2013–14 LEARNING SEMINAR: JACQUET–LANGLANDS

The aim of the seminar this year is to understand the Jacquet–Langlands correspondence, including the representation theory lying behind it from a general point of view (not just GL_2), and how it combines with the trace formula to address local-global compatibility. The prerequisites are familiarity with holomorphic modular forms (including Hecke eigenforms), class field theory from the adelic viewpoint, quaternion algebras (over general fields, and arithmetic aspects over local and global fields), basic notions related to connected reductive groups over fields (parabolic subgroups, conjugacy theorems, etc.), and the basic structure and representation theory of connected compact Lie groups.

For illustrative purposes we'll focus on GL_n, but the general picture is important for conceptual understanding. We will be starting a few weeks late, due to logistical reasons. We meet on Wednesdays, 2pm–4pm, 383-N.

1. Fall quarter: Representation theory

We begin by discussing the global and local aspects of representation theory: adelic, p-adic fields, and archimedean local fields ($\mathbf{R}$ and $\mathbf{C}$).

References:

(BH) Bushnell–Henniart, *The local Langlands correspondence for* GL(2).

(BZ) Bernstein–Zelevinsky, *Induced representations of reductive* $\mathfrak{p}$*-adic groups. I*, Ann. ENS, 1977.

(D) DeBacker, `http://www.math.lsa.umich.edu/~smdbackr/MATH/notes.pdf`.

(K1) Knapp, *Introduction to the Langlands Program*, 1997.

(K2) Knapp, *Representation theory of semisimple Lie groups: an overview based on examples.*

(LW) Loeffler–Weinstein, *On the computation of local components of a newform.*

Lecture 1 (October 16, Akshay): For a connected reductive group G over a number field k, automorphic representations as irreducible subrepresentations of $L^2(G(k)\backslash G(\mathbf{A}_k))$. For GL_2 over $\mathbf{Q}$, describe the injection from the set of holomorphic eigenforms (modulo near equivalence) into the set of automorphic representations. Discuss the relationship between classical Hecke algebras and convolution algebras/group actions. Decomposition of adelic representations as tensor product over places (Flath).

Lecture 2 (October 23, Brian C.): Smooth representations of p-adic reductive groups versus unitary representations (statements only). Admissibility, duality, and induction. Relationship between Hecke algebras and K-isotypic G-representations (generalization of Proposition on page 38 of [BH]). Explicit structure of the unramified Hecke algebra (give abstract statement in general, explicit example of GL_n).

Lecture 3 (October 30, November 6; Iurie): More on p-adic reductive group representations: discuss principal series, parabolic induction, and Jacquet functors in general and illustrate for GL_n: see §2 and especially Theorem 2.9 of [BZ]. Also see [D].

LECTURE 4 (November 13, Macky): Spherical representations, spherical Hecke algebra, Satake isomorphism (statement in general quasi-split case, made explicit for GL_n).

LECTURE 5 (November 20, Niccolo): Discuss supercuspidal representations: general definition as in BZ (also see Ch. 3 in [BH], and [D]). Illustrate for GL_2 by induction from a finite field (Ch. 3 of [BH], maybe also [LW]). State local Langlands for GL_n over non-archimedean local fields and its compatiblity with parabolic induction (partial reference: §8 of [K1], and §30–32 of [BH] for ε-factors).

LECTURE 6 (December 4, Zhiwei): $(\mathfrak{g}, K)$-modules and unitary representations, parabolic induction, statement and sketch of proof that every irreducible representation is a subquotient of principal series. Definition of discrete series.

LECTURES 7, 8 (December 11, Akshay): Discrete series for real Lie groups: examples for GL_2, discuss existence only when $\mathrm{rank}(G) = \mathrm{rank}(K)$ (no proof). State the Langlands classification without proof, and discuss the statement of local Langlands for GL_n over $\mathbf{R}$ and $\mathbf{C}$. Reference: [K2].

2. WINTER QUARTER: TRACE FORMULA

Next, we prove the Jacquet-Langlands correspondence.

Reference:

- (B) Badulescu, lecture notes on the Jacquet-Langlands correspondence. (Provides details on functional analysis aspects.)
- (GJ) Gelbart–Jacquet, *Forms of* GL(2) *from the analytic point of view* (in volume 1 of the Corvallis proceedings).

LECTURE 9 (January 15, Zeb): Return to classical automorphic forms for GL_2 over $\mathbf{Q}$: classification as holomorphic or Maass forms using representation theory discussed already. The representation-theoretic formulation of the Ramanujan-Petersson conjecture and Selberg's eigenvalue conjecture.

LECTURE 10 (January 22, Akshay): Eisenstein series I. (As a starting point, look at §5 of [GJ]).

LECTURE 11 (January 29, Akshay): Eisenstein series II.

LECTURES 12, 13 (February 5, 12 Jeremy): Trace formula for compact quotients. Prove trace formula in adelic form on a division algebra definite at ∞ ([GJ], §1). Discuss application to class numbers.

No meeting on February 19 (perfectoid workshop at MSRI)

LECTURE 14 (February 26, Iurie): Trace formula for non-compact quotients I. Truncation of Eisenstein series and the Maass-Selberg relations ([GJ], §5).

LECTURE 15 (March 5, Iurie): Trace formula for non-compact quotients II. Statement and proof of the trace formula ([GJ], §6).

LECTURE 16 (March 12, Brian C.): ℓ-adic Lefschetz trace formula and Honda–Tate theory. Computing trace of T_p by mod-p techniques, and relation with abelian varieties over finite fields.

3. SPRING QUARTER: GALOIS REPRESENTATIONS

As an application, we prove the compatibility between Frobenius eigenvalues on the Galois side and Hecke eigenvalues on the automorphic side.

Reference:

(C) Clozel, *Nombre de points des variétés de Shimura sur un corps fini* (d'aprés R. Kottwitz).
(CCO) Chai, Conrad, Oort, *Complex multiplication and lifting problems*, AMS, 2014.

Lecture 17 (April 2, Brian L., Hilaf): Honda–Tate I: Overview of the proof of Honda–Tate bijection (including the role of p-divisible groups and Dieudonné modules), discussion of the role of p-part of Tate's isogeny theorem (see [CCO] for everything).

Lecture 18 (April 9, Brian L., Hilaf): Honda–Tate II: continuation of last time and application to proving formula (20) in [C] (for the case of ordinary elliptic curves).

Lecture 19 (April 16, Dan L.): Local–global compatibility for GL_2

Lecture 20 (April 23, Zhiwei): Jacquet–Langlands correspondence ([GJ], §8).

Lecture 21 (April 30, Akshay): The "big picture" on the trace formula.

Lecture 22 (May 7, Macky): Matching of orbital integrals ([GJ], §8)

Lecture 23 (May 14, Zeb, Evan): Hecke trace and class numbers I

Lecture 24 (May 21, Zeb, Evan): Hecke trace and class numbers II

Lecture 25 (May 28, Niccolò): Comparison of trace formulas

Lecture 26 (June 4, TBA): Overflow