15 Lecture 15: Points and lft maps

15.1 A noetherian property

Let A be an affinoid algebraic over a non-archimedean field k and $X = \mathrm{Spa}(A, A^0)$. For any $x \in X$, the stalk $\mathscr{O}_{X,x}$ is the limit of the directed system $\{A_i\}$ of affinoid algebraic corresponding to rational domains $\mathrm{Spa}(A_i, A_i^0) \subset X$ around x. In particular, if x is a classical point (i.e., corresponds to a maximal ideal of A) then this limit likewise compute the stalk at x on $\mathrm{Sp}(A)$, so by classical rigid geometry [BGR, 7.3.2/7] we know that $\mathscr{O}_{X,x}$ is noetherian for such x (with completion that naturally coincides with the completion of A at the maximal ideal associated to x). But for more general x we don't have the link to rigid geometry and hence it is less clear if the stalk is noetherian. Oddly, this result does not seem to be addressed in the literature, so now we give Temkin's elegant proof of:

Proposition 15.1.1 *For any $x \in X = \mathrm{Spa}(A, A^0)$, the local ring $\mathscr{O}_{X,x}$ is Noetherian.*

Proof. The local ring at a point x in an affinoid adic space $\mathrm{Spa}(A, A^+)$ for a classical affinoid algebra (A, A^+) is the direct limit of a directed system (A_i) for rational domains $\mathrm{Spa}(A_i, A^+)$ that contain x, and the transition maps $A_i \to A_j$ are flat by classical rigid-analytic geometry [BGR, 7.3.2/6]. Let $x_i \in \mathrm{Spec}(A_i)$ be the prime coming from the maximal ideal of the local limit $\mathscr{O}_{X,x}$, so the directed system of algebraic localizations

$$B_i = (A_i)_{x_i}$$

has the same (local) limit. Let $\mathfrak{m}_i$ denote the maximal ideal of B_i.

The advantage of considering the directed system B_i is that since these are local noetherian rings with flat local transition maps, by [EGA, 0_{III}, 10.3.1.3] the limit is noetherian if for sufficiently large i we have

$$\mathfrak{m}_i B_j = \mathfrak{m}_j$$

for all $j \geq i$. (This is the mechanism by which the noetherian property of henselizations of local noetherian rings is proved in [EGA], for example.)

Observe that $\dim(B_i) \leq \dim(A)$ for all i since for a maximal ideal $\mathfrak{n}$ of A_i containing x_i and the point $\xi \in \mathrm{Sp}(A_i) \subset \mathrm{Sp}(A) = X$ corresponding to $\mathfrak{n}$ we have $\dim(B_i) \leq \dim(A_i)_{\mathfrak{n}} = \dim \mathscr{O}_{X,\xi} \leq \dim(A)$ (using that the local noetherian rings $(A_i)_{\mathfrak{n}}$ and $\mathscr{O}_{X,\xi}$ have the same completion and hence the same dimension). The dimension formula for flat local maps of local noetherian rings gives that

$$\dim(B_j) = \dim(B_i) + \dim(B_j/\mathfrak{m}_i B_j) \geq \dim(B_i)$$

for all $j \geq i$. Hence, the dimensions of the B_i's are monotone in i and bounded above, so for some large enough i_0 we have $\dim(B_i) = \dim(B_{i_0})$ for all $i \geq i_0$.

Returning to the dimension formula, the local noetherian fiber algebras $B_i/\mathfrak{m}_{i_0} B_j$ are 0-dimensional for all $i \geq i_0$. Thus, these fiber algebras have nilpotent maximal ideal, so to conclude that $\mathfrak{m}_i = \mathfrak{m}_{i_0} B_i$ for all $i \geq i_0$ it suffices to show that all fiber algebras of $\mathrm{Spec}(A_i) \to \mathrm{Spec}(A_{i_0})$ (e.g., the closed fiber, whose algebra admits $B_i/\mathfrak{m}_{i_0} B_i$ as an algebraic localization) are reduced.

More generally, for any map $A \to A'$ of classical affinoid algebras corresponding to an affinoid subdomain $\mathrm{Sp}(A') \hookrightarrow \mathrm{Sp}(A)$, and any y' in $\mathrm{Spec}(A')$ and its image y in $\mathrm{Spec}(A)$ we claim that for the local map

$$(*) \quad A_x \longrightarrow A'_{y'}$$

all fibers algebras are geometrically regular, i.e., regular and remain so after any finite extension of the base field of the fiber algebra. In other words, we claim that $(*)$ is a regular map of Noetherian rings.

Pick a maximal ideal $\mathfrak{m}'$ of A' containing y', so it contracts to a maximal ideal $\mathfrak{m}$ of A containing y. The map $(*)$ is a localization (on source and target) of the map

$$A_{\mathfrak{m}} \longrightarrow A'_{\mathfrak{m}'},$$

so it suffices to show that the latter flat local map is regular. We know that the induced map on completions is an isomorphism by the classical theory, and to check the regularity property for a map of Noetherian rings it suffices to do after composing with a faithfully flat extension of the target (since a noetherian ring is regular if it admits a faithfully flat regular extension ring). Hence, it suffices to show the map from $A_{\mathfrak{m}}$ to its completion is a regular map. But this latter regularity is a consequence (and a key part in the proof) of the excellence of affinoid algebras, an important result due to Kiehl. □

15.2 Points as morphisms

Given a morphism of adic spaces $f : X \to Y$, and a point $y \in Y$, it is natural to want to view the fiber $f^{-1}(y)$ as the underlying topological space of an "adic space fiber". However, matters are not quite as for schemes because in the context of adic spaces a field (equipped with extra structure) generally does not give rise to a one-point space! So the correct expectation is that $f^{-1}(y)$ should coincide with the fiber over the unique closed point in a suitably-defined "adic space fiber" over $\mathrm{Spa}(\kappa(y), \kappa(y)^+)$ where $\kappa(y)$ is the residue field of $\mathscr{O}_{Y,y}$ and $\kappa(y)^+$ is its valuation ring (the image of $\mathscr{O}^+_{Y,y}$ in $\kappa(y)$); of course, we could also replace $\kappa(y)$ and $\kappa(y)^+$ with their completions for this purpose. Strictly speaking, to make sense of $\mathrm{Spa}(\kappa(y), \kappa(y)^+)$ we have to know that $(\kappa(y), \kappa(y)^+)$ is a Huber pair, and we saw in 14.6 that this holds when y is an analytic point.

Since we have not discussed fiber products of adic spaces (a topic which has some subtleties, since in contrast with schemes it will not always exist), the preceding discussion merely serves as geometric motivaton for the algebraic considerations that follow. In effect, just as a point of an affine scheme can be interpreted in terms of maps from the given ring to a field, we seek a similarly intepretation for points of affinoid adic spaces.

Let (A, A^+) be a Huber pair, $X := \mathrm{Spa}(A, A^+)$ and $\mathrm{Spa}(A', A'^+)$ a rational domain of X containing a point $x \in X$. Let $\mathfrak{p}_x$ be the support of x in A and set $\kappa(x) := \mathrm{Frac}(A/\mathfrak{p}_x)$ as a valued field using v_x. The following diagram is commutative:

$$\begin{array}{ccc}
\kappa(x) & \hookrightarrow & \kappa'(x) \\
\| & & \| \\
\mathrm{Frac}(A/\mathfrak{p}_x) & & \mathrm{Frac}(A'/\mathfrak{p}'_x) \\
\downarrow v_x & \swarrow v'_x & \\
\Gamma_x & &
\end{array}$$

Recall that $\kappa(x)$ and $\kappa'(x)$ have the same completion, as discussed in Lecture 14. In any global situation $\kappa(x)$ will mean $\mathscr{O}_x/\mathfrak{m}_x$ or $(\mathscr{O}_x/\mathfrak{m}_x)^\wedge$ (the context should make the intent clear). The completion convention is, given the last observation, the safest to work with as it is computed using any affinoid open subspace around x.

Let us now assume x is an analytic point of X, so every generization of x is analytic (as analyticity is an open condition) and $(\kappa(x), \kappa(x)^+)$ is a Huber pair. By Proposition 8.3.10, all generizations of x in X are *vertical* generizations.

Analyticity of x ensures that $\kappa(x)$ with its v_x-topology is topologically a rank-1 valued field, where the rank-1 valuation ring is the subring $\kappa(x)^0$ of power-bounded elements (generally bigger than $\kappa(x)^+$). Recall that there is a unique rank-1 vertical generization $\eta_x \in \mathrm{Spv}(\kappa(x))$ of x that lies in the fiber through x for $X \to \mathrm{Spec}(A)$, and that in general (by §4.2 in Lecture 4 and [Mat, 10.1]) the set

$$\{\text{generizations of } x \text{ in } \mathrm{Spv}(\kappa(x))\}$$

is in bijection with the set of valuation subrings R of $\kappa(x)$ containing $\kappa(x)^+$ as well as with the set $\mathrm{Spec}(\kappa(x)^+)$ via the correspondence assigning to any prime ideal $\mathfrak{p}$ of $\kappa(x)^+$ the localization $R = \kappa(x)^+_{\mathfrak{p}}$.

The set of all generizations of x in X (not just in $\mathrm{Spv}(A)$!) is therefore

$$\mathrm{Spa}(\kappa(x), \kappa(x)^+) = \{\kappa(x)^+ \subset R \subset \kappa(x)^0\}$$

under which x is the unique closed point and η_x is the unique generic point (corresponding to $\kappa(x)^0$); this does *not* contain the generic point of $\mathrm{Spv}(\kappa(x))$ (i.e., the trivial valuation) since the latter is not continuous on A and so does not lie in X inside $\mathrm{Spv}(A)$. We are now going to focus on $\mathrm{Spa}(\kappa(x), \kappa(x)^+)$; as the topological space of all generizations of x in X (with the subspace topology from X) it is an analogue of $\mathrm{Spec}(\mathscr{O}_{X,x})$ in the theory of schemes.

Since for analytic $x \in X$ as above the valuation subring $\kappa(x)^+$ is an open subring of the rank-1 valuation ring $\kappa(x)^0$, we are motivated to make:

Definition 15.2.1 An *affinoid field* (k, k^+) is a pair consisting in a complete nonarchimedean field k and an open valuation subring k^+ of k^0 (so $\mathrm{Frac}(k^+) = k$ due to openness of k^+ in k).

Remark 15.2.2 Any affinoid field (k, k^+) is a Huber pair. Indeed, since k^0 is an intermediate valuation ring between k^+ and $k = \mathrm{Frac}(k^+)$, necessarily k^0 arises from a valuation on the residue field of k^+ by [Mat, 10.1] and so the maximal ideal k^{00} of k^0 is contained in that of k^+. Thus, if $\varpi \in k^0$ is a pseudo-uniformizer and $x \in k^\times$ then $\varpi^n/x \in k^{00} \subset k^+$ for all large n (largeness depending on x), so $v(\varpi)^n \leq v(x)$ where v denotes the valuation for k^+. It follows that $v(\varpi^n k^+) \leq v(x)$ for all large n, so the open ideals $\varpi^n k^+$ in k^+ are a base for the v-topology on k, thereby verifying the Huber property.

Proposition 15.2.3 *Let (A, A^+) be any Huber pair, and consider the set*

$$\Sigma_A := \{(A, A^+) \xrightarrow{\varphi} (k, k^+) \mid \varphi(A) \subset k \text{ generates a dense subfield}\}$$

where (k, k^+) is some affinoid field, or equivalently the set of continuous valuations v on A for which $v(A^+) \leq 1$ and the associated completion of $\mathrm{Frac}(A/\mathfrak{p}_v)$ has the topology of a rank-1 valued field.

There is a natural bijection

$$\Sigma_A \simeq |\mathrm{Spa}(A, A^+)_{\mathrm{an}}|$$

assigning to any $\varphi \in \Sigma_A$ the valuation v_φ that is the image under $\mathrm{Spa}(\varphi)$ of the unique closed point in $\mathrm{Spa}(k, k^+)$.

Proof. This is immediate from the preceding discussion since the valuation on k associated to k^+ is the unique closed point in $\mathrm{Spa}(k, k^+)$. □

As a corollary we can immediately globalize by working with affinoid open subspaces:

Corollary 15.2.4 *For an adic space, the set $|X_{\mathrm{an}}|$ of analytic points is in natural bijection with the set of morphisms*

$$\{f : \mathrm{Spa}(k, k^+) \to X \mid \kappa(x) \subset k \text{ is dense}\}$$

from the adic spectra of affinoid fields to X by assigning to any such f the image of the unique closed point of $\mathrm{Spa}(k, k^+)$.

This corollary is most useful when *all* points of X are analytic. Since any map of adic spaces carries analytic points to analytic points, if X is equipped with a morphism to an adic space S whose points are all analytic then the same holds for X. For instance, this happens when $S = \mathrm{Spa}(K, K^+)$ for an affinoid field (K, K^+), the most important example being when $K^+ = K^0$.

15.3 Locally of finite type maps

In this section we discuss what a locally finite type morphism of adic spaces should be. One wrinkle is that classical rigid geometry over a complete nonarchimedean field k does not provide much guidance, since (in hindsight) *all* morphisms in classical rigid geometry are locally of finite type when upgraded to morphisms of adic spaces (since the adic spaces themselves will be locally of finite type over $\mathrm{Spa}(k, k^0)$).

Let (A, A^+) and (B, B^+) be *complete* Huber pairs, with $X := \mathrm{Spa}(B, B^+)$ and $S := \mathrm{Spa}(A, A^+)$, and let $f : X \to S$ be a morphism. One context in which we certainly want to say that f is "finite type" is if f can be factored as a "closed immersion" into some relative unit polydisc over S. On the level of Huber pairs, this amounts to having a commutative diagram

$$\begin{array}{ccc} (A\langle y_1, \ldots, y_n\rangle, A\langle y_1, \ldots, y_n\rangle^+) & \longrightarrow & (B, B^+) \\ \uparrow & \nearrow & \\ (A, A^+) & & \end{array}$$

where $A\langle y_1, \ldots, y_n\rangle \to B$ is a topological quotient map (i.e., B is topologically the quotient of $A\langle y_1, \ldots, y_n\rangle$ by a closed ideal) and the open image B' of $A\langle y_1, \ldots, y_n\rangle^+$ in B is a ring of definition whose integral closure is B^+. (Here it is no different to work with the open subring $A^+\langle y_1, \ldots, y_n\rangle$ whose integral closure in $A\langle y_1, \ldots, y_n\rangle$ in $A\langle y_1, \ldots, y_n\rangle^+$ by definition.)

We should allow "varying polyradii" in these relative polydiscs, much as in classical rigid geometry we consider polydiscs given by conditions $\{|y_i| \le r_i\}$ for $r_i \in \sqrt{|k^\times|}$ (not requiring $r_i \in |k^\times|$). So for nonempty finite subsets $T_1, \ldots, T_n \subset A$ such that $T_i A \subset A$ is open we should also allow ourselves to use the topological A-algebra

$$A\langle X\rangle_T := \{\sum a_\nu X^\nu \in A[\![X]\!] \mid \text{for every } U, a_\nu \in T^\nu U \text{ for all but finitely many } \nu\}$$

where U varies through additive open subgroups of A, having as a base of opens

$$U\langle X\rangle_T := \{\sum a_\nu X^\nu \in A\langle X\rangle_T \mid a_\nu \in T^\nu U \text{ for all } \nu\}$$

and ring of definition $A^+\langle X\rangle_T$. We recall that $A\langle X\rangle_T$ is Huber, complete when A is complete, and is initial among Huber rings over A equipped with an ordered n-tuple of elements $x_1, \ldots, x_n$ such that $t_i x_i$ is power-bounded for all $t_i \in T_i$ and for all $1 \le i \le n$.

Finally, given a Huber pair (A, A^+), we define $A\langle X\rangle_T^+$ to be the integral closure of the ring of definition $A^+\langle X\rangle_T \subset A\langle X\rangle_T$.

Definition 15.3.1 Given a map $(A, A^+) \to (B, B^+)$ of complete Huber pairs, we say it is *of finite type* if there exists $T_1, \ldots, T_n$ as above and a map of Huber pairs

$$(A\langle X\rangle_T, A\langle X\rangle_T^+) \to (B, B^+)$$

such that the map on Huber rings is an open surjection and the map $A\langle X\rangle_T^+ \to B^+$ is integral; equivalently, $B = A\langle X\rangle_T/J$ for some closed ideal J and B^+ is the integral closure in B of the open image of $A^+\langle X\rangle_T$.

Example 15.3.2 For a complete Huber pair (A, A^+) and an element $s \in A$ and a non-empty finite subset $T = \{t_1, \ldots, t_n\} \subset A$ such that $T \cdot A \subset A$ is open, consider the rational subdomain

$$A\langle T/s\rangle = A\langle X_1, \ldots, X_n\rangle_T/\overline{(sX_i - t_i)}$$

with $A\langle T/s\rangle^+$ as usual. Then $(A\langle T/s\rangle, A\langle T/s\rangle^+)$ is of finite type over (A, A^+).

It is important to note that when the base ring is Tate (e.g., a non-archimedean field), the role of the polyradii can be suppressed:

Proposition 15.3.3 *If A is Tate then for (B, B^+) of finite type over (A, A^+) we can always choose $T_i = \{1\}$ for all i; i.e., for any $T_1, \ldots, T_n$ there exists a continuous open A-algebra surjection $A\langle Y\rangle \twoheadrightarrow A\langle X\rangle_T$ such that $A^+\langle Y\rangle \to A^+\langle X\rangle_T$ is integral.*

Proof. The idea is to identify $\mathrm{Spa}(A\langle X\rangle_T, A\langle X\rangle_T^+)$ with a rational domain in some $\mathrm{Spa}(A\langle Y\rangle, A\langle Y\rangle^+)$, as then we can apply Example 15.3.2 to conclude.

First we shall make a "change of coordinates" to bound the T_i's relative to a ring of definition, as follows. Let $A_0 \subset A$ be a ring of definition, and pick a topologically nilpotent unit $\varpi \in A$. Let $m \geq 0$ be an integer large enough so that $\varpi^m t_i \in A_0 \subset A$ for all $t_i \in T_i$ $(1 \leq i \leq n)$. By replacing X_i with $\varpi^{-m}X_i$ and replacing T_i with $\varpi^m T_i$ for all i, we may thereby assume $t_i \in A_0$ for all $t_i \in T_i$ and for all i.

Recall that $A\langle X\rangle_T$ is universal among Huber rings over A equipped with an ordered n-tuple of elements $x_1, \ldots, x_n$ such that $t_i x_i$ is power-bounded for all $t_i \in T_i$ and all i. We have rigged that all elements of each T_i are power-bounded, and the only open ideal in A is the unit ideal (by the Tate hypothesis!), so $T_i A = A$ for all i. Letting $T_i X_i$ denote the finite subset of elements $t_i X_i \in A\langle X\rangle$, we get a finite subset

$$\Sigma := T_1 \cup \cdots \cup T_n \cup T_1X_1 \cup \cdots \cup T_nX_n \subset A\langle X\rangle^0$$

that generates an open ideal of $A\langle X\rangle$ (even the unit ideal). Therefore it makes sense to form the completed rational localization

$$(A\langle X\rangle)\langle \Sigma/1\rangle$$

that is a Huber ring and uniquely isomorphic to $A\langle X\rangle_T$ as a topological $A\langle X\rangle$-algebra via their universal properties as topological A-algebras. It is straightforward to chase the "integral structures" to check that this achieves the original idea with a rational domain in a relative unit polydisc over $\mathrm{Spa}(A, A^+)$. □

As an application of the preceding proposition, we get a first result towards characterizing adic spaces associated to affinoid algebras over a non-archimedean field k among general adic spaces over $\mathrm{Spa}(k, k^0)$:

Corollary 15.3.4 *Let k be a nonarchimedean field, A a k-affinoid algebra, and (A, A^+) is a Huber pair, so $A^+ \subset A^0$ is an open subring. The map $(k, k^0) \to (A, A^+)$ is of finite type if and only if $A^+ = A^0$.*

Proof. Since there exists an open surjection

$$k\langle X\rangle \twoheadrightarrow k\langle X\rangle/J \simeq A$$

and we have seen that the associated map $k\langle X\rangle^0 \to A^0$ is integral, the "if" direction is verified. Conversely, assume $(k, k^0) \to (A, A^+)$ is of finite type, so by Proposition 15.3.3 over the base (k, k^0) there exists an open map

$$k\langle X\rangle \twoheadrightarrow A$$

which induces an integral map

$$k\langle X\rangle^0 \to A^+.$$

But composing it with the inclusion $A^+ \subset A^0$ is known to give an integral map (as always for integral maps, such as surjections, between k-affinoid algebras), so $A^+ \to A^0$ is integral. Since A^+ integrally closed in A, we get $A^+ = A^0$ as desired. □

Definition 15.3.5 A morphism of adic spaces $f : X \to Y$ is *locally of finite type* if, for any $x \in X$, there exists an affinoid open subspaces U around x and V around $f(x)$ such that $f(U) \subseteq V$ and the induced homomorphism of complete Huber pairs $(\mathscr{O}_Y(V), \mathscr{O}_Y^+(V)) \to (\mathscr{O}_X(U), \mathscr{O}_X^+(U))$ is of finite type as in Definition 15.3.1.

By Example 15.3.2, the open immersion of a rational domain into an affinoid adic space is locally of finite type. Thus, in general *every* open immersion of adic spaces is locally of finite type since the rational domains give a base of open subspaces of an affinoid adic space.

It is clear that in order for the notion "locally of finite type" to be useful, in the affinoid setting it has to coincide with "of finite type". The analogous fact for schemes is quite elementary, but for adic spaces the proof is much harder (involving tricky approximation arguments) and is an important theorem of Huber:

Theorem 15.3.6 *Assume that (A, A^+) and (B, B^+) are complete Huber pairs such that each of A and B is strongly Noetherian, or each has Noetherian ring of definition. Then a map $(A, A^+) \to (B, B^+)$ which is locally of finite type is in fact of finite type.*

The only published proof appears to be [H3, Thm. 3.8.15], and unfortunately [H3] is difficult to obtain (e.g., the only copy in the United States is in the University of Michigan math library). In a later version of these notes I will include a proof (in English).

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