_K^\pi$ as a G-submodule of $V \otimes \widetilde{V}$, where the action of G on $V \otimes \widetilde{V}$ is given by $g \cdot (v \otimes \lambda) = (\pi(g)v) \otimes \lambda$, and the action of G on $\mathcal{H}e_K^\pi$ is the one induced by left multiplication. But $V \otimes \widetilde{V} \cong \bigoplus V$ as G-modules, so since $\mathcal{H}e_K^\pi$ is a submodule, $\mathcal{H}e_K^\pi \cong \bigoplus V$ as well. The map

$$h * e_K^\pi \mapsto \sigma(h * e_K^\pi)w :\ \mathcal{H}e_K^\pi \to \mathcal{H}e_K^\pi \cdot w$$

is then a G-homomorphism whose image, which is also isomorphic to a direct sum of copies of V (since it is a factor module of $\mathcal{H}e_K^\pi$), contains w. Since $\sigma(e_K^\pi)w$ was an arbitrary element of $e^\pi W$, we are done. ■

Lemma 10.1.17. *The functor*

$$e^\pi : \mathfrak{R}(G) \to \mathfrak{R}(G)$$

defined by $W \mapsto e^\pi W$ *is exact.*

Proof. It is clear that e^π is left-exact. Suppose that $\beta\colon (\sigma_1, W_1) \to (\sigma_2, W_2)$ is a surjection of smooth G-modules. For all $w_2 \in W_2$ there exists a $w_1 \in W_1$ such that $\beta(w_1) = e^\pi w_2$, so

$$\beta(e^\pi w_1) = e^\pi \beta(w_1) = e^\pi e^\pi w_2 = e^\pi w_2.$$

■

Lemma 10.1.18. *Let $(\pi, V) \in \mathfrak{R}(G)$ be a finite irreducible representation, and let $(\sigma, W) \in \mathfrak{R}(G)$. The following statements are equivalent:*

(1) *$e^\pi\colon W \to W$ is the zero map.*
(2) *W has no subquotient isomorphic to (π, V).*

Proof. Suppose that $e^\pi\colon W \to W$ is the zero map. Since e^π is exact, e^π is zero on every submodule and (hence) every subquotient of W. Since e^π is the identity on V, the representation (π, V) cannot occur as a subquotient.

Now suppose that $e^\pi W \neq \{0\}$. In this case, Lemma 10.1.16 tells us that (π, V) occurs as a direct summand of W. ■

Corollary 10.1.19. *Let $(\pi, V) \in \mathfrak{R}(G)$ be a finite irreducible representation, and let $(\sigma, W) \in \mathfrak{R}(G)$. Then*

(1) *$e^\pi W$ is the V-isotypic submodule of W,*
(2) *$e^\pi e^{\pi'} = e^{\pi'} e^\pi = 0$ for all finite irreducible smooth representations (π', V') not equivalent to V, and*
(3) *$(1 - e^\pi)W$ is the unique G-complement to $e^\pi W$.*

Proof. The first two statements are immediate consequences of Lemma 10.1.16 and Lemma 10.1.18.

Suppose that W' is another G-complement to $e^\pi W$. That is, we have

$$W = e^\pi W \oplus (1 - e^\pi)W \quad \text{and} \quad W = e^\pi W \oplus W'.$$

It will be enough to show that $W' \subset (1 - e^\pi)W$. Since e^π is exact, $e^\pi W' \subset e^\pi W$, so since $W' \cap e^\pi W = \{0\}$, we have that $e^\pi W' = \{0\}$. Thus for all $w' \in W'$ we have $e^\pi w' = 0$, so $w' = (1 - e^\pi)w' \in (1 - e^\pi)W$. ■

Corollary 10.1.20. *Any finite irreducible $(\pi, V) \in \mathfrak{R}(G)$ is both projective and injective in $\mathfrak{R}(G)$.*

Remark 10.1.21. Corollary 10.1.20 is a specific case of Lemmas 8.4.4 and 8.4.6.

Proof. Since $\operatorname{Hom}_G(V, W) = \operatorname{Hom}_G(V, e^\pi W)$, it follows from Lemma 10.1.16 that (π, V) is projective. Since $\operatorname{Hom}_G(W, V) = \operatorname{Hom}_G(e^\pi W, V)$, it follows from Lemma 10.1.16 that (π, V) is injective. ■

Corollary 10.1.22. *Any finite representation $(\sigma, W) \in \mathfrak{R}(G)$ has a canonical decomposition*

$$W = \bigoplus_\pi e^\pi W,$$

where the sum runs over the set of isomorphism classes of irreducible finite representations $(\pi, V) \in \mathfrak{R}(G)$. Therefore, any finite smooth representation of G is semisimple.

Remark 10.1.23. Corollary 10.1.22 is a generalization of Corollary 5.1.2.

Proof. Clearly the sum $W' := \bigoplus_\pi e^\pi W \subset W$ is direct. Consider the quotient representation W/W'. Let (π, V) be any irreducible finite smooth representation of G. Since e^π is exact and $e^\pi W \subset W'$, we have that $e^\pi(W/W') = 0$. Thus W/W' has no finite irreducible subquotient. But since W is finite, so is any subquotient of W/W', so W/W' must be zero. ■

10.2. **A consequence.**

Definition 10.2.1. Let $\mathcal{A}$ be an abelian category with full subcategories $\mathcal{A}_i$ indexed by I. We say that $\mathcal{A} = \prod_I \mathcal{A}_i$ is a *splitting of the category* $\mathcal{A}$ provided that:

(1) each object V of $\mathcal{A}$ has a unique decomposition
$$V = \bigoplus_I V_i,$$
where each $V_i \in \mathcal{A}_i$, and

(2) for $V_i \in \mathcal{A}_i$ and $V_j \in \mathcal{A}_j$, we have $\mathrm{Hom}_{\mathcal{A}}(V_i, V_j) = \{0\}$ when $i \neq j$.

The results of the previous subsection tell us that for any finite irreducible representation (π, V) we can write
$$\mathfrak{R}(G) = \mathfrak{R}(G)(\pi) \times \mathfrak{R}(G)(\text{no-}\pi),$$
where $\mathfrak{R}(G)(\pi)$ is the full subcategory of $\mathfrak{R}(G)$ whose objects are direct sums of copies of (π, V) (i.e., an object in $\mathfrak{R}(G)$ belongs to $\mathfrak{R}(G)(\pi)$ provided that each of its irreducible subquotients is equivalent to (π, V)), and $\mathfrak{R}(G)(\text{no-}\pi)$ denotes the full subcategory of $\mathfrak{R}(G)$ consisting of those representations for which no irreducible subquotient is equivalent to (π, V). Corollary 10.1.22 also shows that
$$\mathfrak{R}_f(G) = \prod_{\pi \in \mathrm{Irr}_f(G)} \mathfrak{R}(G)(\pi),$$
where $\mathfrak{R}_f(G)$ is the full subcategory of $\mathfrak{R}(G)$ consisting of the finite representations, and $\mathrm{Irr}_f(G)$ is the subset of $\mathrm{Irr}(G)$ consisting of equivalence classes of finite representations.

We will soon show (Corollary 11.2.3) that when G is the group of k-points of a connected reductive group defined over k, we have a splitting
$$\mathfrak{R}(G^1) = \mathfrak{R}_{nf}(G^1) \times \prod_{\pi \in \mathrm{Irr}_f(G^1)} \mathfrak{R}(G^1)(\pi).$$
Here $\mathfrak{R}_{nf}(G^1)$ denotes the full subcategory of $\mathfrak{R}(G^1)$ consisting of those representations all of whose nonzero subquotients are not finite.

10.3. **Consequences for reductive groups.** Let G be the group of k-rational points of a connected reductive group.

Since G^1 is a normal subgroup of G, G acts on $\mathrm{Irr}(G^1)$ by $g{\cdot}\sigma(h) = \sigma(g^{-1}hg)$ for $\sigma \in \mathrm{Irr}(G^1)$, $g \in G$, and $h \in G^1$. Since this action factors through the finite group $G/Z(G)G^1$, the G-orbits in $\mathrm{Irr}(G^1)$ are finite.

Let $V \in \mathfrak{R}(G)$, $\sigma \in \mathrm{Irr}(G^1)$, and $g \in G$. A standard calculation shows that $\pi(g) \cdot V(\sigma) = V(g \cdot \sigma)$, where $V(\sigma)$ is the σ-isotypic component of $\mathrm{res}_{G^1} V$.

Lemma 10.3.1. *Let $(\pi_1, V_1),\ (\pi_2, V_2) \in \mathfrak{R}(G)$ be irreducible supercuspidal representations.*

(1) *The elements of* $\mathrm{Irr}(G^1)$ *occurring in* $\mathrm{res}_{G^1} \pi_1$ *form a single G-orbit.*
(2) $\mathrm{res}_{G^1} \pi_1$ *is semisimple and of finite length.*
(3) *The following are equivalent.*
    (a) $\mathrm{res}_{G^1} \pi_1 = \mathrm{res}_{G^1} \pi_2$.
    (b) *There exists a* $\chi \in \mathbf{X}(G)$ *such that* $\pi_1 = \pi_2 \otimes \chi$.
    (c) $\mathrm{Hom}_{G^1}(\pi_1, \pi_2) \neq \{0\}$.

Proof. Since each (π_i, V_i) is supercuspidal, by Theorem 7.3.1 we have that $\mathrm{res}_{G^1} \pi_i$ is finite. Corollary 10.1.22 gives us a decomposition

$$\mathrm{res}_{G^1} \pi_i = \bigoplus_{\sigma \in \mathrm{Irr}_f(G^1)} V_i(\sigma), \tag{7}$$

where $V_i(\sigma)$ is the σ-isotypic submodule in $\mathrm{res}_{G^1} V_i$.

(1) Since (π_1, V_1) is irreducible, Equation (7) and the identity $\pi(g) \cdot V(\sigma) = V(g \cdot \sigma)$ show that the set of σ such that $V_1(\sigma)$ is nonzero must form a single G-orbit.
(2) Corollary 10.1.22 shows that $\mathrm{res}_{G^1} \pi_1$ is semisimple. Since every compact open subgroup of G is contained in G^1, Corollary 7.3.5 gives that $\mathrm{res}_{G^1} \pi_1$ is admissible; by (1) and Lemma 5.1.6, $\mathrm{res}_{G^1} \pi_1$ is of finite length.
(3) Since every element of $\mathbf{X}(G)$ is trivial on G^1, we have that (3b) implies (3a). Since (3a) implies (3c), it is enough to show that (3c) implies (3b).

    Suppose that (3c) is true. From (2) and Schur's lemma, $\mathrm{Hom}_{G^1}(V_2, V_1)$ is a finite-dimensional complex vector space. We let G act on $\mathrm{Hom}_{G^1}(V_2, V_1)$ by

$$g \cdot f = \pi_1(g) \circ f \circ \pi_2(g)^{-1}$$

    for $g \in G$ and $f \in \mathrm{Hom}_{G^1}(V_2, V_1)$. If $g \in G^1$ then $g \cdot f = f$ for all $f \in \mathrm{Hom}_{G^1}(V_2, V_1)$, so the action of G on $\mathrm{Hom}_{G^1}(V_2, V_1)$ factors through the lattice G/G^1. Thus we have a commuting family of invertible operators on a finite-dimensional complex vector space, so by Exercise 10.3.2, there exist a nonzero eigenvector $h \in \mathrm{Hom}_{G^1}(V_2, V_1)$ and a character χ of G/G^1 such that

$$g \cdot h = \chi(g)h$$

    for all $g \in G$. By construction, $h \in \mathrm{Hom}_G(\pi_2 \otimes \chi, \pi_1)$, so the two irreducible representations are equivalent since $h \neq 0$.

■

Exercise 10.3.2. Show that any commuting family of invertible operators on a finite-dimensional complex vector space has a common nonzero eigenvector (use Exercises 2.1.8 and 3.3.6).

Definition 10.3.3. Define an equivalence relation on the irreducible supercuspidal representations of G by setting $\pi_1 \sim \pi_2$ if π_1 is equivalent to $\pi_2 \otimes \chi$ for some $\chi \in \mathbf{X}(G)$. For an irreducible supercuspidal representation $\pi \in \mathfrak{R}(G)$, let $[\pi]$ denote the equivalence class of π under the above relation.

Let (π, V) be an irreducible supercuspidal representation of G. As above, we write

$$\operatorname{res}_{G^1} \pi = \bigoplus_1^m V(\sigma_i)$$

where $V(\sigma_i)$ denotes the σ_i-isotypic component of $\operatorname{res}_{G^1} \pi$, σ_i being an irreducible finite representation of G^1. Let $W \in \mathfrak{R}(G)$. As G^1-modules, we have

$$\operatorname{res}_{G^1} W = e^{\sigma_1} W \oplus e^{\sigma_2} W \oplus \cdots \oplus e^{\sigma_m} W \oplus W'$$

where $W' = \prod_1^m (1 - e^{\sigma_i}) W$ is the unique G^1-complement to $e^{\sigma_1} W \oplus e^{\sigma_2} W \oplus \cdots \oplus e^{\sigma_m} W$. Note that W' can also be characterized as the unique G^1-submodule of W all of whose irreducible subquotients are not equivalent to any of the σ_i.

Proposition 10.3.4. (1) *The G^1-submodule W' is G-invariant.*
(2) *For each i and all $g \in G$, we have $\pi(g) \circ e^{\sigma_i} = e^{(g \cdot \sigma_i)} \circ \pi(g)$.*

Proof. (1) Let $g \in G$. If (τ, U) is an irreducible subquotient of W' then $g \cdot \tau$ is an irreducible subquotient of $\pi(g) W'$, so since $\{\sigma_i\}$ is a G-orbit, there is no i such that $g \cdot \tau$ is equivalent to σ_i. It follows that $e^{\sigma_i} \pi(g) W' = \{0\}$ for all i. By the uniqueness of W', then, we have that $\pi(g) W' = W'$.
(2) Let $w \in W$, and write

$$w = w_1 + \cdots + w_m + w',$$

where $w_i \in e^{\sigma_i} W$ and $w' \in W'$. Fix $i \in \{1, \cdots, m\}$. Let $g \in G$, and suppose that $g \cdot \sigma_i = \sigma_j$, so

$$\pi(g) e^{\sigma_i} \pi(g^{-1}) w = \pi(g) \pi(g^{-1}) w_j = w_j = e^{\sigma_j} w$$

since $\pi(g^{-1}) w_j \in V(\sigma_i)$ and $\pi(g^{-1}) w' \in W'$.

∎

By Proposition 10.3.4, the map

$$e^\pi := \sum_1^m e^{\sigma_i} \colon W \to W$$

is a G-endomorphism. Since $e^{\sigma_i} e^{\sigma_j} = 0$ for $i \neq j$, we have that e^π is idempotent. Thus we have a decomposition $W = e^\pi W \oplus (1 - e^\pi) W$ as G-representations. As before, this shows that

$$\mathfrak{R}(G) = \mathfrak{R}(G)^{[\pi]} \times \mathfrak{R}(G)(\text{no-}[\pi])$$

where $\mathfrak{R}(G)^{[\pi]}$ consists of those representations in $\mathfrak{R}(G)$ all of whose irreducible subquotients are elements of $[\pi]$ and $\mathfrak{R}(G)(\text{no-}[\pi])$ consists of those representations in $\mathfrak{R}(G)$ all of whose irreducible subquotients do not belong to $[\pi]$.

11. SPLITTING $\mathfrak{R}(G)$, II

In this section, G is again the group of k-points of a connected reductive algebraic group defined over k.

11.1. **An application of Bernstein's uniform admissibility theorem to finite irreducible representations.**

Corollary 11.1.1. *Let $K \subset G^1 \subset G$ be a compact open subgroup of G. There exists a compact open subset $S = S(K)$ of G^1 such that for all $(\sigma, V) \in \mathrm{Irr}_f(G^1)$ and all $v \in V$, the support of the function $p_{K,v} : G^1 \to V$ given by*

$$p_{K,v}(g) = e_K \sigma(g) e_K v$$

is contained in S.

Proof. If $K' \subset K$ is another compact open subgroup then the identity $e_K e_{K'} = e_{K'} e_K = e_K$ allows one to show that $\mathrm{supp}(p_{K,v}) \subset \mathrm{supp}(p_{K',v}) \cdot K$. Thus it is enough to show that the corollary is valid for a normal compact open subgroup K of K_0 that has an Iwahori decomposition with respect to $P_\emptyset = M_\emptyset N_\emptyset$. By Corollary **??**, we can choose an $N = N(K)$ such that $\dim_{\mathbb{C}}(V^K) \leq N$ for all $(\sigma, V) \in \mathrm{Irr}_f(G^1)$.

Fix a representation $(\sigma, V) \in \mathrm{Irr}_f(G^1)$. We first show that for all $t \in T^+ \cap G^1$ and all $v \in V$ we have $e_K \sigma(t^N) e_K v = 0$. This is obvious if $V^K = e_K V = \{0\}$, so we may assume that V^K is nonzero. Fix a nonzero $v \in V^K$, so $p_{K,v}(g) = e_K \sigma(g) v$ for all $g \in G^1$. Let $\lambda \in \mathrm{Hom}_{\mathbb{C}}(V^K, \mathbb{C})$, so since (σ, V) is finite,

$$\lambda(\sigma(g)v) = (e_K \lambda)(\sigma(g)v) = \lambda(e_K \sigma(g) v) = 0$$

for all $g \in G$ outside some compact set. Since $\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}}(V^K, \mathbb{C})) \leq N$, there exists an $N' \in \mathbb{Z}_{\geq 1}$ such that $e_K \sigma(t^{(N'-1)}) v \neq 0$ and $e_K \sigma(t^{N'}) v = 0$. We want $N' \leq N$. It suffices to prove that the elements of the set

$$\{e_K \sigma(t^i) v \,|\, 0 \leq i \leq N' - 1\}$$

are linearly independent. Indeed, let $c_0, c_1, \ldots, c_{(N'-1)} \in \mathbb{C}$ such that

$$0 = \sum_{i=0}^{N'-1} c_i \cdot \sigma(e_K t^i e_K) v.$$

By Lemma **??**, we have that for all $1 \leq j \leq N' - 1$,

$$0 = \sigma(e_K t e_K)^j \cdot \sum_{i=0}^{N'-1} c_i \cdot \sigma(e_K t^i e_K) v = \sum_{i=j}^{N'-1} c_{(i-j)} \cdot \sigma(e_K t^i e_K) v.$$

This implies that $c_0 = c_1 = \cdots = c_{(N'-1)} = 0$.

Identifying $T^+ \cap G^1$ with $\mathbb{Z}^\ell_{\geq 0}$ for some ℓ, let $S_N \subset T^+ \cap G^1$ be the finite set

$$S_N = \{(a_1, a_2, \ldots, a_\ell) \in \mathbb{Z}^\ell_{\geq 0} \,|\, a_1, a_2, \ldots, a_\ell \leq N\},$$

and for $1 \leq j \leq \ell$ let $t_j \in S_N$ be the jth unit coordinate $(0, \ldots, 1, \ldots, 0)$. Let $t \in (T^+ \cap G^1) \smallsetminus S_N$, so there is some $1 \leq j \leq \ell$ such that $t \cdot t_j^{(-N)} \in T^+ \cap G^1$. For all $v \in V$ we have

$$\sigma(e_K t e_K)v = \sigma(e_K t t_j^{(-N)} e_K)\sigma(e_K t_j^N e_K)v = 0.$$

Thus $\operatorname{supp}(p_{K,v}) \cap (T^+ \cap G^1) \subset S_N$ for all $v \in V$.

In general, for $g \in G^1$, we can find $w \in \omega$, $t \in T^+ \cap G^1$, and $k_1, k_2 \in K_0$ such that $e_K g e_K = e_K k_1 w t k_2 e_K$. Since k_1, k_2, and w normalize K, they commute with e_K, so $\sigma(e_K g e_K) = 0$ (that is, $\sigma(e_K g e_K) : V \to V$ is the zero map) if and only if $\sigma(e_K t e_K) = 0$. Consequently, for all $v \in V$, the support of the map

$$g \mapsto \sigma(e_K g e_K)v$$

is contained in $S = K_0 \omega S_N K_0$. ∎

Corollary 11.1.2. *Let K be a compact open subgroup of G. There exist only a finite number of inequivalent finite irreducible representations of G^1 with K-fixed vectors.*

Proof. Let $S = S(K)$ be the compact open subset of G^1 given in Corollary 11.1.1. Let $(\pi, V) \in \operatorname{Irr}_f(G^1)$ and choose nonzero $v \in V^K$ and $\lambda \in \widetilde{V}^K$, so for all $g \in G$,

$$m_{\lambda,v}(g) = \lambda(\pi(g)v) = (e_K\lambda)(\pi(g)e_K v) = \lambda(e_K \pi(g) e_K v)$$

and for all $k_1, k_2 \in K$,

$$m_{\lambda,v}(k_1^{-1} g k_2) = (k_1\lambda)(\pi(gk_2)v) = m_{\lambda,v}(g).$$

Thus $m_{v,\lambda} \in C(S//K)$. Since $C(S//K)$ is finite-dimensional and the matrix coefficients of inequivalent finite irreducible representations are linearly independent (Corollary 8.3.4), the corollary follows. ∎

Corollary 11.1.3. *Let K be a compact open subgroup of G. There are only finitely many equivalence classes $[\pi]$ of irreducible supercuspidal representations of G with K-fixed vectors.*

Proof. Since the equivalence class of an irreducible supercuspidal representation is determined by its restriction to G^1, this follows from Corollary 11.1.2. ∎

Remark 11.1.4. Corollary 11.1.3 also holds for discrete series representations of G. That is, up to twisting by an unramified character, there are only finitely many discrete series representations of G with K-fixed vectors.

11.2. **Further splitting the categories $\mathfrak{R}(G)$ and $\mathfrak{R}(G^1)$.**

Definition 11.2.1. Recall from Subsection 10.2 that $\mathfrak{R}_f(G^1)$ denotes the full subcategory of $\mathfrak{R}(G^1)$ consisting of the finite representations, and $\mathfrak{R}_{nf}(G^1)$ denotes the full subcategory of $\mathfrak{R}(G^1)$ consisting of those representations all of whose irreducible subquotients are not finite.

Note that all of the irreducible subquotients of a finite representation are finite. Conversely, if $(\sigma, W) \in \mathfrak{R}(G^1)$ is a representation whose irreducible subquotients are all finite, then clearly

$$W = \bigoplus_{(\pi,V) \in \operatorname{Irr}_f(G^1)} e^\pi W,$$

so (σ, W) is finite.

Corollary 11.2.2. *The category* $\mathfrak{R}(G^1)$ *splits into*

$$\mathfrak{R}(G^1) = \mathfrak{R}_f(G^1) \times \mathfrak{R}_{nf}(G^1).$$

Proof. First note that for $(\pi, V) \in \mathrm{Irr}_f(G^1)$ and $(\sigma, W) \in \mathfrak{R}_{nf}(G^1)$, we have $\mathrm{Hom}_{G^1}(V, W) = \mathrm{Hom}_{G^1}(W, V) = 0$, so by Corollary 10.1.19, the same is true for any $(\pi, V) \in \mathfrak{R}_f(G^1)$.

Let $(\sigma, W) \in \mathfrak{R}(G^1)$. We need to show that there is a unique decomposition

$$W = W_f \oplus W_{nf}$$

where $W_f \in \mathfrak{R}_f(G^1)$ and $W_{nf} \in \mathfrak{R}_{nf}(G^1)$ are subrepresentations of W.

Recall that for $(\pi, V) \in \mathrm{Irr}_f(G^1)$, we have a projection map $e^\pi \colon W \to W$. If K is a compact open subgroup of G that fixes w, then by definition, $e^\pi w = \sigma(e_K^\pi)w$. Thus by Corollary 11.1.2, $e^\pi w = 0$ for all but finitely many π. We may therefore define a G^1-map $e^f \colon W \to W$ by

$$w \longmapsto \sum_{\pi \in \mathrm{Irr}_f(G^1)} e^\pi w.$$

Since $e^{\pi_1} e^{\pi_2} = 0$ for inequivalent $\pi_1, \pi_2 \in \mathrm{Irr}_f(G^1)$, it follows that $e^f e^f = e^f$. Thus

$$W = e^f W \oplus (1 - e^f) W$$

as G^1-modules.

Since $e^\pi e^f = e^f e^\pi = e^\pi$ for $(\pi, V) \in \mathrm{Irr}_f(G^1)$, we have

$$e^f W = \bigoplus_{(\pi, V) \in \mathrm{Irr}_f(G^1)} e^\pi W.$$

It follows that $e^f W$ is a maximal submodule of W whose irreducible subquotients are all finite. As for $(1 - e^f)W$, since $e^\pi(1 - e^f) = (e^\pi - e^\pi e^f) = e^\pi - e^\pi = 0$ for all $(\pi, V) \in \mathrm{Irr}_f(G^1)$, it follows from Lemma 10.1.18 that $(1 - e^f)W$ has no irreducible finite subquotient.

Lastly, we turn to the uniqueness of the decomposition. Suppose that there is another decomposition

$$W = W'_f \oplus W'_{nf}$$

for some subrepresentations $W'_f \in \mathfrak{R}_f(G^1)$ and $W'_{nf} \in \mathfrak{R}_{nf}(G^1)$ of W. Since W'_f is finite, the map e^f acts as the identity on W'_f; thus $W'_f \subset W_f$. On the other hand, $e^f W'_{nf}$ is zero: otherwise, there would exist $(\pi, V) \in \mathrm{Irr}_f(G^1)$ such that $e^\pi W'_{nf} \neq 0$, so by Lemma 10.1.18, W'_{nf} would have an irreducible subquotient isomorphic to (π, V), a contradiction. Thus $W'_{nf} \subset W_{nf}$ as well, so the decomposition is unique. ∎

The fact that $\mathfrak{R}_f(G^1)$ splits (cf. Subsection 10.2) implies that $\mathfrak{R}(G^1)$ splits even farther:

Corollary 11.2.3. *The category* $\mathfrak{R}(G^1)$ *splits into*

$$\mathfrak{R}(G^1) = \mathfrak{R}_{nf}(G^1) \times \prod_{\pi \in \mathrm{Irr}_f(G^1)} \mathfrak{R}(G^1)(\pi).$$

Definition 11.2.4. Let $\mathfrak{R}_{\rm sc}(G)$ denote the full subcategory of $\mathfrak{R}(G)$ consisting of those representations whose every irreducible subquotient is supercuspidal, and let $\mathfrak{R}_{\rm ind}(G)$ denote the full subcategory of $\mathfrak{R}(G)$ consisting of those representations for which no irreducible subquotient is supercuspidal. (This notation is motivated by Corollary 7.3.3.)

Corollary 11.2.5. *The category $\mathfrak{R}(G)$ splits into*

$$\mathfrak{R}(G) = \mathfrak{R}_{\rm sc}(G) \times \mathfrak{R}_{\rm ind}(G).$$

Proof. Let $V \in \mathfrak{R}(G)$. Corollary 11.2.2 gives a unique decomposition

$$\mathrm{res}_{G^1} V = V_f \oplus V_{nf}$$

where $V_f \in \mathfrak{R}_f(G^1)$ and $V_{nf} \in \mathfrak{R}_{nf}(G^1)$. Since this decomposition is unique, G stabilizes V_f and V_{nf}, so $V = V_f \oplus V_{nf}$ as G-modules. By Theorem 7.3.1, an irreducible representation of G is supercuspidal if and only if its restriction to G^1 is finite, so $V_f \in \mathfrak{R}_{\rm sc}(G)$ and $V_{nf} \in \mathfrak{R}_{\rm ind}(G)$. ∎

Corollary 11.2.6. *The category $\mathfrak{R}(G)$ splits into*

$$\mathfrak{R}(G) = \mathfrak{R}_{\rm ind}(G) \times \prod_{\pi \in \mathrm{Irr}_{\rm sc}(G)/\sim} \mathfrak{R}(G)^{[\pi]}.$$

Here $\mathrm{Irr}_{\rm sc}(G)$ is the set of isomorphism classes of irreducible supercuspidal representations of G. Recall that for two irreducible supercuspidal representations π_1, π_2 of G we write $\pi_1 \sim \pi_2$ if there exists a $\chi \in \mathbf{X}(G)$ such that π_1 is equivalent to $\pi_2 \otimes \chi$, and that we write $[\pi]$ for the equivalence class of $\pi \in \mathrm{Irr}_{\rm sc}(G)$.

Proof. By Corollary 11.2.5, it is enough to show

$$\mathfrak{R}_{\rm sc}(G) = \prod_{\pi \in \mathrm{Irr}_{\rm sc}(G)/\sim} \mathfrak{R}(G)^{[\pi]}.$$

Let $V \in \mathfrak{R}_{\rm sc}(G)$. Write

$$\mathrm{res}_{G_0} V = \bigoplus_{(\sigma) \in \mathrm{Irr}_f(G^1)/G} V^{(\sigma)}$$

where the sum is over G-orbits in $\mathrm{Irr}_f(G^1)$ and $V^{(\sigma)}$ denotes the direct summand of $\mathrm{res}_{G^1} V$ whose irreducible G^1-subquotients all belong to the orbit (σ). The G-representation $V^{(\sigma)}$ is supercuspidal, and from Lemma 10.3.1 any two irreducible G-subquotients in $V^{(\sigma)}$ are equivalent with respect to $\sim$.

Corollary 11.2.3 shows that if $V \in \mathfrak{R}(G)^{[\pi_1]}$ and $W \in \mathfrak{R}(G)^{[\pi_2]}$ where $\pi_1 \not\sim \pi_2$ then $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_G(W, V) = 0$. ∎

12. A THEOREM OF HOWE

In this section, G is again the k-points of a connected reductive algebraic group defined over k. Our goal is to prove the following result, due to Howe.

Theorem 12.0.7. *If $(\pi, V) \in \mathfrak{R}(G)$, then π is finitely generated and admissible if and only if π has finite length.*

We need some preparation before we can prove the theorem.

Lemma 12.0.8. *Let P be a standard parabolic subgroup of G with a Levi decomposition $P = MN$, and let K be a normal subgroup of K_0. If $(\pi, V) \in \mathfrak{R}(G)$ is generated by its K-fixed vectors, then V_N is generated by its $(K \cap M)$-fixed vectors.*

Proof. Since $G = PK_0$ and K is a normal subgroup of K_0, we have that V is spanned over $\mathbb{C}$ by

$$\{\pi(p)v \,|\, v \in V^K \text{ and } p \in P\}.$$

Therefore, V_N is spanned over $\mathbb{C}$ by

$$\{(\pi(p)v)_N \,|\, v \in V^K \text{ and } p \in P\}.$$

Writing $p = mn$ for $m \in M$ and $n \in N$, we have $(\pi(p)v)_N = \pi_N(m)(v_N)$. Moreover, if $v \in V^K$ then $v_N \in (V_N)^{K \cap M}$. The lemma follows. ■

Lemma 12.0.9. *Let $K \subset G$ be a compact open subgroup, and let $(\pi, V) \in \mathfrak{R}(G)$ be a representation that has a subquotient which has nonzero K-fixed vectors. Then V has nonzero K-fixed vectors.*

Proof. Let $W_1 \subset W \subset V$ be G-submodules such that $(W/W_1)^K \neq \{0\}$. By Lemma 5.2.1, the sequence

$$0 \longrightarrow W_1^K \longrightarrow W^K \longrightarrow (W/W_1)^K \longrightarrow 0$$

is exact, so $W^K \subset V^K$ is nonzero. ■

Lemma 12.0.10. *Let K be a normal compact open subgroup of K_0 that has an Iwahori factorization with respect to all standard parabolic subgroups of G. Let $(\pi, V) \in \mathfrak{R}(G)$. If V^K generates V, then every supercuspidal subquotient of V has nonzero K-fixed vectors.*

Proof. First let (π', V') be an irreducible supercuspidal subquotient of (π, V). In this case, Corollary 11.2.6 gives us a decomposition

$$V = V^{[\pi']} \oplus V^{\text{no-}[\pi']}$$

as G-representations, where $W := V^{[\pi']} \in \mathfrak{R}(G)^{[\pi']}$ is nonzero. Since V is generated by its K-fixed vectors, W is generated by its K-fixed vectors, so $W^K \neq \{0\}$. Write $\operatorname{res}_{G^1} \pi' = \bigoplus V'(\sigma_i)$, where the sum runs over a simple G-orbit of finite irreducible representations σ_i of G^1, and $V'(\sigma_i)$ is isomorphic to a direct sum of copies of σ_i. We can also write $W = \bigoplus W(\sigma_i)$ as G^1-modules, with the sum running over the same set of σ_i; since $K \subset G^1$, this shows that some σ_i has K-fixed vectors. Thus V' has K-fixed vectors.

Now let (σ, W) be any supercuspidal subquotient of (π, V). Any irreducible subquotient π' of W is an irreducible supercuspidal subquotient of (π, V), so π' has nonzero K-fixed vectors. Thus σ has nonzero K-fixed vectors by Lemma 12.0.9. ■

Corollary 12.0.11. *Let K and (π, V) be as in Lemma 12.0.10. If V^K generates V, then every subquotient of V has nonzero K-fixed vectors.*

Proof. First let (π', V') be an irreducible subquotient of (π, V). If π' is supercuspidal then it has nonzero K-fixed vectors by Lemma 12.0.10, so we may assume that π' is not supercuspidal. Choose a standard parabolic P with Levi decomposition $P = MN$ for which π'_N is supercuspidal. From Lemma 12.0.8 we have that V_N is generated by its $(K \cap M)$-fixed vectors. Since V'_N is a subquotient of V_N, it follows from Lemma 12.0.10 (as applied to the group M and the compact open subgroup $K \cap M \lhd K_0 \cap M$) that $(V'_N)^{(K\cap M)}$ is nonzero. By Jacquet's Lemma (Theorem 6.3.2), we have that $(V')^K$ surjects onto $(V'_N)^{K\cap M}$, so V' has nonzero K-fixed vectors.

Any subquotient (σ, W) of (π, V) has an irreducible subquotient which is either supercuspidal or not supercuspidal. In either case, Lemmas 12.0.9 and 12.0.10 and the above imply that W has nonzero K-fixed vectors. ■

Corollary 12.0.12. *Let K be a normal compact open subgroup of K_0 that has an Iwahori factorization with respect to all standard parabolic subgroups of G. Let $(\pi, V) \in \mathfrak{R}(G)$. If V^K generates V, then every submodule of V is generated by its K-fixed vectors.*

Proof. Let W be a submodule of V, and let W' be the submodule of W generated by W^K. We need to show that $W' = W$. Since $W^K = W'^K$ and taking K-fixed vectors is exact, every subquotient of W/W' has no nonzero K-fixed vectors. Since every subquotient of W/W' is a subquotient of V, it follows from Corollary 12.0.11 that $W = W'$. ■

Proof of Theorem 12.0.7. "$\Rightarrow$": Suppose that V is generated by $v_1, v_2, \ldots, v_r$. There exists a compact open normal subgroup K of K_0 admitting an Iwahori factorization with respect to all standard parabolics such that $v_i \in V^K$ for all i. That is, V is generated by V^K. Since V is admissible, the dimension of V^K is finite.

Let

$$\{0\} = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_s = V$$

be any proper filtration of V. From Corollary 12.0.12, we must have that $V_i^K \subsetneq V_j^K$ for all $i < j$. Consequently, $s \leq \dim_{\mathbb{C}}(V^K) < \infty$, so the length of V is at most $\dim_{\mathbb{C}}(V^K)$.

"$\Leftarrow$": This is true by Exercises 5.2.2 and 5.1.10 and Corollary 7.3.5. ■

13. The Bernstein center for cuspidal components

In this section, G is again the k-points of a connected reductive algebraic group defined over k. Fix an irreducible, supercuspidal $(\pi, V) \in \mathfrak{R}(G)$ and a finite, irreducible $(\sigma, W) \in \mathfrak{R}(G^1)$ occurring in $\mathrm{res}_{G^1} \pi$. In this section, we shall describe both the set $\mathrm{Irr}(\mathfrak{R}(G)^{[\pi]})$ of equivalence classes of simple objects in $\mathfrak{R}(G)^{[\pi]}$ and the center of the category $\mathfrak{R}(G)^{[\pi]}$, defined as follows.

Definition 13.0.13. Let $\mathcal{A}$ be a category. We define the *center* $\mathfrak{z}(\mathcal{A})$ of $\mathcal{A}$ to be the collection $\mathrm{End}(\mathrm{Id}_{\mathcal{A}}) = \mathrm{Hom}(\mathrm{Id}_{\mathcal{A}}, \mathrm{Id}_{\mathcal{A}})$ (cf. Definition 2.5.1). That is, an element z of $\mathfrak{z}(\mathcal{A})$ is a rule which associates to each object $V \in \mathcal{A}$ a map $z_V : V \to V$ such that for any two objects $V_1, V_2 \in \mathcal{A}$ and any morphism $f : V_1 \to V_2$, we have $z_{V_2} \circ f = f \circ z_{V_1}$.

Example 13.0.14. Let G be a unimodular t.d.-group and let $(\pi, V) \in \mathfrak{R}(G)$ be a finite irreducible representation. Then by Lemma 10.1.15, the natural transformation $e^\pi : W \to W$ is in the center of $\mathfrak{R}(G)$.

Remark 13.0.15. When $\mathcal{A}$ is an abelian category and $\mathfrak{z}(\mathcal{A})$ is a set, $\mathfrak{z}(\mathcal{A})$ is a commutative ring with unit. (I am not sure under what conditions $\mathfrak{z}(\mathcal{A})$ is a set.)

Exercise 13.0.16. Let R be a (not necessarily commutative) ring with identity, and let $\mathcal{A}$ be the category of R-modules. Show that $\mathfrak{z}(\mathcal{A})$ is isomorphic to the center of R as rings. (The center of R is the subring of all elements $r \in R$ such that $rs = sr$ for all $s \in R$.)

13.1. **A first categorical equivalence.** Recall that G acts on $\mathfrak{R}(G^1)$ by

$$g \cdot \sigma' := \sigma' \circ \mathrm{Int}(g^{-1})$$

for $(\sigma', W') \in \mathfrak{R}(G^1)$ and $g \in G$, where $\mathrm{Int}(g^{-1}) : G \to G$ is conjugation by g^{-1}.

Definition 13.1.1.

$$G_\sigma := \{g \in G \mid g \cdot \sigma \text{ is equivalent to } \sigma\}.$$

Remark 13.1.2. Since $G^1 Z(G) \leq G_\sigma \leq G$ and $G/G^1 Z(G)$ is finite and abelian, G_σ is a normal subgroup of G and G/G_σ is finite and abelian.

Definition 13.1.3. We let $\mathfrak{R}(G_\sigma)^\sigma$ denote the full subcategory of $\mathfrak{R}(G_\sigma)$ consisting of those representations W for which $\mathrm{res}_{G^1} W$ is σ-isotypic.

Remark 13.1.4. Since the map

$$f \mapsto \sum_{\overline{g} \in G_\sigma/G^1} f(g^{-1}) : \ \mathrm{res}_{G^1}(\text{c-}\mathrm{Ind}_{G^1}^{G_\sigma} \sigma) \longrightarrow \bigoplus_{\overline{g} \in G_\sigma/G^1} (g \cdot \sigma, W)$$

is an isomorphism, it follows that $\text{c-}\mathrm{Ind}_{G^1}^{G_\sigma} \sigma \in \mathfrak{R}(G_\sigma)^\sigma$.

Lemma 13.1.5. *The functors*

$$(\tau, U) \longmapsto \mathrm{Ind}_{G_\sigma}^G(\tau) : \ \mathfrak{R}(G_\sigma)^\sigma \longrightarrow \mathfrak{R}(G)^{[\pi]}$$

and

$$(\pi', V') \longmapsto (\mathrm{res}_{G_\sigma} \pi', e^\sigma V') : \ \mathfrak{R}(G)^{[\pi]} \longrightarrow \mathfrak{R}(G_\sigma)$$

define an equivalence of categories.

Proof. Denote the above functors by α and β, respectively.

First we show that $\beta \circ \alpha$ is naturally isomorphic to the identity. Let $(\tau, U) \in \mathfrak{R}(G_\sigma)^\sigma$, so $\beta \circ \alpha(U) = e^\sigma \mathrm{res}_{G^1} \mathrm{Ind}_{G_\sigma}^G U$. The map

$$f \mapsto \sum_{\overline{g} \in G/G_\sigma} f(g^{-1}) : \ \mathrm{res}_{G_\sigma} \mathrm{Ind}_{G_\sigma}^G U \longrightarrow \bigoplus_{\overline{g} \in G/G_\sigma} (g \cdot \tau, U)$$

is an isomorphism of G_σ-modules, so $f \mapsto f(1)$ is a natural isomorphism $e^\sigma \mathrm{res}_{G_\sigma} \mathrm{Ind}_{G_\sigma}^G U \cong (\tau, U)$.

We now show that $\alpha \circ \beta$ is naturally isomorphic to the identity. Let $(\pi', V') \in \mathfrak{R}(G)^{[\pi]}$. By Frobenius reciprocity, the map

$$e^\sigma \in \mathrm{Hom}_{G_\sigma}(\mathrm{res}_{G_\sigma} V', e^\sigma V') \quad \text{defines a map} \quad E^\sigma \in \mathrm{Hom}_G(V', \mathrm{Ind}_{G_\sigma}^G e^\sigma V')$$

given by

$$E^\sigma(v)(g) = e^\sigma(\pi'(g)v) = \pi'(g) \cdot e^{(g^{-1}\cdot\sigma)}v.$$

Since $\pi'(g^{-1})(e^{(g\cdot\sigma)}V') = e^\sigma V'$, the map $v \mapsto E^\sigma(v)(g^{-1})$ restricts to an isomorphism $e^{(g\cdot\sigma)}V' \cong (g\cdot(e^\sigma\pi'), e^\sigma V')$ for all $g \in G$. This combined with the decomposition

$$\operatorname{res}_{G_\sigma} V' = \bigoplus_{\overline{g}\in G/G_\sigma} e^{g\cdot\sigma}V'$$

and the above identification of $\operatorname{res}_{G_\sigma} \operatorname{Ind}_{G_\sigma}^G e^\sigma V'$ with $\bigoplus_{\overline{g}\in G/G_\sigma}(g\cdot(e^\sigma\pi'), e^\sigma V')$ show that E^σ is a (natural) isomorphism. ∎

Remark 13.1.6. Since $\mathfrak{R}(G)^{[\pi]}$ is naturally equivalent to $\mathfrak{R}(G_\sigma)^\sigma$, the centers $\mathfrak{z}(\mathfrak{R}(G)^{[\pi]})$ and $\mathfrak{z}(\mathfrak{R}(G_\sigma)^\sigma)$ are naturally isomorphic as $\mathbb{C}$-algebras. If we denote the map between centers by $z \mapsto z^\sigma$, then unwinding the definitions in Lemma 13.1.5, we find that $z_U^\sigma(u) = (z_{\operatorname{Ind}_{G_\sigma}^G U}(f_u))(1)$, where $f_u \in \operatorname{Ind}_{G_\sigma}^G U$ is defined by

$$f_u(g) = \begin{cases} \sigma(g)u & \text{if } g \in G_\sigma \\ 0 & \text{if } g \notin G_\sigma. \end{cases}$$

Also note that for all $\psi \in \mathbf{X}(G)$ we have $z^\sigma_{e^\sigma(\pi\otimes\psi)} = e^\sigma z_{\pi\otimes\psi} = z(\pi\otimes\psi)\cdot \operatorname{Id}_{e^\sigma(\pi\otimes\psi)}$ where $z(\pi\otimes\psi) \in \mathbb{C}$ is the scalar by which z acts on $\pi\otimes\psi$.

13.2. **A second categorical equivalence.** Unfortunately, it is **not** true (in general) that there exists a representation of G_σ whose restriction to G^1 is σ. However, there is always a projective representation $G_\sigma \to \mathrm{PGL}(W)$ lifting the *projectivization* of σ (that is, the composition of $\sigma : G^1 \to \mathrm{GL}(W)$ with the projection $\mathrm{GL}(W) \to \mathrm{PGL}(W)$), as follows: by definition, $g^{-1}\cdot\sigma \cong \sigma$ when $g \in G_\sigma$, so there is a $P(g) \in \mathrm{GL}(W)$ such that

$$g^{-1}\cdot\sigma = P(g)\sigma P(g)^{-1};$$

by Schur's lemma, the operator $P(g)$ is determined up to an element of $\mathbb{C}^\times$, so its image in $\mathrm{PGL}(W)$ is well-defined. We therefore have a homomorphism $P : G_\sigma \to \mathrm{PGL}(W)$ which extends the projectivization of σ (recall that $g\cdot\sigma$ is defined to be $\sigma\circ\operatorname{Int}(g^{-1})$). There may not exist a way to normalize $P(g)$ so that $P(hg) = P(h)P(g)$ in $\mathrm{GL}(W)$ for all $h, g \in G_\sigma$, since P would extend σ to G_σ in that case.

Definition 13.2.1. Define a subgroup $\widetilde{G}_\sigma$ of $G_\sigma \times \mathrm{GL}(W)$ by

$$\widetilde{G}_\sigma := \{(g, P) \mid g \in G_\sigma \text{ and } P \in \mathrm{GL}(W) \text{ such that } g^{-1}\cdot\sigma = P\sigma P^{-1}\}.$$

We do not endow $\widetilde{G}_\sigma$ with a topology.

Remark 13.2.2. (1) The group $\widetilde{G}_\sigma$ is a central extension

$$1 \to \mathbb{C}^\times \to \widetilde{G}_\sigma \to G_\sigma \to 1$$

of G_σ by $\mathbb{C}^\times$, where the first map is $z \mapsto (1, z)$ and the second is projection onto the first factor.

(2) The map

$$g_1 \mapsto (g_1, \sigma(g_1)) : \; G^1 \to \widetilde{G}_\sigma$$

is an embedding of G^1 into $\widetilde{G}_\sigma$. It follows that the representation $(\widetilde{\sigma}, \widetilde{W})$ of $\widetilde{G}_\sigma$ given by $\widetilde{W} = W$ and $\widetilde{\sigma}(g, P)w = Pw$ is an extension of σ, so it is irreducible.

Definition 13.2.3. Let $\mathcal{B}$ be the category of representations of $\widetilde{G}_\sigma$ on which $G^1 \hookrightarrow \widetilde{G}_\sigma$ acts trivially and each $z \in \mathbb{C}^\times \hookrightarrow \widetilde{G}_\sigma$ acts by z^{-1}.

Lemma 13.2.4. *The categories $\mathcal{B}$ and $\mathfrak{R}(G_\sigma)^\sigma$ are equivalent.*

Proof. This is a glorified version of the standard result that the category of $\mathbb{C}$-vector spaces is equivalent to the category whose objects are the (σ, W)-isotypic semisimple representations of G^1; this equivalence is realized by the functors $V \mapsto V \otimes W$ (V a $\mathbb{C}$-vector space) and $W' \mapsto \mathrm{Hom}_{G^1}(W, W')$ (W' a representation of G^1).

Define a functor $\alpha \colon \mathcal{B} \to \mathfrak{R}(G_\sigma)^\sigma$ by $\alpha(\widetilde{\tau}, \widetilde{U}) = (\tau, \widetilde{W} \otimes_{\mathbb{C}} \widetilde{U})$, where

$$\tau(g_\sigma)(\widetilde{w} \otimes \widetilde{u}) := P\widetilde{w} \otimes \widetilde{\tau}(g_\sigma, P)\widetilde{u}$$

for any element $(g_\sigma, P) \in \widetilde{G}_\sigma$ in the fiber of g_σ. Since any other element in the fiber of g_σ has the form (g_σ, zP) for some $z \in \mathbb{C}^\times$, the representation τ is well-defined. Finally, for any $g_1 \in G^1$ we have

$$\tau(g_1)(\widetilde{w} \otimes \widetilde{u}) = \sigma(g_1)\widetilde{w} \otimes \widetilde{\tau}(g_1, \sigma(g_1))\widetilde{u} = \sigma(g_1)\widetilde{w} \otimes \widetilde{u},$$

so our construction indeed yields an object in $\mathfrak{R}(G_\sigma)^\sigma$.

In the opposite direction, define a functor $\beta \colon \mathfrak{R}(G_\sigma)^\sigma \to \mathcal{B}$ by

$$\beta(\tau, U) = (\widetilde{\tau}, \mathrm{Hom}_{G^1}(\widetilde{W}, U)),$$

where, for $f \in \mathrm{Hom}_{G^1}(\widetilde{W}, U)$, we define

$$(\widetilde{\tau}(g_\sigma, P)f)\widetilde{w} := \tau(g_\sigma)f(P^{-1}\widetilde{w})$$

for $(g_\sigma, P) \in \widetilde{G}_\sigma$. To verify that this construction generates an object of $\mathcal{B}$, we note that

$$(\widetilde{\tau}(1, z)f)(\widetilde{w}) = \tau(1)f(z^{-1}\widetilde{v}) = z^{-1}f(\widetilde{w})$$

for $z \in \mathbb{C}^\times$ and

$$(\widetilde{\tau}(g_1, \sigma(g_1))f)(\widetilde{w}) = \tau(g_1)f(\sigma(g_1)^{-1}\widetilde{w}) = f(\sigma(g_1)\sigma(g_1)^{-1}\widetilde{w}) = f(\widetilde{w})$$

for $g_1 \in G^1$ and $(g_\sigma, P) \in \widetilde{G}_\sigma$.

It is left to the reader to verify that $\alpha \circ \beta$ and $\beta \circ \alpha$ are naturally isomorphic to the identity functors. ∎

Remark 13.2.5. A representation $(\pi', V') \in \mathfrak{R}(G)^{[\pi]}$ corresponds to $\mathrm{Hom}_{G^1}(\widetilde{W}, e^\sigma V') \in \mathcal{B}$ under the equivalences of categories given in Lemma 13.1.5 and the proof of Lemma 13.2.4.

Remark 13.2.6. Since $\mathfrak{R}(G_\sigma)^\sigma$ is equivalent to $\mathcal{B}$, $\mathfrak{z}(\mathfrak{R}(G_\sigma)^\sigma)$ is naturally isomorphic to $\mathfrak{z}(\mathcal{B})$ as a $\mathbb{C}$-algebra. If we denote the map between centers by $z^\sigma \mapsto z^{\mathcal{B}}$, then for all $\tilde{w} \in \widetilde{W}$ we have

$$\tilde{w} \otimes z^{\mathcal{B}}_{\widetilde{U}}(\tilde{u}) = z^{\sigma}_{\widetilde{W} \otimes \widetilde{U}}(\tilde{w} \otimes \tilde{u}).$$

Also note that for all $\psi \in \mathbf{X}(G)$ we have

$$z^{\mathcal{B}}_{\mathrm{Hom}_{G^1}(\widetilde{W}, e^\sigma(\pi \otimes \psi))} = z(\pi \otimes \psi) \cdot \mathrm{Id}_{\mathrm{Hom}_{G^1}(\widetilde{W}, e^\sigma(\pi \otimes \psi))}$$

where $z \mapsto z^\sigma \mapsto z^{\mathcal{B}}$.

13.3. **An algebraic structure for** $\mathrm{Irr}(\mathfrak{R}(G)^{[\pi]})$**.** As noted above, we have a central extension

$$1 \to \mathbb{C}^\times \to \widetilde{G}_\sigma \to G_\sigma \to 1.$$

which gives us a central extension

$$1 \to \mathbb{C}^\times \to \widetilde{G}_\sigma / G^1 \to G_\sigma / G^1 \to 1.$$

Let $\widetilde{M_\sigma} = \widetilde{G}_\sigma / G^1$, so the category $\mathcal{B}$ can be described as the category of $\widetilde{M_\sigma}$-modules on which each $z \in \mathbb{C}^\times$ acts by z^{-1}.

Lemma 13.3.1. *If $(\widetilde{\pi}, \widetilde{V})$ is an irreducible representation of $\widetilde{M_\sigma}$ on which $z \in \mathbb{C} \hookrightarrow \widetilde{M_\sigma}$ acts by z^{-1}, then the natural map*

$$\mathbb{C} \to \mathrm{End}_{\widetilde{M_\sigma}}(\widetilde{V})$$

is an isomorphism.

Proof. By Remark 2.1.6, it suffices to show that the dimension of $\widetilde{V}$ is countable. Let $\widetilde{v} \in \widetilde{V}$ be nonzero, so $\{\widetilde{\pi}(m)\widetilde{v} \ : \ m \in \widetilde{M_\sigma}\}$ spans $\widetilde{V}$. Since $\mathbb{C}^\times \hookrightarrow \widetilde{M_\sigma}$ acts by scalars and since $\widetilde{M_\sigma}/\mathbb{C}^\times \cong G_\sigma/G^1$ is countable, the result follows. ∎

Let $\widetilde{C}$ be the center of $\widetilde{M_\sigma}$, so by Lemma 13.3.1, $\widetilde{C}$ acts by a character on any irreducible representation of $\widetilde{M_\sigma}$ in the category $\mathcal{B}$. Let $\mathrm{pr}_1 : \widetilde{M_\sigma} \to G_\sigma/G^1$ be the projection map, and let $M := G/G^1$, $M_\sigma := G_\sigma/G^1$, and $C := \mathrm{pr}_1(\widetilde{C})$. Then we have

$$Z(G)G^1/G^1 \leq C \leq M_\sigma \leq M,$$

so $\widetilde{M_\sigma}/\widetilde{C} \cong M_\sigma/C$ is finite and abelian.

Lemma 13.3.2. *For every homomorphism $\chi : \widetilde{C} \to \mathbb{C}^\times$ for which $\chi(z) = z^{-1}$ for all $z \in \mathbb{C}^\times$, there exists a unique irreducible representation of $\widetilde{M_\sigma}$ having (nontrivial) central character χ.*

Proof. Consider the central extension

$$1 \to \widetilde{C} \to \widetilde{M_\sigma} \to \widetilde{M_\sigma}/\widetilde{C} \to 1.$$

As $\widetilde{M_\sigma}/\widetilde{C}$ is finite and abelian, from Section 3.5.1 it is enough to show that the bimultiplicative form

$$\langle \cdot, \cdot \rangle : \widetilde{M_\sigma}/\widetilde{C} \times \widetilde{M_\sigma}/\widetilde{C} \to \mathbb{C} \quad \text{given by} \quad \langle \overline{m}_1, \overline{m}_2 \rangle = \chi((m_1, m_2)) = \chi(m_1 m_2 m_1^{-1} m_2^{-1})$$

is nondegenerate. Let $(g_i, P_i) \in \widetilde{G}_\sigma$ be a lift of m_i for $i = 1, 2$. Since the derived group of G is a subgroup of G^1, we have $(g_1, g_2) \in G^1$, so since

$$\sigma((g_1, g_2)) \cdot \sigma \cdot \sigma((g_1, g_2))^{-1} = (g_1, g_2)^{-1} \cdot \sigma = (P_1, P_2)\sigma(P_1, P_2)^{-1},$$

we have $(P_1, P_2) = z(g_1, g_2) \cdot \sigma((g_1, g_2))$ for some $z(g_1, g_2) \in \mathbb{C}^\times$. Thus $(m_1, m_2) = z(g_1, g_2) \in \mathbb{C}^\times \hookrightarrow \widetilde{M}_\sigma$, so $\langle \overline{m}_1, \overline{m}_2 \rangle = z(g_1, g_2)^{-1}$; in particular, $z(g_1, g_2)$ does not depend on the choice of lifts of $\overline{m}_1$ and $\overline{m}_2$. If $(g_1, P_1) \in \widetilde{G}_\sigma$ has the property that $z(g_1, g_2) = 1$ for all $(g_2, P_2) \in \widetilde{G}_\sigma$, then the commutator $((g_1, P_1), (g_2, P_2))$ in $\widetilde{G}_\sigma$ is contained in the image of G^1, so the image of (g_1, P_1) in $\widetilde{M}_\sigma$ is contained in $\widetilde{C}$. This shows that $\langle \cdot, \cdot \rangle$ is nondegenerate. ∎

Corollary 13.3.3. *Let $\psi, \psi' \in \mathbf{X}(G)$. The representation $\pi \otimes \psi$ is equivalent to $\pi \otimes \psi'$ if and only if $\operatorname{res}_C \psi = \operatorname{res}_C \psi'$.*

Proof. Let χ and χ' be the central characters of the irreducible representations

$$\operatorname{Hom}_{G^1}(\widetilde{W}, e^\sigma(\pi \otimes \psi)) \text{ and } \operatorname{Hom}_{G^1}(\widetilde{W}, e^\sigma(\pi \otimes \psi'))$$

of $\widetilde{M}_\sigma$, respectively. It is clear that if $(g, P) \in \widetilde{G}_\sigma$ is a lift of an element of $\widetilde{C}$ then $\chi'(g, P) = \chi(g, P) \cdot \psi(g)^{-1}\psi'(g)$, so $\chi = \chi'$ if and only if $\operatorname{res}_C \psi = \operatorname{res}_C \psi'$. The result now follows from Lemmas 13.1.5, 13.2.4, and 13.3.2 (and cf. Remark 13.2.5). ∎

Since M/C is finite, the group

$$F := \{\psi \in \mathbf{X}(G) \mid \operatorname{res}_C \psi = 1\}$$

is finite. Let $\mathbf{T}$ denote the $\mathbb{C}$-torus for which $\mathbf{T}(\mathbb{C}) = \operatorname{Hom}(M, \mathbb{C}^\times) = \mathbf{X}(G)$. Explicitly, since $M \cong \mathbb{Z}^n$ is a lattice, $\operatorname{Hom}(M, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^n$, so $\mathbf{T} \cong (\mathbf{G}_m)^n = \operatorname{Spec}(\mathbb{C}[T_1, T_1^{-1}, \dots, T_n, T_n^{-1}])$. Corollary 13.3.3 shows that $\operatorname{Irr}(\mathfrak{R}(G)^{[\pi]})$ is a principal homogeneous space for the torus $(\mathbf{T}/F)(\mathbb{C}) = \mathbf{X}(G)/F = \operatorname{Hom}(C, \mathbb{C}^\times)$ (recall that $C \supset Z(G)G^1/G^1$ is a full-rank sublattice of the lattice M). In this way, we endow $\operatorname{Irr}(\mathfrak{R}(G)^{[\pi]})$ with the structure of a complex algebraic variety.

13.4. A realization of $\mathfrak{z}(\mathfrak{R}(G)^{[\pi]})$.

Here we present a description of the center of $\mathfrak{R}(G)^{[\pi]}$.

Lemma 13.4.1. *The map*

$$z \mapsto (\pi' \mapsto z(\pi'))$$

identifies $\mathfrak{z}(\mathfrak{R}(G)^{[\pi]})$ with the $\mathbb{C}$-algebra of regular functions on the algebraic variety $\operatorname{Irr}(\mathfrak{R}(G)^{[\pi]})$.

Proof. We know that $\mathfrak{z}(\mathfrak{R}(G)^{[\pi]})$ is natural isomorphic to $\mathfrak{z}(\mathcal{B})$ as a $\mathbb{C}$-algebra, so we begin by studying $\mathfrak{z}(\mathcal{B})$. Let $\mathbb{C}[\mathbb{C}^\times \hookrightarrow \widetilde{M}_\sigma]$ denote the group algebra of $\widetilde{M}_\sigma$, with $z \in \mathbb{C}^\times \hookrightarrow \widetilde{M}_\sigma$ identified with $z^{-1} \in \mathbb{C}$. Thus the category $\mathcal{B}$ is naturally equivalent to the category of $\mathbb{C}[\mathbb{C}^\times \hookrightarrow \widetilde{M}_\sigma]$-modules. As $\mathbb{C}[\mathbb{C}^\times \hookrightarrow \widetilde{M}_\sigma]$ has a unit element, by Exercise 13.0.16, $\mathfrak{z}(\mathcal{B})$ is naturally isomorphic as a $\mathbb{C}$-algebra to the center of $\mathbb{C}[\mathbb{C}^\times \hookrightarrow \widetilde{M}_\sigma]$, which is $\mathbb{C}[\mathbb{C}^\times \hookrightarrow \widetilde{C}]$.

The natural surjection $\widetilde{C} \to C$ has kernel $\mathbb{C}^\times \hookrightarrow \widetilde{C}$, so it induces an isomorphism

$$\mathbb{C}[\mathbb{C}^\times \hookrightarrow \widetilde{C}] \to \mathbb{C}[C]$$

of $\mathbb{C}$-algebras, where $\mathbb{C}[C]$ is the group algebra of C. As $\mathbb{C}[C]$ is the ring of regular functions on $\mathbf{T}/F$, each element of $\mathbb{C}[C]$ is completely determined by its values on $(\mathbf{T}/F)(\mathbb{C})$. Since

$\mathrm{Irr}(\mathfrak{R}(G)^{[\pi]})$ is a principal homogeneous space for $(\mathbf{T}/F)(\mathbb{C})$, by Remarks 13.1.6 and 13.2.6, the map $z \mapsto (\psi \mapsto z(\pi \otimes \psi))$ identifies $\mathfrak{z}(\mathfrak{R}(G)^{[\pi]})$ with the ring of regular functions on $\mathrm{Irr}(\mathfrak{R}(G)^{[\pi]})$.

■

Warning: What follows has not been edited or read seriously.

13.5. **The B-Schur lemma and progenerators.** Put $B := \mathbb{C}[G/G^1]$, the ring of regular functions on $\mathbf{X}(G)$. We define the universal character $\chi_{\mathsf{un}}\colon G/G^1 \to B$ by $\chi_{\mathsf{un}}(g) = b_g$ where $b_g(\chi) = \chi(g)$ for $\chi \in \mathbf{X}(G)$. However, it is perhaps better to think of χ_{un} as the regular representation, that is, as a map from G/G^1 to $\mathrm{End}(B)$ where $\chi_{\mathsf{un}}(g)b = bb_g$.

We put $V_B := V \otimes_{\mathbb{C}} B$. This is a (G, B)-module. The action of G is given by $\pi_B = \pi \otimes \chi_{\mathsf{un}}$:

$$\pi_B(g)(v \otimes b) = \pi(g)v \otimes bb_g.$$

The action of B on V_B is given by translation: $b'(v \otimes b) = b \otimes b'b$. Note that if K is a compact open subgroup of G, then V_B^K is the free B-module $V^K \otimes B$.

For $\psi \in \mathbf{X}(G)$ we have the G-module morphism

$$\mathrm{sp}_\psi\colon (\pi_B, V \otimes B) \to (\pi \otimes \psi, V)$$

which sends $v \otimes b$ to $b(\psi)v$. The kernel of this map is $\mathfrak{m}_\psi V_B$ where $\mathfrak{m}_\psi$ is the maximal ideal

$$\{b \in B : \ b(\psi) = 0\}$$

in B.

Lemma 13.5.1. *If $T \in \mathrm{End}_{(B,G)}(V_B)$, then there exists $b_T \in B$ such that T acts on V_B by multiplication by b_T.*

Proof. Since sp_ψ is a G-map, T induces a G-morphism $T_\psi\colon (\pi \otimes \psi, V) \to (\pi \otimes \psi, V)$. As $\pi \otimes \psi$ is irreducible, from Schur's lemma we have that T_ψ is a scalar multiple of Id_V.

Fix a compact open subgroup K of G. Since T commutes with the action of G, we have a map

$$T(K)\colon V^K \otimes B \to V^K \otimes B$$

which can be represented by a square matrix with entries in B.

However, for all $\psi \in \mathbf{X}(G)$, we have $T_\chi(K)\colon V^K \to V^K$ is given by scalar multiplication. As B is the ring of regular functions on $\mathbf{X}(G)$, we conclude that there is a $b \in B$ so that $T(K)$ is scalar multipliciation by b. By varying K, the lemma follows. ∎

Lemma 13.5.2. *The representation $(\pi_B, V_B) \in \mathfrak{R}(G)^{[\pi]}$ is a progenerator for $\mathfrak{R}(G)^{[\pi]}$.*

An object $\Pi \in \mathfrak{R}(G)^{[\pi]}$ is called a *progenerator* provided that Π is projective and finitely generated and every object in $\mathfrak{R}(G)^{[\pi]}$ occurs as a quotient of a direct sum of copies of Π.

Proof. After fixing a set of representatives for G/G^1 we have

$$\pi \otimes \mathbb{C}[G/G^1] \cong \text{c-Ind}_{G^1}^G(\mathrm{res}_{G^1} \pi))$$

under the G-equivariant map $f \mapsto \sum_{G/G^1} f(g) \otimes \bar{g}$.

Set $\Pi := \text{c-Ind}_{G^1}^G(\mathrm{res}_{G^1} \pi)$. Since $\mathrm{res}_{G^1} \pi$ has finite length, Π is finitely generated. We now show that Π is projective, that is,

$$X \mapsto \mathrm{Hom}_G(\Pi, X)$$

is exact on $\mathfrak{R}(G)^{[\pi]}$. Note that

$$\begin{aligned}\operatorname{Hom}_G(\Pi, X) &= \operatorname{Hom}_G(\widetilde{X}, \widetilde{\Pi}) \\ &= \operatorname{Hom}_G(\widetilde{X}, \text{c-Ind}_{G^1}^G \operatorname{res}_{G^1} \tilde{\pi}) \\ &= \operatorname{Hom}_{G^1}(\operatorname{res}_{G^1} \widetilde{X}, \operatorname{res}_{G^1} \tilde{\pi}) \\ &= \operatorname{Hom}_{G^1}(\operatorname{res}_{G^1} \pi, \operatorname{res}_{G^1} X).\end{aligned}$$

As $\operatorname{res}_{G^1} X$ is semisimple for every object in $\mathfrak{R}(G)^{[\pi]}$, we have that Π is projective.

Finally, to see that Π is a generator, it is enough to show that given $X \in \mathfrak{R}(G)^{[\pi]}$, there is a nonzero map from Π to X. Let $X' \subset X$ be a finitely generated G-subspace of X. Any irreducible subquotient of X' looks like $\pi \otimes \psi$ for some $\psi \in \mathbf{X}(G)$. Since Π is projective and $\operatorname{sp}_\psi \Pi = \pi \otimes \psi$, we conclude that there is a nonzero map from Π to X'. Thus $\operatorname{Hom}_G(\Pi, X)$ is nonzero. ■

14. Casselman's perfect pairing

14.1. **Renormalization.** We have discussed the material in this subsection, but the details of the proof have not yet coalesced – essentially, you need a right invariant measure on $P \setminus G$ to make this all work.

We re-normalize induction and Jacqueting as follows. If $P = MN$ is a parabolic subgroup of G and $\sigma \in \mathfrak{R}(M)$ while $\pi \in \mathfrak{R}(G)$, then

$$i_P^G \sigma = \operatorname{Ind}_P^G \delta_P^{1/2} \sigma$$

and

$$r_P^G \pi = \delta_P^{-1/2} \pi_N.$$

We have a "new" version of Frobenius reciprocity:

$$\operatorname{Hom}_G(\pi, i_P^G \sigma) = \operatorname{Hom}_M(r_P^G \pi, \sigma).$$

It is also true that

$$(i_P^G(\sigma))\tilde{} \cong i_P^G(\tilde{\sigma}).$$

14.2. **Statement of the result.** Suppose $\Theta \subset \Delta$. We set $P = P_\theta$, $M = M_\theta$, $N = N_\theta$, and $T_M = T_\theta$. For all $\varepsilon > 0$ we set

$$T_M^+(\varepsilon) := \{t \in T_\theta^+ \mid |\alpha(t)| < \varepsilon \text{ for all } \alpha \in \Delta \setminus \theta\}.$$

Let $\bar{P} = M\bar{N}$ denote the parabolic subgroup opposite $P = MN$.

Example 14.2.1. For example, for $\mathrm{GL}_n(k)$ and $P = MN$ corresponding to the partition $(n_1, n_2, \ldots, n_\ell)$ of n, an element of $T_M^+(\varepsilon)$ looks like

$$\operatorname{diag}(\underbrace{\varpi^{k_1}, \ldots, \varpi^{k_1}}_{n_1}, \underbrace{\varpi^{k_2}, \ldots, \varpi^{k_2}}_{n_2}, \ldots, \underbrace{\varpi^{k_{\ell-1}}, \ldots, \varpi^{k_{\ell-1}}}_{n_{\ell-1}}, \underbrace{\varpi^{k_\ell}, \ldots, \varpi^{k_\ell}}_{n_\ell})$$

with $(k_i - k_{i+1}) > -\log_q(\varepsilon)$ for $1 \le i \le (\ell - 1)$.

Theorem 14.2.2 (Casselman's perfect pairing)**.** *Suppose $(\pi, V) \in \mathfrak{R}(G)$ is admissible. There exists a unique M-invariant nondegenerate bilinear pairing $\langle\, ,\, \rangle_P$ on $r_P^G V \times r_{\bar{P}}^G \widetilde{V}$ satisfying the following condition:*

For all $(v, \lambda) \in V \times \widetilde{V}$ there exists an $\varepsilon > 0$ such that for all $t \in T_M^+(\varepsilon)$ we have

$$\langle r_P^G(t) j_P(v), j_{\bar{P}}(\lambda)\rangle_P = \delta_P^{-1/2}(t) \cdot \lambda(\pi(t)v).$$

Here $j_P \colon V \to V_N$ is the quotient map.

Before beginning the proof of this theorem, we prove some straightforward consequences.

Corollary 14.2.3. *Suppose $(\pi, V) \in \mathfrak{R}(G)$ is admissible. We have*

$$(r_P^G V)\widetilde{\ } \cong r_{\bar{P}}^G \widetilde{V}.$$

Proof. Since V is admissible, both $r_P^G V$ and $\widetilde{V}$ are admissible. Consequently, both $(r_P^G V)\widetilde{\ }$ and $r_{\bar{P}}^G \widetilde{V}$ are admissible.

Since $\langle\, ,\, \rangle_P$ is nondegenerate and M-invariant, for each compact open subgroup K of M the M-equivariant map

$$\mu \mapsto (w \mapsto \langle w, \mu\rangle_P)$$

from $r_{\bar{P}}^G \widetilde{V}$ to $\operatorname{Hom}(r_P^G V, \mathbb{C})$ induces an injective map from $(r_{\bar{P}}^G \widetilde{V})^K$ to $\operatorname{Hom}((r_P^G V)^K, \mathbb{C})$. However, since $(r_P^G V)\widetilde{\ }$ and $r_{\bar{P}}^G \widetilde{V}$ are admissible, the induced map must be surjective. Moreover, the fact that they are admissible also tells us that

$$\operatorname{Hom}((r_P^G V)^K, \mathbb{C}) \cong ((r_P^G V)\widetilde{\ })^K.$$

Since K was arbitrary, the result follows. ∎

Corollary 14.2.4. *Suppose $(\sigma, W) \in \mathfrak{R}(M)$ and $(\pi, V) \in \mathfrak{R}(G)$ are admissible representations. We have*

$$\operatorname{Hom}_G(i_P^G \sigma, V) \cong \operatorname{Hom}_M(\sigma, r_{\bar{P}}^G V).$$

Proof. Since (π, V) and (σ, W) are both admissible, we have $\widetilde{\widetilde{V}} \cong V$ and $\widetilde{\widetilde{W}} \cong W$. Thus

$$\begin{aligned}
\operatorname{Hom}_M(\sigma, r_{\bar{P}}^G V) &= \operatorname{Hom}_M((r_{\bar{P}}^G V)\widetilde{\ }, \widetilde{\sigma}) \\
&= \operatorname{Hom}_M(r_P^G \widetilde{V}, \widetilde{\sigma}) \\
&= \operatorname{Hom}_G(\widetilde{V}, i_P^G \widetilde{\sigma}) \\
&= \operatorname{Hom}_G(\widetilde{V}, (i_P^G \sigma)\widetilde{\ }) \\
&= \operatorname{Hom}_G(i_P^G \sigma, V).
\end{aligned}$$

∎

14.2.1. *A proof of uniqueness.* Let $\mathcal{S}$ denote the set of $\mathbb{C}$-valued sequences on $\mathbb{Z}_{\geq 0}$. We define the translation operator

$$T\colon \mathcal{S} \to \mathcal{S}$$

by $(Ts)(n) = s(n+1)$ for $n \in \mathbb{Z}_{\geq 0}$ and $s \in \mathcal{S}$.

Definition 14.2.5. A sequence $s \in \mathcal{S}$ is called T-*finite* provided that

$$\mathcal{S}_s := \langle T^m s \mid m \in \mathbb{Z}_{\geq 0}\rangle$$

is finite dimensional.

If $s \in \mathcal{S}$ is T-finite, then there exists a finite indexing set I and $(a_i, d_i) \in \mathbb{C} \times \mathbb{Z}_{\geq 0}$ indexed by $i \in I$ so that

$$\mathcal{S}_s = \bigoplus_I \ker(\mathrm{res}_{\mathcal{S}_s}(T - a_i)^{d_i}).$$

In other words, $\mathcal{S}_s$ is a direct sum of generalized eigenspaces for the action of T on $\mathcal{S}_s$.

Example 14.2.6. If $a \in \mathbb{C}$ and $d = 1$, then $s = (a, a^2, a^3, a^4, \ldots)$ is an element of the one-dimensional generalized eigenspace $\ker(T - a)$.

Exercise 14.2.7. Show that an element s of the generalized eigenspace $\ker(T - a)^d$ can be characterized as follows. If $a = 0$, then $s(n) = 0$ for all $n \geq d$. If $a \neq 0$, then $s(n) = p(n) \cdot a^n$, where p is a polynomial of degree less than d.

If $s \in \mathcal{S}$ is T-finite, then we can write

$$s = s_0 + s^0$$

where $s_0(n) = 0$ for all n sufficiently large and

$$s^0(n) = \sum_I p_i(n) \cdot a_i^n$$

for a finite indexing set I. Here the p_i are polynomials indexed by I, and the $a_i \in \mathbb{C}^\times$ are indexed by I.

Lemma 14.2.8. *Fix $(\bar{v}, \bar{\lambda}) \in r_P^G V \times r_{\bar{P}}^G \widetilde{V}$. For $t \in T_M$, the sequence*

$$s(n) := \langle r_P^G(t^n)\bar{v}, \bar{\lambda}\rangle_P$$

is T-finite. Moreover, each generalized eigenvalue for the action of T on $\mathcal{S}_s$ is nonzero.

Proof. Fix a compact open subgroup K of M so that $\bar{v}$ is K-fixed. Since $r_P^G V$ is admissible, the space $(r_P^G V)^K$ is finite dimensional. Since $t \in T_M$, we have t is in the center of M and so $r_P^G(t)\bar{v} \in (r_P^G V)^K$ — in fact, the action of $r_P^G(t)$ on $(r_P^G V)^K$ is invertible.

Consider the T-module map from $(r_P^G V)^K$ to $\mathcal{S}$ which sends $\bar{v}'$ to $s_{\bar{v}'}$ where $s_{\bar{v}'}(n)$ is given by

$$\langle r_P^G(t^n)\bar{v}', \bar{\lambda}\rangle_P.$$

(Here T acts through t on $(r_P^G V)^K$.) Since $(r_P^G V)^K$ is finite dimensional, $\mathcal{S}_s$ is finite dimensional. Since $r_P^G(t)$ acts invertibly on $(r_P^G V)^K$, the generalized eigenvalues for the action of T on $\mathcal{S}_s$ are nonzero. ∎

A proof of the uniqueness of the pairing. Suppose $\langle\ ,\ \rangle'_P$ is another M-invariant nondegenerate pairing satisfying the condition of Theorem 14.2.2. Fix $(v,\lambda) \in V \times \widetilde{V}$. It will be enough to show

$$\langle j_P(v), j_{\bar{P}}\lambda\rangle_P = \langle j_P(v), j_{\bar{P}}\lambda\rangle'_P.$$

Since both pairings satisfy the condition of Theorem 14.2.2, there is an $\varepsilon > 0$ such that for all $t \in T_M^+(\varepsilon)$ we have

$$\langle r_P^G(t) j_P(v), j_{\bar{P}}\lambda\rangle_P = \langle r_P^G(t) j_P(v), j_{\bar{P}}\lambda\rangle'_P.$$

Fix $t \in T_M^+(\varepsilon)$. For $n \in \mathbb{Z}_{\geq 0}$ define

$$s(n) := \langle r_P^G(t^n) j_P(v), j_{\bar{P}}\lambda\rangle_P \text{ and } s'(n) := \langle r_P^G(t^n) j_P(v), j_{\bar{P}}\lambda\rangle'_P.$$

By hypothesis, we have $s(n) = s'(n)$ for all $n > 0$. We need to show that $s(0) = s'(0)$.

From Lemma 14.2.8 we can write

$$s(n) = \sum_I p_i(n) a_i^n \text{ and } s'(n) = \sum_{I'} p'_{i'}(n)(a'_{i'})^n$$

for a finite indexing set I (resp., I'), polynomials p_i indexed by I (resp. $p'_{i'}$ indexed by I'), and $a_i \in \mathbb{C}^\times$ indexed by I (resp. $a'_{i'} \in \mathbb{C}^\times$ indexed by I'). Since $s(n) = s'(n)$ for all $n > 0$, we conclude that $s(0) = s'(0)$. ■

14.3. **A proof of existence.**

14.3.1. *The map s_P^K.* Suppose K is a compact open subgroup of G which has an Iwahori factorization with respect to $P = MN$. We begin by constructing a section $s_P^K \colon (r_P^G V)^{K\cap M} \to V^K$.

Definition 14.3.1. For $t \in T_M^+$, we define

$$\varphi_t^K := \delta_P^{-1/2}(t) \cdot e_K t e_K.$$

From Lemma **??** it follows that $\varphi_t^K * \varphi_{t'}^K = \varphi_{tt'}^K$ for $t, t' \in T_M^+$. Moreover, for all $v \in V^K$ we have $\varphi_t^K v \in V^K$ and

$$j_P(\varphi_t^K \cdot v) = r_P^G(t) \cdot j_P(v). \tag{8}$$

Indeed,

$$\begin{aligned} j_P(\varphi_t^K \cdot v) &= \delta_P^{-1/2}(t) \cdot j_P(e_K t e_K \cdot v) \\ &\qquad \text{since } e_K v = v \\ &= \delta_P^{-1/2}(t) \cdot j_P(e_{K^+} e_{K^-} e_{K^0} \pi(t) v) \\ &\qquad \text{since } t \text{ is in the center of } M \text{ and } j_P \text{ is a } P\text{-module map} \\ &= \delta_P^{-1/2}(t) \cdot j_P(\pi(t) e_{t^{-1}K^-} e_{K^0} \cdot v) \\ &\qquad \text{since } {}^{t^{-1}}K^- \subset K^- \text{ and } e_{\bar{P}\cap K} v = v \\ &= r_P^G(t) \cdot j_P(v). \end{aligned}$$

Lemma 14.3.2. *There exists an $\varepsilon > 0$ such that*

(1) *for all* $t \in T_M^+(\varepsilon)$ *we have*

$$V^K \cap \ker(j_P) \subset \ker(\pi(\varphi_t^K)),$$

(2) *for all* $t \in T_M^+(\varepsilon)$ *we have* j_P *maps* $\varphi_t^K V^K$ *isomorphically onto* $(r_P^G(V))^{K\cap M}$*, and*

(3) *for all* $t, t' \in T_M^+(\varepsilon)$ *we have* $\varphi_t^K V^K = \varphi_{t'}^K V^K$.

Proof. We begin by choosing ε. For all $v \in \ker j_P$ there exists a compact open subgroup $N' \leq N$ so that $e_{N'}v = 0$. Since V^K is finite dimensional, we can find a compact open subgroup $N' \leq N$ so that $e_{N'}v = 0$ for all $v \in V^K \cap \ker j_P$. Choose $\varepsilon > 0$ so that for all $t \in T_M^+(\varepsilon)$ we have ${}^tN' \leq K \cap N$.

"(1)" Fix $t \in T_M^+(\varepsilon)$. For $v \in V^K \cap \ker(j_P)$, we have

$$\begin{aligned}\varphi_t^K \cdot v &= \delta_P^{-1/2}(t) \cdot e_K t e_K \cdot v \\ &= \delta_P^{-1/2}(t) \cdot e_{K^-}\pi(t) \cdot (e_{{}^{t^{-1}}K^+} \cdot v) \\ &= 0\end{aligned}$$

since ${}^{t^{-1}}K^+ \geq N'$ if and only if $K^+ \geq {}^tN'$.

"(2)" Fix $t \in T_M^+(\varepsilon)$. We first show that $j_P\colon \varphi_t^K V^K \to (r_P^G(V))^{K\cap M}$ is injective. Suppose $v \in V^K$ such that $(j_P \circ \varphi_t^K)v = 0$. We need to show that $\varphi_t^K \cdot v = 0$. From Equation (8) we have

$$0 = r_P^G(t) \cdot j_P(v).$$

This implies that $j_P(v) = 0$. From Lemma 14.3.2 (1), we conclude that $v \in \ker(\varphi_t^K)$.

We now show that $j_P\colon \varphi_t^K V^K \to (r_P^G(V))^{K\cap M}$ is surjective. Suppose $w \in (r_P^G(V))^{K\cap M}$. Since t is in the center of M, we have $r_P^G(t)^{-1}w \in (r_P^G(V))^{K\cap M}$. Thanks to Jacquet's Lemma (Theorem 6.3.2), there exists a $v_t \in V^K$ so that $j_P(v_t) = r_P^G(t)^{-1}w$. Put $v = \varphi_t^K v_t$. We have $v \in \varphi_t^K V^K$ and $j_P(v) = j_P(\varphi_t^K v_t) = w$.

"(3)" Without loss of generality, $\varepsilon < 1$. Fix $t, t' \in T_M^+(\varepsilon)$. We have

$$\varphi_{tt'}^K V^K = \varphi_t^K \varphi_{t'}^K V^K \subset \varphi_t^K V^K$$

and

$$\varphi_{tt'}^K V^K = \varphi_{t't}^K V^K = \varphi_{t'}^K \varphi_t^K V^K \subset \varphi_{t'}^K V^K.$$

Since $tt' \in T_M^+(\varepsilon)$, we conclude from Lemma 14.3.2 (2) that

$$\varphi_t^K V^K = \varphi_{tt'}^K V^K = \varphi_{t'}^K V^K.$$

■

Definition 14.3.3. Suppose ε is chosen as in Lemma 14.3.2 and $t \in T_M^+(\varepsilon)$. We define $S_P^K := \varphi_t^K(V^K) \subset V^K$ and let $s_P^K\colon (r_P^G V)^{K\cap M} \to S_P^K$ denote the inverse to $j_P\colon S_P^K \to (r_P^G(V))^{K\cap M}$.

Corollary 14.3.4. *If* K *is a compact open subgroup of* G *which has an Iwahori factorization with respect to* $P = MN$*, then*

$$V^K = S_P^K \oplus (V^K \cap \ker(j_P)).$$

Proof. From Jacquet's lemma (Theorem 6.3.2), we have $j_P(V^K) = (r_P^G V)^{(K\cap M)}$. Therefore, the result follows from the fact that $j_P(S_P^K) = (r_P^G V)^{(K\cap M)}$. ∎

Note that S_P^K and s_P^K are independent of the choice of t; we would also like to understand how they depend on the choice of K.

Lemma 14.3.5. *Suppose K_1, K_2 are two compact open subgroups of G both having Iwahori decomposition with respect to $P = MN$. If $K_1 \leq K_2$, then*

$$s_P^{K_2} = e_{K_2} \circ s_P^{K_1}.$$

Proof. Choose ε so that the statements of Lemma 14.3.2 are valid for both K_1 and for K_2 in the role of K.

Fix $t \in T_M^+(\varepsilon)$. Choose $w \in (r_P^G(V))^{K_2\cap M}$. From Jacquet's Lemma (Theorem 6.3.2) and the fact that t lies in the center of M, we can find a $v_t \in V^{K_2}$ so that $j_P(v_t) = r_P^G(t^{-1})w$. We have

$$\begin{aligned} s_P^{K_2}(w) &= \varphi_t^{K_2} \cdot v_t = \delta_P^{-1/2}(t) \cdot e_{K_2} t e_{K_2} \cdot v_t \\ &= \delta_P^{-1/2}(t) \cdot e_{K_2}(e_{K_1} t e_{K_1}) e_{K_2} \cdot v_t = e_{K_2}\varphi_t^{K_1} \cdot v_t \\ &= e_{K_2} s_P^{K_1}(w). \end{aligned}$$

∎

14.4. **Defining the pairing $\langle\, ,\, \rangle_P$.** By considering

$$T_M^-(\varepsilon) := \{t^{-1} \,|\, t \in T_M^+(\varepsilon)\}$$

and the parabolic $\bar{P} = M\bar{N}$ opposite $P = MN$, we may define $s_{\bar{P}}^K$ and $S_{\bar{P}}^K$ for the contragredient representation.

Lemma 14.4.1. *Suppose K is a compact open subgroup of G having an Iwahori decomposition with respect to $P = MN$. For all $v \in S_P^K$ and for all $\lambda \in \widetilde{V}^K \cap \ker(j_{\bar{P}})$ we have $\lambda(v) = 0$.*

Proof. Choose $v \in S_P^K$ and $\lambda \in \widetilde{V}^K \cap \ker(j_{\bar{P}})$.

Choose ε so that the statements of Lemma 14.3.2 are valid for both V, with respect to $T_M^+(\varepsilon)$ and K, and for $\widetilde{V}$, with respect to $T_M^-(\varepsilon)$ and K. Choose $t \in T_M^+(\varepsilon)$.

Since $S_P^K = \pi(\varphi_t^K)V^K$, there exists $v' \in V^K$ so that $v = \pi(\varphi_t^K)v'$. We have

$$\lambda(v) = \lambda(\pi(\varphi_t^K)v') = (\pi(\varphi_{t^{-1}}^K)\lambda)(v')$$

Since $\lambda \in \widetilde{V}^K \cap \ker(j_{\bar{P}})$, from Lemma 14.3.2 (1) we have $\pi(\varphi_{t^{-1}}^K)\lambda = 0$. ∎

Lemma 14.4.2. *Suppose $w \in r_P^G V$ and $\mu \in r_{\bar{P}}^G(\widetilde{V})$. Choose a compact open subgroup K having Iwahori factorization with respect to $P = MN$ so that w and μ are both $(K \cap M)$-fixed. The number $\langle s_P^K w, s_{\bar{P}}^K \mu\rangle$ is independent of K.*

Proof. Suppose K' is another compact open subgroup of G such that K' has an Iwahori factorization with respect to $P = MN$ and w and μ are $(K' \cap M)$-fixed. We need to show

$$\langle s_P^K w, s_{\bar{P}}^K \mu\rangle = \langle s_P^{K'} w, s_{\bar{P}}^{K'} \mu\rangle.$$

Without loss of generality, we may assume $K' \leq K$.

We consider

$$
\begin{aligned}
\langle s_P^K w, s_{\bar{P}}^K \mu\rangle - \langle s_P^{K'} w, s_{\bar{P}}^{K'} \mu\rangle \\
\text{from Lemma 14.3.5} \\
&= \langle e_K s_P^{K'} w, e_K s_{\bar{P}}^{K'} \mu\rangle - \langle s_P^{K'} w, s_{\bar{P}}^{K'} \mu\rangle \\
&= \langle s_P^{K'} w, e_K s_{\bar{P}}^{K'} \mu\rangle - \langle s_P^{K'} w, s_{\bar{P}}^{K'} \mu\rangle \\
&= \langle s_P^{K'} w, (e_K - 1) s_{\bar{P}}^{K'} \mu\rangle
\end{aligned}
$$

But $(1 - e_K) s_{\bar{P}}^{K'} \mu$ is in $\widetilde{V}^{K'} \cap \ker(j_{\bar{P}})$. From Lemma 14.4.1 we conclude that

$$\langle s_P^K w, s_{\bar{P}}^K \mu\rangle = \langle s_P^{K'} w, s_{\bar{P}}^{K'} \mu\rangle.$$

■

We now know that the following definition makes sense.

Definition 14.4.3. Suppose $w \in r_P^G V$ and $\mu \in r_{\bar{P}}^G(\widetilde{V})$. We define

$$\langle w, \mu\rangle_P := \langle s_P^K w, s_{\bar{P}}^K \mu\rangle$$

where K is any compact open subgroup having an Iwahori factorization with respect to $P = MN$ so that w and μ are both $(K \cap M)$-fixed.

Exercise 14.4.4. Show that $\langle\, ,\, \rangle_P$ is M-invariant. Recall that $\delta_P(m) = \delta_{\bar{P}}(m)^{-1}$ for all $m \in M$.

Lemma 14.4.5. *The pairing $\langle\, ,\, \rangle_P$ on $r_P^G V \times r_{\bar{P}}^G \widetilde{V}$ is nondegenerate.*

Proof. Suppose $0 \neq w \in r_P^G V$. Let K be a compact open subgroup having an Iwahori factorization with respect to $P = MN$ so that w is $(K \cap M)$-fixed. Since the restriction of $\langle\, ,\, \rangle$ to $V^K \times \widetilde{V}^K$ is nondegenerate, there is a $\lambda \in \widetilde{V}^K$ so that $\lambda(s_P^K w) \neq 0$. From Lemma 14.4.1, we know $\lambda \notin \ker(j_{\bar{P}})$. From Corollary 14.3.4 we have

$$V^K = S_P^K \oplus (V^K \cap \ker(j_P)) \text{ and } \widetilde{V}^K = S_{\bar{P}}^K \oplus (\widetilde{V}^K \cap \ker(j_{\bar{P}})).$$

Thus, there is a $\mu \in r_{\bar{P}}^G \widetilde{V}$ so that $\langle s_P^K w, s_{\bar{P}}^K \mu\rangle \neq 0$. ■

14.5. **Completing the proof of Theorem 14.2.2.** To finish the proof of Theorem 14.2.2, we need to show that $\langle\, ,\, \rangle_P$ has the property:

For all $(v, \lambda) \in V \times \widetilde{V}$ there exists an $\varepsilon > 0$ such that for all $t \in T_M^+(\varepsilon)$ we have

$$\langle r_P^G(t) j_P(v), j_{\bar{P}}(\lambda)\rangle_P = \delta_P^{-1/2}(t) \cdot \lambda(\pi(t)v).$$

Suppose $(v, \lambda) \in V \times \widetilde{V}$. Choose a compact open subgroup of K having an Iwahori factorization with respect to $P = MN$ so that v and λ are both K-fixed. Choose ε so that the statements of Lemma 14.3.2 are valid for both V, with respect to $T_M^+(\varepsilon)$ and K, and $\widetilde{V}$, with respect to $T_M^-(\varepsilon)$ and K. For all $t \in T_M^+(\varepsilon)$ we have

$$
\begin{aligned}
\lambda(\pi(t)v) &= \lambda(e_K \pi(t) e_K v) \\
&= \delta_P(t) \lambda(\varphi_t^K(v))
\end{aligned}
$$

Since $\lambda - s_{\bar{P}}^K j_{\bar{P}}\lambda \in \ker(j_{\bar{P}}) \cap \widetilde{V}^K$, from Lemma 14.4.1 we have

$$\begin{aligned}\lambda(\pi(t)v) &= \delta_P(t) \cdot \langle \varphi_t^K(v), s_{\bar{P}}^K j_{\bar{P}}(\lambda)\rangle \\ &= \delta_P^{1/2}(t) \cdot \langle s_P^K r_P^G(t) j_P(v), s_{\bar{P}}^K j_{\bar{P}}(\lambda)\rangle \\ &= \delta_P^{1/2}(t) \cdot \langle r_P^G(t) j_P(v), j_{\bar{P}}(\lambda)\rangle_P.\end{aligned}$$

15. Casselman's square integrability criterion

Suppose that $\pi \in \mathfrak{R}(G)$ is admissible and P is a standard parabolic subgroup of G with a Levi decomposition $P = MN$. From Jacquet's lemma (Theorem 6.3.2) $r_P^G\pi$ is admissible. For each compact open subgroup K of M and each $t \in T_M$ we have ${}^tK = K$. Thus we may write

$$(r_P^G\pi)^K = \bigoplus_{\chi \in \hat{T}_M} (r_P^G\pi)_\chi^K$$

where χ is a smooth character of T_M and $(r_P^G\pi)_\chi^K$ is the generalized eigenspace

$$\{v \in r_P^G\pi^K \mid \text{there exists } d \in \mathbb{Z}_{\geq 0} \text{ such that } (\pi_N(t)\delta_P^{-1/2}(t) - \chi(t))^d v = 0 \text{ for all } t \in T_M\}.$$

If $K' \leq K$ is another compact open subgroup of M, then

$$(r_P^G\pi)_\chi^K \subset (r_P^G\pi)_\chi^{K'}.$$

Hence

$$(r_P^G\pi)_\chi := \varinjlim_{K' \leq K} (r_P^G\pi)_\chi^{K'}$$

(where the colimit is taken over compact open subgroups of M contained in K) makes sense and we can write

$$r_P^G\pi = \bigoplus_{\chi \in \hat{T}_M} (r_P^G\pi)_\chi$$

since colimits commute with colimits. If the space $(r_P^G\pi)_\chi$ in nonzero, then we call χ a *normalized exponent* of π relative to $P = MN$.

In this section we prove:

Theorem 15.0.1 (Casselman's Square integrability criterion)**.** *Suppose $\pi \in \mathfrak{R}(G)$ is irreducible with unitary central character. π is square integrable modulo $Z(G)$ if and only if for all standard proper parabolics P with Levi decomposition $P = MN$ and for all normalized exponents χ of π relative to $P = MN$ we have $|\chi| < 1$ on $T_M^+ \smallsetminus T_G^+$.*

Proof. Fix a proper parabolic subgroup $P = MN$ of G. Define an equivalence relation on $T_\emptyset^+$ by

$$t_1 \sim t_2 \text{ provided that } t_1 t_2^{-1} \in T_G.$$

We have $T_M^+/\sim \cong \mathbb{Z}_{\geq 0}^d$ for some $d \in \mathbb{Z}_{\geq 0}$. We also have

$$G = \coprod_{t \in T_\emptyset^+/\sim\, ;\, w \in \omega} K_0 Z t w K_0,$$

and if K has an Iwahori factorization with respect to $P = MN$, then

$$[KZ\colon KZ \cap {}^t(KZ)] = [K\colon K \cap {}^tK] = \delta_{P_\emptyset}(t).$$

"$\Rightarrow$" Let χ be a normalized exponent of π with respect to $P = MN$. Suppose $v \in V$ such that $0 \neq r_P^G v$ is an eigenvector for χ. Choose $\lambda \in \widetilde{V}$ such that $\langle r_P^G v, r_{\bar{P}}^G \lambda\rangle_P \neq 0$. Choose a compact open subgroup K of G with an Iwahori factorization with respect to $P = MN$ so that $(v, \lambda) \in V^K \times \widetilde{V}^K$. From Theorem 14.2.2 there is an $s \in T_M^+$ so that for all $t \in T_M^+$ we have

$$\begin{aligned}\lambda(\pi(st)v) &= \delta_P^{-1/2}(st) \cdot \langle r_P^G(st) r_P^G v, r_{\bar{P}}^G \lambda\rangle_P \\ &= \delta_P^{-1/2}(st)\chi(st) \cdot \langle r_P^G v, r_{\bar{P}}^G \lambda\rangle_P\end{aligned}$$

We then have

$$\begin{aligned}\infty &> \int_{G/Z} |m_{\lambda,v}(g)|^2 \, dg^* \\ &\geq \sum_{t \in T_M^+/\sim} |m_{\lambda,v}(st)|^2 \cdot \mathrm{meas}_{dg^*}(KZstK) \\ &= \sum_{t \in T_M^+/\sim} \left|\langle r_P^G v, r_{\bar{P}}^G \lambda\rangle \cdot \chi(st)\delta_P^{-1/2}(st)\right|^2 \cdot \delta_P(st) \cdot \mathrm{meas}_{dg^*}(KZ) \\ &= \mathrm{meas}_{dg^*}(KZ) \cdot \left|\langle r_P^G v, r_{\bar{P}}^G \lambda\rangle_P \cdot \chi(s)\right|^2 \cdot \sum_{t \in T_M^+/\sim} |\chi(t)|^2 .\end{aligned}$$

Let $t_1, t_2, \ldots, t_m$ be a generating set for $T_M^+/\sim$, that is, for all $t \in T_M^+/\sim$ there exist $k_1, k_2, \ldots, k_m \in \mathbb{Z}_{\geq 0}$ such that $t = \prod t_i^{k_i}$. We then have

$$\infty > \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \prod_{i=1}^{n} |\chi(t_i)|^{2k_i} .$$

For the last item to converge, we must have $|\chi(t)| < 1$ for all $t \in T_M^+ \smallsetminus T_G^+$.

"$\Leftarrow$" Suppose $(v, \lambda) \in V \times \widetilde{V}$. Choose a compact open subgroup $K \leq K_0$ having an Iwahori decomposition with respect to every standard parabolic $P = MN$. From the Cartan decomposition, there exist $g_1, g_2, \ldots, g_m \in \mathrm{stab}_G(K)$ so that

$$G = \coprod_{1 \leq i,j \leq m; t \in T_\emptyset^+/\sim} KZg_i t g_j K.$$

Thus, it is enough to show

$$\sum_{t \in T_\emptyset^+/\sim} |m_{v,\lambda}(g_i t g_j)|^2 \, \delta_{P_\emptyset}(t) < \infty$$

for each pair (i, j) with $1 \leq i, j \leq m$. After replacing v by $\pi(g_j)v$ and λ by $\widetilde{\pi}(g_i^{-1})\lambda$, we may assume that $g_i = g_j = 1$.

From Theorem 14.2.2, there is an $s_0 \in T_\emptyset^+$ so that

$$m_{v,\lambda}(s_0 t) = \delta_{P_\emptyset}^{-1/2}(s_0 t) \cdot \langle r_{P_\emptyset}^G(s_0 t) r_{P_\emptyset}^G v, r_{\bar{P}}^G \lambda\rangle_P$$

for all $t \in T_\emptyset^+$. After writing $r_{P_\emptyset}^G v$ as a sum of generalized eigenvectors for the action of $T_\emptyset$ on $r_{P_\emptyset}^G V$, we see that there is a polynomial Q (regarded as a function on $T_M \cong \mathbb{Z}^d$ for some d) so that

$$\langle r_{P_\emptyset}^G(s_0 t) r_{P_\emptyset}^G(v), r_{\bar{P}}^G \lambda\rangle_P = \sum_\chi Q(s_0 t)\chi(s_0 t)$$

where the sum is over normalized exponents of π with respect to $P_\emptyset = M_\emptyset N_\emptyset$. Since $\mathrm{res}_{T_\emptyset^+ \smallsetminus T_G^+} |\chi| < 1$, we conclude that

$$\sum_{t \in T_\emptyset^+/\sim} |m_{v,\lambda}(s_0 t)|^2 \, \delta_{P_\emptyset}(s_0 t) = \sum_{t \in T_\emptyset^+/\sim} \left|Q^2(s_0 t)\right| |\chi(s_0 t)|^2$$

converges.

We have taken care of the "interior of the cone $T_\emptyset^+$". For each $\alpha \in \Delta$, we can find a finite subset

$$\{s_{\alpha,1}, s_{\alpha,2}, \ldots, s_{\alpha,j_\alpha}\}$$

of $T_\emptyset^+$ so that

$$T_\emptyset^+/\sim = s_0 T_{\emptyset+}/\sim \ \amalg \left(\coprod_{\alpha \in \Delta} \coprod_{1 \le i \le j_\alpha} s_{\alpha,i} T_{\{\alpha\}}^+/\sim \right).$$

By repeating the above argument, for each $\alpha \in \Delta$ and each $s_{\alpha,i}$ we can find an $s_0^\alpha \in T_{\{\alpha\}}^+$ so that

$$\sum_{t \in T_{\{\alpha\}}^+/\sim} |m_{v,\lambda}(s_{\alpha,i} s_0^\alpha t)|^2 \cdot \delta_{P_{\{\alpha\}}}(s_{\alpha,i} s_0^\alpha t)$$

converges. The result follows by induction. ■

16. Restriction Induction

16.1. **Statement of main result.** In this section, we shall prove the following result.

Theorem 16.1.1. *Suppose $P = MN$ and $Q = LU$ are two standard parabolic subgroups of G and $(\sigma, W) \in \mathfrak{R}(M)$. As a representation of L,*

$$r_Q^G i_P^G \sigma$$

has a filtration for which the associated graded pieces are

$$i_{L \cap w^{-1} P w}^L w^{-1} \cdot r_{wQw^{-1} \cap M}^M \sigma$$

where w runs over representatives in G for the double coset space $W_M \backslash W / W_L \cong P \backslash G / Q$.

Remark 16.1.2. In each double coset of $W_M \backslash W / W_L$ it is possible to choose a representative w so that $L \cap w^{-1} P w$ is a standard parabolic in L and $wQw^{-1} \cap M$ is a standard parabolic in M.

16.2. **The setting.** We now introduce a take on induction that will prove useful in our attack on Theorem 16.1.1.

Suppose G is the usual kind of group. That is, G has a neighborhood basis of the identity consisting of compact open subgroups and for each compact open subgroup K of G, G/K is countable.

We now present a few definitions.

Definition 16.2.1. We let Δ_G denote the set of left invariant complex measures on G.

We have $\Delta_G = \mathbb{C} \cdot d_\ell g$. The set Δ_G carries two natural G actions. Namely, if $g \in G$ and $d\mu \in \Delta_G$, then $R_g \cdot d\mu = \delta_G(g^{-1})d\mu$ and $L_g \cdot d\mu = d\mu$. Unless otherwise specified, we shall always treat Δ_G as a left (resp. right) G-module via right translations (resp. left translations).

We recall that $\mathcal{H}_G$ denotes the set of locally constant compactly supported measures on G. As in subsection 9.2, $\mathcal{H}_G$ is an algebra with respect to convolution, and it carries natural left and right actions of G: $g \cdot d\mu = L_{g^{-1}} \cdot d\mu$ and $d\mu \cdot g = d\mu \cdot R_{g^{-1}}$. We treat $C_c^\infty(G)$ as a left G-module via right translation and as a right G-module via left translation. We give $C_c^\infty(G) \otimes_{\mathbb{C}} \Delta_G$ the structure of a G-bimodule via the diagonal and the actions described above. With these conventions, there is a G-bimodule isomorphism $C_c^\infty(G) \otimes_{\mathbb{C}} \Delta_G \cong \mathcal{H}_G$ which sends $f \otimes d_\ell x$ to the measure $f(x^{-1})d_\ell(x^{-1})$.

Suppose (π_1, V_1) and (π_2, V_2) are two left G-modules. We define

$$V_1 \otimes_G V_2 := (V_1 \otimes_{\mathbb{C}} V_2)/\langle gv_1 \otimes gv_2 - v_1 \otimes v_2 \rangle.$$

Note that $V_1 \otimes_G V_2$ is simply $(V_1 \otimes_{\mathbb{C}} V_2)_G$, the coinvariants of $V_1 \otimes_{\mathbb{C}} V_2$ with respect to the diagonal action of G. In particular $V_1 \otimes_G \mathbb{C} \cong (V_1)_G$.

On the other hand, if V_1 is a right G-module and V_2 is a left G-module, then

$$V_1 \otimes_G V_2 = (V_1 \otimes_{\mathcal{H}_G} V_2)$$

Example 16.2.2. We examine the case when $V_1 = \mathcal{H}_G$. Suppose $d\mu_1, d\mu_2 \in \mathcal{H}_G$ and $v_2 \in V_2$. Treating $\mathcal{H}_G$ as a right G-module, we have

$$L(d\mu_1)d\mu_2 \otimes v_2 = (d\mu_1 * d\mu_2) \otimes v_2 = (d\mu_2 \cdot d\mu_1) \otimes v_2 = d\mu_2 \otimes \pi_2(d\mu_1)v_2.$$

If we treat $\mathcal{H}_G$ as a left G-module, then we have

$$R(d\mu_2)d\mu_1 \otimes v_2 = (d\mu_1 * d\mu_2) \otimes v_2 = d\mu_2 \cdot d\mu_1 \otimes v_2 = d\mu_1 \otimes \pi_2(d\mu_2)v_2.$$

Note that although we have not defined $V_1 \otimes_{\mathcal{H}_G} V_2$ for two left G-modules V_1 and V_2,

Lemma 16.2.3. *Suppose $(\pi, V) \in \mathfrak{R}(G)$. We have $\mathcal{H}_G \otimes_{\mathcal{H}_G} V \cong V$ and $(\mathcal{H}_G \otimes_{\mathbb{C}} V)_G \cong V$ under the maps induced by $d\mu \otimes v \mapsto \pi(d\mu)v$.*

Proof. One checks that the maps induced by $d\mu \otimes v \mapsto \pi(d\mu)v$ are well defined and G-equivariant[6] Since (π, V) is a smooth representation, for every $v \in V$ there exists a compact open subgroup K

[6] That is $(d\mu_1 d\mu_2) \otimes v$ and $d\mu_1 \otimes \pi(d\mu_2)v$ get mapped to the same place *and* $R(g)d\mu \otimes \pi(g)v$ and $d\mu \otimes v$ do as well.

of G such that $e_K w = w$. Let dk denote the normalized Haar measure on K. Since $\pi(dk) = e_K$, we see that the map is surjective.

Suppose $\sum_i (d\mu_i \otimes v_i) \in \mathcal{H}_G \otimes_G V$ such that $\sum_i \pi(d\mu_i)v_i = 0$. We can choose a compact open subgroup K of G and a normalized Haar measure dk on K so that $L(d\mu_i)dk = d\mu_i$ for each i. Thus

$$\sum_i d\mu_i \otimes v_i = \sum_i L(d\mu_i)dk \otimes v_i = \sum_i dk \otimes \pi(d\mu_i)v_i = 0$$

and so the map is injective. ∎

16.3. **A proposition of Tate.** Suppose H is a closed subgroup of G. We treat $C_c^\infty(G)$ (resp. Δ_H) as a right (resp. left) H-module via left (resp. right) translations. We treat $C_c^\infty(G) \otimes_{\mathcal{H}_H} (W \otimes_{\mathbb{C}} \Delta_H)$ as a left G-module via $g \cdot (\phi \otimes (w \otimes d\mu)) = g\phi \otimes (gw \otimes gd\mu) = R_g\phi \otimes (\sigma(g)w \otimes R_g d\mu)$.

Lemma 16.3.1. *If K is a compact open subgroup of G and $(\sigma, W) \in \mathfrak{R}(H)$, then*

$$(C_c^\infty(G) \otimes_{\mathcal{H}_H} (W \otimes_C \Delta_H))^K \cong \bigoplus_{\overline{g} \in H\backslash G/K} C_c(H/(H \cap {}^gK)) \otimes_{\mathcal{H}_H} (W \otimes_{\mathbb{C}} \Delta_H).$$

Proof. We have

$$\begin{aligned}
(C_c^\infty(G) \otimes_{\mathcal{H}_H} (W \otimes_{\mathbb{C}} \Delta_H))^K &= e_K \left(C_c^\infty(G) \otimes_{\mathcal{H}_H} (W \otimes_{\mathbb{C}} \Delta_H)\right) \\
&= e_K \left((C_c^\infty(G)e_K \oplus C_c^\infty(G)(1 - e_K)) \otimes_{\mathcal{H}_H} (W \otimes_{\mathbb{C}} \Delta_H)\right) \\
&= C_c(G/K) \otimes_{\mathcal{H}_H} (W \otimes_{\mathbb{C}} \Delta_H) \\
&= \bigoplus_{\overline{g} \in H\backslash G/K} C_c(HgK/K) \otimes_{\mathcal{H}_H} (W \otimes_{\mathbb{C}} \Delta_H) \\
&\cong \bigoplus_{\overline{g} \in H\backslash G/K} C_c(H/H \cap {}^gK) \otimes_{\mathcal{H}_H} (W \otimes_{\mathbb{C}} \Delta_H).
\end{aligned}$$

∎

Proposition 16.3.2 (Tate)**.** *Suppose H is a closed subgroup of G and $(\sigma, W) \in \mathfrak{R}(H)$.*

(1) *We have an isomorphism*

$$C_c^\infty(G) \otimes_{\mathcal{H}_H} (W \otimes_{\mathbb{C}} \Delta_H) \cong \text{c-Ind}_H^G(\sigma)$$

via the map which sends $f \otimes (w \otimes d_\ell h)$ to $(g \mapsto \int_H f(h^{-1}g)\sigma(h)w\, d_\ell h)$.

(2) *We have*

$$\mathrm{Hom}_H(\mathcal{H}_G, W)^\infty \cong \mathrm{Ind}_H^G(\sigma).$$

Proof. We begin with statement (1). Since both sides of the equation are smooth representations, it is enough to show that for each compact open subgroup K of G, the K-fixed vectors of the left-hand side are isomorphic to the K-fixed vectors of the right hand side.

Fix a compact open subgroup K of G. From Lemma 16.3.1, we have that the K-fixed vectors of the left-hand side are isomorphic to

$$\bigoplus_{\overline{g} \in H\backslash G/K} C_c(H/H \cap {}^gK) \otimes_{\mathcal{H}_H} (W \otimes_{\mathbb{C}} \Delta_H).$$

If we knew that

$$C_c(H/H \cap {}^gK) \otimes_{\mathcal{H}_H} (W \otimes_{\mathbb{C}} \Delta_H) \cong W^{H \cap {}^gK},$$

then from Lemma 5.5.2 we'd be done.

By putting $G = H$ in Lemma 16.3.1, we are asking that

$$(C_c^\infty(H) \otimes_{\mathcal{H}_H} (W \otimes_{\mathbb{C}} \Delta_H))^{K_H} \cong W^{K_H}$$

for all compact open subgroups K_H of H. However, this is true for all compact open subgroups K_H of H if and only if

$$C_c^\infty(H) \otimes_{\mathcal{H}_H} (W \otimes_{\mathbb{C}} \Delta_H) \cong W$$

which is true if and only if

$$(C_c^\infty(H) \otimes_{\mathbb{C}} \Delta_H) \otimes_{\mathcal{H}_H} W \cong W$$

which is true if and only if

$$\mathcal{H}_H \otimes_{\mathcal{H}_H} W \cong W,$$

the conclusion of Lemma 16.2.3.

We now turn our attention to statement (2). Note that for all $V \in \mathfrak{R}(G)$ we have

$$\begin{aligned} \mathrm{Hom}_H(\mathrm{res}_H V, W) &= \mathrm{Hom}_H(\mathcal{H}_G \otimes_{\mathcal{H}_G} V, W) = \mathrm{Hom}_H((\mathcal{H}_G \otimes_{\mathbb{C}} V)_G, W) \\ &= \mathrm{Hom}_{G \times H}(\mathcal{H}_G \otimes_{\mathbb{C}} V, W) = \mathrm{Hom}_G(V, \mathrm{Hom}_H(\mathcal{H}_G, W)) \\ &= \mathrm{Hom}_G(V, \mathrm{Hom}_H(\mathcal{H}_G, W)^\infty). \end{aligned}$$

Since, up to isomorphism, $\mathrm{Ind}_H^G W$ is the unique object in $\mathfrak{R}(G)$ for which

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) = \mathrm{Hom}_H(\mathrm{res}_H V, W),$$

we are done. ■

Lemma 16.3.3. *If* $(\sigma, W) \in \mathfrak{R}(H)$*, then*

$$((\text{c-Ind}_H^G W) \otimes_{\mathbb{C}} \Delta_G)\widetilde{\ } \cong \text{c-Ind}_H^G((W \otimes_{\mathbb{C}} \Delta_H)\widetilde{\ }).$$

Proof. From Proposition 16.3.2 we have

$$\text{c-Ind}_H^G(W \otimes_{\mathbb{C}} \Delta_G) \cong \mathcal{H}_G \otimes_{\mathcal{H}_H} (W \otimes_{\mathbb{C}} \Delta_H).$$

So, it is enough to show

$$(\mathcal{H}_G \otimes_{\mathcal{H}_H} W)\widetilde{\ } \cong \mathrm{Ind}_H^G(\widetilde{W}).$$

From Proposition 16.3.2 we have

$$\begin{aligned} \mathrm{Ind}_H^G \widetilde{W} &\cong \mathrm{Hom}_H(\mathcal{H}_G, \widetilde{W})^\infty \\ &\cong \mathrm{Hom}_H(\mathcal{H}_G, \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C}))^\infty \\ &\cong \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_G \otimes_{\mathcal{H}_H} W, \mathbb{C})^\infty \\ &\cong (\mathcal{H}_G \otimes_{\mathcal{H}_H} W)\widetilde{\ }. \end{aligned}$$

■

16.4. **An application: forms on G/H.** In Proposition 16.3.2 if we take $W = \widetilde{\Delta}_H$, then then

$$C_c^\infty(G) \otimes_{\mathcal{H}_H} \mathbb{C} \cong \text{c-Ind}_H^G(\widetilde{\Delta}_H)$$

and the latter object may be idenitified (non-canonically) with $\text{c-Ind}_H^G \delta_H$. Tensoring the isomorphism of 16.3.2(a) with $\Delta_G \otimes_{\mathbb{C}} \widetilde{\Delta_G}$, we have the relation

$$\mathcal{H}_G \otimes_{\mathcal{H}_H} (W \otimes_{\mathbb{C}} \Delta_H \otimes_{\mathbb{C}} \widetilde{\Delta_G}) \cong \text{c-Ind}_H^G W.$$

If we take $W = \widetilde{\Delta}_H \otimes_{\mathbb{C}} \Delta_G$ and plug this into the above relation, then we have

$$\mathcal{H}_G \otimes_{\mathcal{H}_H} \mathbb{C} \cong \text{c-Ind}_H^G(\widetilde{\Delta}_H) \otimes_{\mathbb{C}} \widetilde{\Delta}_G$$

and the latter object may be idenitified (non-canonically) with $\text{c-Ind}_H^G(\delta_H) \otimes \delta_G$.

Since

$$\mathrm{Hom}_G(\mathcal{H}_G \otimes_{\mathcal{H}_H} \mathbb{C}, \mathbb{C}) \cong \mathrm{Hom}_{G\times H}(\mathcal{H}_G, \mathbb{C}) \cong \mathbb{C},$$

we have that $\mathrm{Hom}_G(\text{c-Ind}_H^G(\delta_H) \otimes \delta_G, \mathbb{C})$ is one-dimensional. Let

$$\int_{H\backslash G} dg/dh\colon\ \text{c-Ind}_H^G(\delta_H) \otimes \delta_G \to \mathbb{C}$$

be a nonzero G-equivariant map; this is unique up to scaling by an element of $\mathbb{C}^\times$. We can fix $d_\ell h$ and $d_r g$ so that the following diagram commutes.

$$\begin{tikzcd} & \mathbb{C} & \\ C_c^\infty(G) \arrow[ur, "\int_G d_r g"] \arrow[rr, "\varphi"'] & & \text{c-Ind}_H^G(\delta_H) \otimes \delta_G \arrow[ul, "\int_{H\backslash G} dg/dh"'] \end{tikzcd}$$

Here $(\varphi(f))(g) = \delta_G(g) \int_H f(h^{-1}g) \cdot \delta_H(h)\, d_\ell h$. Note that $\int_{H\backslash G} dg/dh$ takes positive values on

$$\{f \in \text{c-Ind}_H^G(\delta_H^{-1}) \otimes \delta_G \mid f \geq 0 \text{ and } f \neq 0\}$$

The lemma below follows immediately from the above discussion

Lemma 16.4.1. *If $(\sigma, W) \in \mathfrak{R}(H)$ is unitary with positive definite Hermitian form $(\ ,\ )_W$, then*

$$\text{c-Ind}_H^G(W \otimes \delta_H^{-1/2}) \otimes \delta_G^{1/2}$$

is unitary with respect to the postive definite Hermitian form

$$(f_1, f_2) := \int_{H\backslash G} (f_1(g), f_2(g))_W\, dg/dh.$$

16.5. **ℓ-spaces.**

Definition 16.5.1. An ℓ-space is a Hausdorff space such that each point has a neighborhood basis consisiting of compact open subsets.

Remark 16.5.2. An open (resp., closed, resp. locally closed (that is, the intersection of a closed and open set)) subset of an ℓ-space is again an ℓ-space.

Definition 16.5.3. For an ℓ-space X, we let $C_c^\infty(X)$ denote the set of localy constant, compactly supported, complex valued functions on X.

Exercise 16.5.4. If X is an ℓ-space and $f \in C_c^\infty(X)$, then $f(X)$ is finite and $\{x \in X \mid f(x) \neq 0\}$ is open.

Lemma 16.5.5. *Suppose $Z \subset X$ is closed. Let $U = X \setminus Z$. We have*

$$0 \to C_c^\infty(U) \to C_c^\infty(X) \to C_c^\infty(Z) \to 0$$

is exact. The first map is extension by zero and the last map is restriction.

Proof. We only need to check that the last map is surjective. Suppose $f \in C_c^\infty(Z)$. Without loss of generality, $f = [C']$ where C' is a compact open subset of Z and $[C']$ denotes the characteristic function of C'. Since we can choose a compact open subset C in X such that $C' = C \cap Z$, the result follows. ■

Suppose G is a topological ℓ-group and $H \leq G$. Fix $(\sigma, W) \in \mathfrak{R}(H)$.

Definition 16.5.6. If $U \subset G$ is open and $HU = U$, then we define

$$(\text{c-Ind}_H^G(W))_U := \{f \colon G \to W \mid f \in \text{c-Ind}_H^G(W) \text{ and } \operatorname{supp}(f) \subset U\}.$$

Lemma 16.5.7. *If U is open, then*

$$0 \to (\text{c-Ind}_H^G(W))_U \to (\text{c-Ind}_H^G W) \to C_c^\infty(G \setminus U) \otimes_H (W \otimes \Delta_H) \to 0$$

is exact.

Proof. We have

$$0 \to C_c^\infty(U) \to C_c^\infty(G) \to C_c^\infty(G \setminus U) \to 0$$

is exact and so

$$C_c^\infty(U) \otimes_H (W \otimes \Delta_H) \to \text{c-Ind}_H^G(W) \to C_c^\infty(G \setminus U) \otimes_H (W \otimes \Delta_H) \to 0$$

is exact. We need to show

$$\alpha \colon C_c^\infty(U) \otimes_H (W \otimes \Delta_H) \to \text{c-Ind}_H^G(W)$$

is injective with image $\text{c-Ind}_H^G(W)_U$.

We first consider the case when U is closed. In this case the injective map $C_c^\infty(U) \to C_c^\infty(G)$ has a left inverse $f \mapsto f \cdot [U]$. Thus, α is injective and

$$\begin{aligned}
\mathrm{Im}(\alpha) &= \mathrm{Im}(C_c^\infty(U) \otimes_H (W \otimes \Delta_H) \to C_c^\infty(G) \otimes_H (W \otimes \Delta_H)) \\
&= \mathrm{Im}(C_c^\infty(U) \otimes_H (W \otimes \Delta_H) \overset{[U]\otimes 1}{\to} C_c^\infty(G) \otimes_H (W \otimes \Delta_H)) \\
&= \mathrm{Im}(\text{c-Ind}_H^G(W) \overset{[U]}{\to} \text{c-Ind}_H^G(W)) \\
&= (\text{c-Ind}_H^G W)_U
\end{aligned}$$

In general, we consider the projection map $pr\colon G \to H\backslash G$. Since $H\backslash G$ is a directed union of compact open subsets, by looking at pr^{-1} of compact open subsets in $pr(U)$ we can write U as a directed union of closed, open subsets C of U for which $HC = C$. Thus

$$C_c^\infty(U) = \varinjlim_{C} C_c^\infty(C),$$

and the result follows immediately from the case when U is closed. ■

Now we can analyze $r_Q^G i_P^G W$ with $P = MN$, $Q = LU$ standard parabolics and W a smooth representation of M.

Proof of 16.1.1. First, $Q/G\backslash P$ is finite. We can order the double cosets $\theta_1 = Qw_1P, \ldots, \theta_t = Qw_tP$ in such a way that $\theta_1, \theta_1 \cup \theta_2, \theta_1 \cup \theta_2 \cup \theta_3, \ldots, \theta_1 \cup \cdots \cup \theta_t$ are all open in G. This fact is verified by the following lemma found in [**?**] on page 7, which is proved using the Baire category theorem.

Lemma 16.5.8. *Suppose G is an ℓ-space and G/K is countable for all compact open subgroups $K \subset G$. Suppose G acts continuously on X. Assume G has finitely many orbits. Then there exists an open orbit X_0, and for every $x_0 \in X_0$, the map $G \to X_0$ given by $g \mapsto gx_0$ is open, so that $X_0 \cong G/\operatorname{stab}(x_0)$ as topological spaces.*

We now continue with our proof of 16.1.1. Let $U = \theta_1, U_2 = \theta_1 \cup \theta_2, \ldots, U_t = \theta_1 \cup \cdots \theta_t$. From the above results,

$$(\text{c-Ind}_P^G W)_{U_1} \subset (\text{c-Ind}_P^G W)_{U_2} \subset \cdots \subset (\text{c-Ind}_P^G W)_{U_t} = \text{c-Ind}_H^G W.$$

These are left Q-modules. Take Jacquet modules with respect to U; by exactness of this operation, we have

$$[(\text{c-Ind}_P^G W)_{U_1}]_U \subset \cdots \subset [(\text{c-Ind}_P^G W)_{U_t}]_U = (\text{c-Ind}_P^G W)_U.$$

We consider successive quotients of this filtration. By exactness,

$$\frac{[(\text{c-Ind}_P^G W)_{U_j}]_U}{[(\text{c-Ind}_P^G W)_{U_{j-1}}]_U} = \left[\frac{(\text{c-Ind}_P^G W)_{U_j}}{(\text{c-Ind}_P^G W)_{U_{j-1}}}\right]_U,$$

and both of these are

$$(C_c^\infty(\theta_j) \otimes_{\mathcal{H}_P} W'$$

where $W' = W \otimes_{\mathbb{C}} \Delta_P$. Now simplify

$$C_c^\infty(\theta_j) \otimes_{\mathcal{H}_P} W' = C_c^\infty(PwQ) \otimes_{\mathcal{H}_P} W'$$

($w = w_j$).

We show that

$$C_c^\infty(PwQ) \otimes_{\mathcal{H}_P} W' \cong \text{c-Ind}_{Q\cap w^{-1}Pw}^{Q} W''.$$

(W' is a representation of P; composing with $w^{-1}Pw \to P$, $W'' = W'$.) Replace Q with wQw^{-1} and still call it Q. (Of course, wQw^{-1} is possibly nonstandard parabolic.) With the new Q, we are looking to prove

$$C_c^\infty(PQ) \otimes_{\mathcal{H}_P} W' \cong \text{c-Ind}_{P\cap Q}^{Q} W'.$$

Now $C_c^\infty(P \times Q) = C_c^\infty(P) \otimes_{\mathbb{C}} C_c^\infty(Q)$ and $PQ = (P \times Q)/(P \cap Q)$. From before,

$$\begin{aligned}
C_c^\infty(PQ) &= C_c^\infty((P \times Q)/(P \cap Q)) \\
&= \text{c-Ind}_{Q\cap P}^{Q\times P} 1 \\
&= C_c^\infty(Q \times P) \otimes_{\mathcal{H}_{Q\cap P}} \delta_{Q\cap P}^{-1} \\
&= (C_c^\infty(Q) \otimes_{\mathbb{C}} C_c^\infty(P)) \otimes_{\mathcal{H}_{Q\cap P}} \delta_{Q\cap P}^{-1}.
\end{aligned}$$

So

$$\begin{aligned}
C_c^\infty(PQ) \otimes_{\mathcal{H}_P} W' &= C_c^\infty(Q) \otimes_{\mathcal{H}_{Q\cap P}} \delta_{Q\cap P} \otimes_{\mathcal{H}_{Q\cap P}} C_c^\infty(P) \otimes_{\mathcal{H}_P} W' \\
&= C_c^\infty(Q) \otimes_{\mathcal{H}_{Q\cap P}} (W' \otimes \delta_{Q\cap P}) \\
&= \text{c-Ind}_{Q\cap P}^{Q} W'.
\end{aligned}$$

To complete the proof, we need to calculate $(\ )_U$. To this end, note that

$$\begin{aligned}
{}_U &= \mathbb{C} \otimes_U C_c^\infty(Q) \otimes_{P\cap Q} (W \otimes \delta_{P\cap Q}) \\
&= \delta_Q \otimes C_c^\infty(Q/U) \otimes_{P\cap Q} (W \otimes \delta_{P\cap Q}) \\
&\cong \delta_Q \otimes (C_c^\infty(L) \otimes_{P\cap Q} W \otimes \delta_{P\cap Q}),
\end{aligned}$$

where the final isomorphism is noncanonical. Now $U \cap P$ and $Q \cap N$ are normal in $Q \cap P$; so is their product.

Define $P_L :=$ (image of $Q \cap P$ in $L = Q/U) = (Q \cap P)/(U \cap P)$ and $Q_M :=$ (image of $Q \cap P$ in $M = P/N) = (Q \cap P)/(Q \cap N)$. Put $J = (Q \cap P)/(U \cap P) \cdot (Q \cap N)$. This is a group. It is isomorphic to a Levi component of a parabolic of G. We have short exact sequences

$$\begin{aligned}
1 \longrightarrow Q \cap N/U \cap N \longrightarrow P_L \longrightarrow J \longrightarrow 1 \\
1 \longrightarrow U \cap P/U \cap N \longrightarrow Q_M \longrightarrow J \longrightarrow 1
\end{aligned}$$

where the first terms are the unipotent radicals of P_L and Q_M, respectively, and the final terms are Levi components of P_L and Q_M, respectively.

Now

$$\begin{aligned}
C_c^\infty(L) \otimes_{P\cap Q} W \otimes \delta_{P\cap Q} &= [C_c^\infty(L) \otimes_{U\cap P} W \otimes \delta_{P\cap Q}]_{P_L} \\
&= [C_c^\infty(L) \otimes_{\mathbb{C}} (W \otimes \delta_{P\cap Q})_{U\cap P}]_{P_L}.
\end{aligned}$$

So

$$\begin{aligned}\delta_Q \otimes (C_c^\infty(L) \otimes_{Q\cap P} W \otimes \delta_{P\cap Q}) &= \delta_Q \otimes [C_c^\infty(L) \otimes_{\mathbb{C}} (W \otimes \delta_{P\cap Q})_{U\cap P}]_{P_L} \\ &= \delta_Q \otimes [C_c^\infty(L) \otimes_{\mathbb{C}} (W \otimes \delta_{P\cap Q})_{(U\cap P)/(U\cap N)}]_{P_L}.\end{aligned}$$

Note that $(W \otimes \delta_{P\cap Q})_{(U\cap P)/(U\cap N)}$ is the Jacquet module for the representation W of M and that everything after the δ_Q is c-$\mathrm{Ind}_{P_L}^L ?$.

Putting in all the relevant modulus functions, we get that the piece of $r_Q^G i_P^G W$ corresponding to our given double coset QP (new Q) is given by

$$i_{P_L}^L r_{Q_M}^M (\delta_Q^{1/2} \delta_{Q_M}^{1/2} \delta_P^{1/2} \delta_{Q\cap P}^{-1} \delta_{P_L}^{1/2}) W.$$

It suffices to show that the product of modulus functions is trivial as a quasi-character on J.

We claim that

$$(*) \qquad \delta_{Q_M}^{1/2} \delta_{Q\cap P}^{-1} \delta_{P_L}^{1/2} = \underbrace{\delta_{Q_M}^{1/2} \delta_{Q\cap P}^{-1/2}}_{m_{Q\cap N}^{1/2}} \underbrace{\delta_{Q\cap P}^{-1/2} \delta_{P_L}^{1/2}}_{m_{U\cap P}^{1/2}}$$

where, for $x \in Q \cap P$,

$$m_{U\cap P}(x) = \left|\det(x^{-1}, \mathrm{Lie}(U \cap P))\right|_F, m_{Q\cap N}(x) \quad = \left|\det(x^{-1}, \mathrm{Lie}(Q \cap N))\right|_F.$$

The equalities in $(*)$ come from exact sequences $1 \to Q \cap N \to Q \cap P \to Q_M \to 1$, etc. It remains to show that

$$\delta_Q m_{Q\cap N} \delta_P m_{U\cap P} = 1.$$

Q is the conjugate of a standard parabolic by an element of the Weyl group. Furthermore, $L \supset M_0$, $M \supset M_0$, $J \cong L \cap M \supset M_0$, $B = M_0 N_0$. We can restrict these quasi-characters to $M_0 \subset J$.

We need that for every $x \in M_0$,

$$(**) \qquad \det(x, \mathrm{Lie}(Q/Q \cap N)) = \det(x, \mathrm{Lie}(P/U \cap P))^{-1}.$$

Both sides are products of roots on M_0. Get that a root α contributes to the left-hand side of $(**)$ if and only if α is not in $\overline{U}$ and not in N. A root α contributes to the right-hand side of $(**)$ if and only if α is not in $\overline{N}$ and not in N. It is easy to see that α contributes to the left-hand side if and only if $-\alpha$ contributes to the right-hand side.

We conclude that $r_Q^G i_P^G \sigma$ has a filtration for which the associated graded pieces are given by

$$i_{L\cap w^{-1}Pw}^L \circ w \circ r_{wQw^{-1}}^M \sigma$$

where $w \in W_M \backslash W / W_L$. ■

17. Some applications of Theorem 16.1.1

17.1. **Some preliminary results.**

Lemma 17.1.1. *Suppose that P is a proper parabolic subgroup of G with a Levi decomposition $P = MN$. Suppose that $\sigma \in \mathfrak{R}(M)$. Each irreducible subquotient of $i_P^G \sigma$ is not supercuspidal.*

Proof. Suppose $V_1 \subset V_2$ are two G-subrepresentations of $i_P^G\sigma$ such that V_2/V_1 is an irreducible supercuspidal representation. Since irreducible supercuspidal representations are projective (Lemma 8.4.4), from Frobenius reciprocity we have

$$0 \neq \mathrm{Hom}_G(V_2/V_1, i_P^G\sigma) = \mathrm{Hom}_M(r_P^G(V_2/V_1), \sigma).$$

However, since V_2/V_1 is supercuspidal, we have $r_P^G(V_2/V_1) = 0$, a contradiction. ■

Corollary 17.1.2. *Suppose P is a proper parabolic subgroup of G with Levi decomposition $P = MN$ and $\sigma \in \mathfrak{R}(M)$. If $i_P^G\sigma$ is not of finite length, then there exists a proper parabolic subgroup Q of G with a Levi decomposition $Q = LU$ such that the L-representation $r_Q^G i_P^G\sigma$ is not of finite length.*

Proof. From Lemma 17.1.1, for each irreducible subquotient τ of i_P^G, there exists a standard parabolic Q such that $r_Q^G\tau$ is nonzero. Since there are only a finite number of standard parabolics and a nonfinite number of Jordan-Holder factors, the result follows. ■

17.2. **Some actual applications.** Suppose P and Q are standard parabolic subgroups of G with (standard) Levi decompositions $P = MN$ and $Q = LU$. Suppose $(\sigma, W) \in \mathfrak{R}(M)$ is irreducible supercuspidal.

Lemma 17.2.1. *Suppose $w \in N_G(T_\emptyset)$ is a representative for an element of the double coset space $W_M\backslash W/W_L \cong P\backslash G/Q$. We have*

$$i^L_{L\cap w^{-1}Pw^{-1}} w^{-1} \cdot r^M_{wQw^{-1}\cap M}\sigma = \begin{cases} \{0\} & wQw^{-1}\cap M \neq M \\ i^L_{L\cap w^{-1}Pw} w^{-1}\cdot\sigma & wQw^{-1}\cap M = M. \end{cases}$$

Proof. If $wQw^{-1}\cap M \neq M$, then $wQw^{-1}\cap M$ is a proper parabolic subgroup of M. Since σ is a supercuspidal representation of M, we have $r^M_{wQw^{-1}\cap M}\sigma = \{0\}$. On the other hand, if $wQw^{-1}\cap M = M$, then $r^M_{wQw^{-1}\cap M}\sigma = r^M_M\sigma = \sigma$. ■

Lemma 17.2.2. *Suppose (π, V) is an irreducible subquotient of $i_P^G\sigma$ and $\tau \in \mathfrak{R}(L)$ is irreducible and supercuspidal. If π occurs as a subrepresentation of $i_Q^G\tau$, then there exists a representative w in G of an element of the double coset space $W_M\backslash W/W_L$ such that*

$$w^{-1}Mw = L \text{ and } w^{-1}\cdot\sigma \cong \tau.$$

Proof. Since π occurs as a subrepresentation of $i_Q^G\tau$, from Frobenius reciprocity we have $\mathrm{Hom}_L(r_Q^G\pi, \tau) \neq 0$ and so τ occurs as an irreducible subquotient in $r_Q^G i_P^G\sigma$. From Theorem 16.1.1 we have that τ occurs in $i^L_{L\cap w^{-1}Pw} w^{-1}\cdot r^M_{wQw^{-1}\cap M}\sigma$ for some double coset representative w. From Lemma 17.2.1 we have that τ occurs in $i^L_{L\cap w^{-1}Pw} w^{-1}\cdot\sigma$ and $wQw^{-1}\cap M = M$. As τ is supercuspidal, from Lemma 17.1.1 we conclude that $L\cap w^{-1}Pw = L$. Since L and M are standard, we conclude that $L = w^{-1}Mw$ and $w^{-1}\cdot\sigma = \tau$. ■

Corollary 17.2.3. *If π is an irreducible subquotient of $i_P^G\sigma$, then there exists a parabolic P' with Levi decomposition $P' = MN'$ such that π occurs as a subrepresentation of $i_{P'}^G\sigma$.*

Proof. Suppose that π is an irreducible subquotient of $i_P^G\sigma$. From Corollary 7.3.3, there exists a standard parabolic Q' with a Levi decomposition $Q' = L'U'$ and $\tau \in \mathfrak{R}(L')$ supercuspidal and irreducible such that π is a subrepresentation of $i_{Q'}^G\tau$. From Lemma 17.2.2, there exists a representative w for an element of $W_M\backslash W/W_{L'}$ such that $w^{-1}\cdot\sigma \cong \tau$ and $w^{-1}Mw = L'$. If we set $P' = wQ'w^{-1}$, the result follows. ■

We can restate the above results in a some what fancier way.

Definition 17.2.4. Suppose H is a subgroup of G, $\sigma \in \mathfrak{R}(H)$, and $g \in G$. We define the representation $g\cdot\sigma$ of gHg^{-1} by $g\cdot\sigma(\ell) = \sigma(g^{-1}\ell g)$ for $\ell \in gHg^{-1}$.

Definition 17.2.5. Suppose P_i is a parabolic subgroup of G with a Levi decomposition $P_i = M_iN_i$. Let $\sigma_i \in \mathfrak{R}(M_i)$ be irreducible and supercuspidal. We call (M_i, σ_i) a cuspidal datum and write

$$(M_1, \sigma_1) \sim (M_2, \sigma_2)$$

provided that there exists $g \in G$ such that $gM_1g^{-1} = M_2$ and $g\cdot\sigma_2 = \sigma_1$.

The following corollary is a recasting of our prior results.

Corollary 17.2.6. *Suppose $\pi \in \mathfrak{R}(G)$ is irreducible and (M_1, σ_1) and (M_2, σ_2) are two cuspidal data. If P_i is a parabolic subgroup of G with Levi decomposition $P_i = M_iN_i$ and σ_i is an irreducible subquotient of $r_{P_i}^G\pi$, then $(M_1, \sigma_1) \sim (M_2, \sigma_2)$.*

We also have:

Corollary 17.2.7. *Suppose $\pi \in \mathfrak{R}(G)$ is irreducible. Up to the relation $\sim$, there exists a unique cuspidal datum (M', σ') such that π occurs as a subrepresentation of $i_{P'}^G\sigma'$.*

17.3. **The Bernstein spectrum.** In this subsection, we introduce the definition of $\mathfrak{B}(G)$, the *Bernstein spectrum* of G.

Definition 17.3.1. If M is the Levi component of a parabolic subgroup of G, then we let $\mathcal{P}(M)$ denote the finite set of parabolic subgroups of G which have M as a Levi component.

Definition 17.3.2. Suppose (M_1, τ_1) and (M_2, τ_2) are two cuspidal data. We write $(M_1, \tau_1) \approx (M_2, \tau_2)$ provided that there exist $g \in G$ and $\chi \in \mathbf{X}(M_2)$ such that

(1) ${}^gM_1 = M_2$ and
(2) $g\cdot\tau_1 \cong \tau_2 \otimes \chi$.

Remark 17.3.3. The relation $\approx$ is an equivalence relation, and if (M_1, τ_1) and (M_2, τ_2) are two cuspidal data for which $(M_1, \tau_1) \sim (M_2, \tau_2)$, then $(M_1, \tau_1) \approx (M_2, \tau_2)$.

Definition 17.3.4. If (M, τ) is a cuspidal datum, then $[M, \tau]$ denotes the equivalence class of (M, τ) with respect to $\approx$.

Definition 17.3.5. The set of equivalence classes $[M, \tau]$ is called the *Bernstein spectrum* of G and is denoted $\mathfrak{B}(G)$.

17.4. **Defining** $\mathfrak{R}(G)^{[L,\sigma]}$**.**

Lemma 17.4.1. *Suppose $(\pi, V) \in \mathfrak{R}(G)$. If for each cuspidal datum (L,σ) and for all parabolics $Q \in \mathcal{P}(L)$ we have*

$$\{0\} = e^{\sigma} \circ r_Q^G \pi \in \mathfrak{R}(L)^{[\sigma]},$$

then $V = \{0\}$.

Proof. If $V \neq \{0\}$, then there exists a parabolic subgroup Q which is minimal with respect to the property $r_Q^G\pi \neq \{0\}$. Let $Q = LU$ be a Levi decomposition of Q. For any proper parabolic subgroup P of L, we have

$$r_P^L r_Q^G \pi = \{0\}.$$

Therefore, $r_Q^G\pi \in \mathfrak{R}_{\mathrm{sc}}(L)$. From Corollary 11.2.6, the discussion following Definition 10.3.3, and our hypothesis, it follows that $r_Q^G\pi = \{0\}$, a contradiction. ■

It will be convenient to introduce the following notation.

Definition 17.4.2. If L is a Levi component of a parabolic subgroup of G, then for $\sigma \in \mathfrak{R}(L)$ we define

$$i_L^G\sigma = \bigoplus_{Q \in \mathcal{P}(L)} i_Q^G\sigma$$

and for $\pi \in \mathfrak{R}(G)$ we define

$$r_L^G\pi = \bigoplus_{Q \in \mathcal{P}(L)} r_Q^G\pi.$$

Lemma 17.4.3. *Fix a cuspidal datum (L,σ). For $(\pi, V) \in \mathfrak{R}(G)$, the following statements are equivalent.*

1. *π is a subrepresentation of an element of $i_L^G(\mathfrak{R}(L)^{[\sigma]})$.*
2. *π is a subquotient of an element of $i_L^G(\mathfrak{R}(L)^{[\sigma]})$.*
3. *If (M,τ) is a cuspidal datum such that $(M,\tau) \notin [L,\sigma]$, then $e^{\tau} \circ r_P^G\pi = \{0\}$ for all parabolic subgroups $P \in \mathcal{P}(M)$.*
4. *Each irreducible subquotient of π occurs as a subrepresentation of $i_Q^G(\sigma \otimes \chi)$ for some $\chi \in \mathbf{X}(L)$ and some parabolic $Q \in \mathcal{P}(L)$.*
5. *If $N_G(L)^{\sigma} := \mathrm{stab}_{N_G(L)}\,\sigma$, then for all parabolic subgroups $Q \in \mathcal{P}(L)$ we have $r_Q^G\pi \in \bigoplus_{N_G(L)/N_G(L)^{\sigma}} \mathfrak{R}(L)^{[g\cdot\sigma]}$. Moreover, the natural G-homomorphism*

$$\varphi\colon (\pi, V) \to \bigoplus_{Q \in \mathcal{P}(L)} i_Q^G \circ e^{\sigma} \circ r_Q^G\pi$$

is injective.

Remark 17.4.4. In item (5) above, the map φ is obtained as follows. Suppose $Q \in \mathcal{P}(L)$. The projection map $e^{\sigma}\colon \mathfrak{R}(L) \to \mathfrak{R}(L)^{[\sigma]}$ induces an element of $\mathrm{Hom}_L(r_Q^G\pi, e^{\sigma} \circ r_Q^G\pi)$. From Frobenius reciprocity, we then have an element $\varphi_Q \in \mathrm{Hom}_G(\pi, i_Q^G \circ e^{\sigma} \circ r_Q^G\pi)$.

Proof. It is clear that $(5) \Rightarrow (1) \Rightarrow (2)$. We shall show $(2) \Rightarrow (3) \Rightarrow (5)$ and $(3) \Leftrightarrow (4)$.

"$(2) \Rightarrow (3)$": Since the Jacquet functor is exact and preserves direct sums, it is enough to show that if $\sigma' \in \mathfrak{R}(L)^{[\sigma]}$ and $(M,\tau) \notin [L,\sigma]$, then $e^\tau \circ r_P^G \circ i_Q^G \sigma' = \{0\}$ for all parabolic subgroups $P \in \mathcal{P}(M)$ and $Q \in \mathcal{P}(L)$.

Without loss of generality, we may assume that P and Q are in standard position. From Theorem 16.1.1 we know that $e^\tau \circ r_P^G \circ i_Q^G \sigma'$ has a filtration with associated graded pieces

$$e^\tau \circ i^M_{M \cap w^{-1}Qw} w^{-1} \cdot r^L_{L \cap wPw^{-1}} \sigma'$$

Since σ' is supercuspidal, we have that $R^L_{L \cap wPw^{-1}} \sigma'$ is zero unless $L \cap wPw^{-1} = L$. Since τ is supercuspidal, from Lemma 17.1.1 we have that $e^\tau(i_{M \cap w^{-1}Qw} w^{-1} \cdot \sigma') = 0$ unless $M \cap w^{-1}Qw = M$. Thus, the only way the associated graded piece can be nonzero is if $wMw^{-1} = L$ and $w \cdot \tau = \sigma$. But this means that $(M,\tau) \in [L,\sigma]$, a contradiction.

"$(3) \Rightarrow (5)$": Let Q be a parabolic subgroup of G in $\mathcal{P}(L)$. We first show that $r_Q^G \pi \in \mathfrak{R}_{\mathrm{sc}}(L)$. Indeed, if $r_Q^G \pi \notin \mathfrak{R}_{\mathrm{sc}}(L)$, then there exists a parabolic subgroup $P \in \mathcal{P}(M)$ such that $P \cap L$ is a proper parabolic subgroup of L which is minimal with respect to the property that $\{0\} \neq r_P^G \pi = r^L_{P \cap L} r_Q^G \pi$. Thus $r_P^G \pi \in \mathfrak{R}_{\mathrm{sc}}(M)$ and so from Corollary 11.2.6 and the discussion following Definition 10.3.3 there exists a cuspidal datum (M,τ) for which $e^\tau \circ r_P^G \pi \neq \{0\}$. Since M is not conjugate to L, this contradicts (3).

Therefore, we have $r_Q^G \pi \in \mathfrak{R}_{\mathrm{sc}}(L)$. In fact, it follows from the above paragraph and (3) that

$$r_Q^G \pi \in \bigoplus_{g \in N_G(L)/N_G(L)L^\sigma} \mathfrak{R}(L)^{[g \cdot \sigma]}.$$

Finally, from Lemma 17.4.1 we need to show that for each cuspidal datum (M,τ) and for all parabolic subgroups $P \in \mathcal{P}(M)$ we have

$$\{0\} = e^\tau \circ r_P^G(\ker \varphi).$$

Since $\ker \varphi \subset \pi$, if $(M,\tau) \notin [L,\sigma]$, then this follows from (3).

Thus, we must show that for any parabolic subgroup $Q' \in \mathcal{P}(L)$ we have

$$e^\sigma \circ r^G_{Q'}(\ker \varphi) = \{0\}.$$

Note that if Q is any parabolic subgroup of G in $\mathcal{P}(L)$ and

$$\varphi_Q \colon \pi \to i_Q^G \circ e^\sigma \circ r_Q^G \pi,$$

is the natural map, then

$$\ker \varphi = \bigcap_{Q \in \mathcal{P}(L)} \ker \varphi_Q.$$

Thus, to show that $e^\sigma \circ r^G_{Q'}(\ker \varphi) = \{0\}$ for all $Q' \in \mathcal{P}(L)$, it will be enough to show that $e^\sigma \circ r_Q^G(\ker \varphi_Q) = \{0\}$ for all $Q \in \mathcal{P}(L)$.

Fix $Q \in \mathcal{P}(L)$. Since e^σ and r_Q^G are exact functors, we have the exact sequence

$$0 \to e^\sigma \circ r_Q^G(\ker \varphi_Q) \to e^\sigma \circ r_Q^G \pi \to e^\sigma \circ r_Q^G \circ i_Q^G(e^\sigma \circ r_Q^G \pi).$$

From Theorem 16.1.1 the last term is isomorphic (as an L-module) to $e^\sigma \circ r_Q^G \pi$. Thus, $e^\sigma \circ r_Q^G(\ker \varphi_Q) = \{0\}$.

"(4) $\Rightarrow$ (3)": If (M,τ) is a cuspidal datum and $(M,\tau)\notin[L,\sigma]$, then from Theorem 16.1.1 we have $e^{\tau}\circ r_P^G\circ i_Q^G(\sigma\otimes\chi)=\{0\}$ for all $\chi\in\mathbf{X}(L)$ and for all parabolic subgroups $P\in\mathcal{P}(M)$. Thus, if every irreducible subquotient of π occurs as a subrepresentation of $i_Q^G(\sigma\otimes\chi)$ for some $\chi\in\mathbf{X}(L)$ and some parabolic $Q\in\mathcal{P}(L)$, then since e^{τ} and r_Q^G are both exact, we have $e^{\tau}\circ r_P^G\pi=\{0\}$.

"(3) $\Rightarrow$ (4)": Let π' be an irreducible subquotient of π. There exist a cuspidal datum (M,τ) and a parabolic $P\in\mathcal{P}(M)$ such that π' occurs as a subrepresentation of $i_P^G\tau$. This implies that $e^{\tau}\circ r_P^G\pi\neq\{0\}$. Thus we have $(M,\tau)\in[L,\sigma]$. Free to conjugate by elements of G, for some parabolic $Q\in\mathcal{P}(L)$ we have $e^{\sigma}\circ r_Q^G\pi'\neq 0$. This implies $\mathrm{Hom}_L(r_Q^G\pi',e^{\sigma}\circ r_Q^G\pi')\neq 0$. Therefore, for some $\chi\in\mathbf{X}(L)$, we have $\mathrm{Hom}_L(r_Q^G\pi',\sigma\otimes\chi)\neq 0$. From Frobenius reciprocity, we have that π' occurs as a subrepresentation of $i_Q^G(\sigma\otimes\chi)$. ■

Lemma 17.4.5. *Suppose (L_1,σ_1) and (L_2,σ_2) are two cuspidal data. Suppose $[L_1,\sigma_1]\neq[L_2,\sigma_2]$.*

(1) *If $(\pi,V)\in\mathfrak{R}(G)$ satisfies any of the five statements of Lemma 17.4.3 with respect to both (L_1,σ_1) and (L_2,σ_2), then $V=\{0\}$.*
(2) *If $\pi_i\in\mathfrak{R}(G)$ satisfies any of the five statments of Lemma 17.4.3 with respect to (L_i,σ_i), then $\mathrm{Hom}_G(\pi_1,\pi_2)=0$.*

Proof. We begin with the first statement. From Lemma 17.4.3 (3) we have

$$e^{\tau}\circ r_P^G\pi=\{0\}$$

for each cuspidal datum (M,τ) and each parabolic subgroup P of G in $\mathcal{P}(M)$. From Lemma 17.4.1 we have that $\pi=\{0\}$.

For the second statement, suppose that $f\in\mathrm{Hom}_G(\pi_1,\pi_2)$. Since $f(\pi_1)$ satisfies all of the statements of Lemma 17.4.3 with respect to both (L_1,σ_1) and (L_2,σ_2), we have that $f(\pi_1)=\{0\}$. ■

Definition 17.4.6. Let (L,σ) be a cuspidal datum. We let $\mathfrak{R}(G)^{[L,\sigma]}$ denote the full subcategory of $\mathfrak{R}(G)$ whose objects are smooth representations of G that satisfy any of the equivalent statements listed in Lemma 17.4.3.

17.5. **The Bernstein decomposition.**

Definition 17.5.1. A representation $V\in\mathfrak{R}(G)$ is said to be *split* if we can write

$$V=\bigoplus_{[L,\sigma]\in\mathfrak{B}(G)}V([L,\sigma])$$

with $V([L,\sigma])\in\mathfrak{R}(G)^{[L,\sigma]}$.

If $[L,\sigma]\in\mathfrak{B}(G)$, then we define $\mathrm{Irr}_{[L,\sigma]}(G)$ to be the set of equivalence classes of irreducible representations in $\mathfrak{R}(G)^{[L,\sigma]}$.

Lemma 17.5.2. *If $V_1,V_2\in\mathfrak{R}(G)$ with V_2 split and $V_1\subset V_2$, then V_1 is split.*

Proof. Since V_2 is split, we can write

$$V_2 = \bigoplus_{[L,\sigma]\in\mathfrak{B}(G)} V_2([L,\sigma])$$

with $V_2([L,\sigma]) \in \mathfrak{R}(G)^{[L,\sigma]}$. We define

$$V_1([L,\sigma]) = V_2([L,\sigma]) \cap V_1$$

and consider

$$W = V_1/(\oplus_{[L,\sigma]\in\mathfrak{B}(G)} V_1([L,\sigma])).$$

In order to establish the result, it will be enough to show that W has no irreducible subquotients.

Fix $[M,\tau] \in \mathfrak{B}(G)$. Let $\mathfrak{B}(G)' = \mathfrak{B}(G) \smallsetminus \{[M,\tau]\}$. Consider the projection operator

$$\mathrm{pr}_{[M,\tau]} \colon V_2 \to \bigoplus_{[L,\sigma]\in\mathfrak{B}(G)'} V_2([L,\sigma]).$$

This projection induces a map

$$\overline{\mathrm{pr}}_{[M,\tau]} \colon W \to (\bigoplus_{[L,\sigma]\in\mathfrak{B}(G)'} V_2([L,\sigma]))/(\bigoplus_{[L,\sigma]\in\mathfrak{B}(G)'} V_1([L,\sigma])).$$

Since V_2 is split, we have $\ker(\mathrm{pr}_{[M,\tau]}) = V_2([M,\tau])$. This implies that the kernel of the map

$$\mathrm{res}_{V_1} \mathrm{pr}_{[M,\tau]} \to \bigoplus_{[L,\sigma]\in\mathfrak{B}(G)'} V_2([L,\sigma])$$

is $V_1([M,\tau])$. Therefore, $\overline{\mathrm{pr}}_{[M,\tau]}$ injects W into a subquotient of

$$\bigoplus_{[L,\sigma]\in\mathfrak{B}(G)'} V_2([L,\sigma]).$$

Therefore, any irreducible subquotient of W belongs to[7]

$$\bigcup_{[L,\sigma]\in\mathfrak{B}(G)'} \mathrm{Irr}_{[L,\sigma]}(G).$$

As $[M,\tau]$ was arbitrary and

$$\mathrm{Irr}(G) = \bigcup_{[L,\sigma]\in\mathfrak{B}(G)} \mathrm{Irr}_{[L,\sigma]}(G),$$

we conclude that every irreducible subquotient of W is the zero representation. ■

Lemma 17.5.3. *If $(\pi, V) \in \mathbb{R}(G)$ and $v \in V$, then $e^\sigma r_L^G v = 0$ for almost all $[L,\sigma] \in \mathfrak{B}(G)$.*

Proof. Suppose $v \in V$. Since, up to conjugation, there are only finitely many Levi subgroups of G, it will be enough to show that for a fixed L, $e^\sigma r_L^G v = 0$ for almost all $\sigma \in \mathfrak{R}_{\mathrm{sc}}(L)$ (up to twisting by an element of $\mathbf{X}(L)$.

Choose a compact open subgroup of G for which v is K-fixed. Since $r_L^G v$ is $K \cap M$-fixed, from Corollary 11.1.2 there are only finitely many pairs $[L,\sigma]$ for which $e^\sigma r_L^G v \neq 0$. ■

[7]Literally: is an element of an element of

Lemma 17.5.4. *For each $V \in \mathfrak{R}(G)$, the map from V to*

$$\bigoplus_{[L,\sigma]} \bigoplus_{P \in \mathcal{P}(L)} i_P^G e^\sigma r_P^G \pi$$

given by

$$v \mapsto \sum_{[L,\sigma]} \sum_{P \in \mathcal{P}(L)} (g \mapsto e^\sigma r_P^G \pi(g) v)$$

is injective

Proof. The kernel of the map will be a G-module V' for which

$$e^\sigma r_L^G V' = \{0\}$$

for all $[L, \sigma] \in \mathfrak{B}$. Such a G-module must be trivial. ∎

Exercise 17.5.5. Define $i_{\mathrm{sc}} \colon \mathfrak{R}(G) \to \mathfrak{R}(G)$ by

$$i_{\mathrm{sc}}(V) := \bigoplus_{[L,\sigma]\,,\, L \neq G} \bigoplus_{P \in \mathcal{P}(L)} i_P^G \circ e^\sigma \circ r_P^G V$$

for $V \in \mathfrak{R}(G)$. Show that the kernel of i_{sc} is $\mathfrak{R}(G)_{\mathrm{sc}}$.

The following theorem follows from Lemma 17.5.2 and Lemma 17.5.4.

Theorem 17.5.6.

$$\mathfrak{R}(G) = \prod_{[L,\sigma] \in \mathfrak{B}(G)} \mathfrak{R}(G)^{[L,\sigma]}.$$

18. Langlands' classification

18.1. **Tempered representations.**

Definition 18.1.1. A representation $\pi \in \mathfrak{R}(G)$ is said to be tempered provided that π is admissible and for all standard parabolics P with Levi decomposition $P = MN$ and for all normalized exponents of π relative to $P = MN$ we have $|\chi| \leq 1$ on T_M^+.

Remark 18.1.2. If π is tempered, then every normalized exponent of π relative to G is unitary. In particular, if π is irreducible and tempered, then its central character is unitary.

Remark 18.1.3. If $\pi \in \mathfrak{R}(G)$ is irreducible and square integrable modulo $Z(G)$, then from Casselman's square integrability criterion (Theorem 15.0.1) and Corollary 7.3.5, π is tempered.

Our immediate goal is to understand how tempered representations behave with respect to induction and Jacqueting.

Lemma 18.1.4. *Suppose $\pi \in \mathfrak{R}(G)$ is tempered. Suppose P is a standard parabolic with Levi decomposition $P = MN$ and χ is a normalized exponent of π with respect to $P = MN$. If χ is a unitary character of T_M, then*

$$(r_P^G \pi)_\chi$$

is a tempered representation of M.

Proof. The standard parabolics in M come from standard parabolics Q in G that are contained in P. If Q has Levi decomposition $Q = LU$, then the standard parabolic $Q \cap M$ of M has Levi decomposition $L(Q \cap M)$. Thus, it follows from the transitivity of Jacqueting that any normalized exponent χ_Q of $(r_P^G \pi)_\chi$ with respect to $Q \cap M$ is a normalized exponent of π with respect to Q. Consequently, since π is tempered, we have that $|\chi_Q| \le 1$ on

(9) $$\{t \in T_L \mid |\alpha(t)| \le 1 \text{ for all } \alpha \in \Delta\}.$$

We need to show that $|\chi_Q| \le 1$ on the set

$$\{t \in T_L \mid |\alpha(t)| \le 1 \text{ for all } \alpha \in \Delta_M\}.$$

Here Δ_M is θ where $M = M_\theta$. If t belongs to this set, then there exists $t_1 \in T_M$ and t_2 in the set defined by statement (9) such that $t = t_1 t_2$. Thus, it will be enough to show that $|\chi_Q| \le 1$ on T_M. However, this follows immediately from the fact that

$$\operatorname{res}_{T_M} \chi_Q = \chi$$

and χ is a unitary character. ∎

Lemma 18.1.5. *Suppose P is a standard parabolic subgroup of G with Levi decomposition $P = MN$. If $\sigma \in \mathfrak{R}(M)$ is tempered, then $i_P^G \sigma$ is tempered.*

Proof. Fix a standard parabolic subgroup Q of G with Levi decomposition $Q = LU$. Fix a normalized exponent χ of $i_P^G \sigma$ relative to Q. We must show that $|\chi| \le 1$ on

(10) $$\{t \in T_L \mid |\alpha(t)| \le 1 \text{ for all } \alpha \in \Delta\}.$$

From Theorem 16.1.1, as an L-module, $(r_Q^G i_P^G \sigma)_\chi$ has a filtration with associated graded pieces

$$(i^L_{L \cap w^{-1}Pw} w^{-1} \cdot r^M_{wQw^{-1} \cap M} \sigma)_\chi$$

where w runs over representatives in G for the double coset space $W_M \backslash W / W_L \cong P \backslash G / Q$. From Remark 16.1.2 we can assume that $L \cap w^{-1}Pw$ is a standard parabolic in L and $wQw^{-1} \cap M$ is a standard parabolic in M.

Since σ is a tempered representation of M, we have that every normalized exponent of σ with respect to $wQw^{-1} \cap M$ has absolute value less than or equal to one on the set

$$\{t \in T_{wLw^{-1} \cap M} \mid |\alpha(t)| \le 1 \text{ for all } \alpha \in \Delta_M\}.$$

Consequently, every character χ of $T_{L \cap w^{-1}Mw}$ for which $(w^{-1} \cdot r^M_{wQw^{-1} \cap M} \sigma)_\chi \neq \{0\}$ has $|\chi| \le 1$ on

$$\{t \in T_{L \cap w^{-1}Mw} \mid |\alpha(t)| \le 1 \text{ for all } \alpha \in \Delta_{w^{-1}Mw}\}.$$

Therefore, every character ψ of T_L for which $(i^L_{w^{-1}Pw \cap L} w^{-1} \cdot r^M_{wQw^{-1} \cap M} \sigma)_\psi \neq \{0\}$ has $|\psi| \le 1$ on the subset

$$\{t \in T_L \mid |\alpha(t)| \le 1 \text{ for all } \alpha \in \Delta\}$$

of the set

$$\{t \in T_{L \cap w^{-1}Mw} \mid |\alpha(t)| \le 1 \text{ for all } \alpha \in \Delta_{w^{-1}Mw}\}.$$

∎

Lemma 18.1.6. *Suppose $\pi \in \mathfrak{R}(G)$ is irreducible and tempered. If for each proper standard parabolic P of G with Levi decomposition $P = MN$ and for each normalized exponent χ of π with respect to $P = MN$ we have χ is not unitary, then π is square integrable modulo the center of G.*

Proof. Suppose P is a standard proper parabolic subgroup of G with Levi decomposition $P = MN$. Let χ be a normalized exponent of π relative to $P = MN$. In order to use the square integrability criterion (Theorem 15.0.1), we must show that $|\chi| < 1$ on T_M^+.

Suppose first that P is a maximal standard proper parabolic subgroup of G. Since π is tempered, we have $|\chi| \leq 1$ on $T_M^+ \setminus T_G^+$. Thus, since T_M/T_G is free of rank one, we have that either $|\chi| = 1$ on T_M^+ or $|\chi| < 1$ on $T_M^+ \setminus T_G^+$. By hypothesis, the latter condition must hold.

Let $P_1, P_2, \ldots, P_m$ denote the standard proper maximal parabolic subgroups of G than contain P. Suppose that $P_i = M_i N_i$ is the Levi decomposition of P_i. We have $M \subset M_i$ for each i. Thus, if χ is an exponent of π relative to $P = MN$, then its restriction χ_i to T_{M_i} is an exponent of π with respect to $P_i = M_i N_i$.

Suppose $t \in T_M^+ \setminus T_G^+$. There exist $t_i \in T_{M_i}^+$ such that $t = t_1 t_2 \cdots t_m$. We have

$$|\chi|(t) = |\chi|(t_1)\,|\chi|(t_2)\cdots|\chi|(t_m) = |\chi_1|(t_1)\,|\chi_2|(t_2)\cdots|\chi_m|(t_m) < 1.$$

■

Theorem 18.1.7. *Suppose $\pi \in \mathfrak{R}(G)$ is irreducible. We have that π is tempered if and only if there exist a standard parabolic P with Levi decomposition $P = MN$ and an irreducible square integrable modulo $Z(M)$ representation $\sigma \in \mathfrak{R}(M)$ for which π is a subrepresentation of $i_P^G\sigma$.*

Remark 18.1.8. It follows from this theorem that every irreducible tempered representation is unitary. Consequently, every tempered representation is unitary.

Proof. "$\Leftarrow$": This is Lemma 18.1.5.

"$\Rightarrow$": Choose a standard parabolic P with Levi decomposition $P = MN$ which is minimal with respect to the property: there exists a normalized exponent of π relative to $P = MN$ which is unitary. Since G has this property, P must exist. Let χ denote the unitary normalized exponent. Lemma 18.1.4 tells us that $(r_P^G\pi)_\chi$ is tempered. Since π is irreducible, from Lemma 6.3.1 we have that $(r_P^G\pi)_\chi$ is finitely generated. From Lemma 3.3.5, we can choose an irreducible quotient σ of $(r_P^G\pi)_\chi$. From Lemma 18.1.6 and the minimality of $P = MN$, it follows that σ is square integrable modulo the center of M. The result now follows from Frobenius Reciprocity. ■

18.2. **Real exponents.**

18.2.1. *Some definitions.* We begin with some standard definitions. Suppose T denotes the group of k-rational points of a k-split torus $\mathbf{T}$ of $\mathbf{G}$.

Definition 18.2.1. We define $\mathbf{X}_*(T)$ to be the group of one parameter subgroups of T, that is, the set of algebraic homomorphisms from GL_1 to $\mathbf{T}$.

It is standard to denote the group $\mathrm{Rat}(T)$ by $X^*(T)$. We have a perfect pairing $\langle\,,\,\rangle\colon X_*(T) \times X^*(T) \to \mathbb{Z}$ defined by the equation

$$\chi \circ \lambda(t) = t^{\langle \lambda, \chi \rangle}$$

for $\chi \in \mathbf{X}^*(T)$ and $\lambda \in \mathbf{X}_*(T)$.

Recall that

$$T^0 = \bigcap_{\chi \in \mathbf{X}^*(T)} \ker |\chi|$$

(see Definitions 7.2.1 and 7.2.3).

Exercise 18.2.2. Show that the map $\lambda \to \lambda(\varpi)$ induces an isomorphism of $\mathbf{X}_*(T)$ with T/T^0.

Suppose P is a standard parabolic subgroup with a Levi decomposition $P = MN$.

We define

$$\mathfrak{a}_M := \mathbf{X}_*(T_M) \otimes_{\mathbb{Z}} \mathbb{R},$$

and, because it plays a central role, we set $\mathfrak{a}_\emptyset := \mathfrak{a}_{M_\emptyset}$. Similarly, we define

$$\mathfrak{a}_M^* := \mathbf{X}^*(T_M) \otimes_{\mathbb{Z}} \mathbb{R},$$

and

$$\mathfrak{a}_\emptyset^* := \mathfrak{a}_{M_\emptyset}^*.$$

We extend the perfect pairing $\langle\,,\,\rangle$ on $\mathbf{X}_*(T_M) \times \mathbf{X}^*(T_M)$ to a pairing on $\mathfrak{a}_M \times \mathfrak{a}_M^*$. Since W acts on both $X_*(T_\emptyset)$ and $X^*(T_\emptyset)$, it naturally acts on both $\mathfrak{a}_\emptyset$ and $\mathfrak{a}_\emptyset^*$.

We identify $\mathfrak{a}_M^*$ with the subspace of W_M-fixed elements in $\mathfrak{a}_\emptyset^*$. With respect to this identification, the projection map from $\mathfrak{a}_\emptyset^*$ to $\mathfrak{a}_M^*$ which sends $\nu \in \mathfrak{a}_\emptyset^*$ to

$$\frac{1}{|W_M|} \sum_{w \in W_M} w\nu$$

agrees with the natural restriction map from $\mathfrak{a}_\emptyset^*$ to $\mathfrak{a}_M^*$.

We have a natural inclusion from $\mathfrak{a}_M$ into $\mathfrak{a}_\emptyset$. Moreover, if $\mathfrak{a}^M$ denotes the perpendicular in $\mathfrak{a}_\emptyset$ to $\mathfrak{a}_M^*$ ($\subset \mathfrak{a}_\emptyset^*$), then $\mathfrak{a}_\emptyset = \mathfrak{a}_M \oplus \mathfrak{a}^M$.

Note that the set of roots Φ injects into $\mathfrak{a}_\emptyset^*$. For $\nu \in \mathfrak{a}_\emptyset^*$, we write $\nu \geq 0$ if ν can be written as a linear combination of simple roots with nonnegative coefficients. We write $\nu > 0$ if $\nu \geq 0$ and $\nu \neq 0$. For $\nu, \nu' \in \mathfrak{a}_\emptyset^*$, we write $\nu \geq \nu'$ (resp. $\nu > \nu'$) provided that $\nu - \nu' \geq 0$ (resp. $\nu - \nu' > 0$). (Note: If $T_G \neq \{1\}$, then any element of $\mathfrak{a}_\emptyset^*$ which projects to a nontrivial element in $\mathfrak{a}_G^*$ will fail to be "positive".)

We define

$$C_P := \{\nu \in \mathfrak{a}_M^* \mid \langle \check{\alpha}, \nu \rangle > 0 \text{ for all } \alpha \in \Delta \setminus \Delta_M\}$$

and

$$\overline{C}_P := \{\nu \in \mathfrak{a}_M^* \mid \langle \check{\alpha}, \nu \rangle \geq 0 \text{ for all } \alpha \in \Delta \setminus \Delta_M\}$$

Here $\check{\alpha}$ denotes the unique one parameter subgroup in $\mathfrak{a}_\emptyset$ for which

$$w_\alpha(\nu) = \nu - \langle \check{\alpha}, \nu \rangle \alpha$$

for the simple reflection w_α corresponding to α. We set $C := C_{P_\emptyset}$ and $\overline{C} := \overline{C}_{P_\emptyset}$; C is usualy called the *standard Weyl chamber* in $\mathfrak{a}_\emptyset^*$.

Exercise 18.2.3. (1) For the group $\mathrm{GL}_n(k)$, let M be the standard parabolic corresponding to the partition $n = n_1 + n_2 + \cdots + n_k$. Explicitly realize a_M^* and the map $a_\emptyset^* \to a_M^*$.

(2) For the groups $\mathrm{GL}_2(k)$ and $\mathrm{SL}_3(k)$, draw a picture illustrating the sets $\mathfrak{a}_M^*$, C_P, and $\overline{C}_P$ for all possible standard parabolics P with Levi decompositions $P = MN$.

Exercise 18.2.4. Let φ denote the projection map $\mathfrak{a}_\emptyset^* \to \mathfrak{a}_M^*$. Show that if $\nu \in C := C_{P_\emptyset}$, then $\nu \geq \varphi(\nu)$. Show that φ takes $\overline{C} := \overline{C}_{P_\emptyset}$ onto $\overline{C}_P$ and φ takes the set $\{\nu \in \mathfrak{a}_\emptyset^* \,|\, \nu \geq 0\} \neq C$ onto

$$\{\nu \in \mathfrak{a}_M^* \mid \mathrm{res}_{\{x \in \mathfrak{a}_M \,|\, \alpha \in \Delta \Rightarrow \alpha(x) \leq 0\}}\, \nu \leq 0\}$$

so that $\nu \geq 0$ implies $\varphi(\nu) \geq 0$.

18.2.2. *Characters of M.* Suppose P is a standard parabolic subgroup of G with Levi decomposition $P = MN$. We now wish to show that the set $\mathfrak{a}_M^*$ is in natural bijective correspondence with the set $\mathbf{X}^+(M)$ of positive real-valued unramified characters of M.

We have $T_M/T_M^0 = T_M/(T_M \cap M_0)$ is a full rank sublattice of M/M^0 of finite index. Consequently, for every $\chi \in \mathbf{X}^*(T_M)$, there exists a unique $\chi' \in \mathbf{X}^+(M)$ such that $\chi'(t) = |\chi(t)|_k$ for all $t \in T_M$. We therefore get a map from $\mathfrak{a}_M^* = \mathbf{X}^*(T_M) \otimes_{\mathbb{Z}} \mathbb{R}$ to $\mathbf{X}^+(M)$ by sending $\chi \otimes r$ to $(m \mapsto \chi'(m)^r)$.

In the reverse direction, suppose that $\psi \in \mathbf{X}^+(M)$. We define $||\psi|| : \mathbf{X}_*(T_M) \otimes \mathbb{R} \to \mathbb{R}$ by sending $\lambda \otimes r \in \mathbf{X}^*(T_M) \otimes \mathbb{R}$ to $r \cdot \log_q(\psi(\lambda(\varpi^{-1})))$.

These two maps are inverses of each other. To simplify our notation, we will think of $\nu \in \mathfrak{a}_M^*$ as being both an element of $\mathfrak{a}_M^*$ and as an element of $\mathbf{X}^+(M)$.

18.2.3. *Real exponents.*

Definition 18.2.5. Suppose $\pi \in \mathfrak{R}(G)$ is admissible and P is a standard parabolic with Levi decomposition $P = MN$. An element $\nu \in \mathfrak{a}_M^*$ ($\subset \mathfrak{a}_\emptyset^*$) is called a *real exponent* of π relative to $P = MN$ provided that there exists a normalized exponent χ of π relative to $P = MN$ such that $\nu(\lambda \otimes s) = |\chi|\,(\lambda \otimes s) := s \cdot \log_q(|\chi(\lambda(\varpi^{-1}))|_{\mathbb{C}})$.

Exercise 18.2.6. Show that $\pi \in \mathfrak{R}(G)$ is tempered if and only if π is admissible and for each standard parabolic subgroup P with Levi decomposition $P = MN$ and for each real exponent $\nu \in \mathfrak{a}_M^*$ of π relative to $P = MN$ we have $\nu \geq 0$.

18.3. **The Langlands' classification theorem.** Fix a standard parabolic subgroup P of G with Levi decomposition $P = MN$. Let σ denote a tempered representation of M. Choose $\nu \in C_P$.

Thinking of ν as an element of $\mathbf{X}^+(M)$, we look at the representation

$$i_P^G(\sigma \otimes \nu).$$

It will turn out that this representation has a unique irreducible quotient (called the *Langlands' quotient*) . Moreover, a different choice of data (P', σ', ν') produces the same Langlands' quotient if and only if $P' = P$, $\sigma' \cong \sigma$ and $\nu' = \nu$. Finally, we will show that any irreducible smooth representation occurs as a Langlands' quotient.

18.4. **Some preliminaries.** Fix a standard parabolic subgroup P of G with Levi decomposition $P = MN$. Let σ denote a tempered representation of M. Choose $\nu \in C_P$.

Lemma 18.4.1. *Suppose Q is a standard parabolic subgroup of G with Levi decomposition $Q = LU$. If $\nu' \in \mathfrak{a}_L^* \subset \mathfrak{a}_\emptyset^*$ is a real exponent of $i_P^G(\sigma \otimes \nu)$ relative to $\overline{Q} = L\overline{U}$ (the parabolic opposite $Q = LU$), then $\nu' \leq \nu$.*

The proof is very similar to that of Lemma 18.1.5.

Proof. As an L-module, $r_{\overline{Q}}^G i_P^G(\sigma \otimes \nu)$ has a filtration with associated graded pieces

$$i_{L\cap w^{-1}Pw}^L\, w^{-1} \cdot r_{w\overline{Q}w^{-1}\cap M}^M(\sigma \otimes \nu)$$

for $w \in W_M \backslash W / W_L$. We need to show that if χ is a smooth character of T_L for which

$$(i_{L\cap w^{-1}Pw}^L w^{-1} \cdot r_{w\overline{Q}w^{-1}\cap M}^M(\sigma \otimes \nu))_\chi \neq \{0\},$$

then $\nu - |\chi| \geq 0$.

Any real exponent of $\sigma \otimes \nu$ relative to $w\overline{Q}w^{-1} \cap M$ has the form

$$\nu + \mu \in \mathfrak{a}_{wLw^{-1}\cap M}^*$$

(where $\nu \in \mathfrak{a}_M^* \subset \mathfrak{a}_{wLw^{-1}\cap M}^*$). Since σ is a tempered representation of M, we have $\mu \geq 0$ in $\mathfrak{a}_M^*$ with respect to $w\overline{P}_\emptyset w^{-1} \cap M$. That is, μ is a non negative linear combination of simple roots of M which are positive with respect to $w\overline{P}_\emptyset w^{-1} \cap M$. Therefore, if ψ is a smooth character of $T_{L\cap w^{-1}Mw}$ such that

$$(w^{-1} \cdot r_{w\overline{Q}w^{-1}\cap M}^M)_\psi \neq \{0\},$$

then $|\psi| \in \mathfrak{a}_{L\cap w^{-1}Mw}^*$ has the form $w^{-1}\nu + w^{-1}\mu$ where μ is a non negative linear combination of simple roots of $w^{-1}Mw$ which are negative with respect to $P_\emptyset \cap w^{-1}Mw$.

If ψ is a smooth character of T_L for which

$$(i_{L\cap w^{-1}Pw}^L w^{-1} \cdot r_{w\overline{Q}w^{-1}\cap M}^M(\sigma \otimes \nu))_\psi \neq \{0\},$$

then, from the above, $|\psi| \in \mathfrak{a}_L^*$ looks like

$$\overline{w^{-1}\nu} + \overline{w^{-1}\mu}$$

where $\overline{x}$ denotes the image of $x \in \mathfrak{a}_{L\cap w^{-1}Mw}^*$ in $\mathfrak{a}_L^*$ under the natural projection map. We have $\overline{w^{-1}\mu} \leq 0$ in $\mathfrak{a}_L^*$.

Since $\nu \in C$, we have $\nu \geq \overline{\nu}$ and $\nu - w^{-1}\nu \geq 0$. Thus

$$\begin{aligned} \nu - |\chi| &\geq \overline{\nu} - \overline{w^{-1}\nu} - \overline{w^{-1}\mu} \\ &\geq \overline{\nu} - \overline{w^{-1}\nu} \qquad (11) \\ &\geq 0 \end{aligned}$$

■

Definition 18.4.2. If $\sigma \in \mathfrak{R}(M)$ is admissible, then we define

$$\sigma_\nu = \oplus \sigma_\chi$$

where the sum is over those characters χ of T_M for which $|\chi| = \nu$.

We let $\overline{P} = M\overline{N}$ denote the parabolic opposite $P = MN$.

Corollary 18.4.3.
$$(r^G_{\overline{P}} i^G_P(\sigma \otimes \nu))_\nu \cong \sigma \otimes \nu.$$

Proof. As an M-representation, $r^G_{\overline{P}} i^G_P(\sigma \otimes \nu)$ has a filtration with associated graded pieces
$$i^M_{M \cap w^{-1}Pw} w^{-1} \cdot r^M_{w\overline{P}w^{-1} \cap M}(\sigma \otimes \nu)$$
for $w \in W_M \backslash W / W_M$. If we can show that for $w \notin W_M$ and $\chi \in \hat{T}_M$ we have
$$(i^M_{M \cap w^{-1}Pw} w^{-1} \cdot r^M_{w\overline{P}w^{-1} \cap M}(\sigma \otimes \nu))_\chi \neq \{0\}$$
implies $|\chi| < \nu$, then we shall be done.

Fix $w \notin W_M$ and χ as in the previous paragraph. From Equation (11), it is enough to show that the projection of $\nu - w^{-1}\nu$ into $\mathfrak{a}^*_M$ is not zero.

Since $\nu \in C_P$, by looking at the reduced expression of w as a product of simple reflections w_α for $\alpha \in \Delta$ we see that that there exists $\beta \in \Delta \smallsetminus \Delta_M$ such that $\nu - w^{-1}\nu \geq \nu - w_\beta \nu$. (The length of the reduced expression of w is equal to the minimal number of hyperplanes we must cross as we travel from ν to $w^{-1}\nu$.)

However, for all $\alpha \in \Delta \smallsetminus \Delta_M$, we have $\langle \check{\alpha}, \nu \rangle > 0$. This implies
$$\begin{aligned} \nu - w_\alpha \nu &= \nu - (\nu - \langle \check{\alpha}, \nu \rangle \alpha) \\ &= \langle \check{\alpha}, \nu \rangle \\ &> 0. \end{aligned}$$
Consequently, the image of $\nu - w_\alpha \nu$ in $\mathfrak{a}^*_M$ is strictly positive.

Thus, the image of $\nu - w^{-1}\nu \geq \nu - w_\beta \nu$ in $\mathfrak{a}^*_M$ under projection is strictly positive. ∎

Lemma 18.4.4.
$$\mathrm{Hom}_G(i^G_P(\sigma \otimes \nu), i^G_{\overline{P}}(\sigma \otimes \nu)) \cong \mathbb{C}.$$

Proof. Let χ_0 denote the central character of $\sigma \otimes \nu$. we have $\nu = |\mathrm{res}_{T_M}(\chi_0)|$. Therefore,
$$\begin{aligned} \mathrm{Hom}_G(i^G_P(\sigma \otimes \nu), i^G_{\overline{P}}(\sigma \otimes \nu)) &= \mathrm{Hom}_M(r^G_{\overline{P}} i^G_P \sigma \otimes \nu, \sigma \otimes \nu) \\ &= \mathrm{Hom}_M((r^G_{\overline{P}} i^G_P \sigma \otimes \nu)_{\chi_0}, \sigma \otimes \nu) \end{aligned}$$
Since, from Corollary 18.4.3, we have
$$(r^G_{\overline{P}} i^G_P(\sigma \otimes \nu))_{\chi_0} \subset (r^G_{\overline{P}} i^G_P(\sigma \otimes \nu))_\nu \cong \sigma \otimes \nu,$$
it follows that
$$\begin{aligned} \mathrm{Hom}_G(i^G_P(\sigma \otimes \nu), i^G_{\overline{P}}(\sigma \otimes \nu)) &= \mathrm{Hom}_M((r^G_{\overline{P}} i^G_P(\sigma \otimes \nu))_\nu, \sigma \otimes \nu) \\ &= \mathrm{Hom}_M(\sigma \otimes \nu, \sigma \otimes \nu) \end{aligned}$$
which, from Schur's lemma, is isomorphic to $\mathbb{C}$. ∎

Theorem 18.4.5. *Choose $0 \neq \alpha \in \mathrm{Hom}_G(i^G_P(\sigma \otimes \nu), i^G_{\overline{P}}(\sigma \otimes \nu))$. The G-module $i^G_P(\sigma \otimes \nu)/\ker(\alpha)$ is the unique irreducible quotient of $i^G_P(\sigma \otimes \nu)$ and the unique irreducible submodule of $i^G_{\overline{P}}(\sigma \otimes \nu)$.*

Once this theorem is proved, the following definition will make sense.

Definition 18.4.6. We define the *Langlands' quotient* $J(P,\sigma,\nu)$ to be the unique irreducible quotient of $i_P^G(\sigma\otimes\nu)$.

Proof. Choose $0\neq\alpha\in\mathrm{Hom}_G(i_P^G(\sigma\otimes\nu), i_{\overline{P}}^G(\sigma\otimes\nu))$.

We first show that $i_P^G(\sigma\otimes\nu)$ has a unique irreducible quotient. It will be enough to show that every proper G-submodule of $i_P^G(\sigma\otimes\nu)$ is contained in $\ker(\alpha)$. For this, it is enough to show that if $v\in i_P^G(\sigma\otimes\nu)$ such that $\langle G\cdot v\rangle\neq i_P^G(\sigma\otimes\nu)$, then $v\in\ker(\alpha)$. From the construction of the maps in the proof of Frobenius Reciprocity, it will be enough to show that for each $g\in G$ the image of $(gv)_{\overline{N}}\in r_{\overline{P}}^G i_P^G(\sigma\otimes\nu)$ under the projection onto the direct summand $(r_{\overline{P}}^G i_P^G(\sigma\otimes\nu))_\nu$ is zero.

Fix such a v. Since we can realize the contragredient of $i_P^G(\sigma\otimes\nu)$ as $i_P^G(\widetilde{\sigma}\otimes\widetilde{\nu})$ where $\widetilde{\sigma},\widetilde{\nu}$ denote the contragredients of σ and ν, respectively, there exists $0\neq\lambda\in i_P^G(\widetilde{\sigma}\otimes\widetilde{\nu})$ such that for all $g_1,g_2\in G$ we have $(g_1\lambda)(g_2v)=0$. By replacing λ with $g\lambda$ for some $g\in G$, we may assume that $\lambda(1)\in\widetilde{\sigma}\otimes\widetilde{\nu}$ is nonzero. Then, for all compact opens subgroups N' of N we have

$$(e_{N'}\lambda)(1)=\lambda(1).$$

Consequently, the element λ_N of $r_P^G i_P^G(\widetilde{\sigma}\otimes\widetilde{\nu})$ has nonzero image in $\widetilde{\sigma}\otimes\widetilde{\nu}\subset r_P^G i_P^G(\widetilde{\sigma}\otimes\widetilde{\nu})$.

From Casselman's perfect pairing (Theorem 15.0.1), we have that the contragredient of $r_P^G i_P^G(\widetilde{\sigma}\otimes\widetilde{\nu})$ is isomorphic to $r_{\overline{P}}^G i_P^G(\sigma\otimes\nu)$. Let χ_0 denote the central character of $\sigma\otimes\nu$. From Corollary 18.4.3 we have that

$$(r_P^G i_P^G(\widetilde{\sigma}\otimes\widetilde{\nu}))_{\widetilde{\chi}_0}$$

which is the contragredient of

$$(r_{\overline{P}}^G i_P^G(\sigma\otimes\nu))_{\chi_0}$$

is isomorphic to

$$\widetilde{\sigma}\otimes\widetilde{\nu}.$$

Therefore the only copy of $\widetilde{\sigma}\otimes\widetilde{\nu}$ in $r_P^G i_P^G(\widetilde{\sigma}\otimes\widetilde{\nu})$ occurs as a direct summand. Therefore, without loss of generality, we may assume that $\lambda_N\in\widetilde{\sigma}\otimes\widetilde{\nu}$.

If we denote by $\langle\,,\,\rangle_N$ Casselman's perfect pairing between

$$r_{\overline{P}}^G i_P^G(\sigma\otimes\nu)\ \text{ and }\ r_P^G i_P^G(\widetilde{\sigma}\otimes\widetilde{\nu}),$$

then for fixed $g\in G$ and for all $m\in M$, we have that

$$\langle m\lambda_N,(gv)_{\overline{N}}\rangle_N=0.$$

(This is true because $(m\lambda)(gv)=0$ for all $m\in M$.) Thus if φ denotes the map from $i_P^G(\sigma\otimes\nu)\to\sigma\otimes\nu$ obtained by composing Jacqueting with projection:

$$i_P^G(\sigma\otimes\nu)\to r_{\overline{P}}^G i_P^G(\sigma\otimes\nu)\to(r_{\overline{P}}^G i_P^G(\sigma\otimes\nu))_\nu\cong\sigma\otimes\nu$$

then

$$\langle m\lambda_N,\varphi(gv)\rangle_N=0.$$

This shows that the component of $(gv)_{\overline{N}}$ in the direct summand $\sigma \otimes \nu$ of $r_P^G i_P^G(\sigma \otimes \nu)$ is zero. Since this was true for all $g \in G$, we conclude that $v \in \ker(\alpha)$.

Finally, we note that the image of α in $i_P^G(\sigma \otimes \nu)$ is irreducible. The fact that it is unique follows from the fact that its contragredient, $i_P^G(\widetilde{\sigma} \otimes \widetilde{\nu})$, has a unique irreducible quotient from the above. ■

Remark 18.4.7. In the proof of Theorem 18.4.5, we actually showed that $\ker(\alpha)$ is contained in $\ker(\varphi \circ \alpha)$.

18.5. Uniqueness of the datum (P, σ, ν).

Lemma 18.5.1. *Suppose $\pi \in \mathfrak{R}(G)$ is irreducible. We have that $\pi \cong J(P, \sigma, \nu)$ if and only if*

(1) *for all standard parabolics Q of G with Levi decomposition $Q = LU$ and for each real exponent $\nu' \in \mathfrak{a}_L^* \subset \mathfrak{a}_\emptyset^*$ of π relative to $\overline{Q} = L\overline{U}$ (the parabolic opposite $Q = LU$) we have $\nu' \leq \nu$ and*

(2)
$$(r_P^G \pi)_\nu \cong \sigma \otimes \nu.$$

Proof. "⇒": Since $J(P, \sigma, \nu)$ is a quotient of $i_P^G(\sigma \otimes \nu)$, Item (1) follows from Lemma 18.4.1. As for Item (2), from the exactness of

$$0 \to \ker(\alpha) \to i_P^G(\sigma \otimes \nu) \to \pi \to 0$$

we have the exact sequence

$$0 \to (r_P^G \ker(\alpha))_\nu \to (r_P^G i_P^G(\sigma \otimes \nu))_\nu \to (r_P^G \pi)_\nu \to 0.$$

Since $(r_P^G i_P^G(\sigma \otimes \nu))_\nu \cong \sigma \otimes \nu$, it will be enough to show that $(r_P^G \ker(\alpha))_\nu = \{0\}$. However, this follows from Remark 18.4.7.

"⇐": From Item 2 we have that $r_P^G \pi$ surjects onto $(r_P^G \pi)_\nu$ which is isomorphic to $\sigma \otimes \nu$. Therefore, from Frobenius reciprocity we have that π is a subrepresentation of $i_P^G(\sigma \otimes \nu)$. The result now follows from the uniqueness of $J(P, \sigma, \nu)$. ■

Lemma 18.5.2. *Suppose (P', σ', ν') is another datum. (The standarad parabolic P' has a Levi decomposition $P' = M'N'$.) We have*

$$J(P, \sigma, \nu) \cong J(P', \sigma', \nu')$$

if and only if

$$P = P',\ \sigma \cong \sigma',\ \textit{and}\ \nu = \nu'.$$

Proof. "⇐": There is nothing to prove.

"⇒": Since $J(P, \sigma, \nu)$ has ν as a real exponent with respect to the parabolic opposite $P = MN$ and $J(P', \sigma', \nu')$ has ν' as a real exponent with respect to the parabolic opposite $P' = M'N'$, we conclude from Lemma 18.5.1 (1) that $\nu \leq \nu'$ and $\nu' \leq \nu$. Therefore, $\nu = \nu'$ which implies $P = P'$. Finally, we have

$$\sigma \otimes \nu \cong (r_P^G J(P, \sigma, \nu))_\nu \cong (r_P^G J(P, \sigma', \nu)_\nu \cong \sigma' \otimes \nu.$$

This implies that $\sigma \cong \sigma'$. ■

18.6. **Surjectivity.** In this subsection, we show that every irreducible smooth representation of G occurs as a Langlands' quotient.

Note that

$$\overline{C} = \coprod C_P$$

where the union is over the set of standard parabolics P with Levi decomposition $P = MN$.

We fix a W-invariant inner form on $\mathfrak{a}_\emptyset^*$. As usual, this gives us a W-invariant metric on $\mathfrak{a}_\emptyset^*$.

Definition 18.6.1. For $\nu \in \mathfrak{a}_\emptyset^*$ we denote by ν_0 the unique element of $\overline{C}$ which is closest to ν.

Remark 18.6.2. If $\nu \in \mathfrak{a}_\emptyset^*$ and $\nu_0 \in C_P$ for some standard parabolic P with a Levi decomposition $P = MN$, then we have that $(\nu_0 - \nu)$ is perpendicular to $\mathfrak{a}_M^*$. It follows that we can write

$$(\nu_0 - \nu) = \sum_{\alpha \in \Delta_M} c_\alpha \alpha$$

with $c_\alpha \in \mathbb{R}_{\geq 0}$. Thus $\nu_0 \geq \nu$.

Remark 18.6.3. Suppose $\nu_0 \in C_P$ as in Remark 18.6.2. An element $\mu \in \mathfrak{a}_\emptyset^* \smallsetminus \overline{C}$ has $\mu_0 = \nu_0$ if and only if the image of μ under the projection from $\mathfrak{a}_\emptyset^*$ to $\mathfrak{a}_M^*$ is ν_0.

Lemma 18.6.4. *Suppose $\mu, \nu \in \mathfrak{a}_\emptyset^*$. If $\nu \geq \mu$, then $\nu_0 \geq \mu_0$.*

Proof. Since $\nu_0 \geq \nu$ and $(\nu_0)_0 = \nu_0$, without loss of generality we may assume that $\nu = \nu_0$. Let Q denote the standard parabolic with Levi decomposition $Q = LU$ for which $\mu_0 \in C_Q$. If $\overline{x}$ denotes the image of $x \in \mathfrak{a}_\emptyset^*$ under the projection map $\mathfrak{a}_\emptyset^* \to \mathfrak{a}_L^*$, then $\nu_0 \geq \overline{\nu}_0 \geq \overline{\mu} = \mu_0$. ∎

Lemma 18.6.5. *Suppose P is a standard parabolic subgroup of G with Levi decomposition $P = MN$. Let $\nu = \nu_0 \in C_P$. Denote by $x \mapsto \overline{x}$ the projection map from $\mathfrak{a}_\emptyset^*$ to $\mathfrak{a}_M^*$. If $\mu \in \mathfrak{a}_\emptyset^*$ such that $\overline{\mu} = \nu$, then $\mu_0 \geq \nu$. Moreover, if $\mu_0 = \nu$, then $\mu \leq \nu$.*

Proof. Since $\overline{\mu} = \nu_0$, there exists a subset I of Δ_M such that

$$\mu = \nu_0 + \sum_{\alpha \in I} c_\alpha \alpha - \sum_{\alpha \in \Delta_M \smallsetminus I} c_\alpha \alpha$$

with $c_\alpha \in \mathbb{R}_{\geq 0}$. Since

$$\mu \geq \nu_0 - \sum_{\alpha \in \Delta_M \smallsetminus I} c_\alpha \alpha,$$

from Lemma 18.6.4 we have $\mu_0 \geq (\nu_0 - \sum_{\alpha \in \Delta_M \smallsetminus I} c_\alpha \alpha)_0 = \nu_0$. If $\mu_0 = \nu_0$, then $\nu_0 = \mu_0 \geq \mu$. ∎

Lemma 18.6.6. *Suppose $\pi \in \mathfrak{R}(G)$ is irreducible and P (resp. Q) is a standard parabolic with Levi decomposition $P = MN$ (resp. $Q = LU$). If $\nu \in C_P$ is a real exponent of π relative to $\overline{Q} = L\overline{U}$, the parabolic opposite $Q = LU$, then $Q \subset P$ and ν is a real exponent of π relative to the parabolic $\overline{P} = M\overline{N}$ opposite $P = MN$.*

Proof. If $\nu \in C_P \subset \mathfrak{a}_\emptyset^*$ is a real exponent of π relative to $\overline{Q} = L\overline{U}$, then $\nu \in \mathfrak{a}_L^*$ implies that $\mathfrak{a}_M^* \subset \mathfrak{a}_L^*$ which implies $T_M \subset T_L$ which implies $M \supset L$ which implies $P \supset Q$. Now

$$\begin{aligned} \{0\} \neq (r^G_{\overline{Q}}\pi)_\nu &= (r^M_{\overline{Q}\cap M} r^G_{\overline{P}}\pi)_\nu \\ &= r^M_{\overline{Q}\cap M}(r^G_{\overline{P}}\pi)_\nu \end{aligned}$$

Therefore, ν is a real exponent of π relative to $\overline{P} = M\overline{N}$. ■

Theorem 18.6.7. *For each $\pi \in \mathfrak{R}(G)$ which is irreducible, there exists a standard parabolic P with Levi decomposition $P = MN$, an irreducible tempered representation $\sigma \in \mathfrak{R}(M)$, and $\nu \in C_P$ such that $\pi = J(P, \sigma, \nu)$.*

Proof. From Lemma 18.4.5 it will be enough to find a triple (P, σ, ν) as above so that π occurs as a subrepresentation of $i^G_{\overline{P}}(\sigma \otimes \nu)$.

Let $E(\pi)$ denote the set of $\nu \in \overline{C}$ for which there exists a standard parabolic Q with Levi decomposition $Q = LU$ and $\nu \in \mathfrak{a}_L^*$ such that ν is a real exponent for π relative to $\overline{Q} = L\overline{U}$, the parabolic opposite $Q = LU$.

If χ denotes the central character of π, then $|\mathrm{res}_{T_G}\chi| \in \mathfrak{a}_G^* = C_G \subset \overline{C}$, so $E(\pi)$ is not empty. Moreover, since π is irreducible, for each $Q = LU$ as above, the Jacquet module $r^G_{\overline{Q}}$ is admissible and finitely generated, hence, from Theorem 12.0.7 $r^G_{\overline{Q}}\pi$ has finite length. Therefore, the set $E(\pi)$ has finite cardinality.

Choose $\nu \in E(\pi)$ maximal with respect to $\geq$. Suppose $\nu \in C_P$ for some standard parabolic P with Levi decomposition $P = MN$. From Lemma 18.6.6 and the definition of $E(\pi)$, ν is an exponent of π relative to $\overline{P} = M\overline{N}$, the parabolic opposite $P = MN$. Thus, $(r^G_{\overline{P}}\pi)_\nu \in \mathfrak{R}(M)$ is nontrivial and finitely generated, hence it has an irreducible quotient, σ'. Since $r^G_{\overline{P}}\pi$ surjects onto σ' via the maps

$$r^G_{\overline{P}}\pi \to (r^G_{\overline{P}})_\nu \to \sigma',$$

from Frobenius reciprocity we have that π is a subrepresentation of $i^G_{\overline{P}}\sigma'$. Set $\sigma = \sigma' \otimes \nu^{-1}$. It will be enough to show that σ is tempered.

Let $Q \subset P$ be a standard parabolic of G with Levi decomposition $Q = LU$. Let $\overline{Q} = L\overline{U}$ denote the parabolic opposite $Q = LU$. We need to show that if $\mu \in \mathfrak{a}_L^*$ is a real exponent of σ relative to $\overline{Q} \cap M = (L \cap M)(\overline{U} \cap M)$, then $\mu \leq 0$. Note that $\mathfrak{a}_M^* \subset \mathfrak{a}_L^*$.

Suppose $\mu \in \mathfrak{a}_L^*$ is a real exponent of σ relative to $\overline{Q} \cap M = (L \cap M)(\overline{U} \cap M)$. Any such μ looks like $\mu' - \nu$ for some real exponent $\mu' \in \mathfrak{a}_L^*$ of π relative to $\overline{Q} = L\overline{U}$ such that the image $\overline{\mu}'$ in $\mathfrak{a}_M^*$ of μ' under the projection map $\mathfrak{a}_M^* \to \mathfrak{a}_L^*$ is ν. (Note that this latter condition implies that ν and μ' have the same image under projection onto $\mathfrak{a}_G^*$.)

We consider two cases: either $\mu' \in \overline{C}_Q$ or $\mu' \in \mathfrak{a}_L^* \smallsetminus \overline{C}_Q$.

In the first case, we have $\mu' \in E(\pi)$ and $\mu' = (\mu')_0$. Since $\overline{\mu}' = \nu = \nu_0$, from Lemma 18.6.5 we have

$$\mu' = (\mu')_0 \geq \nu_0 = \nu.$$

Thus, since μ', ν are comparable elements of $E(\pi)$ and ν is a maximal element of $E(\pi)$, we have $\mu' = \nu$. Consequently, $\mu = \mu' - \nu = 0$.

In the second case, we have $\mu' \in \mathfrak{a}_L^* \smallsetminus \overline{C}_Q$ and $\overline{\mu}' = \nu_0$. Consequently, we must have $(\mu')_0 = \nu_0$. From Lemma 18.6.5, we have $\mu' \leq \nu$. Consequently, $\mu = \mu' - \nu \leq 0$. ■

19. Solutions

Solution to Exercise 1.0.2

(1) This is clear from the definition.
(2) The statement is clear if $r_1 = 0$ or $r_2 = 0$. If $r_i = p^{\ell_i} \cdot a_i/b_i$ where $p \nmid a_i$ and $p \nmid b_i$ for $i = 1, 2$, then $r_1 r_2 = p^{\ell_1+\ell_2} \cdot a_1 a_2/(b_1 b_2)$, so since $p \nmid a_1 a_2$ and $p \nmid b_1 b_2$, we have $|r_1 \cdot r_2|_p = |r_1|_p \cdot |r_2|_p$.
(3) Again suppose $r_i = p^{\ell_i} \cdot a_i/b_i$ for $i = 1, 2$ as before, and assume that $\ell_1 \geq \ell_2$. Then

$$r_1 + r_2 = p^{\ell_2} \frac{p^{\ell_1-\ell_2} a_1 b_2 + b_1 a_2}{b_1 b_2}.$$

If $\ell_1 > \ell_2$ then p does not divide the numerator or the denominator, so $|r_1 + r_2|_p = p^{-\ell_2} = \max(|r_1|_p, |r_2|_p)$. If $\ell_1 = \ell_2$ then p may divide the numerator but not the denominator, so $|r_1 + r_2|_p \leq p^{-\ell_2}$.

Solution to Exercise 1.0.3

We showed in the solution to Exercise 1.0.2(3) that if $|r_1|_p \neq |r_2|_p$ then the inequality is an equality. The converse is false: set $r_1 = r_2 = 0$. If p is odd, there are also nontrivial counterexamples: take $r_1 = r_2 = 1$, for instance. If p is two, however, and $r_i = 2^\ell \cdot a_i/b_i$ have the same 2-adic norm for $i = 1, 2$, then $a_1 b_2 + b_1 a_2$ is even, so the 2-adic norm of

$$r_1 + r_2 = p^\ell \frac{a_1 b_2 + b_1 a_2}{b_1 b_2}$$

is strictly smaller than $\max(|r_1|_2, |r_2|_2)$. Thus $|r_1 + r_2|_2 = \max(|r_1|_2, |r_2|_2)$ implies either $|r_1|_2 \neq |r_2|_2$ or $r_1 = r_2 = 0$.

Solution to Exercise 1.0.4

Since the map $x \mapsto x^\alpha$ is an increasing function, it is clear that all three parts of Exercise 1.0.2 are satisfied by $|\cdot|_p^\alpha$. For $r > 0$, let $B_r(p, \alpha) = \{x \in \mathbb{Q} \mid |x|_p^\alpha < r\}$ be the ball of radius r around 0 in the metric $|\cdot|_p^\alpha$. Then $B_{r^\alpha}(p, \alpha) = B_r(p, 1)$, which shows that both metrics complete to $\mathbb{Q}_p$.

If $\alpha > 1$ then $|1 + 1|^\alpha = 2^\alpha > 2 = |1|^\alpha + |1|^\alpha$, so $|\cdot|^\alpha$ does not satisfy the triangle inequality. We claim that $|\cdot|^\alpha$ does satisfy the triangle inequality for $\alpha < 1$. Suppose that $\alpha \in (0, 1]$, and let $a, b \in \mathbb{R}$. We have $|a + b| \leq |a| + |b|$, so it suffices to show that $(a + b)^\alpha \leq a^\alpha + b^\alpha$ for $a, b \in \mathbb{R}_{\geq 0}$. Dividing through by $(a + b)^\alpha$, we may assume that $a, b \in [0, 1]$ and that $a + b = 1$. Then $a^\alpha \geq a$ and $b^\alpha \geq b$, so $a^\alpha + b^\alpha \geq 1$, as required.

Solution to Exercise 1.0.5

Fix $n \in \mathbb{Z}$, and let $m < n$. Since $\wp^m$ is compact, $\wp^m/\wp^n$ is finite. The result follows since $k/\wp^n = \bigcup_{m<n} \wp^m/\wp^n$.

Solution to Exercise 1.0.8

Let G be an ℓ-group and suppose $A \subset G$ has more than one element. Using translations, we see that every point of G has a basis consisting of compact open subsets. The Hausdorff condition implies that if $x, y \in A$ are distinct then we can choose a compact open neighborhood K of x such that $y \notin A$. Now K is closed (again because G is Hausdorff) and open, so $G \setminus K$ is also closed and open, whence $A = (A \cap K) \coprod (A \setminus K)$ is disconnected.

The converse sometimes goes under the name of van Dantzig's theorem. See, for instance, Theorem 1.34 in *The Structure of Compact Groups* by Karl Heinrich Hofmann and Sidney A. Morris.

Solution to Exercise 2.0.9

(1) Suppose G has a countable base of open sets, and let $K \subset G$ be an open subgroup. For $g \in G$, let $m_g : G \to G$ be the multiplication map $h \mapsto gh$. Then m_g is a homeomorphism with inverse $m_{g^{-1}}$, so $gK = m_g(K)$ is open for all $g \in G$. If $\{U_i \mid i \in \mathbb{N}\}$ is a countable base of G, then for all $g \in G$, there is some $i \in \mathbb{N}$ such that $U_i \subset gK$; since the cosets $\{gK \mid g \in G\}$ are disjoint, there are therefore a countable number of them.

(2) We already know that the sets K_n, $n \geq 1$ form a countable neighborhood base of $1 \in k^\times$. Therefore $\{xK_n \mid n \geq 1,\ x \in k^\times\}$ is a basis for the topology of $k^\times$. But $\mathbb{Q}^\times$ is dense in $k^\times$, so for any $n \geq 1$ and any $x \in k^\times$, there is a $q \in \mathbb{Q}^\times$ such that $q^{-1}x \in K_n$; thus there are only countably many cosets xK_n above.

Solution to Exercise 2.1.8

Let $A \in \mathrm{End}_G(V)$. Since V is finite-dimensional, A has a nonzero eigenvalue λ. Thus the G-map $A - \lambda$ has nontrivial kernel, so since V is irreducible, $A - \lambda = 0$, i.e., $A = \lambda$.

Solution to Exercise 2.2.1

Choose an open neighborhood $U \subset \mathbb{C}^\times$ of 1 which does not contain any nontrivial subgroup of $\mathbb{C}^\times$: for instance, one can take $U = \{z \in \mathbb{C} \mid |z - 1| < \frac{1}{2}\}$. Then $\psi^{-1}(U)$ is an open neighborhood of the identity in G, so we can find a compact open subgroup $K \subset \psi^{-1}(U)$. But now $\psi(K) \subset U$ is a subgroup, so by our choice of U we have $\psi(K) = \{1\}$, or in other symbols $K \subset \ker \psi$. This implies that $\ker \psi$ is open, so in particular ψ is smooth.

For a non-unitary character of $k^\times$, choose a uniformizer $\pi \in k^\times$, or equivalently an isomorphism $k^\times \cong R^\times \times \mathbb{Z}$. Then for any $z \in \mathbb{C}^\times$, there is a unique homomorphism $\psi : k^\times \to \mathbb{C}^\times$ such that $\psi(R^\times) = \{1\}$ and $\psi(\pi) = z$. In particular, we can choose $z \in \mathbb{C}^\times \setminus S^1$.

Solution to Exercise 2.2.2

First consider the projection $R \to \mathfrak{f}$, which is a ring homomorphism, so that it induces a homomorphism $R^\times \to \mathfrak{f}^\times$ on units. It is not hard to see that this map is surjective, since $R^\times = R \setminus \wp$. The kernel of this map consists of elements of $R^\times$ which difffer from 1 by an element of $\wp$, i.e. $1 + \wp$, so that $R^\times/(1 + \wp) \cong \mathfrak{f}^\times$.

The proof of the second assertion is more interesting. Choose a uniformizer $\pi \in \wp$ and consider the map $R \to (1 + \wp^k)/(1 + \wp^{k+1})$ which sends $x \mapsto x\pi^k \pmod{1 + \wp^{k+1}}$. The calculation

$$(1 + x\pi^k)(1 + y\pi^k) = 1 + (x + y)\pi^k + xy\pi^{2k}$$

shows that this is a homomorphism with respect to addition on R and multiplication on the target. The kernel consists of those $x \in R$ such that $x\pi^k \in \wp^{k+1}$, which is just $\wp$, so $R/\wp \cong (1+\wp^k)(1+\wp^{k+1})$ as claimed.

The fact that $\#R^\times/(1+\wp^m) = (q-1)q^{m-1}$ now follows from an induction on the short exact sequences

$$1 \to (1+\wp)/(1+\wp^m) \to R^\times/(1+\wp^m) \to R^\times/(1+\wp) \to 1 \text{ and } 1 \to (1+\wp^{m-1})/(1+\wp^m) \to R^\times/(1+\wp^m)$$

Solution to Exercise 2.2.4

Let $\psi \in \mathbf{X}(k^\times)$. Since ψ is unramified, we can use the isomorphism of $k^\times$ with $\mathbb{Z}\times R^\times$ to think of ψ as a character of $\mathbb{Z}$, that is, a group homomorphism $\mathbb{Z} \to \mathbb{C}^\times$. Since $\mathbb{Z}$ is cyclic, the group of all such characters can be identified with $\mathbb{C}^\times$ via the map $\psi \mapsto \psi(1)$. These identifications are compatible with the map $\psi \mapsto \psi(\varpi)$.

Solution to Exercise 2.3.1

To begin, we have

$$\pi(x)\pi(y) = \begin{pmatrix} 1 & v(x) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & v(y) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & v(x)+v(y) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & v(xy) \\ 0 & 1 \end{pmatrix} = \pi(xy)$$

so (π, V) is a representation of $k^\times$. Since $\operatorname{res}_{R^\times}\pi$ is trivial, (π, V) is smooth. Suppose that $\mathbb{C}\cdot\begin{pmatrix} a \\ b \end{pmatrix}$ is a proper $k^\times$-submodule of V. Then

$$\pi(\varpi)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ b \end{pmatrix} = c\begin{pmatrix} a \\ b \end{pmatrix}$$

for some $c \in \mathbb{C}^\times$. If $b \neq 0$ then $c = 1$, so $a+b = a$ and $b = 0$, a contradiction. Thus $b = 0$, so the only proper $k^\times$-submodule of V is $\mathbb{C}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Thus (π, V) cannot completely decompose.

Solution to Exercise 2.4.1

Let $h \in G$ be such that $\psi(h) \neq 1$. By translation invariance of the Haar measure we have

$$\psi(h)\int_G \psi(g)\,dg = \int_G \psi(hg)\,dg = \int_G \psi(g)\,dg,$$

which implies that the integral is zero.

Solution to Exercise 2.4.2

(1) Let $y \in R^\times$ and $v \in V$. We have

$$\pi(y)e_\psi(v) = \pi(y)\int_{R^\times}\overline{\psi}(x)\pi(x)\,dx = \int_{R^\times}\overline{\psi}(x)\pi(yx)\,dx.$$

Since dx is a Haar measure, we can translate by y^{-1} to show that

$$\pi(y)e_\psi(v) = \int_{R^\times}\overline{\psi}(y^{-1}x)\pi(x)\,dx = \overline{\psi}(y^{-1})e_\psi(v)$$

which is just $\psi(y)e_\psi(v)$ since $\operatorname{res}_{R^\times}\psi$ is unitary.

(2) Let $v \in V$. We have

$$e_{\psi'}\cdot e_\psi(v) = \int \overline{\psi'}(y)\pi(y)\int\overline{\psi}(x)\pi(x)\,dx\,dy = \iint \overline{\psi'}(y)\overline{\psi}(x)\pi(yx)\,dx\,dy.$$

Since dy is a Haar measure, this is

$$\iint \overline{\psi'}(yx^{-1})\overline{\psi}(x)\pi(y)\,dx\,dy = \int \overline{\psi'}(y)\pi(y)\int \overline{\psi}(x)\psi'(x)v\,dx\,dy.$$

We claim that $f(v) := \int \overline{\psi}(x)\psi'(x)v\,dx = 0$. Indeed, by part (1) above, for every $y \in R^\times$ we have $\psi(y)f(v) = \psi'(y)f(v)$, so if $f(v) \neq 0$ then $\psi(y) = \psi'(y)$. Alternatively, $\int \overline{\psi}(x)\psi'(x)\,dx$ is just the Hermetian product of two nonequivalent irreducible representations ψ and ψ' of a finite quotient group of $R^\times$, and is therefore zero. In any case, this implies that $e_{\psi'} \cdot e_\psi(v) = 0$ as required.

(3) In part (2) we showed that for $v \in V$,

$$e_\psi \cdot e_\psi(v) = \int \overline{\psi}(y)\pi(y)\int \overline{\psi}(x)\psi(x)v\,dx\,dy = \int \overline{\psi}(y)\pi(y)\int v\,dx\,dy = e_\psi(v).$$

Solution to Exercise 2.5.6

Denote the functor $(\pi, V) \mapsto \mathrm{Hom}_{k^\times}(\mathbb{C}_{\widetilde{\psi}} \otimes_{\mathbb{C}} B, V)$ by F. Since V is ψ-isotypic, we have that $\mathrm{Hom}_{R^\times}(\mathbb{C}_\psi, V) \cong \mathrm{res}_{R^\times} V$ naturally. Exactness and faithfulness of F are then a direct consequence of Lemma 2.5.4. It remains to show that F induces a bijection of objects and maps. Define a functor G from the category of B-modules to $\mathfrak{R}^\psi(k^\times)$ as follows. Let M be a B-module, and let $m \in M$ and $x \in k^\times$. Set $\pi(x)m := \widetilde{\psi}(x)\,\mathrm{ev}_x \cdot m$, and set $G(M) := (\pi, M)$. Note that $G(M)$ is indeed a smooth $k^\times$-representation since B is a $\mathbb{C}$-algebra and $\widetilde{\psi}(x)\,\mathrm{ev}_x = 1$ when $\psi(x) = 1$. With these identifications, the map $\varphi \mapsto \varphi(1 \otimes 1)$ is a natural isomorphism of $\mathrm{Hom}_{k^\times}(\mathbb{C}_{\widetilde{\psi}} \otimes_{\mathbb{C}} B, M)$ with M as B-modules, for any B-module M. The same map is a $k^\times$-isomorphism of $\mathrm{Hom}_{k^\times}(\mathbb{C}_{\widetilde{\psi}} \otimes_{\mathbb{C}} B, V)$ to V, for any smooth representation V. Thus G is the inverse functor to F, so F defines an equivalence of categories.

Solution to Exercise 2.5.7

These three categories are actually isomorphic. A representation of $\mathbb{Z}$ is a complex vector space V together with a homomorphism $\mathbb{Z} \to \mathrm{GL}(V)$, which is uniquely determined by the image of 1, and moreover can be any $T \in \mathrm{GL}(V)$. So we see that the objects correspond to pairs (V, T) of the kind just described. A morphism of $\mathbb{Z}$-representations on vector spaces V and W is a linear map $\varphi : V \to W$ which intertwines the $\mathbb{Z}$-actions, meaning that if $S \in \mathrm{GL}(V)$ and $T \in \mathrm{GL}(W)$ are the images of 1 then $\varphi \circ T^n = S^n \circ \varphi$ for all $n \in \mathbb{Z}$. But this equivalent to $\varphi \circ T = S \circ \varphi$, so $\mathcal{R}(\mathbb{Z})$ is really the same as (i.e. isomorphic to as a category) the category of pairs described in the exercise. Also, $\mathcal{R}(\mathbb{Z})$ is isomorphic to $\mathbb{C}[t, t^-1]$-mod because $\mathbb{C}[t, t^1]$ is isomorphic to the group alegbra $\mathbb{C}[\mathbb{Z}]$ via the unique $\mathbb{C}$-algebra homomorphism which sends $t \mapsto 1$, and one knows that the category of representations of a discrete group G is isomorphic to $\mathbb{C}[G]$-mod.

Solution to Exercise 3.0.9

We inductively define a character Λ of k/R by defining compatible characters of $\wp^n/R$ for each $n < 0$. Note that for any $n \in \mathbb{Z}$, the map $x + \wp^n \mapsto \varpi x + \wp^{n+1}$ defines an isomorphism $\wp^{n-1}/\wp^n \to \wp^n/\wp^{n+1}$, so $\wp^{n-1}/\wp^n \cong \mathfrak{f} \cong \mathbb{Z}_p^m$ for each n, where $|\mathfrak{f}| = p^m$.

Let $\Lambda_0 : R/R \to \mathbb{C}^\times$ be the trivial homomorphism. Let $n < 0$, and suppose that there exists a character $\Lambda_n : \wp^n/R \to \mathbb{C}^\times$. Let $\overline{s}_1, \ldots, \overline{s}_m$ be m order-p generators of $\wp^{n-1}/\wp^n$ — i.e., $\wp^{n-1}/\wp^n \cong \langle \overline{s}_1 \rangle \oplus \cdots \oplus \langle \overline{s}_m \rangle$ — and let s_i be any lift in $\wp^{n-1}$ of $\overline{s}_i$. For $1 \leq i \leq m$, let $\alpha_i \in \mathbb{C}^\times$ be an mth root of $\Lambda_n(ms_i)$; choose $\alpha_i \neq 1$. Set

$$\Lambda_{n-1}(d_1 s_1 + \cdots + d_m s_m + s) := \alpha_1^{d_1} \cdots \alpha_m^{d_m} \cdot \Lambda_n(s)$$

where each $d_i \in \mathbb{Z}$ and $s \in \wp^n$. If

$$d_1 s_1 + \cdots + d_m s_m + s = d_1' s_1 + \cdots + d_m' s_m + s'$$

then both sides of the above equation project onto the same element of $\wp^{n-1}/\wp^n$, so for each i, $d_i' - d_i = k_i \cdot m$ for some $k_i \in \mathbb{Z}$. Thus $s - s' = \sum_i k_i \cdot ms_i$, so $\Lambda_n(s - s') = \prod_i \Lambda_n(ms_i)^{k_i} = \prod_i \alpha_i^{k_i \cdot m}$. Thus

$$\alpha_1^{d_1} \cdots \alpha_m^{d_m} \cdot \Lambda_n(s) = \alpha_1^{d_1'} \cdots \alpha_m^{d_m'} \cdot \Lambda_n(s')$$

which proves that Λ_{n-1} is well-defined. Clearly Λ_{n-1} is then a (nontrivial) homomorphism which agrees with Λ_n on $\wp^n/R$.

We can now define $\Lambda : k/R \to \mathbb{C}^\times$ by $\Lambda(x) = \Lambda_n(x)$, where $n < 0$ is any integer such that $x \in \wp^n$. This induces a nontrivial smooth character of k.

Alternatively, this claim follows from Lemma 3.5.11: Since $\mathbb{C}^\times$ is an injective object in the category of abelian groups, we can extend any nontrivial homomorphism $R/\wp = \mathfrak{f} \to \mathbb{C}^\times$ to all of $k/\wp$.

Solution to Exercise 3.0.10

First we do the exercise for $\mathfrak{f}$. Let $\overline{\Lambda} : \mathfrak{f} \to \mathbb{C}^\times$ be a nontrivial additive character of $\mathfrak{f}$. We claim that the map $\overline{\varphi} : \overline{x} \mapsto \overline{\Lambda}_{\overline{x}}$ is an isomorphism of $\mathfrak{f}$ with $\widehat{\mathfrak{f}}$. Clearly $\overline{\varphi}$ is a homomorphism. Suppose that $\overline{\Lambda}_{\overline{x}} = 1$ and that $\overline{x} \neq 0$. Then for all $\overline{y} \in \mathfrak{f}$, $\overline{\Lambda}_{\overline{x}}(\overline{y}\overline{x}^{-1}) = \overline{\Lambda}(\overline{y}) = 1$, which contradicts the assumption that $\overline{\Lambda}$ is nontrivial. Thus $\overline{\varphi}$ is injective.

We prove surjectivity by counting. Let p be the characteristic of $\mathfrak{f}$, so $q = p^n$ for some $n > 0$, and $\mathfrak{f} \cong \mathbb{F}_p^n$ as additive groups. Since any finite subgroup of S^1 is cyclic, the image of any homomorphism $\Lambda' : \mathfrak{f} \to \mathbb{C}^\times$ must be contained in the subgroup of the pth-roots of unity, so we can think of Λ' as a $\mathbb{Z}$-module homomorphism $\mathbb{F}_p^n \to \mathbb{F}_p$. But any such $\mathbb{Z}$-module homomorphism is also an $\mathbb{F}_p$-vector space homomorphism, so we can identify $\widehat{\mathfrak{f}}$ with the vector space dual $(\mathbb{F}_p^n)^*$ of $\mathbb{F}_p^n$. Since $(\mathbb{F}_p^n)^*$ has $p^n = q$ elements, $\overline{\varphi}$ must also be a surjection. (Or one can use the representation theory of finite groups: to wit, if G is any finite abelian group, then there are $|G|$ irreducible representations of G, each of which is a unitary character.)

Denote the map $x \mapsto \Lambda_x$ by φ. As above, φ is an injective group homomorphism; we must show that it is surjective. Replacing Λ by Λ_{ϖ^m} for some $m \in Z$ if necessary, we may assume that Λ is trivial on $\wp$ but not on R. Let $\Lambda' \in \widehat{k}$ be any character, and assume without loss of generality that Λ' is trivial on $\wp$ also. Thus Λ and Λ' induce characters $\overline{\Lambda}$ and $\overline{\Lambda}'$ on $\mathfrak{f} = R/\wp$, $\overline{\Lambda}$ being nontrivial. Consequently, by what we showed above, we can find an $x_0 \in R$ such that $\Lambda_{x_0} = \Lambda'$ on R. Then $(\Lambda_{-x_0}\Lambda')_{\varpi^{-1}} = \Lambda_{-x_0\varpi^{-1}}\Lambda'_{\varpi^{-1}}$ is trivial on $\wp$, so we can find an $x_1 \in R$ such that $\Lambda_{x_1} = \Lambda_{-x_0\varpi^{-1}}\Lambda'_{\varpi^{-1}}$ on R, i.e., $\Lambda_{x_0+\varpi x_1}$ agrees with Λ' on $\varpi^{-1}R$. Continuing in this fashion, we can find $x_0, x_1, x_2, \ldots \in R$ such that $\Lambda_{x_0+\varpi x_1+\cdots+\varpi^m x_m} = \Lambda'$ on $\varpi^{-m}R$, for each

$m > 0$. Setting $x = x_0 + \varpi x_1 + \varpi^2 x_2 + \cdots$ (which exists since k is complete), then, we have that $\Lambda_x = \Lambda'$ as required.

Next we show that φ is continuous. Let $B(K, U) = \{\Lambda' \in \widehat{k} \mid \Lambda'(K) \subset U\}$ be an element of the subbasis for the topology of $\widehat{k}$, where $U \subset \mathbb{C}$ is any set (recall that $\mathbb{C}$ has the discrete topology) and $K \subset k$ is compact. Let $x \in \varphi^{-1}(B(K, U))$, i.e., $\Lambda(xK) \subset U$. Suppose that Λ is trivial on $\wp$, and note that $K \subset \wp^m$ for some $m \in \mathbb{Z}$, since $\{K \cap \wp^m \mid m \in \mathbb{Z}\}$ is an open cover of K. Thus $\wp^{|m|+1} K \subset \wp$, so $\Lambda((x + \wp^{|m|+1})K) = \Lambda(xK)$, i.e., $x + \wp^{|m|+1} \subset \varphi^{-1}(B(K, U))$. Thus φ is continuous.

It remains to show that φ^{-1} is continuous; it suffices to show that for all $m \in \mathbb{Z}$, there exist K and U such that $B(K, U) = \varphi(\wp^m)$. Choose $m \in \mathbb{Z}$, and suppose again that Λ is trivial on $\wp$ but not on R. Set $K = \wp^{-m+1}$ and $U = \{1\}$, so $\Lambda_x \in B(K, U)$ if and only if Λ_x is trivial on $\wp^{-m+1}$, which is true if and only if $v(x) \geq m$, i.e., $x \in \wp^m$.

Solution to Exercise 3.0.11

Let W be a finite-dimensional k-vector space, and let $\varphi : W^* \to \widehat{W}$ be given by $\varphi(\lambda)(v) = \Lambda(\lambda(v))$. It is clear that φ is an injective homomorphism. Moreover, if we give $\widehat{W}$ the structure of a k-vector space by setting $(x \cdot \chi)(v) = \chi(xv)$ for $x \in k$, $\chi \in \widehat{W}$, and $v \in W$, then φ is a linear map. If $W = k^n$ then $\widehat{W} = \operatorname{Hom}(\bigoplus_1^n k, \mathbb{C}^\times) = \bigoplus_1^n \operatorname{Hom}(k, \mathbb{C}^\times) = \widehat{k}^n$. This shows that any character of W is unitary. We effectively showed that $\dim_k \widehat{k} = 1$ in the previous exercise, so $\dim_k \widehat{W} = n = \dim_k W^*$, so φ must be an isomorphism. One shows that φ is also a homeomorphism using the homeomorphisms $\widehat{k} \cong k \cong k^*$, $W^* \cong (k^*)^n$, and $\widehat{W} \cong \widehat{k}^n$.

Solution to Exercise 3.0.12

Define a map $f : A^\Gamma \to ((\widehat{A})_\Gamma)^\wedge$ by $f(a)(\lambda) = \lambda(a)$. If $\lambda = \gamma \cdot \lambda' \cdot (\lambda')^{-1}$ for some $\lambda' \in \widehat{A}$ and $\gamma \in \Gamma$ then

$$f(a)(\lambda) = \lambda'(\gamma^{-1}a - a) = 1$$

since $\gamma^{-1}a = a$. Thus f is well-defined.

Suppose that $f(a) = 1$, so $\lambda(a) = 1$ for all $\lambda \in \widehat{A}$. This shows that $a = 1$ since the natural map $A \to \widehat{\widehat{A}}$ is injective (to see this, use Lemma 3.5.11). Thus f is injective.

Let $\Lambda \in ((\widehat{A})_\Gamma)^\wedge$. The projection $\widehat{A} \to (\widehat{A})_\Gamma$ allows us to extend Λ to $\widehat{A}$, so by hypothesis, there is some $a \in A$ such that $\Lambda(\lambda) = \lambda(a)$ for all $\lambda \in \widehat{A}$. We must show that $a \in A^\Gamma$. Suppose that there were some $\gamma \in \Gamma$ such that $\gamma a \neq a$, and find some $\lambda \in \widehat{A}$ such that $\lambda(\gamma a - a) \neq 1$. Since Λ is trivial on $\widehat{A}(\Gamma)$, we have

$$1 = \Lambda(\gamma^{-1}\lambda \cdot \lambda^{-1}) = \lambda(\gamma \cdot a - a),$$

a contradiction. Thus $a \in A^\Gamma$.

Solution to Exercise 3.1.2

For $v \in V$, we have

$$\begin{aligned} e_K^2 v &= \int_K \pi(x_1) \int_K \pi(x_2) v \, dx_2 \, dx_1 \\ &= \int_K \left(\int_K \pi(x_1 x_2) v \, dx_2 \right) dx_1 \\ &= \int_K \left(\int_K \pi(x_2) v \, dx_2 \right) dx_1 \\ &= \int_K (e_K v) \, dx_1 = e_K v \end{aligned}$$

so e_K is a projection operator. Thus $e_K V \subset V^K$; the other inclusion is clear since our Haar measure dx is normalized. For any $v \in V$, we have $v = (1-e_K)v + e_K v$, so $V = (1-e_K)V + V^K$. If $v \in (1-e_K)V \cap V^K$ then $v = w - e_K w$ for some $w \in V$, and $v = e_K v = e_K(w - e_K w) = e_K w - e_K w = 0$, so the sum is direct. Since dx is a Haar measure, we see that for $x \in K$ and $v \in V$, $e_K(\pi(x)v) = e_K v = \pi(x) \cdot e_K v$, so $(1-e_K)V$ and V^K are both K-modules.

For $\lambda \in \widetilde{V}$ and $v \in V$, we have

$$(e_K \lambda)(v) = \int_K (\widetilde{\pi}(x)\lambda) v \, dx = \int_K \lambda(\pi(x^{-1})v) \, dx = \lambda \left(\int_K \pi(x^{-1}) v \, dx \right) = \lambda(e_K v).$$

We are free to move the v and the λ in and out of the integral sign because the integral is a finite sum and everything is linear, and $d(x^{-1}) = dx$ by the uniqueness of the normalized Haar measure. The above identity combined with the decompositions $V = (1-e_K)V \oplus V^K$ and $\widetilde{V} = (1-e_K)\widetilde{V} \oplus \widetilde{V}^K$ show that the restriction map $\widetilde{V}^K \to \mathrm{Hom}_{\mathbb{C}}(V^K, \mathbb{C})$ is injective. It remains to point out that if we extend any $\lambda \in \mathrm{Hom}_{\mathbb{C}}(V^K, \mathbb{C})$ to all of $V = (1-e_K)V \oplus V^K$ in the obvious way, then $\lambda \in \widetilde{V}^K$.

Solution to Exercise 3.3.6

We proceed by induction on the dimension of V. Clearly any one-dimensional representation is irreducible.

Suppose that $n := \dim_{\mathbb{C}} V > 1$, and assume that any complex representation of G with dimension less than n has an irreducible subrepresentation. Suppose that (π, V) is not irreducible. Let $W \subset V$ be a nonzero proper subrepresentation of V. Since $\dim_{\mathbb{C}} W < n$, by the inductive hypothesis, W has an irreducible subrepresentation.

Solution to Exercise 3.3.10

For convenience, define $I(K, v) := \int_K \psi^{-1}(x) \cdot \pi(x) w \, dx$ for a compact open subgroup $K \subset F$ and a vector $v \in V$.

First we show that if $I(K, w) = 0$ then $I(K', w) = 0$ for every compact open subgroup $K' \supset K$. Indeed, for any $y \in K'$, we have

$$\int_{yK} \psi^{-1}(x) \cdot \pi(x) w \, dx = \int_K \psi^{-1}(yx) \cdot \pi(yx) w \, dx = \psi^{-1}(y) \cdot \pi(y) I(K, w) = 0.$$

Choosing a (finite) set $\{y\}$ of coset representatives for K'/K, we have

$$I(K',w) = \sum \int_{yK} \psi^{-1}(x) \cdot \pi(x) w \, dx = 0.$$

Let W be the set of all $w \in V$ for which $I(K,w) = 0$ for some compact open subgroup $K \subset F$. Let $w \in W$, and let K be such that $w \in I(K,w)$. Since $\{K_n\}_{n=1}^{\infty}$ is an open cover of K, we have that $K \subset K_n$ for some $n \geq 1$. Let $f \in F$, and let n be large enough that K_n contains K and f. Then

$$\int_{K_n} \psi^{-1}(x)\pi(xf)w \, dx = \psi(f) \int_{K_n} \psi^{-1}(x)\pi(x)w \, dx = 0,$$

which shows that W is an F-submodule of V. In addition, for $v \in V$, we have

$$\begin{aligned} &\int_{K_n} \psi^{-1}(x)\pi(x)(\pi(f)v - \psi(f)v) \, dx \\ &\qquad = \int_{K_n} \psi^{-1}(x)\pi(xf) \, dx - \psi(f) \int_{K_n} \psi^{-1}(x)\pi(x) \, dx = 0, \end{aligned}$$

which proves that $V(F,\psi) \subset W$.

Now let $w \in W$, and choose a $K \subset F$ such that $I(K,w) = 0$. Assume without loss of generality that $\int_K dx = 1$. Then we have

$$\begin{aligned} w &= w - \int_K \psi^{-1}(x)\pi(x)w \, dx \\ &= \int_K (w - \psi^{-1}(x)\pi(x)w) \, dx \\ &= \int_K (\pi(x) - \psi(x)) \cdot (-\psi^{-1}(x)w) \, dx. \end{aligned}$$

The last integral is a finite sum of the form $\sum_i c_i(\pi(x_i) - \psi(x_1))v_i$, so $w \in V(F,\psi)$.

Solution to Exercise 3.4.1

Let $\widetilde{\chi}, \widetilde{\chi}'$ be two characters of SZ which restrict to χ on Z. Then $\widetilde{\chi}(\widetilde{\chi}')^{-1}$ is trivial on Z, so it reduces to a character on $SZ/Z = S \cong k$. Since $\Lambda(s) := \chi([0,0,s])$ is also a smooth character of k, by Exercise 3.0.10 there is some $\widehat{s} \in k$ such that

$$\chi([0,0,s\widehat{s}]) = \Lambda_{\widehat{s}}(s) = \widetilde{\chi}([s,0,0])(\widetilde{\chi}')^{-1}([s,0,0]).$$

for all $s \in k$. Rearranging, this gives

$$\widetilde{\chi}'([s,0,0]) = \chi([0,0,-s\widehat{s}])\widetilde{\chi}([s,0,0]) = \widetilde{\chi}_{\widehat{s}}([s,0,0])$$

for all $s \in k$, i.e., $\widetilde{\chi}' = \widetilde{\chi}_{\widehat{s}}$. Thus $\widehat{S}$ acts transitively.

We have that $\widetilde{\chi}_{\widehat{s}} = \widetilde{\chi}_{\widehat{s}'}$ if and only if $\chi([0,0,s(\widehat{s} - \widehat{s}')]) = 1$ for all $s \in k$. Since χ is nontrivial, it follows that $\widehat{s} = \widehat{s}'$. Thus the action is simply transitive.

Solution to Exercise 3.5.1

Let $a, b, c \in H$; by abuse of notation, write a, b, c for their images in P too. We have that

$$\begin{aligned}\langle ab, c\rangle\langle a, c\rangle^{-1}\langle b, c\rangle^{-1} &= \chi(abcb^{-1}a^{-1}c^{-1})\chi(cac^{-1}a^{-1})\chi(cbc^{-1}b^{-1}) \\ &= \chi(abcb^{-1}c^{-1}a^{-1})\chi(cbc^{-1}b^{-1}) \\ &= \chi(abcb^{-1}c^{-1}(cbc^{-1}b^{-1})a^{-1}) \\ &= \chi(1) = 1\end{aligned}$$

since $cbc^{-1}b^{-1} \in Z$. As for the second variable, since $\langle\cdot,\cdot\rangle$ is alternating, we have

$$\langle a, bc\rangle = \langle bc, a\rangle^{-1} = (\langle b, a\rangle\langle c, a\rangle)^{-1} = \langle a, b\rangle\langle a, c\rangle.$$

Solution to Exercise 3.5.3

(1) We calculate

$$[a, b, c][d, e, f][a, b, c]^{-1}[d, e, f]^{-1} = [0, 0, ea - bd] \in Z.$$

(2) If $p \in H/Z$ is nonzero then the above equation makes it clear that there is some $q \in H/Z$ such that $\langle p, q\rangle \neq 1$ since χ is nontrivial.

(3) The only thing to show is that the image $\overline{S}$ of S in P is a maximal isotropic subgroup (the proof for $\widehat{S}$ is the same). Since S is abelian, it is clear that $\overline{S}$ is isotropic. Let $[s, 0, 0] \in \overline{S}$ and $[a, b, 0] \in H/Z$. We have

$$\langle [s, 0, 0], [a, b, 0]\rangle = \chi([0, 0, bs])$$

so since χ is nontrivial, either $b = 0$ or $\langle [s, 0, 0], [a, b, 0]\rangle \neq 1$ for some $s \in k$.

(4) Recall that χ is a nontrivial character of $Z \cong k$, and that $P \cong k \oplus k$. Let n be the unique integer such that χ is trivial on $\wp^n$ but not on $\wp^{n-1}$. Let $K_m = \wp^m \oplus \wp^m$ for $m \in \mathbb{Z}$, so it is clear that

$$K_m^\perp = \{[d, e, 0] \mid \chi(ea - bd) = 1 \text{ for all } a, b \in \wp^m\} = \wp^{n-m} \oplus \wp^{n-m}.$$

For any $q \in P$, we have that

$$(qK_m)^\perp = \{p \in P \mid \langle p, qp'\rangle = \langle p, q\rangle\langle p, p'\rangle = 1 \text{ for all } p' \in K_m\} = \{q\}^\perp \cap K_m^\perp$$

since if $p \in (qK_m)^\perp$ then $\langle p, q\rangle\langle p, 1\rangle = \langle p, q\rangle = 1$. Since $\{q\}^\perp$ is the kernel of the map $p \mapsto \langle p, q\rangle$, $\{q\}^\perp$ is closed, so $(qK_m)^\perp$ is also a compact open subgroup.

Now let $K \subset P$ be any compact open subgroup, and find an m such that $K_m \subset K$. Then K/K_m is finite, so K is a finite union of subsets q_iK_m. Thus $K^\perp = \bigcap_i (q_iK_m)^\perp$ is also a compact open subgroup.

Solution to Exercise 3.5.6

Let K be a compact open subgroup of H, let $f \in \operatorname{Ind}_{SZ}^H(\widetilde{\chi})^K$, and let $s\widehat{t}z \in K$, where $s \in S, \widehat{t} \in \widehat{S}$, and $z \in Z$. We have $\widehat{s} \cdot s\widehat{t}z = sz \cdot (s^{-1}\widehat{s}s\widehat{s}^{-1})\widehat{s\widehat{t}}$, so

$$f(\widehat{s}) = f(\widehat{s} \cdot s\widehat{t}z) = \widetilde{\chi}(sz) \cdot \langle s^{-1}, \widehat{s}\rangle \cdot f(\widehat{s\widehat{t}})$$

for all $\widehat{s} \in \widehat{S}$. Assume that $K \cap (SZ) \subset \ker \widetilde{\chi}$, and set $\widehat{t} = 1$ and $z = 1$. Thus $f(\widehat{s}) = 0$ when there is an $s \in K \cap S$ such that $\langle s^{-1}, \widehat{s} \rangle \neq 1$, i.e., $f(\widehat{s}) = 0$ outside of the set

$$\{\widehat{s} \in \widehat{S} \mid \langle s^{-1}, \widehat{s} \rangle = 1 \text{ for all } s \in K \cap S\}.$$

By Remark 3.5.2(1), this set is compact, and it is clearly determined by K. Setting $s = 1$ and $z = 1$, we also see that f is locally constant with respect to $K \cap \widehat{S}$, so the dimension of $\mathrm{Ind}_{SZ}^{H}(\widetilde{\chi})^K$ must be finite.

The above proof shows that the restriction of any $f \in \mathrm{Ind}_{SZ}^{H}(\widetilde{\chi})$ to $\widehat{S}$ is locally constant and compactly supported, so $C_c^\infty(\widehat{S}) \cong \mathrm{Ind}_{SZ}^{H}(\widetilde{\chi})$ as complex vector spaces.

Solution to Exercise 3.5.9

This is nearly identical to the solution for Exercise 3.0.9. Any element of S' has a representation as $(s_1')^{r_1}(s_2')^{r_2}\cdots(s_n')^{r_n}$ for $r_i \in \mathbb{Z}$ that is unique up to translations of r_i by d_i. It follows that if $s_1^{r_1}s_2^{r_2}\cdots s_n^{r_n}z$ and $s_1^{r_1'}s_2^{r_2'}\cdots s_n^{r_n'}z'$ are two representations of the same element of SZ then for each i, we have $r_i - r_i' = m_i d_i$ for some $m_i \in \mathbb{Z}$. Since $s_i^{m_i d_i} \in Z$, we have

$$\begin{aligned} 1 &= (z')^{-1}z \cdot s_n^{-r_n'}\cdots s_2^{-r_2'}s_1^{-r_1'} \cdot s_1^{r_1}s_2^{r_2}\cdots s_n^{r_n} \\ &= (z')^{-1}z \cdot s_1^{m_1 d_1} \cdot s_n^{-r_n'}\cdots s_2^{-r_2'} \cdot s_2^{r_2}\cdots s_n^{r_n} \\ &= \cdots = (z')^{-1}z \cdot s_1^{m_1 d_1}s_2^{m_2 d_2}\cdots s_n^{m_n d_n} \end{aligned}$$

so $z^{-1}z' = s_1^{m_1 d_1}s_2^{m_2 d_2}\cdots s_n^{m_n d_n}$. Since each α_i is a d_ith root of $\chi(s_i^{d_i})$, we have

$$\alpha_1^{r_1}\alpha_2^{r_2}\cdots\alpha_n^{r_n}\chi(z) = \alpha_1^{r_1'}\alpha_2^{r_2'}\cdots\alpha_n^{r_n'}\chi(z')$$

which shows that $\widetilde{\chi}$ is well-defined. For any i, j, we have $\langle s_i', s_j' \rangle = 1$ since S' is isotropic, so $\chi(s_i^{r_i}s_j^{r_j'}s_i^{-r_i}s_j^{-r_j'}) = 1$, which shows that $\widetilde{\chi}$ is a character. It is obvious that $\widetilde{\chi}$ agrees with χ on Z.

Solution to Exercise 4.0.15

Let (π, V) be a finite-dimensional representation of $\mathrm{GL}_n(k)$. Then π has open kernel (choose a finite spanning set for V and intersect their stabilizers), so in particular $K = \ker \pi \cap N_\emptyset$ is open in $N_\emptyset$. By the calculation in Example 4.0.14 we have $N = \cup_{t \in T} {}^tK$, so in fact $\pi|_{N_\emptyset}$ is trivial. Now it suffices to show that the conjugates of $N_\emptyset$ generate $\mathrm{SL}_n(k)$: certainly they generate a normal subgroup, and the only (closed) proper normal subgroups of $\mathrm{SL}_n(k)$ are subgroups of the center, which is finite.

Solution to Exercise 5.1.5

First note that for any $\varphi \in \mathrm{Hom}_K(W_\sigma, V)$ we have that $\varphi(W_\sigma) \subset V(\sigma)$, so $\mathrm{Hom}_K(W_\sigma, V) = \mathrm{Hom}_K(W_\sigma, V(\sigma))$. Suppose first that $\dim_{\mathbb{C}} V(\sigma) < \infty$. Choose a decomposition $V(\sigma) = \bigoplus_1^{m(\sigma)} W_\sigma$ of $V(\sigma)$ into a direct sum of copies of W_σ. Then we have

$$\mathrm{Hom}_K(W_\sigma, V(\sigma)) \cong \mathrm{Hom}_K\left(W_\sigma, \bigoplus_1^{m(\sigma)} W_\sigma\right) = \bigoplus_1^{m(\sigma)} \mathrm{Hom}_K(W_\sigma, W_\sigma) = \bigoplus_1^{m(\sigma)} \mathbb{C}$$

so $\dim_{\mathbb{C}}(\operatorname{Hom}_K(W_\sigma, V)) = m(\sigma)$. If $m(\sigma) = \infty$ then the middle equality above no longer holds, but one can still choose infinitely many linearly independent K-maps $W_\sigma \to V(\sigma)$.

Solution to Exercise 5.1.10

Let V be a smooth G-module that is not finitely generated. Let $v_1 \in V$ be nonzero, and set $V_1 = \langle v_1 \rangle$ (i.e., V_1 is the G-module generated by v_1). Choose $v_2 \in V \setminus V_1$, and set $V_2 = \langle v_1, v_2 \rangle$. Continuing in this fashion, we construct an infinite proper filtration

$$\{0\} = V_1 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots$$

of V, which shows that V does not have finite length.

Solution to Exercise 5.2.2

We prove the statement by induction on the length of V. If V has length 1 then V is simple, so by hypothesis, V is admissible.

Suppose that the length of V is n, and that the statement is true for all smooth representations of length less than n. Let

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$$

be a filtration of V such that each V_i/V_{i-1} is simple. Note that the length of V_{n-1} is $n-1$. We have an exact sequence

$$0 \longrightarrow V_{n-1} \longrightarrow V \longrightarrow V/V_{n-1} \longrightarrow 0$$

which gives rise to an exact sequence

$$0 \longrightarrow V_{n-1}^K \longrightarrow V^K \longrightarrow (V/V_{n-1})^K \longrightarrow 0$$

for any compact open subgroup $K \subset G$. By the inductive hypothesis, V_{n-1}^K is finite-dimensional, and $(V/V_{n-1})^K$ is finite-dimensional since V/V_{n-1} is simple. Thus $\dim_{\mathbb{C}} V^K < \infty$, so V is admissible.

Solution to Exercise 5.3.1

Let $f \in C_c^\infty(G, V)$, let $S \subset G$ be the support of f, and for $g \in G$ let K_g be a compact open subgroup of G such that $f(gx) = f(g)$ for all $x \in K_g$. The set $\{gK_g \mid g \in S\}$ is an open cover of S, so it has a finite subcover $\{g_i K_{g_i} \mid i = 1, \ldots, n\}$. Setting $K = \bigcap_{i=1}^n K_{g_i}$, we have that $f \in C_c^\infty(G, V)^K = C_c(G/K, V)$.

Define $\varphi : C_c^\infty(G) \otimes V \to C_c^\infty(G, V)$ by $\varphi(f \otimes v)(x) = f(x)v$. Setting $g \cdot (f \otimes v) := (g \cdot f) \otimes v$, we see that φ is a G-map. First we show that φ is surjective. Let $f \in C_c^\infty(G, V)$, so $f \in C_c(G/K, V)$ for some compact open subgroup K. Let $g_1, \ldots, g_n \in G$ be such that f is supported on $g_1 K \sqcup \cdots \sqcup g_n K$, and let $v_i = f(g_i)$. For each i, let $[g_i K] \in C_c^\infty(G)$ be the characteristic function of $g_i K$. Thus $f = \varphi([g_1 K] \otimes v_1 + \cdots + [g_n K] \otimes v_n)$.

It remains to show that φ is injective. Indeed, suppose that $\varphi(f_1 \otimes v_1 + \cdots + f_n \otimes v_n) = 0$ for some $f_i \in C_c^\infty(G)$ and $v_i \in V$. Let K be a compact open subgroup such that each $f_i \in C_c(G/K)$. Thus each f_i is a linear combination of characteristic functions of cosets of G/K, so we may assume that each $f_i = [g_i K]$ for some $g_i \in G$, and that $g_i K = g_j K \implies i = j$. But then

$$\varphi(f_1 \otimes v_1 + \cdots + f_n \otimes v_n)(g_i) = v_i = 0,$$

which shows that φ is injective.

Solution to Exercise 5.3.2

Let $f \in C_c^\infty(G,V)$, and let $K \subset G$ be a compact open subgroup such that $f \in C_c(G/K,V)$. Choose $x_1,\dots,x_n \in G$ such that f is supported on $x_1K \cup \cdots \cup x_nK$. Let $K' \subset G$ be the compact open subgroup $\bigcap_{i=1}^n x_iKx_i^{-1}$, so if $g \in K'$ and $x \in G$, then $x \in x_iK$ if and only if $gx \in x_iK$. Therefore, if $x \notin x_1K \cup \cdots \cup x_nK$ then $f(gx) = f(x) = 0$. On the other hand, suppose that $x = x_iy$ for some $y \in K$, and let $z \in K$ be such that $g = x_izx_i^{-1}$. Then we have

$$f(gx) = f(x_izx_i^{-1} \cdot x_iy) = f(x_izy) = f(x_i) = f(x),$$

so $f(gx) = f(x)$ for all $x \in G$.

Solution to Exercise 5.3.3

Let K be any compact open subgroup of G such that $f \in C_c(G/K,V)$, and let $g_1,\dots,g_n \in G$ be elements such that f is supported on $g_1K \sqcup \cdots \sqcup g_nK$. Then we have

$$\int_G f(g)\,d_\ell g = \sum_{i=1}^n \int_{g_iK} f(g)\,d_\ell g = \sum_{i=1}^n f(g_i)\int_{g_iK} d_\ell g = \sum_{i=1}^n f(g_i)\cdot \mathrm{meas}_{d_\ell g}(K).$$

Solution to Exercise 5.4.1

(1) Let $d'_\ell g$ be any other left Haar measure on G, so $d'_\ell g = c\cdot d_\ell g$ for some $c \in \mathbb{R}_{>0}$. Then for any Borel set $S \subset G$ and any $x \in G$, we have

$$\mathrm{meas}_{d'_\ell g}({}^{x^{-1}}S) = c\cdot \mathrm{meas}_{d_\ell g}({}^{x^{-1}}S) = c\cdot \delta_G(x)\,\mathrm{meas}_{d_\ell g} S = \delta_G(x)\,\mathrm{meas}_{d'_\ell g} S.$$

(2) Let $x,y \in G$, and let $S \subset G$ be some Borel set such that $\mathrm{meas}_{d_\ell g} S \neq 0,\infty$. We have

$$\delta_G(xy)\,\mathrm{meas}_{d_\ell g} S = \mathrm{meas}_{d_\ell g}\left({}^{(xy)^{-1}}S\right) = \delta_G(y)\delta_G(x)\,\mathrm{meas}_{d_\ell g} S$$

so δ_G is a character. Let $K \subset G$ be any compact open subgroup. For any $x \in K$, ${}^{x^{-1}}K = K$, so since $\mathrm{meas}_{d_\ell g} K = \mathrm{meas}_{d_\ell g}({}^{x^{-1}}K)$ is nonzero and noninfinite, we must have $\delta_G(x) = 1$.

(3) Define a measure d_rg on G by setting $\int_S d_rg := \int_S \delta_G(g)d_\ell g$ for a Borel set $S \subset G$. For any $x \in G$, we have

$$\begin{aligned}\int_{Sx} d_rg = \int_{Sx}\delta_G(g)d_\ell g = \int_{x^{-1}Sx}\delta_G(x^{-1}g)d_\ell g = \delta_G(x^{-1})\int_S \delta_G(g)d^xg\\ = \delta_G(x^{-1})\int_S \delta_G(g)\delta_G(x)d_\ell g = \int_S d_rg\end{aligned}$$

which shows that d_r is a right Haar measure. In addition,

$$\begin{aligned}\int_{xSx^{-1}} d_rg = \int_{xSx^{-1}}\delta_G(g)\,d_\ell g = \int_S \delta_G(g)\,d^{x^{-1}}g\\ = \delta_G(x)^{-1}\int_S \delta_G(g)\,d_\ell g = \delta_G(x)^{-1}\int_S d_rg\end{aligned}$$

which shows that the modulus character of d_rg is δ_G^{-1}. (Or one could note that $\delta_G(g)^{-1}d_rg$ and $\delta'_G(g)d_rg$ are both left Haar measures, where δ'_G is the modulus character of d_rg.)

Solution to Exercise 5.4.4

Let $S \subset k$ be a Borel set, and let $y \in k$. We have

$$\operatorname{meas}_{d(xg)}(y+S) = \operatorname{meas}_{dg}(xy+xS) = \operatorname{meas}_{dg}(xS) = \operatorname{meas}_{d(xg)} S,$$

which shows that $d(xg)$ is a Haar measure, and therefore $d(xg) = c \cdot dg$ for some $c \in \mathbb{R}_{>0}$. It remains to calculate c. Suppose that $v(x) \leq 0$. We have

$$\operatorname{meas}_{d(xg)} R = \operatorname{meas}_{dg} \wp^{v(x)} = [\wp^{v(x)} : R] \operatorname{meas}_{dg} R = q^{-v(x)} \operatorname{meas}_{dg} R = |x| \cdot \operatorname{meas}_{dg} R.$$

The proof where $v(x) > 0$ is the same, replacing $[\wp^{v(x)} : R]$ with $[R : \wp^{v(x)}]^{-1}$.

Solution to Exercise 5.4.6

(1) First note that the only compact subgroup of $\mathbb{R}_{>0}$ is trivial. Let $\chi : N \to \mathbb{R}_{>0}$ be a smooth character, so $\chi(K) = \{1\}$ for any compact subgroup $K \subset N$. Since N is the increasing union of compact subgroups, χ must be trivial. Since δ_P restricts to a smooth character of N, we have $\delta_P(mn) = \delta_P(m)\delta_P(n) = \delta_P(m)$.

(2) Let $K \subset G$ be a compact open subgroup with an Iwahori decomposition with respect to $P_\theta = M_\theta N_\theta$, and let K_1 be a compact open subgroup of $K \cap {}^tK$ which also has an Iwahori decomposition with respect to $P_\theta = M_\theta N_\theta$. Define $K^\pm, K^0, K_1^\pm$, and K_1^0 as usual. Then we have

$$\delta_{P_\theta}(t) = \frac{\operatorname{meas}(K^0K^+)}{\operatorname{meas}({}^tK{}^tK^+)} = \frac{[K^0K^+ : K_1^0K_1^+] \cdot \operatorname{meas}(K_1^0K_1^+)}{[{}^tK^{0t}K^+ : K_1^0K_1^+] \cdot \operatorname{meas}(K_1^0K_1^+)} = \frac{[K^0K^+ : K_1^0K_1^+]}{[{}^tK^{0t}K^+ : K_1^0K_1^+]}.$$

One also calculates that

$$[K^0K^+ : K_1^0K_1^+] = \frac{[K^0 : K_1^0][K^+ : K_1^+]}{[K^0 \cap K^+ : K_1^0 \cap K_1^+]} = [K^0 : K_1^0][K^+ : K_1^+]$$

(since $M \cap N$ is trivial), and similarly for $[{}^tK^{0t}K^+ : K_1^0K_1^+]$. Since t is in the center of M_θ, we have ${}^tK^0 = K^0$. Thus

$$\delta_{P_\theta}(t) = \frac{[K^0 : K_1^0][K^+ : K_1^+]}{[{}^tK^0 : K_1^0][{}^tK^+ : K_1^+]} = \frac{[K^+ : K_1^+]}{[{}^tK^+ : K_1^+]}.$$

Much like in Example 5.4.5, then, we calculate that

$$\delta_{P_\theta}(t) = \prod_{\alpha \in \Phi^+} |\alpha(t)|^{-1} = \delta_{P_\emptyset}(t)$$

since $\alpha(t) = 1$ for all $\alpha \in \theta$ by the definition of T_θ.

This proof deserves an example. Let $G = \mathrm{GL}_3(k)$, and write $\Phi^+ = \{\alpha, \beta, \alpha+\beta\}$. Suppose that $\theta = \{\alpha\}$, so we can take

$$P_\theta = \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix} \qquad M_\theta = \begin{pmatrix} * & * & \\ * & * & \\ & & * \end{pmatrix} \qquad N_\theta = \begin{pmatrix} 1 & & * \\ & & * \\ & & 1 \end{pmatrix}$$

and

$$K = \begin{pmatrix} 1+\wp & \wp & \wp \\ \wp & \wp & \wp \\ \wp & \wp & 1+\wp \end{pmatrix} \quad K^- = \begin{pmatrix} 1 & & \\ \wp & & \\ \wp & 1 & \end{pmatrix} \quad K = \begin{pmatrix} 1+\wp & \wp & \\ \wp & \wp & \\ & & 1+\wp \end{pmatrix} \quad K^+ = \begin{pmatrix} 1 & & \wp \\ & & \wp \\ & & 1 \end{pmatrix}.$$

Suppose that $t = \mathrm{diag}(\varpi^d, \varpi^d, \varpi^e)$, and let $D = |d-e|$. Then

$$ {}^tK = \begin{pmatrix} 1+\wp & \wp & \wp^{e-d} \\ \wp & \wp & \wp^{e-d} \\ \wp^{d-e} & \wp^{d-e} & 1+\wp \end{pmatrix} \qquad \text{so we can take} \qquad K_1 = \begin{pmatrix} 1+\wp^D & \wp^D & \wp^D \\ \wp^D & \wp^D & \wp^D \\ \wp^D & \wp^D & 1+\wp^D \end{pmatrix}. $$

Then we have

$$ \frac{[K^+ : K_1^+]}{[{}^tK^+ : K_1^+]} = \frac{[\wp : \wp^D]}{[\wp^{e-d} : \wp^D]} \frac{[\wp : \wp^D]}{[\wp^{e-d} : \wp^D]} = \prod_{\gamma\in\Phi^+} |\gamma(t)|^{-1}. $$

(3) Let $g \in G$. By the Cartan decomposition, we can write $g = x_1 t x_2$, where $x_1, x_2 \in K_0$ and $t \in T^+$. Since K_0 is compact, $\delta_G(x_1) = \delta_G(x_2) = 1$, so it suffices to show that $\delta_G(t) = 1$. Let $K \subset G$ be a compact open subgroup with an Iwahori decomposition with respect to $P_\emptyset = M_\emptyset N_\emptyset$, and let $K_1 \subset G$ be a compact open subgroup of $K \cap {}^tK$ with an Iwahori factorization with respect to $P_\emptyset = M_\emptyset N_\emptyset$. Define $K^\pm, K^0, K_1^\pm, K_1^0$ as before. A calculation similar to the one in part (2) shows that

$$ \delta_G(t) = \frac{[K^+ : K_1^+][K^- : K_1^-]}{[{}^tK^+ : K_1^+][{}^tK^- : K_1^-]}. $$

In part (2), we also showed that

$$ \frac{[K^+ : K_1^+]}{[{}^tK^+ : K_1^+]} = \prod_{\alpha\in\Phi^+} |\alpha(t)|^{-1}, $$

and similarly,

$$ \frac{[K^- : K_1^-]}{[{}^tK^- : K_1^-]} = \prod_{\alpha\in\Phi^-} |\alpha(t)|^{-1} = \prod_{\alpha\in\Phi^-} |\alpha(t)|, $$

which completes the proof.

(4) Denote the map $m \mapsto |\det(\mathrm{Ad}(m^{-1})|_{\mathfrak{n}})| : M \to \mathbb{R}_{>0}$ by δ. Since δ is a smooth character, as in part (3), it suffices to show that $\delta(t) = \delta_P(t)$ for $t \in T^+$. Let $K \subset G$ be a compact open subgroup with an Iwahori decomposition with respect to $P = MN$, and define $K_1, K^\pm, K^0, K_1^\pm, K_1^0$ as before. In part (2), we effectively showed that

$$ \delta_P(t) = \frac{[K^0 : K_1^0]}{[{}^tK^0 : K_1^0]} \frac{[K^+ : K_1^+]}{[{}^tK^+ : K_1^+]} = \frac{[K^0 : K_1^0]}{[{}^tK^0 : K_1^0]} \prod_{\alpha\in\theta_1} |\alpha(t)|^{-1}, $$

where $\theta_1 \subset \Phi^+$ is the set of roots whose eigenspaces make up the Lie algebra $\mathfrak{n}$. By part (3), M is unimodular, so $[K^0 : K_1^0] = [{}^tK^0 : K_1^0]$. Since t acts diagonally on the root spaces in $\mathfrak{n}$ by $\alpha(t)$, we therefore have $\delta(t) = \delta_P(t)$.

Solution to Exercise 7.1.3

Let $W = V_1/V_2$ be an irreducible subquotient of V, where $V_2 \subset V_1 \subset V$ are subrepresentations. Thus the center Z of G acts by a character χ on W. Choose a nonzero $\overline{v} \in W$, and let $v \in V_1$ be any lift of $\overline{v}$. Find $\overline{\lambda} \in \widetilde{W}$ such that $\overline{\lambda}(\overline{v}) = 1$ — we can do this since $\overline{v} \in W^K$ for some compact open subgroup $K \subset G$, and $\widetilde{W}^K = \mathrm{Hom}_{\mathbb{C}}(W^K, \mathbb{C})$. Let $\lambda \in \widetilde{V_1}$ be the image of $\overline{\lambda}$ under the natural map $\widetilde{W} \to \widetilde{V_1}$. By Corollary 5.2.3, the restriction map $\widetilde{V} \to \widetilde{V_1}$ is surjective, so λ extends to a linear map $V \to \mathbb{C}$. By hypothesis, the support $\operatorname{supp}(m_{\lambda,v})$ of the matrix coefficient $m_{\lambda,v}$ is compact. Since Z acts by χ on W, for $z \in Z$, we have that $\pi(z)v = \chi(z)v + w$, where $w \in V_2$. Thus

$$m_{\lambda,v}(z) = \lambda(\pi(z)v) = \lambda(\chi(z)v + w) = \lambda(\chi(z)v) = \chi(z) \neq 0$$

since λ is trivial on V_2 by definition. Thus $Z \subset \operatorname{supp}(m_{\lambda,v})$, so since Z is a closed subset of a compact set, Z is compact.

Solution to Exercise 7.2.6

(1) Let $\chi \in \operatorname{Rat}(G)$, so $|\chi| : G \to \mathbb{R}_{>0}$ is a continuous character. Since the only compact subgroup of $\mathbb{R}_{>0}$ is trivial, $|\chi|$ must be trivial on every compact open subgroup of G.

(2) Since G^1 is an intersection of normal subgroups (the kernel of any homomorphism is normal), it is also a normal subgroup. For each $\chi \in \operatorname{Rat}(G)$, the character $|\chi| : G \to \mathbb{R}_{>0}$ is continuous, so $\ker|\chi| = |\chi|^{-1}(1)$ is closed; thus G^1 is closed as well. Since G^1 is a subgroup that contains a compact open subgroup, it must also be open. Any Haar measure on G restricts to a Haar measure on G^1, so G^1 is unimodular since G is.

(3) When $G = \mathrm{GL}_n(k)$, $G^1 = \ker|\det|$, so $G/G^1 \cong \operatorname{Im}|\det| = \{q^m \mid m \in \mathbb{Z}\} \cong \mathbb{Z}$. Since $Z(G) = k^\times$, the k-split part of $Z(G)$ has rank 1.

(4) If $a \cdot 1_n \in Z(G)$ (where $1_n \in \mathrm{GL}_n(k)$ is the identity) then $|\det(a \cdot 1_n)| = |a|^n$, so $|\det|(Z(G)) = \{q^{mn} \mid m \in \mathbb{Z}\}$ is the index-n subgroup of G/G^1. Thus $G/(Z(G)G^1) = (G/G^1)/(Z(G)/(Z(G) \cap G^1)) \cong \mathbb{Z}/n\mathbb{Z}$.

(5) In our case, $Z(G) \cap G^1$ is the compact group $R^\times \cdot 1_n$.

Solution to Exercise 7.3.7

(1) Suppose that the support C of $m_{\lambda,v}$ is compact modulo the center Z of G for some nonzero $\lambda \in \widetilde{V}$ and $v \in V$. Let $\lambda' \in \widetilde{V}$ and $v' \in V$ be arbitrary. By Corollary 7.3.6, both V and $\widetilde{V}$ are irreducible, so we can find $g_i, h_j \in G$ and $c_i, d_j \in \mathbb{C}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ such that

$$\lambda' = \sum_{i=1}^{m} c_i \cdot \widetilde{\pi}(g_i)\lambda \quad \text{and} \quad v' = \sum_{j=1}^{n} d_j \cdot \pi(h_j)v.$$

Thus for $x \in G$,

$$m_{\lambda',v'}(x) = \lambda'(\pi(g)v') = \sum_{i,j} c_i d_j \lambda(\pi(g_i^{-1} x g_j)v),$$

so the support of $m_{\lambda',v'}$ is contained in

$$\bigcup_{i,j} g_i C g_j^{-1},$$

which is also compact modulo Z.

(2) Let (π, V) denote the representation $\text{c-Ind}_K^G \sigma$, and suppose that (π, V) is irreducible. Choose a nonzero $w \in W$, and let $K_1 \subset K$ be a compact open subgroup such that $w \in W^{K_1}$. Define $f_w \in \text{c-Ind}_K^G \sigma$ by

$$f_w(x) = \begin{cases} \sigma(x)w & \text{if } x \in K \\ 0 & \text{otherwise.} \end{cases}$$

Thus f_w is fixed by K_1, which is a compact open subgroup of G as well. Define $\lambda \in \mathrm{Hom}_{\mathbb{C}}(V^{K_1}, \mathbb{C})$ by $\lambda(f) = f(1)$ and extend λ to all of V, so $\lambda \in \widetilde{V}$. Note that $\lambda(f_w) = w \neq 0$. For $x \in G$, we have

$$m_{\lambda, f_w}(x) = \lambda(\pi(x) \cdot f_w) = f_w(x)$$

which is zero unless $x \in K$. Thus the support of m_{λ, f_w} is compact modulo Z, so by part (1), (π, V) is supercuspidal.

Solution to Exercise 8.2.5

Let $(\pi, V) \in \mathfrak{R}(G)$, and let $\omega : G \to \mathbb{C}^\times$ be a smooth character. First we claim that if $\pi \otimes \omega$ is admissible, then π is admissible. Let $K \subset G$ be a compact open subgroup on which ω is trivial, and let $K' \subset G$ be any compact open subgroup. Then since ω is trivial on K, any vector $v \in V$ is fixed by $K' \cap K$ under π if and only if v is fixed by $K' \cap K$ under $\pi \otimes \omega$. Thus $V^{K' \cap K} \supset V^{K'}$ is finite-dimensional.

By the previous paragraph, we may assume that (π, V) is an irreducible representation that is square-integrable modulo the center. Suppose that (π, V) is not admissible, so there exists a compact open subgroup $K \subset G$, a vector $v \in V^K$, and elements $g_1, g_2, g_3, \ldots \in G$ such that the vectors $e_K\pi(g_1)v,\ e_K\pi(g_2)v,\ e_K\pi(g_3)v,\ \ldots \in V^K$ are linearly independent. Choose $\lambda \in \widetilde{V}^K = \mathrm{Hom}_{\mathbb{C}}(V^K, \mathbb{C})$ such that $\lambda(e_K\pi(g_i)v) = 1$ for all $i \geq 1$. Let $\overline{K}$ be the image of K in G/Z, so $\overline{K}$ is a compact open subgroup. Then we have

$$\begin{aligned}
\int_{G/Z} |m_{\lambda,v}(g)|^2 \, dg^* &= \int_{G/Z} |\lambda(\pi(g)v)|^2 \, dg^* \\
&= \int_{G/Z} |(e_K\lambda)(\pi(g)v)|^2 \, dg^* \\
&= \int_{G/Z} |\lambda(e_K\pi(g)v)|^2 \, dg^* \\
&= \sum_{\overline{g} \in (G/Z)/\overline{K}} |\lambda(e_K\pi(g)v)|^2 \cdot \mathrm{meas}_{dg^*}(\overline{K}) \\
&\geq \sum_{i=1}^{\infty} |\lambda(e_K\pi(g_i)v)|^2 \cdot \mathrm{meas}_{dg^*}(\overline{K}) \\
&= \infty,
\end{aligned}$$

a contradiction.

Solution to Exercise 9.1.3

Since $e = eee$, the relation $\leq$ is reflexive. If $e = fef$ and $f = efe$ then

$$e = ee = (fef)e = f(efe) = ff = f,$$

so $\leq$ is antisymmetric. If $e\mathcal{H}e \subset f\mathcal{H}f$ and $f\mathcal{H}f \subset g\mathcal{H}g$ then clearly $e\mathcal{H}e \subset g\mathcal{H}g$, so $\leq$ is transitive.

Clearly $0 \leq e$ for all $e \in I$, so by antisymmetry, 0 is the unique minimal element of I.

Solution to Exercise 9.1.5

Let $x \in \mathcal{H}$, so $\{x\} \subset \mathcal{H}$ is a finite subset. By hypothesis, there exists an $e \in I$ such that $exe = x$. Thus

$$\mathcal{H} = \bigcup_{e \in I} e\mathcal{H}e.$$

Let $S \subset I$ be any finite set of idempotents. We can find an element $f \in I$ with the property that $e = fef$ for all $e \in S$, so I is filtered with respect to $\leq$.

Conversely, suppose that $\mathcal{H} = \bigcup_{e \in I} e\mathcal{H}e$, and that I is filtered with respect to $\leq$. Let $S \subset \mathcal{H}$ be any finite subset, and for each $s \in S$, let $e_s \in I$ and $x_s \in \mathcal{H}$ be elements such that $s = e_s x_s e_s$. Find $f \in I$ such that $e_s \leq f$ for all $s \in S$. Then for $s \in S$, we have

$$fsf = f(e_s x_s e_s)f = f(fe_s f)x_s(fe_s f)f = (fe_s f)x_s(fe_s f) = e_s x_s e_s = s.$$

Solution to Exercise 9.1.6

By definition, $h_\alpha \longrightarrow 0$ if and only if for all $e \in I$, there exists an A such that for all $\alpha \geq A$, $h_\alpha \in \mathcal{H}(1-e)$. If $h_\alpha = h(1-e)$ thet $h_\alpha \cdot e = he - he = 0$; conversely, if $h_\alpha \cdot e = 0$ then $h_\alpha = h_\alpha \cdot (1-e) \in \mathcal{H}(1-e)$.

Solution to Exercise 9.2.3

Suppose that $T_1 = \int_G d\mu_1(g)$ and $T_2 = \int_G d\mu_2(g)$. Then

$$T_2(g \mapsto T_1(gf)) = \int_G T_1(R(g_2)f)\, d\mu_2(g_2) = \int_G \int_G f(g_1 g_2)\, d\mu_1(g_1)\, d\mu_2(g_2) = (T_1 * T_2)(f).$$

Solution to Exercise 9.2.4

For $f_1, f_2 \in C_c^\infty(G)$ and $f \in C^\infty(G)$, we have

$$\begin{aligned}\int_G f(g_1)(f_1 * f_2)(g_1)\, dg_1 &= \int_G \int_G f(g_1) f_1(g_1 g_2^{-1}) f_2(g_2)\, dg_2\, dg_1 \\ &= \int_G \int_G f(g_1 g_2) f_1(g_1) f_2(g_2)\, dg_1\, dg_2.\end{aligned}$$

For $(h_1, h_2) \in G$, we have

$$\int_G f(h_1 g h_2^{-1}) f_1(g)\, dg = \int_G f(g) f_1(h_1^{-1} g h_2)\, dg$$

since dg is unimodular.

Solution to Exercise 9.4.3

(1) Let $\overline{v} \in V_e$ such that $e\mathcal{H}\overline{v} = 0$. Let $v \in V$ be a lift of $\overline{v}$, so $e\mathcal{H}e\mathcal{H}v = 0$. Thus $eee\mathcal{H}v = e\mathcal{H}v = 0$, so $\overline{v} = 0$. It follows that $(V_e)_e = V_e$.

(2) Suppose that

$$\varphi(h_1 \otimes ev_1 + \cdots + h_n \otimes ev_n) = h_1 e \cdot v_1 + \cdots + h_n e \cdot v_n = 0.$$

Let $eh \in e\mathcal{H}$. We have

$$\sum_i ehh_i \otimes ev_i = \sum_i ehh_i e \otimes v_i = 1 \otimes \left(eh \sum_i h_i ev_i \right) = 0.$$

(3) First note that for any $f \in \mathrm{Hom}_{e\mathcal{H}e}(W, V)$, $f(w) = f(e \cdot w) = e \cdot f(w)$, so $f(W) \subset eV$. Thus $\mathrm{Hom}_{e\mathcal{H}e}(W, V) = \mathrm{Hom}_{e\mathcal{H}e}(W, eV)$.

Let $f \in \mathrm{Hom}_{\mathcal{H}}(\mathcal{H} \otimes_{e\mathcal{H}e} W, V)$. Define a map $g : W \to V$ by $g(w) = f(e \otimes w)$, so for $ehe \in e\mathcal{H}e$, we have

$$g(ehe \cdot w) = f(e \otimes (ehe)w) = f(ehe \otimes w) = ehe \cdot f(e \otimes w)$$

since f is an $\mathcal{H}$-map.

Conversely, for $g \in \mathrm{Hom}_{e\mathcal{H}e}(W, V)$, define a map $f : \mathcal{H} \otimes_{e\mathcal{H}e} W \to V$ by $f(h \otimes w) = h \cdot g(w)$. This is well-defined since for $eh'e \in e\mathcal{H}e$, we have

$$heh'e \cdot g(w) = f(h(eh'e) \otimes w) = f(h \otimes (eh'e)w) = h \cdot g(ehe' \cdot w).$$

The maps $f \mapsto g$ and $g \mapsto f$ are clearly inverse.

Solution to Exercise 10.1.9

We know that $m \circ \tau^{-1}$ is a $\mathbb{C}$-linear map; we must prove that it respects multiplication. First note that for $v, v', v'' \in V$ and $\lambda, \lambda' \in \widetilde{V}$, we have

$$(\tau(v \otimes \lambda) \circ \tau(v' \otimes \lambda'))(v'') = \tau(v \otimes \lambda)(\lambda'(v'')v') = \lambda(v')\lambda'(v'')v = \tau(v \otimes \lambda(v')\lambda')(v'').$$

This gives the induced multiplication law on $V \otimes_{\mathbb{C}} \widetilde{V}$, so we must show that

$$\deg(\pi)^2 \cdot \check{m}_{\lambda,v} * \check{m}_{\lambda',v'} = \deg(\pi)\lambda(v') \cdot \check{m}_{\lambda',v}.$$

Indeed, for $x \in G$, we have

$$\begin{aligned}
\deg(\pi) \cdot (\check{m}_{\lambda,v} * \check{m}_{\lambda',v'})(x) &= \deg(\pi) \cdot \int_G \check{m}_{\lambda,v}(xg^{-1}) \cdot \check{m}_{\lambda',v'}(g)\, dg \\
&= \deg(\pi) \cdot \int_G m_{\lambda,\pi(x^{-1})v}(g) \cdot \check{m}_{\lambda',v'}(g)\, dg \\
&= \lambda(v') \cdot \lambda'(\pi(x^{-1})v) \\
&= \lambda(v') \cdot \check{m}_{\lambda',v}(x).
\end{aligned}$$

Solution to Exercise 10.3.2

Let $\mathcal{A} = \{A_i \mid i \in I\}$ be a commuting family of invertible operators on a finite-dimensional complex vector space V. The set $\mathcal{A}$ defines a representation π of the abelian group $\bigoplus_I \mathbb{Z}$ on V by setting $\pi((1)_i) = A_i$. Thus we can reformulate the question as follows: let (π, V) be a finite-dimensional complex representation of an abelian group G. Then there is a nonzero eigenvector which is common to $\pi(g)$ for all $g \in G$.

By Exencise 3.3.6, there is an irreducible subrepresentation $W \subset V$ of G. Since G is abelian, $\pi(g) \in \mathrm{End}_G(W)$ for all $g \in G$. It follows from Exercise 2.1.8 that W is one-dimensional, so any nonzero element of W is an eigenvector of each $\pi(g)$.

Solution to Exercise 13.0.16

Let Z be the center of R. Any $z \in Z$ defines an endomorphism φ_M^z of any R-module M by $z \mapsto zx$; if $f : M \to N$ is a map of R-modules, we have $f(zx) = zf(x)$, so $f \circ \varphi_M^z = \varphi_N^z \circ f$, so $\varphi^z \in \mathfrak{z}(\mathcal{A})$. It is clear that $\varphi^z = \varphi^{z'} \implies z = z'$ (take $M = R$), so it remains to show that if $\varphi \in \mathfrak{z}(\mathcal{A})$ then $\varphi = \varphi^z$ for some $z \in Z$. Let $\varphi \in \mathfrak{z}(\mathcal{A})$. The map $\varphi_R : R \to R$ is a R-module endomorphism, so $\varphi_R(x) = \varphi_R(x \cdot 1) = xz$, where $z = \varphi_R(1)$. Let $y \in R$, and define a map $\psi^y : R \to R$ of R-modules by $\psi^y(x) = xy$. Since φ is in the center of $\mathcal{A}$, we have

$$yz = \varphi_R \circ \psi^y(1) = \psi^y \circ \varphi_R(1) = zy,$$

so $z \in Z$. Let M be any R-module and let $m \in M$, and define a R-map $f : R \to M$ by $f(x) = xm$. Then

$$\varphi_M(m) = \varphi_M \circ f(1) = f \circ \varphi_R(1) = f(z) = zm = \varphi_M^z(m),$$

so $\varphi = \varphi^z$. Thus $z \mapsto \varphi^z$ is a bijection of sets; it is clear that it is a ring isomorphism.

Solution to Exercise 14.2.7

Suppose $s \in \ker(T - a)^d$. If $a = 0$, then $s(n + d) = 0$ for $n \in \mathbb{Z}_{\geq 0}$, in which case $s(n) = 0$ whenever $n \geq d$.

Now assume $a \neq 0$. We proceed by induction on d. If $d = 1$, then $Ts - as = 0$, so $s(n + 1) - s(n)a = 0$ for all $n \in \mathbb{Z}_{\geq 0}$. Hence $s(n) = s(0)a^n$, and we can take $p = s(0)$, a polynomial of degree $0 < 1$.

Now suppose the result holds for $d \geq 1$ and suppose $s \in \ker(T - a)^{d+1}$. Then $Ts - as \in \ker(T - a)^d$, so $s(n + 1) - s(n)a = p(n)a^n$ for all $n \in \mathbb{Z}_{\geq 0}$ and some $p \in \mathbb{C}[x]$ of degree $< d$. Working from this equation, we deduce that

$$\begin{aligned} s(n) &= \left(\sum_{i=0}^{n-1} p(i)\right) a^{n-1} + s(0)a^n \\ &= \left(\frac{1}{a}\sum_{i=0}^{n-1} p(i) + s(0)\right) a^n. \end{aligned}$$

It suffices to prove that if $p \in \mathbb{C}[x]$ has degree $< d$, and $f(n) := \sum_{i=0}^{n-1} p(i)$, then f is a polynomial function of degree $< d + 1$. Separating the expression for $f(n)$ by degrees, this reduces to the well-known result that $\sum_{i=0}^{n-1} i^k$ is a polynomial function of n of degree $k + 1$.

Solution to Exercise 14.4.4

all $(v, \lambda) \in V \times \widetilde{V}$ there exists an $\varepsilon > 0$ such that for all $t \in T_M^+(\varepsilon)$ we have

Solution to Exercise 16.5.4

This result holds when X is any topological space and $C_c^\infty(X)$ is interpreted as the set of locally constant compactly supported functions. Fix $f \in C_c^\infty(X)$ and let $U_\alpha = f^{-1}(\alpha)$. As f is locally constant, U_α is a union of open sets and hence open. Now

$$f^{-1}(\mathbb{C}^\times) = \bigcup_{\alpha \in \mathbb{C}^\times} U_\alpha$$

is a union of opens and hence open. Furthermore, $f^{-1}(\mathbb{C}^\times) = X \smallsetminus U_0$ and is also closed. Hence $\mathrm{supp}(f) = f^{-1}(\mathbb{C}^\times)$. This set is compact by hypothesis. Furthermore, $\{U_\alpha \mid \alpha \in f(X) \smallsetminus \{0\}\}$ is a disjoint open cover of $\mathrm{supp}(f)$ and hence finite. It follows that $f(X)$ is finite.

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E-mail address: `smdbackr@umich.edu`

UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109