16 Lecture 16: Rigid geometry and perfectoid rings

16.1 Link to rigid geometry

Let k be a complete nonarchimedean field equipped with a nontrivial valuation of rank 1. We denote by $\mathfrak{R}_k$ the category of rigid-analytic space over k. In this section we construct a fully faithful functor $r_k : X \rightsquigarrow X^{\mathrm{ad}}$ from $\mathfrak{R}_k$ to the category $\mathfrak{A}_k$ of locally finite type adic spaces over (k, k^0). It is tempting to think that this should be an equivalence (surely every adic space locally of finite type over $\mathrm{Spa}(k, k^0)$ arises from a rigid-analytic space), but this is false because open subspaces of X^{ad} need not arise from admissible open subsets of X (as we will discuss in Example 16.1.6 with two classes of counterexamples after we have defined the adification functor). Characterizing the essential image of this adification functor looks hopeless, but at least quasi-separatedness (i.e., quasi-compactness of the overlap of any two quasi-compact open subspaces) will be a sufficient criterion for membership in the essential image.

We would like to define the functor $r_k : \mathfrak{R}_k \to \mathfrak{A}_k$ in such a way that it carries open immersions to open immersions interacting well with intersections, and for all admissible open subspaces $\{U_i\}$ of a rigid-analytic space X over k we will show that the open subspaces $\{U_i^{\mathrm{ad}}\}$ cover X^{ad} topologically *if and only if* $\{U_i\}$ is an admissible cover of X (thereby explaining in a new way the real significance of Tate's notion of admissible cover). Moreover, for peace of mind we will build an equivalence of topoi

$$\mathbf{Shv}_{\mathrm{Tate}}(X) \simeq \mathbf{Shv}(|X^{\mathrm{ad}}|)$$

where $|X^{\mathrm{ad}}|$ is a locally spectral and sober topological space, globalizing the affinoid case done in [H1, §4] (whose main results are proved in two ways, initially using quantifier elimination for non-trivially valued algebraically closed fields but then proved in [H1, Rem 4.7ff.] without quanitifier elimination).

Huber carries out this construction in [H2] by using some technology from [SGA4] involving "locally coherent morphisms" between topoi, but it seems a bit heavy to express things this way since the underlying idea of the construction is quite intuitive (namely: bootstrap from the affinoid case via lots of careful gluing). Thus, we will explain the procedure in more down-to-earth terms, though undoubtedly it is expressing exactly the same thing as what Huber does with locally coherent morphisms.

In order to show that the operation $\mathrm{Sp}(A) \mapsto \mathrm{Spa}(A, A^0)$ can be globalized via gluing, we need to carry out some preliminary work entirely in the affinoid setting, beginning with the following result [H1, Cor. 4.3] (motivated by applying [EGA, IV_3, §10] to schemes locally of finite type over a field):

Proposition 16.1.1 *For a k-affinoid algebra A, define $X = \mathrm{Spa}(A, A^0)$ and let $X_0 = \mathrm{Sp}(A)$ be the set of "classical points" inside X. There is a natural bijection from the set of constructible subsets of X to the set of constructible subsets of X_0 (i.e., the Boolean algebra generated by quasi-compact open subsets of X_0) defined by $C \mapsto C_0 := C \cap X_0$. Moreover, in both directions this bijection is inclusion-preserving and commutes with finite unions and intersections and complements, and C is open if and only if C_0 is admissible open.*

Proof. We explain the crucial step, namely that if C is non-empty then C_0 is non-empty. Once this is verified, the rest is quite straightforward. As in [EGA, 0_{III}, 9.1.3], C is a finite union of overlaps $U_i \cap (X - V_i)$ for quasi-compact open $U_i, V_i \subset X$. Hence, for the purpose of showing C_0 is non-empty we may assume $C = U \cap (X - V)$ for some such U and V. Each of U and V is a finite union of rational domains, so we can assume U is a rational domain.

Renaming such U as X, we are reduced to showing that if $V_1, \ldots, V_n$ are rational domains in X whose union contains X_0 then the V_i's cover X. Since $\{V_{i,0}\}$ covers $X_0 = \mathrm{Sp}(A)$, by [BGR, 8.2.2/2] we can refine each V_i through rational localizations so that $\{V_{i,0}\}$ becomes a *rational covering*; i.e.,

$V_i = X((f_1, \ldots, f_n)/f_i)$ for $f_1, \ldots, f_n \in A$ that generate the unit ideal. But then for any $x \in X$ the valuation v_x on A cannot kill every f_i (as the f_i's generate 1), so some $v_x(f_i)$ is non-zero, so by choosing i_0 with $v_x(f_{i_0})$ maximal we have $v_x(f_i) \leq v_x(f_{i_0}) \neq 0$ for all i; i.e., $x \in V_{i_0}$. This shows that the V_i's cover X. $\square$

A much deeper fact, ultimately using that $A\langle T/s\rangle^0$ is the integral closure of $A^0\langle T/s\rangle$ in $A\langle T/s\rangle$, is [H1, Cor. 4.4]:

Proposition 16.1.2 *For k-affinoid A, the equivalence between lattices of quasi-compact open subspaces of* $\mathrm{Sp}(A)$ *and* $\mathrm{Spa}(A, A^0)$ *uniquely extends to an equivalence between categories of sheaves of sets for the Tate topology on* $\mathrm{Sp}(A)$ *and the usual spectral topology on* $\mathrm{Spa}(A, A^0)$.

The essential difficulty in the proof is that one has to grapple with non-quasi-compact open subspaces on both sides. The key idea in the proof is to show that every prime filter of rational domains in $\mathrm{Sp}(A)$ corresponds to the collection of rational domains in $\mathrm{Spa}(A, A^0)$ containing a uniquely determined point; this is a rather subtle valuation-theoretic construction problem in $\mathrm{Cont}(A)$ (see [H1, (4.7.2)]).

Granting the above two results of Huber in the affinoid case, our aim for general rigid-analytic spaces X is to define $r_k(X)$ extending the operation $r_k(\mathrm{Sp}(A)) = \mathrm{Spa}(A, A^0)$ on affinoids in such a manner that two properties hold:

(I) For any open immersion $U \hookrightarrow X$, the induced morphism of adic spaces $r_k(U) \to r_k(X)$ is an open immersion and $r_k(U \cap V) = r_k(U) \cap r_k(V)$ for a second open immersion $V \hookrightarrow X$.

(II) The equivalence between Tate and adic topoi in the affinoid case extends to the general case (visibly unique due to the local nature of sheaves).

The full faithfulness of r_k is *not* dragged along during the construction, but will be handled afterwards. The main subtlety in the proof of full faithfulness is that for a given affinoid open subspace $V \subset Y$ and morphism of adic spaces $f : r_k(X) \to r_k(Y)$, it is not at all obvious why the open subspace $f^{-1}(r_k(V)) \subset r_k(X)$ should have the form $r_k(U)$ for an admissible open subspace of X (note that $f^{-1}(r_k(V))$ might not be quasi-compact!).

Since (II) for affinoid X is Proposition 16.1.2, so let us now verify (I) for affinoid $X = \mathrm{Sp}(A)$ and affinoid U and V. By the Gerritzen-Grauert theorem, U is a finite union of rational domains $U_i \subset X$ with overlaps $U_i \cap U_j$ that are rational in X and hence in each of U_i and U_j. Thus, by Proposition 16.1.1, $\{r_k(U_i)\}$ is an open cover of $r_k(U)$ by rational domains with constructible $r_k(U_i) \cap r_k(U_j)$ that clearly coincides with the rational domain $r_k(U_i \cap U_j)$.

But likewise each $r_k(U_i)$ is a rational domain in $r_k(X)$, and as such the overlap of $r_k(U_i)$ and $r_k(U_j)$ inside $r_k(X)$ also coincides with the rational domain $r_k(U_i \cap U_j)$. Hence, $r_k(U) \to r_k(X)$ is an isomorphism onto the open subspace given by the union of the $r_k(U_i)$'s with overlaps $r_k(U_i \cap U_j)$. If $V \subset X$ is another affinoid open subspace, say with finite rational covering $\{V_\alpha\}$, we see that $r_k(U) \cap r_k(V)$ is covered by the overlaps $r_k(U_i) \cap r_k(V_\alpha) = r_k(U_i \cap V_\alpha)$ for the rational domains $U_i \cap V_\alpha$ that cover the *affinoid* open subspace $U \cap V \subset X$. This shows that

$$r_k(U \cap V) = r_k(U) \cap r_k(V).$$

It now follows, as will be useful for globalization, that if $\{U_i\}$ is *any* (necessarily admissible) finite affinoid open cover of U (not necessarily by rational domains!) then the open subspace $r_k(U) \subset r_k(X)$ is covered by the $r_k(U_i)$'s with overlaps $r_k(U_i) \cap r_k(U_j) = r_k(U_i \cap U_j)$.

Next, we move on to the separated case. Here we have to build r_k as well. As a first step, we will take care of (II): for separated X we will build a sober topological space to be denoted $|X|$ such that the Tate topos of X is naturally identified with the category of sheaves on $|X|$ extending the known affinoid case. (Recall that this property functorially characterizes such a space $|X|$, by [MM, Ch. IX].) For ease of notation, we shall write $|U|$ to denote $|r_k(U)|$ for affinoid U.

For affinoid open $U, V \subset X$, the overlap $U \cap V$ is affinoid since X is separated, and the natural maps $|U \cap V| \rightrightarrows |U|, |V|$ are open immersions by the settled affinoid case. If $W \subset X$ is a third affinoid open subspace then the settled affinoid case applied to W as an ambient space gives

$$|U \cap W| \cap |V \cap W| = |U \cap V \cap W|$$

inside $|W|$. This ensures that if we consider the affinoid adic spaces $r_k(U)$ for *all* affinoid open $U \subset X$ and the open subspaces $|U \cap V| \subset |U|, |V|$ for all additional affinoid open $V \subset X$ then the triple-overlap axiom for gluing is satisfied; this gluing is *defined* to be $r_k(X)$. With this definition, it is easy to check that if $U \hookrightarrow X$ is any admissible open subspace then there is a unique map $r_k(U) \to r_k(X)$ extending the affinoid case and it is an open immersion satisfying $r_k(U) \cap r_k(V) = r_k(U \cap V)$ for any second admissible open subspace $V \hookrightarrow X$.

In order that r_k on separated X is promoted to a functor, we want to show that for a morphism $f : X \to Y$ of rigid-analytic spaces and affinoid opens $U \subset X$ and $V \subset Y$ with $U \subset f^{-1}(V)$, the maps $r_k(U) \to r_k(V)$ actually glue to a well-defined morphism of adic spaces $r_k(X) \to r_k(Y)$. For this purpose Huber uses a procedure based on universal mapping properties with ringed spaces, but we prefer something more tangible:

Proposition 16.1.3 *If $\{U_i\}$ is a collection of admissible open subspaces of X then it is an admissible cover if and only if the open subspaces $r_k(U_i) \subset r_k(X)$ are a covering in the usual sense.*

Proof. If $\{U_i\}$ is an admissible cover of X then for every affinoid open $V \subset X$ there is a *finite* affinoid open cover $\{V_1, \dots, V_n\}$ of V that refines the collection $\{U_i \cap V\}$. Thus, $r_k(V)$ is covered by the $r_k(V_j)$'s due to the settled affinoid case, so if $V_j \subset U_{i(j)}$ then the union of the $r_k(U_{i(j)})$'s contains $r_k(V)$. By design, $r_k(X)$ is covered by the $r_k(V)$'s, so also by the $r_k(U_i)$'s.

The converse is more interesting: assuming $\{r_k(U_i)\}$ is a covering of $r_k(X)$ we have to show that $\{U_i\}$ is an admissible covering of X. Certainly the U_i's cover X set-theoretically (by considering classical points inside $r_k(X)$), so the issue is to verify admissibility. To that end, we choose an affinoid open subspace $V \subset X$ and have to build a finite affinoid open cover of V which refines $\{U_i \cap V\}$. Since $r_k(V)$ is a *quasi-compact* open subspace of $r_k(X)$, some finite collection $r_k(U_1), \dots, r_k(U_n)$ among the given opens covering $r_k(X)$ has union which contains $r_k(V)$. In other words, the open subspaces $r_k(V \cap U_1), \dots, r_k(V \cap U_n)$ cover $r_k(V)$.

For later purposes beyond the separated case, we shall avoid using that $V \cap U_j$ is known to be affinoid in the separated case. Let $\{W_{jh}\}_{h \in H_j}$ be an admissible affinoid covering of $V \cap U_j$, so the settled implication ensusres that $\{r_k(W_{jh})\}_{h \in H_j}$ is an open covering of $r_k(V \cap U_j)$, and hence the $r_k(W_{jh})$'s for all (j, h) constitute an open cover of the affinoid $r_k(V)$. Hence, we get a finite subcover, and the corresponding W_{jh}'s consistute a finite collection of affinoid opens inside V which cover V (as their r_k's do) and refines $\{U_j \cap V\}$. □

Having made r_k into a functor in the separated case, it is clear from the construction that it satisfies (I). From the construction of r_k we define a functor $\mathbf{Shv}_{\mathrm{Tate}}(X) \to \mathbf{Shv}(|r_k(X)|)$ by gluing from the affinoid case. This is seen to be an equivalence by using (II) in the affinoid case and the local nature of sheaves. Thus, (II) is also settled in the separated case.

We can now pass to the general case by using separated open subspaces in place of affinoid open subspaces in the gluing construction, and everything goes through without change (e.g., the proof of Proposition 16.1.3 works verbatim). This completes the construction of the functor

$$r_k : \mathfrak{R}_k \to \mathfrak{A}_k,$$

and it remains to show:

Proposition 16.1.4 *The functor r_k is fully faithful.*

Proof. For a morphism $f : X \to Y$ in $\mathfrak{R}_k$, the map $|r_k(f)|$ on underlying topological spaces recovers f on underlying sets. Hence, to verify faithfulness we ma work *locally* on the target Y and then locally on the source X to reduce to the easy affinoid case.

With faithfulness settled in general, to show that r_k is full we consider a morphism

$$\varphi : r_k(X) \to r_k(Y)$$

in $\mathfrak{A}_k$. The case of affinoid Y is easy by working locally on X (using faithfulness and gluing of solutions locally on X), so the main task is to reduce to affinoid Y. This is a problem because if we try to work with φ-preimages of $r_k(V)$ for affinoid open $V \subset Y$ then we have to confront the problem that a general open subspace of $r_k(X)$ is *not* of the form $r_k(U)$ for admissible open $U \subset X$ (e.g., remove a higher-rank closed point from the closed adic unit disk)!

Consider the collection $\{U_i\}$ of all affinoid opens in Y (or just an admissible cover of such), so $\{r_k(U_i)\}$ is an open cover of $r_k(Y)$. Hence, the open subspaces $\varphi^{-1}(r_k(U_i))$ cover $r_k(X)$, so via rational domains in affinoid open subspaces of $r_k(X)$ coming from affinoid open subspaces of X we can find affinoid opens $V_j \subset X$ such that $\{r_k(V_j)\}$ is an open cover of $r_k(X)$ that refines $\{\varphi^{-1}(r_k(U_i))\}$. Therefore, by Proposition 16.1.3, $\{V_j\}$ is an admissible open covering of X! It therefore suffices to work on each V_j separately in place of X (faithfulness and admissibility will then allow us to make a global solution), and by design each $r_k(V_j) \to r_k(Y)$ factors through some $r_k(U_i)$.

By renaming such a V_j as X, we are reduced to the case when X is affinoid and there exists an affinoid open subspace $U \subset Y$ such that φ carries $r_k(X)$ into $r_k(U) \subset r_k(Y)$. But the resulting morphism $r_k(X) \to r_k(U)$ must arises from a map $X \to U$ via the affinoid case, and its composition with the inclusion of U into Y then clearly does the job. □

Remark 16.1.5 Much as with complex-analytic spaces, any map $r_k(X) \to r_k(Y)$ merely as ringed spaces over k (ignoring topological sheaf aspects and $\mathscr{O}^+$'s) turns out to automatically arise from a (uniquely determined) morphism of adic spaces over $\mathrm{Spa}(k, k^0)$. The key step is to verify locality of the map on stalks at higher-rank points. Huber proves this by arguing with locally coherent morphisms of sites; it can be done by more concrete methods as well; perhaps I will add such a proof into a later version of the notes.

Example 16.1.6 For an affinoid rigid-analytic space X, open subsets $V \subset X^{\mathrm{ad}}$ are not necessarily of the form U^{ad} for an admissible open $U \subset X$. As we noted earlier, a trivial example is to remove a higher-rank closed point, such as a type-5 point from the adic closed unit disk. But we would like to give a more striking example with $X = \mathrm{Sp}(k\langle t\rangle)$ for algebraically closed k. Here X^{ad} is the adic closed unit disc $\mathbf{D}_k$ and if we remove the closure of the Gauss point $\eta \in X^{\mathrm{ad}}$ then we get an open subset $V \subset X^{\mathrm{ad}}$ which is not quasi-compact. In view of our characterization of points of X and their closure relations, V is readily seen to be covered by the pairwise disjoint open subspaces $D_{\overline{a}} = \{|t-a| < 1\}^{\mathrm{ad}}$ for a representative $a \in k^0$ of each $\overline{a} \in \widetilde{k} := k^0/k^{00}$. Hence, $V \simeq Y^{\mathrm{ad}}$ for

$$Y := \coprod_{\overline{a}} \{|t-a| < 1\}.$$

Assume $V \simeq U^{\mathrm{ad}}$ for an admissible open subspace $U \subset X$, so the resulting isomorphism of adic spaces $Y^{\mathrm{ad}} \simeq U^{\mathrm{ad}}$ respecting inclusions into X^{ad} would have to arise from an isomorphism of rigid-analytic spaces $Y \simeq U$ respecting morphisms to X. But clearly V contains all type-1 points, so necessarily

$U = X$. In other words, Y would express X as a (highly) disconnected spaces, which is false. (More concretely, this example expresses the presence of non-admissibility phenomena in Tate's theory.)

Observe that V is itself in the essential image of r_k; it simply does not come from an admissible open subspace of X. (In particular, if $r_k(f)$ is an open immersion then generally f can *fail* to be an open immersion.) This is no accident, as it is a pleasant exercise with the construction of r_k to verify that r_k restricts to an equivalence between *quasi-separated* objects (i.e., those for which any overlap of quasi-compact open subspaces is quasi-compact).

16.2 Perfectoid rings

In Scholze's original paper [Sch], he introduced perfectoid fields and perfectoid algebras over a field. Fontaine's Bourbaki report removed the presence of a ground field, which turned out to be very very useful for later developments. We shall adopt this level of generality.

Definition 16.2.1 A *perfectoid ring* is a complete Tate ring A (Banach with topologically niloptent unit $\varpi \in A$) such that

1. A^0 is bounded (so since automatically open, it is then a ring of definition for A),
2. there exists a topologically nilpotent unit ϖ with $\varpi^p \mid p$ in A^0,
3. $\varphi := \mathrm{Frob}_{A^0/\varpi^p A^0}$ is a surjective.

We are led to give the following:

Definition 16.2.2 A *perfectoid field* is a perfectoid ring K that is a field, and its topology is defined by a rank-1 valuation.

Remark 16.2.3 It is a highly non-trivial matter to determine if a perfectoid ring that is a field is necessarily a perfectoid field. We will never need to use the answer to this problem, but we note that it has been settled in the affirmative by Kedlaya [K].

Example 16.2.4 The typical examples of perfectoid fields are $\mathbf{Q}_p(p^{1/p^\infty})^\wedge$ (with $\varpi = p^{1/p}$) and $\mathbf{Q}_p(\zeta_{p^\infty})$ (with ϖ from the "$\mathbf{Z}/p\mathbf{Z}$" part of $\mathbf{Q}_p(\zeta_{p^2})/\mathbf{Q}_p$ (a $(\mathbf{Z}/p^2\mathbf{Z})^\times$-extension). A basic example of a perfectoid ring that is not a field is:

$$\mathbf{Q}_p\langle T^{1/p^\infty}\rangle = (\bigcup_{n\geq 1} \mathbf{Q}_p\langle T^{1/p^n}\rangle)^\wedge = (\bigcup_n \mathbf{Z}_p[T^{1/p^n}])_p^\wedge[1/p].$$

We begin by proving an important property of all perfectoid rings:

Lemma 16.2.5 *Any perfectoid ring A is reduced.*

Proof. Since $A = A^0[\frac{1}{\varpi}]$, it's enough to show that A^0 is reduced. Choose $a \in A^0$ such that $a^N = 0$. Then $(a/\varpi^n)^N = 0$ for any $n \geq 1$, so a/ϖ^n is power-bounded for all n. Then $a \in \bigcap_n \varpi^n A^0 = 0$. □

Remark 16.2.6 We stress that this argument does not at all prove that any affinoid algebra is reduced. We are crucially using that A^0 has the ϖ-adic topology!

We claim that φ_{K^0/pK^0} is surjective. Pick $x \in K^0$, so

$$x \equiv y^p + \varpi^p z.$$

In turn, $z = w^p + \varpi^p t$, so

$$x \equiv (y + \varpi w)^p + \varpi^{2p} t \bmod pK^0.$$

If we keep going, we get that $x \equiv x_n^p + \varpi^{np}(\ldots) \bmod pK^0$ where $\{x_n\}$ is Cauchy. In characteristic 0, so $|p| \neq 0$, we have $\varpi^{np} \in pK^0$ for $n \gg 0$, so x is a pth power mod pK^0. If the characteristic is p, then $x_n \to y \in K^0$ and $x = y^p$.

In particular, in characteristic p a perfectoid field is just a perfect non-archimedean field by another name. More specifically, we have shown:

Corollary 16.2.7 *If K is a perfectoid field of charateristic p then $\varphi : K^0 \to K^0$ is bijective, so K is perfect and $|K^\times|$ is p-divisible.*

The final part of this corollary is valid in characteristic 0 too:

Proposition 16.2.8 *The group $\Gamma := |K^\times|$ is p-divisible for any perfectoid field.*

Proof. We've already handled the characteristic p case, so we can restrict our attention to characteristic 0. In this case Γ is generated by $|p|$ and $\{|p| < |x| \leq 1\}$. Since K^0/pK^0 has surjective Frobenius,

$$|x| \equiv y^p \pmod{pK^0} \implies |x| = |y^p| = |y|^p$$

by the triangle inequality. Now we only have to check $|p|$. Well, $p = \varpi^p a$ for some $a \in K^0$ so $|p| = |\varpi|^p|a| \in \Gamma^p$ and both $|\varpi|$ and $|a|$ must lie in the interval $(|p|, 1]$, so $|p|$ is also in Γ^p. □

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