10 Lecture 10: Affinoid adic spaces I

10.1 Introduction

The aim of this lecture is to define the topological space underlying affinoid adic spaces, and discuss some essential topological properties as well as a fundamental correspondence in the spirit of the Nullstellensatz. We review the main result from Lecture 9, rephrasing it in a handy way, and define the adic spectrum of a Huber ring A with respect to some additional data attached to A. We shall deduce spectrality of the adic spectrum at once from Corollary 9.3.2, and hence, in the last instance, from spectrality of the continuous valuation spectrum $\mathrm{Cont}(A)$.

In the last section we formulate five topics concerning affinoid adic spectra which we shall address in the next lecture.

10.2 Pro-constructible loci in $\mathrm{Cont}(A)$

Let A be an arbitrary commutative ring (no topology). Recall from §9.1 that for any ideal $J \subset A$ whose radical is the radical of a finitely generated ideal (ie. $\mathrm{Spec}(A) - V(J)$ is qc) we studied the subspace $\mathrm{Spv}(A, J)$ of the valuation spectrum $\mathrm{Spv}(A)$; this consists of the points $v \in \mathrm{Spv}(A)$ admitting no *proper* horizontal specialization supported over $\mathrm{Spec}(A) - V(J)$ (a vacuous constraint when v itself lies over $V(J)$, so mainly of interest when v does not lie over $V(J)$; e.g., $\mathrm{Spv}(A, A)$ is the set of v with no proper horizontal specialization whatsoever, or equivalently $c\Gamma_v = \Gamma_v$). In terms of notation introduced in §9.2, in Definition 9.2.3 we defined

$$\mathrm{Spv}(A, J) := \{v \in \mathrm{Spv}(A) \mid c\Gamma_v(J) = \Gamma_v\}$$

where $c\Gamma_v(J)$ is a convex subgroup of Γ_v containing $c\Gamma_v$ whose properties were summarized in Proposition 9.2.2. Geometrically, the points of $\mathrm{Spv}(A, J)$ not over $V(J)$ are those “at the interior edge” (relative to horizontal specialization) of the set of points of $\mathrm{Spv}(A)$ not over $V(J)$. The assignment

$$r : v \mapsto v|_{c\Gamma_v(J)}$$

gave a continuous spectral retraction of $\mathrm{Spv}(A)$ onto $\mathrm{Spv}(A, J)$, which was used in our deduction of spectrality of $\mathrm{Spv}(A, J)$ from that of $\mathrm{Spv}(A)$.

We now fix a *Huber ring* A, and a couple of definition (A_0, I), so $A_0 \subset A$ is an open bounded subring and I a finitely generated ideal of A_0 such that the topology on A_0 is the I-adic topology.

Recall the fundamental Theorem 9.3.1:

Theorem 10.2.1 *We have:*

$$\mathrm{Cont}(A) = \{v \in \mathrm{Spv}(A, A^{00} \cdot A) \mid v(a) < 1 \ \text{ for all } \ a \in A^{00}\}.$$

Remark 10.2.2 Recall that if A is Tate then A^{00} contains a unit of A, and therefore $A^{00} \cdot A = A$. Hence, if A is Tate then

$$\mathrm{Cont}(A) = \{v \in \mathrm{Spv}(A, A) \mid v(a) < 1 \ \text{ for all } \ a \in A^{00}\}.$$

For general Huber ring A we have

$$\mathrm{Spv}(A, A^{00} \cdot A) = \mathrm{Spv}(A, I \cdot A)$$

where the ideal I of A_0 is finitely generated as such. We fix generators $\{a_1, \ldots, a_n\}$ of I as an ideal of A_0, so these are generators of $I \cdot A$ that are *topologically nilpotent* in A (whereas if A is Tate then $I \cdot A$ can also be generated by the element 1 that is only topologically nilpotent when $\{0\}$ is dense in A). By Corollary 9.3.3, we have the following concrete description of the continuous valuation spectrum:

$$\mathrm{Cont}(A) = \{v \in \mathrm{Spv}(A) \mid \gamma_i := v(a_i) \in \Gamma_v \text{ are all cofinal, } v(a) < 1/\max_i \gamma_i \text{ for all } a \in A_0\}$$

with the understanding that the condition "$v(a) < 1/\max_i \gamma_i$" is vacuously satisfied when $\gamma_i = 0$ for all i.

Example 10.2.3 The following example clarifies the importance of cofinality over the condition "$v(a) < 1$" when v has higher rank (i.e., not trivial and not of rank 1). Assume A is Tate with ring of definition A_0, and let $\varpi \in A^\times$ be a pseudo-uniformizer (i.e., a topologically nilpotent unit). By replacing ϖ with ϖ^m for large enough m we may and do assume $\varpi \in A_0$, so $I := \varpi A_0$ is an ideal of definition of A_0. Since $v(\varpi) \neq 0$ for all valuation v on A (as $\varpi \in A^\times$), we have:

$$\mathrm{Cont}(A) = \{v \in \mathrm{Spv}(A) \mid v(\varpi) \text{ is cofinal, } v(a) < 1/v(\varpi) \text{ for all } a \in A_0\}.$$

Note that v may well be of rank > 1, and the difference between having $v(\varpi) < 1$ and having $v(\varpi)$ be cofinal for Γ_v is substantial. For instance, consider the field

$$A := k(u, x) = k(x)(u) = k(u)(x)$$

where we endow A with the "x-adic topology" to make it a Huber ring (i.e., as the fraction field of the discrete valuation ring $k(u)[x]_{(x)}$). Consider the valuation ring

$$R_v := k[x]_{(x)} + uk(x)[u]_{(u)} \subset A$$

with associated valuation on A denoted as v. Clearly $v(x) < 1$ since $x \in R_v$, but $v(x)$ is not cofinal for $\Gamma_v = a^{\mathbf{Z}} \times b^{\mathbf{Z}}$, with $v(x) = (a, 1)$ and $v(u) = (1, b)$ for any choice of $0 < a, b < 1$. Concretely, x is not topologically nilpotent for v. Thus, v is *not* continuous on A even though $v(x) < 1$ with x a pseudo-uniformizer of the Tate ring A. This also illustrates the fact that the subring of power-bounded elements in a higher-rank valuation ring need *not* coincide with the valuation ring (since $1/x$ is power-bounded because $1/x^n \in uk(x)[u]_{(u)}$ for all $n \geq 1$ yet $1/x \notin R_v$).

One can also make a mixed-characteristic analogue of the preceding example by equipping the field $A' := \mathbf{Q}(u)$ with the p-adic topology and working with the valuation v' corresponding to the rank-2 valuation ring $\mathbf{Z}_{(p)} + u\mathbf{Q}[u]_{(u)}$. In both cases the two valuations we considered are not continuous, since their values respectively on x and p are not cofinal.

Loci cut out by inequalities

Choose $a \in A^0$ and $v \in \mathrm{Cont}(A)$. Experience with rank-1 valuations might lead one to guess that necessarily $v(a) \leq 1$ in Γ_v, but this is false in higher rank:

Example 10.2.4 Let (K, v) be a valued field with valuation ring R, where v is nontrivial and of rank > 1. Assume that v is microbial; i.e., there exists a topologically nilpotent element of $K^\times$, or equivalently, K is Tate. Note that v is continuous on K by design.

By Proposition 9.1.3 (and its proof), there exists a height-1 prime ideal $\mathfrak{p} \subset R$ such that $R_{\mathfrak{p}}$ induces the same topology on K, and $R_{\mathfrak{p}}$ is a rank-1 valuation ring. Thus, $K^0 = R_{\mathfrak{p}}$. Since R is higher-rank, we have

$$R \subsetneq R_{\mathfrak{p}}.$$

Any $a \in R_{\mathfrak{p}} - R \subset K - R$ is a power-bounded element of K which does not satisfy $v(a) \leq 1$ in Γ_v.

Remark 10.2.5 It is not true that height-1 prime ideals always exist in a general valuation ring R that is not a field! For example, let us consider the abelian group

$$\Gamma := (a^{\mathbf{Z}})^{\oplus I}, \quad 0 < a < 1$$

where I is $\mathbf{Z}$ itself, and Γ is correspondingly ordered lexicographically. Concretely, an element $\gamma \in \Gamma$ is a doubly-infinite sequence of powers

$$\gamma = (a^{c_n})_{n \in \mathbf{Z}}$$

where $c_n = 0$ for all but finitely many $n \in \mathbf{Z}$. For any $1 \neq \gamma \in \Gamma$, let n_0 be the minimal integer such that $c_{n_0} \neq 0$. If a *convex* subgroup of Γ contains γ then it must contain the entire n_0-component $a^{\mathbf{Z}}$ and so contains the group of elements (a^{d_n}) where $d_n = 0$ for $n < n_0$. This latter subgroup is manifestly convex, and it is not hard to deduce from this that these are exactly the *nontrivial proper* convex subgroups of Γ. (Consider the quotient of Γ modulo the direct sum of the factors for $n > n_0$.)

Since we can move "arbitrarily far to the left", among *proper* convex subgroups of Γ there are no maximal elements with respect to inclusion. Every totally ordered abelian group G is the value group of a suitable valuation v on the fraction field of the group algebra $k[G]$ (see [AM, Exer. 33, Chapter 5], which is reminiscent of our study of $K = k((u))((x))$ as a rank-2 valued field). Thus, the valuation ring R_v has no height-1 prime ideal due to the inclusion-reversing bijective correspondence between the set of nonzero prime ideals of R and the set of proper convex subgroups of Γ [Wed, Prop. 2.14].

Let us now prove the following:

Lemma 10.2.6 *For any element $a \in A$, the locus*

$$\{v \in \mathrm{Cont}(A) \mid v(a) \leq 1 \text{ in } \Gamma_v\} \subseteq \mathrm{Cont}(A)$$

is constructible.

The preceding discussion shows that if $a \in A^0$ then this constructible subset can be a proper subset due to the behavior at higher-rank valuations!

Proof. Let $J \subset A$ be any ideal whose radical is the radical of a finitely generated ideal of A, and call $X := \mathrm{Spv}(A, J)$. (The case of interest to us is $J = A^{00} \cdot A$.) By Proposition 9.2.5(2), for every $s \in A$ and nonempty finite subset $T \subset A$ such that

$$J \subset \mathrm{rad}(T \cdot A),$$

the subset

$$X(T/s) = \{v \in X \,|\, v(t) \leq v(s) \neq 0, t \in T\}$$

is a qc open of X.

Now set $J = A^{00} \cdot A$. We know that the subset $\mathrm{Cont}(A) \subset X$ is proconstructible, and the above gives that $X(\{a, 1\}/1)$ is a qc open subset of X. The following diagram of sets is cartesian:

$$\begin{array}{ccc} \mathrm{Cont}(A) & \hookrightarrow & \mathrm{Spv}(A, I) \\ \uparrow & & \uparrow \\ \{v \in \mathrm{Cont}(A) \mid v(a) \leq 1\} & \hookrightarrow & X(\{a, 1\}/1) \end{array}$$

Hence, we are reduced to proving that for a spectral space Y, qc open subset $U \subset Y$, and proconstructible subset Z (so Z is spectral), the open subset $U \cap Z \subset Z$ is qc and hence constructible in Z.

Endowing Y and Z with the constructible topology,

$$Z_{\mathrm{cons}} \hookrightarrow Y_{\mathrm{cons}}$$

is a continuous injection between compact Hausdorff spaces, and hence is a closed map. In particular Z is closed in Y in the constructible topology, and being U qc, $Z \cap U$ is closed in Z in the constructible topology. Since Z_{cons} is compact Hausdorff, $Z \cap U$ is compact for the constructible topology. It follows $Z \cap U$ is qc for the initial topology on Z (which has far fewer open sets than for the constructible topology on Z in general). □

10.3 Huber pairs and their adic spectrum

Now we apply the construction in Lemma 10.2.6 many times:

Definition 10.3.1 For a subset $\Sigma \subset A$, define

$$\mathrm{Spa}(A, \Sigma) := \{v \in \mathrm{Cont}(A) \mid v(a) \leq 1 \text{ for all } \ a \in \Sigma\}$$

and call it the *adic spectrum* of the pair (A, Σ).

As a consequence of Lemma 10.2.6, $\mathrm{Spa}(A, \Sigma)$ is proconstructible in $\mathrm{Cont}(A)$ for all subsets $\Sigma \subset A$, and since this latter is spectral, we deduce that $\mathrm{Spa}(A, \Sigma)$ is spectral.

Remark 10.3.2 By the ultrametric inequality, for every topologically nilpotent element $f \in A^{00}$ we have that $v(a) \leq 1$ if and only if $v(a + f) \leq 1$. In particular $\mathrm{Spa}(A, \Sigma) = \mathrm{Spa}(A, \Sigma + A^{00})$ inside $\mathrm{Cont}(A)$.

The subring $\mathbf{Z}[\Sigma + A^{00}] \subset A$ generated by any subset $\Sigma \subset A$ together with A^{00} is an open subring of A, and due to the ultrametric inequality clearly

$$\mathrm{Spa}(A, \Sigma) = \mathrm{Spa}(A, \mathbf{Z}[\Sigma + A^{00}]).$$

For the same reason, if $B \subset A$ is a subring and $v : A \to \Gamma_v \cup \{0\}$ is a valuation on A such that $v(b) \leq 1$ in Γ_v for all $b \in B$ then the integral closure $\widetilde{B}$ of B in A (which contains all nilpotent elements of A!) satisfies

$$v(\widetilde{b}) \leq 1$$

for all $\widetilde{b} \in \widetilde{B}$, as we see by applying the ultrametric inequality to a *monic* polynomial relation for $\bar{b}$ over B, much as how we show that a valuation ring is integrally closed in its fraction field. (Note that such $\widetilde{B}$ is generally *not* normal as a ring in the sense of commutative algebra when A is not a normal ring.) Thus, $\mathrm{Spa}(A, \Sigma) = \mathrm{Spa}(A, A^+)$ where A^+ is the open integrally closed subring of A given by the integral closure of $\mathbf{Z}[\Sigma + A^{00}]$ in A.

Definition 10.3.3 A *Huber pair* is (A, A^+) where A is a Huber ring and $A^+ \subset A^0$ is an open subring of A that is integrally closed in A. The *adic spectrum* of a Huber pair (A, A^+) is

$$\mathrm{Spa}(A, A^+),$$

and is denoted $\mathrm{Spa}(A)$ when A^+ is understood from the context.

Remark 10.3.4 In [H1] and [Wed, §7.3], Huber pairs are called "affinoid rings". The requirement $A^+ \subset A^0$ will play an important technical role beginning in our study of rational domains in §11.4 and adic spectra of completions in §11.5 (it is lurking in the proofs of results we will invoke from [H1, 3.9-3.11] and in the very statement of Proposition 11.6.1).

Before discussing examples, let us give one more definition and discuss an important correspondence, which the reader may regard as to be in the spirit of an adic version of the algebraic "Nullstellensatz".

Definition 10.3.5 Given a Huber ring A, we define

$$\mathcal{G}_A := \{A^+ \subset A \text{ open integrally closed subring}\}$$

and

$$\mathcal{F}_A := \{\text{proconstructible sets of } \mathrm{Cont}(A) \text{ of the form } \mathrm{Spa}(A, \Sigma),\ \Sigma \subset A \text{ subset}\}.$$

Note that we do *not* require $A^+ \subset A^0$ (in contrast with the situation in Remark 10.3.4).

Theorem 10.3.6 *We have a bijective inclusion-reversing correspondence:*

$$\mathcal{G}_A \xrightarrow{\sigma} \mathcal{F}_A$$

sending an open subring of integral elements $A^+ \subset A$ *to* $\mathrm{Spa}(A, A^+)$*, with inverse given by the assigning to any such proconstructible subset* $\mathcal{V} \subset \mathrm{Cont}(A)$ *the subring*

$$\{a \in A \mid v(a) \leq 1 \text{ for all } v \in \mathcal{V}\}.$$

Moreover, if $A^+ \subset A^0$ *then* $\mathrm{Spa}(A, A^+)$ *contains all rank-*1 *points of* $\mathrm{Cont}(A)_{\mathrm{an}}$ *and all trivial valuations of* $\mathrm{Cont}(A)$*.*

Before we discuss the proof of this theorem, we record some interesting consequences. Suppose $A^+ \subset A^0$, so $\mathrm{Spa}(A, A^+)$ contains all rank-1 points and all trivial valuations in $\mathrm{Cont}(A)$. By Proposition 9.1.3 and the discussion in §9.1, each fiber of $\mathrm{Spa}(A, A^+) \to \mathrm{Spec}(A)$ that is not empty has a unique generic point: it is a trivial valuation for open primes in A and it is a rank-1 valuation for non-open primes of A. (It can happen that some fibers over $\mathrm{Spec}(A)$ are empty yet $A \neq 0$; e.g., even the entire space $\mathrm{Spa}(A, A^+)$ can be empty with $A \neq 0$. We will address this issue next time.)

Since all specialization (equivalently, generization) relations inside $\mathrm{Cont}(A)_{\mathrm{an}}$ are vertical (by Proposition 8.3.10), the same holds in the subspace $\mathrm{Spa}(A, A^+)_{\mathrm{an}}$. Thus, every *analytic* point $v \in \mathrm{Spa}(A, A^+)$ admits a unique "minimal-rank" generization η_v in the space $\mathrm{Spa}(A, A^+)_{\mathrm{an}}$ of analytic points. This η_v is a rank-1 point with the same support as v and it defines the unique rank-1 valuation on $\kappa(v)$ yielding the *same* topology as v (due to how η_v is built). This is especially useful when A is Tate, as all perfectoid algebras will be, as then *all* points of $\mathrm{Cont}(A)$ are analytic.

On the other hand, if v is a non-analytic point then $\mathfrak{p}_v$ is open and hence *all* valuations on $\mathrm{Frac}(A/\mathfrak{p}_v)$ are continuous on A. The fiber of $\mathrm{Spa}(A, A^+) \to \mathrm{Spec}(A)$ over an open prime ideal $\mathfrak{p}$ is therefore the Riemann–Zariski space $\mathrm{Spv}(\kappa(\mathfrak{p}), A^+)$ of valuations on $\kappa(\mathfrak{p})$ whose valuation ring contains the image of A^+. This has a unique generic point, namely the trivial valuation on $\kappa(\mathfrak{p})$, and it does *not* recover the v-topology on $\kappa(\mathfrak{p}) = \kappa(v)$ unless v is itself a trivial valuation.

Proof. We first check that all rank-1 analytic points v of $\mathrm{Cont}(A)$ satisy $v(a) \leq 1$ for all $a \in A^0$. (This is false without analyticity: consider a discrete Huber ring A that is the fraction field of a discrete valuation ring, so $A = A^0$ due to the discreteness of the impose topology.) Let (A_0, I) be a pair of definition for A as usual, so by continuity $v(t) < 1$ for all $t \in I$. By analyticity, $v(I)$ is nonzero. Choose $t \in I$ such that $v(t) \neq 0$. Topological nilpotence of elements of I and continuity of v force $v(s) < 1$ for all $s \in I$. Power-boundedness of a provides a large integer m so that $a^j I^m \subseteq I$ for all $j \geq 1$. Hence, $v(a)^j v(t)^m < 1$ for all j. Thus, $v(a)^j < 1/v(t)^m$ in the value group of v for all $j > 0$. Since v is a rank-1 valuation, this forces $v(a) \leq 1$, as desired.

The only difficulty for the bijectivity aspect of σ is to show that if $a \notin A^+$ then there exists $u \in \mathrm{Spa}(A, A^+)$ such that $u(a) > 1$. Before worrying about continuity of valuations, we try to build *some* valuation on A such that the valuation ring in the corresponding residue field contains the image of A^+ but *not* the image of a. This is essentially a problem of building appropriate prime ideals not containing a and then using the fact that local subrings of domains are always dominated by valuation rings. Horizontal specialization will be used at the end to handle continuity issues.

We claim that

$$1/a \notin A^+[1/a]^\times \subset A_a,$$

or in other words $a \notin A^+[1/a]$ inside A_a. (Keep in mind that we are assuming $a \notin A^+$.) Suppose to the contrary that a lies in this subring of A_a, so $a = f/a^n$ inside A_a for some $n \geq 0$ and $f \in A^+$. Hence, $a^n - f = 0$ in A_a, or equivalently $a^m(a^n - f) = 0$ in A for some $m \geq 0$. But then $a^{n+m} - fa^m = 0$ in A, so we have an integral dependence relation for a over A^+, forcing $a \in A^+$, contrary to our hypothesis.

It follows that there exists a prime ideal $\mathfrak{p}$ of $A^+[1/a]$ containing $1/a$. Choose a minimal prime $\mathfrak{q} \subset \mathfrak{p}$ of $A^+[1/a]$, and consider a valuation ring $R \subset \kappa_a(\mathfrak{p}) := \mathrm{Frac}(A^+[1/a]/\mathfrak{q})$ dominating the local domain $(A^+[1/a]/\mathfrak{q})_{\mathfrak{p}/\mathfrak{q}}$. This R is attached to a valuation on $\kappa_a(\mathfrak{p})$ which yields on $A^+[1/a]$ a valuation v whose support is exactly $\mathfrak{q}$, and such that $v(A^+) \leq 1$ and $v(\mathfrak{p}) < 1$ (the latter due to the local domination). As a consequence, $v(1/a) < 1$.

Localizing the *inclusion* $A^+[1/a] \hookrightarrow A_a$ at the minimal prime $\mathfrak{q}$ of $A^+[1/a]$ yields an injection of rings $A^+[1/a]_{\mathfrak{q}} \hookrightarrow (A_a)_{\mathfrak{q}}$, forcing $(A_a)_{\mathfrak{q}}$ to be nonzero. Thus, there is a prime ideal $\mathfrak{Q}$ of A_a whose contraction to $A^+[1/a]$ lies inside $\mathfrak{q}$ and hence is equal to $\mathfrak{q}$ due to the minimality of $\mathfrak{q}$. The resulting injection of domains $A^+[1/a]/\mathfrak{q} \hookrightarrow A_a/\mathfrak{Q}$ yields a corresponding injection between their fraction fields. But valuations on fields always extend to valuations on extension fields (since any local ring inside a field is dominated by a maximal one with full fraction field, and such maximal local rings are valuation rings by Zorn's Lemma). Thus, we can extend v to a valuation v' on A_a.

Restriction of v' yields a valuation w on A satisfying $w(a) > 1$ and $w(A^+) \leq 1$, but in view of the extremely abstract mechanism by which w was constructed it is probably not continuous! To fix this problem, we will work with the horizontal specialization $u = w_{|c\Gamma_w}$. Note that u has no proper horizontal specialization at all, so it certainly lies in $\mathrm{Spv}(A, A^{00} \cdot A)$. Hence, by our "algebraic" characterization of continuity of valuations on Huber rings, u is continuous provided that $u(x) < 1$ for all $x \in A^{00}$. Since $u(y) \leq w(y)$ for all $y \in A$ (in fact, $u(y) = w(y)$ if $w(y) \in c\Gamma_w$, and $u(y) = 0$ otherwise), we have $u(A^+) \leq 1$ and $u(a) = w(a) > 1$ (as $c\Gamma_w$ contains $w(A)_{\geq 1}$). Hence, it suffices to show $w(x) < 1$ for every $x \in A^{00}$. Topological nilpotence of x in A and openness of A^+ in A gives $x^m a \in A^+$ for large m. (Recall that $a \notin A^+$, so such membership in A^+ is a nontrivial condition on x^m.). Thus, $w(x^m a) \leq 1$, so $w(x)^m \leq 1/w(a) < 1$ and hence $w(x) < 1$. We conclude that $u \in \mathrm{Spa}(A, A^+)$ and $u(a) > 1$. □

10.4 Some loose ends for affinoid adic spectra

In the next lecture we will address the following topics that quickly come to mind upon thinking about $\mathrm{Spa}(A, A^+)$:

(1) Give a concrete description of a topological base of qc open "rational domains" of $\mathrm{Spa}(A, A^+)$ and prove that the natural map

$$\mathrm{Spa}(A^\wedge, (A^+)^\wedge) \to \mathrm{Spa}(A, A^+)$$

is a homeomorphism preserving the property of "rational domain" in *both* directions. (This homeomorphism property is an essential technical device in the study of perfectoid spaces and "approximation" arguments with them. It would be a mistake to only define the topological space $\mathrm{Spa}(A, A^+)$ under a completeness hypothesis on A.)

(2) Characterize when $\mathrm{Spa}(A, A^+)$ is empty, and likewise for its subset of analytic points.

(3) For a non-archimedean field k and a k-affinoid algebra A, discuss density of $\mathrm{Sp}(A) \subset \mathrm{Spa}(A, A^0)$ and how $\mathrm{Spa}(A, A^0)$ is related to the Berkovich space $M(A)$ and the Tate topos of $\mathrm{Sp}(A)$.

(4) Why consider the case $A^+ \neq A^0$? (This question is immediately suggested by (3) and occurs to everyone who sees $\mathrm{Spa}(A, A^+)$ defined for the first time, especially if done without first going through a detailed study of $\mathrm{Spv}(A)$ and $\mathrm{Cont}(A)$ culminating in Theorem 10.3.6 as we have done following [H1]. Some simple examples will illustrate why it would be a bad idea to insist on taking $A^+ = A^0$.)

(5) Equip $\mathrm{Spa}(A, A^+)$ with a suitable structure presheaf (initially defined on the "rational domains" in (1)), and discuss when it is a sheaf. Relate morphisms of such "structured locally ringed spaces" to morphisms of pairs (A, A^+).

References

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[H1] R. Huber, *Continuous valuations*.

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