17 Perfectoid rings

Introduction

In Scholze's original paper, he introduces perfectoid fields and perfectoid algebras over a field. Fontaine's Bourbaki report [Fo], removed the assumption of ground field, which turned out to be very very useful. We will adopt that level of generality. We shall discuss all the main results on perfectoid spaces within this level of generalities, which is essential for "relativizing" the theory and making it more flexible for further applications. We note however that the reader should be very cautious on focusing on the case of perfectoid fields and perfectoid algebras over perfectoid fields. This basic case will turn out to be conceptually fundamental in sight of a sophisticated generalization of Krasner's Lemma, which lies at the heart of many ideas in the whole theory. To give an outline of what the field case is about, take any arithmetically profinite extension K of $\mathbf{Q}_p$ with residue field κ. One can attach to it its *field of norms* $\widetilde{K}$, which, non-canonically, is isomorphic to $\kappa((t))$. Fontaine and Wintenberger show that the Galois theory of K is identified with that of $\widetilde{K}$. The completion E of K is a perfectoid field, and the completion of the purely inseparable closure of $\widetilde{K}$ is what we shall call its tilt $E^\flat$. Another way to say that the Galois theories of these fields coincide is that the small étale sites of $K, \widetilde{K}, E, E^\flat$ are (canonically) identified (!). This phenomenon will be likewise true for perfectoid rings and their tilt, and in fact for perfectoid spaces and their tild, as we shall discuss in later lectures.

17.1 Perfectoid rings

Definition 17.1.1 A *perfectoid ring* is a complete Tate ring A (Banach with topologically nilpotent unit $\varpi \in A$) satisfying the following properties:

(1) A^0 is bounded.

(2) There exists a topologically nilpotent unit ϖ with $\varpi^p \mid p$ in A^0.

(3) $\Phi = \mathrm{Frob}_{A^0/\varpi A^0}$ is a surjective.

As a first remark, we note that condition (3) is actually independent of ϖ.

Proposition 17.1.2 *For any complete Tate ring A and nonzero pseudo-uniformizer ϖ satisfying $\varpi^p \mid p$ in A^0, the Frobenius map $\Phi : A^0/\varpi \to A^0/\varpi^p$ is necessarily injective. The surjectivity condition is independent of the choice of such ϖ.*

Proof. Let $x \in A^0$ be satisfying $x^p = \varpi^p a$ for some $a \in A^0$. Then the element $x/\varpi \in A$ lies in A^0 because its p-th power does. Hence Φ is injective as claimed. We claim the surjectivity of Φ is equivalent to surjectivity of the (necessarily injective) Frobenius map:

$$A^0/(p, \varpi^n) \to A^0/(p, \varpi^{np})$$

for any $n \geq 1$. For $n = 1$ this coincides with surjectivity of Φ. Suppose such map is surjective for some n. Then it is surjective for all $1 \leq m < n$. We only need to check surjectivity for all $m > n$. Let $x \in A^0/p$. We can write:

$$x = a^p + \varpi^{np} b, \quad a, b \in A^0/p.$$

Likewise, $b = c^p + \varpi^p u$, for some $c, u \in A^0/p$. Hence:

$$x = (a + \varpi^n c)^p + \varpi^{(n+1)p} u.$$

Surjectivity for $m = n+1$ follows, and the claim is proved. We are now ready to conclude the proof. Suppose ϖ' is another pseudo-uniformizer satisfying $\varpi'^p \mid p$ in A^0. If we take n large enough, then $\varpi^n \in \varpi' A^0$, and surjectivity of $A^0/(p, \varpi^n) \to A^0/(p, \varpi^{np})$ implies surjectivity of $A^0/\varpi' \to A^0/\varpi'^p$. It follows that Φ is an isomorphism for all ϖ satisfying $\varpi^p \mid p$ in A^0, as soon as it holds for one such ϖ. The proof is complete. □

Notation Given Proposition 17.1.2 and its proof, and in the context of the same Proposition, we shall always denote by Φ the Frobenius $\mathrm{Frob}_{A^0/\varpi A^0}$ and likewise Frob_{A^0/pA^0}.

Example 17.1.3 Here is an example of perfectoid ring which doesn't arise as an algebra over a field:

$$A = \mathbf{Z}_p^{\mathrm{cyc}}\langle (p/T)^{1/p^\infty}\rangle^\wedge[1/T].$$

One can take $\varpi := T^{1/p}$, as $\varpi^p = T$ divides p in A^0.

We are now ready to introduce the notion of perfectoid fields. Naively, one could assign such notion by saying that a perfectoid field is a perfectoid ring which, in addition, is a field. However, it is non-trivial to determine if such a field is nonarchimedean with respect to a rank 1 valuation (although that problem has been settled by Kedlaya), and this latter property is quite essential to our purposes, for example when dealing with a crucial Henselian approximation process aiming at "spreading" finite étale algebras over perfectoid rings out of a certain perfectoid field. We, therefore, build this condition into the definition:

Definition 17.1.4 A *perfectoid field* is a perfectoid ring K that is a field and its topology is defined by a rank 1 valuation

$$|\cdot| : K \to \mathbf{R}_{\geq 0}.$$

Example 17.1.5 Some examples and plenty of non-examples:

(1) An infinite family of non-examples is given by any discretely-values nonarchimedean field K of residue characteristic p. Indeed, let K^0 be the valuation ring and ϖ be any nonzero element of its maximal ideal. It follows the quotients K^0/ϖ and K^0/ϖ^p are Artin local rings of different lengths, hence they can never be isomorphic.

(2) We consider $\mathbf{Q}_p(p^{1/p^\infty})^\wedge$ (with $\varpi = p^{1/p}$) and $\mathbf{Q}_p(\zeta_{p^\infty})^\wedge$, with ϖ coming from the $\mathbf{Z}/p\mathbf{Z}$ piece of $\mathbf{Q}_p(\zeta_{p^2})/\mathbf{Q}_p$ (a $(\mathbf{Z}/p^2\mathbf{Z})^\times$-extension). Both are perfectoid fields.

(3) We consider:

$$\mathbf{Q}_p\langle T^{1/p^\infty}\rangle = \varinjlim_{n\geq 1} \mathbf{Q}_p\langle T^{1/p^n}\rangle^\wedge := (\varinjlim_{n} \mathbf{Z}_p[T^{1/p^n}])^\wedge[1/p].$$

and this is *not* perfectoid. However,

$$\mathbf{Q}_p^{\mathrm{cyc}}\langle T^{1/p^\infty}\rangle = \mathbf{Q}_p\langle T^{1/p^\infty}\rangle \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{Z}_p^{\mathrm{cyc}}$$

is. This is also obtained as $A[1/p]$, A being the p-adic completion of $\mathbf{Z}_p^{\mathrm{cyc}}[T^{1/p^\infty}]$.

A few properties.

Lemma 17.1.6 *Any perfectoid ring A is reduced.*

Proof. Since $A = A^0[\frac{1}{\varpi}]$, it's enough to show that A^0 is reduced. Choose $a \in A^0$ such that $a^N = 0$ for $N \gg 1$. Then $(a/\varpi^n)^N = 0$ for any $n \geq 1$ and $N \gg 1$, and so a/ϖ^n is power-bounded for all $n \geq 1$. Then $a \in \bigcap \varpi^n A^0$, which is 0, since ϖ is topologically nilpotent. □

Some preliminary remarks, before moving on.

Remark 17.1.7 Why doesn't this prove that any affinoid algebra is reduced? We are using that A has the ϖ-adic topology. Note also that if A was a perfectoid *algebra* over a perfectoid field K, reducedness of A was a consequence of the following simple observation. Suppose $\varepsilon \in A^0$ was a nilpotent element. Then $K\varepsilon \subset A^0$ is a power-bounded subset which is not bounded. Contradiction.

Remark 17.1.8 In Sholze's first paper [Sch], we define a perfectoid field K of residue characteristic $p > 0$, by considering, rather, $\Phi = \mathrm{Frob}_{K^0/pK^0}$. This notion turns out to be equivalent to our Definition 17.1.4. The only verification needed is to check that Φ is surjective. Pick $x \in K^0$, so that:

$$x \equiv y^p + \varpi^p z, \quad z \in K^0.$$

In turn, $z = w^p + \varpi^p t$, so

$$x \equiv (y + \varpi w)^p + \varpi^{2p} t \mod p.$$

Iterating, we get that $x \equiv x_n^p + \varpi^{np}(\cdots) \mod p$ where x_n is a Cauchy sequence in K^0. If the characteristic of K is 0, so that $|p| \neq 0$, then ϖ^{np} is in pK^0 for $n \gg 0$ and x is a p-th power modulo pK^0, as desired. If the characteristic is $p > 0$, then $x_n \to y \in K^0$ and $x = y^p$.

Therefore, a perfectoid field in characteristic p is perfect.

Proposition 17.1.9 *If K is a perfectoid field of charateristic p, then $\Phi := \mathrm{Frob}_{K^0} : K^0 \to K^0$ is bijective, so K is perfect and $|K^\times|$ is p-divisible.*

Proof. The proof of perfectness of K discussed in Remark 17.1.8, shows that Φ is surjective, hence bijective by the same argument as in Remark 17.1.2. As a direct consequence, if we let $\alpha \in |K^\times|$, then $\alpha^{1/p}$ is also in $|K^\times|$, as we can write $\alpha = |x|$ for some nonzero $x \in K$, and write $x = y^p$ for some nonzero $y \in K$. We conclude. □

In fact, the above p-divisibility result for the value group of a perfectoid field in characteristic $p > 0$ holds in general (!). Not just a curious phenomenon, but rather a first manifestation of a deeply conceptual connection between perfectoid spaces of arbitrary characteristic and residue characteristic $p > 0$, and perfectoid spaces of characteristic $p > 0$.

Proposition 17.1.10 $\Gamma = |K^\times|$ *is p-divisible for any perfectoid field.*

Proof. We've already handled the characteristic p case, so we can restrict our attention to characteristic 0. Γ is generated by $|p|$ and $\{|p| < |x| \leq 1\}$. Since K^0/pK^0 has surjective Frobenius,

$$|x| \equiv y^p \mod pK^0 \quad \text{yields} \quad |x| = |y^p| = |y|^p$$

by the triangle inequality. Now we only have to check the same for $|p|$. We write $p = \varpi^p a$ for some $a \in K^0$. Then, $|p| = |\varpi|^p|a| \in \Gamma^p$ and both $|\varpi|$ and $|a|$ must be between $|p|$ and 1, so $|p|$ is also in Γ^p by convexity. □

The above Remark 17.1.8 and Proposition 17.1.9 can be reproved with no base field assumption, that is, suitably, for perfectoid rings. There's a subtlety to be careful about. We show the following:

Proposition 17.1.11 *Let A be a topological ring with $pA = 0$. The following are equivalent:*

(1) *A is perfectoid.*

(2) *A is a perfect (Frobenius Φ is an isomorphism on A^0) uniform complete Tate ring.*

This is the exact analogue of Proposition 17.1.9. Note that in condition (2) the notion of perfectness is assigned saying Φ is an isomorphism on A^0, where A^0 is defined because we are under the assumption that A is a topological ring.

Proof. Let A be a complete uniform Tate ring. If A is perfect, then just take ϖ to be *any* pseudo-uniformizer. The condition $\varpi^p \mid p$ is trivially satisfied as $p = 0$. If $x \in A$ is power-bounded, then so is x^p, and conversely. This means that $\Phi : A^0 \to A^0$ is an isomorphism. In particular, the Frobenius map $A^0/\varpi \to A^0/\varpi^p$ is surjective. For injectivity, assume $x \in A^0$ with $x^p = \varpi^p a$ for some $a \in A^0$. Write $a = b^p$ for some $b \in A^0$. Then we have:

$$x = \varpi b,$$

as desired. Conversely. Suppose A is perfectoid. Then the Frobenius map $A^0/\varpi \to A^0/\varpi^p$ is an isomorphism, and hence so is $A^0/\varpi^n \to A^0/\varpi^{np}$ by induction. Taking inverse limiits and using completeness, we find that $\Phi : A^0 \to A^0$ is an isomorphism, as desired. □

Note how the last part of the proof is just a reformulation of the argument in Remark 17.1.8. We now remark that if we assume $pA = 0$, then trivially $pA = pA^0$ is a closed ideal in A^0. We'd like to characterize perfectoid rings as being a special class of complete uniform Tate rings, but such characterization in the lack of the assumption $pA = 0$ encounters the difficulty of having, possibly, pA^0 non-closed in A^0.

Proposition 17.1.12 *Let A be a complete uniform Tate ring.*

(1) *If there exists a pseudo-uniformizer $\varpi \in A$ such that $\varpi^p \mid p$ and $\Phi : A^0/p \to A^0/p$ is surjective, then A is perfectoid.*

(2) *If A is perfectoid, then $\Phi : A^0/p \to A^0/p$ is surjective under the additional assumption that the ideal $pA^0 \subset A^0$ is closed.*

Proof. For (1), assume $\Phi : A^0/p \to A^0/p$ is surjective. Then $\Phi : A^0/\varpi \to A^0/\varpi^p$ is surjective as well. Injectivity is automatic by Proposition 17.1.2. For (2), we assume A is perfectoid such that pA^0 is a closed ideal of A^0. We let $\varpi \in A$ be a pseudo-uniformizer such that $\varpi^p \mid p$, and such that $\Phi : A^0/\varpi \to A^0/\varpi^p$ is an isomorphism. By the proof of Proposition 17.1.2, the Frobenius map:

$$A^0/(p, \varpi^n) \to A^0/(p, \varpi^{np})$$

is an isomorphism for all $n \geq 1$. We now take inverse limits over all $n \geq 1$. The proof will be complete upon showing that the natural map:

$$A^0/p \to \varprojlim A^0/(p, \varpi^n)$$

is an isomorphism. Since A^0 is ϖ-adically complete, so is A^0/p (pA^0 is closed!), and hence such map is surjective. We claim A^0/p is also ϖ-adically separated, whence injectivity would follow. We haven't used uniformity at all yet! Suppose x lies in the kernel. Then there exists $y_n, z_n \in A^0$ such that:

$$x = \varpi^n y_n + p z_n, \quad n \geq 1$$

which is to say: $x \in (p, \varpi^n)$ for all $n \geq 1$. ϖ is topologically nilpotent, and since A is uniform, A^0 is bounded and $\varpi^n y_n \to 0$ in A^0 as $n \to \infty$. We find that $x = \lim_{n\to\infty} p z_n$ lies in the closure of pA^0, which is closed. We conclude. □

The natural question one can ask himself is whether or not there exist perfectoid rings A in which pA^0 is not closed as an ideal of A^0. Here is an example:

Example 17.1.13 To be filled in. Just take Example in §7 of the Stacks project's "Examples" and readapt it! I only need to type it.

17.2 The Tilting functor

There is a functor

$$\begin{array}{ccc} \{\text{perfectoid rings}\} & \longrightarrow & \{\text{perfectoid } \mathbf{F}_p\text{-algebras }\} \\ \uparrow & & \uparrow \\ \{\text{perfectoid fields}\} & \longrightarrow & \{\text{perfectoid fields of char. } p\} \end{array}$$

denoted $A \mapsto A^\flat$. Note that the above diagram contains many nontrivial statements in it. There exists such assignment $A \mapsto A^\flat$ sending a perfectoid ring to a perfectoid $\mathbf{F}_p$-algebra, and such assignment is such that such ring A is a field if and only if $A^\flat$ is. More than this, such A is a *perfectoid* field if and only if $A^\flat$ is, which means we'll have to keep track how this functor interacts with valuations.

Theorem 17.2.1 *For K a perfectoid field of characeristic 0, there exists an equivalence of categories:*

$$\{\text{perfectoid } K\text{-algebras}\} \to \{\text{perfectoid } K^\flat\text{-algebras }\}.$$

We shall refer to such equivalence with the name of "tilting equivalence", from now on.

The inverse functor depends on the "untilt" K of $K^\flat$, as for different K one can obtain the same $K^\flat$. Fontaine gave an exhaustive description of all the characteristic 0 fields that give a particular $K^\flat$.

Remark 17.2.2 For a perfectoid field K of characteristic 0, the equivalence

$$\{\text{finite separable } L/K\} \cong \left\{\text{finite separable } L'/K^\flat\right\}$$

is a Theorem due to Fontaine and Wintenberger. The equivalence respects degrees in both directions, and in fact the Galois theories on both sides. We shall expand on this later, as this will turn out to be essential in the sequel.

A consequence of the theorem is that there is a homeomorphism

$$\mathrm{Cont}(K) \simeq \mathrm{Cont}(K^\flat)$$

respecting rational domains in both directions.

Choose $\varpi \in A$ a pseudo-uniformizer such that $\varpi^p \mid p$, so that $\varpi \in A^0$ and A^0 has the ϖ-adic topology.

Definition 17.2.3 We define:

$$A^{\flat 0} = \varprojlim_{\Phi} A^0/\varpi A^0 = \{(\overline{a_n})_{n\geq 0} \mid \overline{a_{n+1}}^p = \overline{a_n}\}.$$

Note that on $A^{\flat 0}$, we have a *canonical* p-th root: if $a = (\overline{a_n})_{n\geq 0}$ then $a^{1/p} := (\overline{a_{n+1}})_{n\geq 0}$.

Lemma 17.2.4 *The multiplicative map:*

$$\varprojlim_{a\mapsto a^p} A^0 \to \varprojlim_{\Phi} A^0/\varpi A^0 =: A^{\flat 0}_{\varpi}$$

sending $(a^{(n)}) \mapsto (a^{(n)} \mod \varpi)_{n\geq 0}$ is a homeomorphism.

Remark 17.2.5 We note that the left side is independent of ϖ, so $A^{\flat 0}_{\varpi}$ is independent of ϖ as a set. What about the additive structure? If $\varpi' \mid \varpi$ in A^0, then we get a commutative diagram

$$\begin{array}{ccc} \varprojlim A^0 & \longrightarrow & A^{\flat 0}_{\varpi} \\ & \searrow & \downarrow \\ & & A^{\flat 0}_{\varpi'} \end{array}$$

But it is easy to see explicitly that replacing ϖ by some power doesn't affect the process at all.

We then define $A = A^0[1/\varpi]$.

Proof. The key is to build a continuous 0-th component of the inverse. We can then apply this construction to the canonical p-th root extraction on the right side.

Let $x = (\overline{x_n}) \in A^{\flat 0}$. We initially pick *any* sequence of lifts (x_n). Then we define

$$x^{\#} = \varinjlim_{n\to\infty} x_n^{p^n}.$$

Let's first check that this is well-defined, i.e. if (x_n') is another sequence of representatives then we claim that

$$(x_n')^{p^n} \equiv x_n^{p^n} \mod \varpi^{n+1}A^0.$$

More generally, we claim that if $t' \equiv t \mod \varpi A^0$ then

$$(t')^{p^i} \equiv t^{p^i} \mod \varpi^{i+1}A^0 \text{ for all} i.$$

For $i = 0$, the claim is trivially true. Assume it's true for $i-1$. Then,

$$(t')^{p^{i-1}} \equiv t^{p^{i-1}} + \varpi^i a$$

and therefore

$$(t')^{p^i} = t^{p^i} + p\varpi^i(\cdots) + \varpi^{ip}a^p.$$

The upshot is that $x_n^{p^n} \mod \varpi^{n+1}A^0$ is intrinsic to x. Also, $(x_{n+1})^p$ is a representative of $\overline{x_{n+1}}^p = \overline{x_n}$, so $x_{n+1}^{p^{n+1}} = (x_{n+1}^p)^{p^n} \equiv x_n^{p^n} \mod \varpi^{n+1}A^0$. This gives the Cauchy property for the ϖ-adic topology.

We have constructed $x \mapsto x^{\#}$ sending $A^{\flat 0} \to A^0$. It is an easy exercise to check the remaining details. ☐

As a first step towards the tilting equivalence, we seek to define some $\varpi^\flat \in A^{\flat 0}$, not a zero-divisor, satisfying the following properties:

(1) $A^{\flat 0}[1/\varpi^\flat]$ is perfectoid using $\varpi^\flat$ with $A^{\flat 0}$ the subring of power-bounded elements.

(2) There is a natural isomorphism:

$$A^{\flat 0}/\varpi^\flat \simeq A^0/\varpi A^0$$

using the 0-th projection $A^{\flat 0} \to A^0/\varpi A^\flat$.

Morally, we want $\varpi^\flat = (\varpi, \varpi^{1/p}, \varpi^{1/p^2}, \ldots)$. Unfortunately we don't know that we have all these pth roots in A^0 (in classical p-adic Hodge theory, when one is working with $\mathcal{O}_{\mathbf{C}_p}$, one can do this concretely). We try to construct an element with similar behavior.

Since $\mathrm{Frob}_{A^0/\varpi^p A^0}$ is surjective, there exists $\varpi_1' \in A^0$ such that

$$(\varpi_1')^p \equiv \varpi \mod \varpi^p A^0.$$

Therefore:

$$\begin{aligned}(\varpi_1')^p &= \varpi + \varpi^p A^0 \\ &= \varpi(1 + \varpi^{p-1}A^0)\end{aligned}$$

Inductively, choose ϖ'_{i+1} such that $(\varpi'_{i+1})^p \equiv \varpi_i \mod \varpi^p A^0$, so that this is an element of $A^{\flat 0}$. Therefore, it comes from a unique element $(\varpi_1, \varpi_2, \ldots) \in \varprojlim_{x \mapsto x^p} A^0$ such that $\varpi_i \equiv \varpi'_i \mod \varpi^p$. We *define* this element to be $\varpi^\flat$.

Lemma 17.2.6 *$\varpi^\flat$ is* not *a zero-divisor in $A^{\flat 0}$ and is topologically nilpotent.*

Proof. Using the multiplicative and topological identification:

$$A^{\flat 0} \simeq \varprojlim A^0/\varpi A^0$$

via

$$\varpi^\flat \leftrightarrow (0, \overline{\varpi_1}, \overline{\varpi_2}, \ldots)$$

we see that $(\varpi^\flat)^{p^n}$ starts with $n-1$ zeros, so $\varpi^\flat$ is topologically nilpotent. To show that $\varpi^\flat$ is *not* a zero-divisor in $A^{\flat 0} \simeq \varprojlim_{x \mapsto x^p} A^0$ (this being a multiplicative identification), it suffices to use that ϖ_0 is not a zero-divisor in A^0 and that A^0 is reduced. □

Example 17.2.7 If K is a perfectoid field, we are going to see that $K^\flat := K^{\flat 0}[1/\varpi^\flat]$ is a perfectoid field of characteristic p with $|K^{\flat \times}| = |K^\times|$, and $|\varpi^\flat| = |\varpi|$.

Remark 17.2.8 In general, $\varpi^\# = \varpi_0$ is always ϖ times a unit of A^0. That implies that the topology on $A^{\flat 0}$ is the $\varpi^\flat$-adic topology.

In fact, we claim that:

$$(\varpi^\flat)^{p^n} A^{\flat 0} = \{(0, \ldots, 0, \ldots) \in \varprojlim A^0/\varpi A^0\}.$$

with zeros in the first $n+1$ slots.

This corresponds exactly to:

$$\{(a^{(0)}, \ldots, a^{(n-1)}, a^{(n)}, \ldots) \in \varprojlim A^0 \mid a^{(n-1)} \equiv 0 \mod \varpi A^0\}.$$

This shows that:

$$A^{\flat 0}/\varpi^\flat \cong A^0/\varpi A^0.$$

Indeed, we just checked that $\varpi^\flat$ generates the kernel, so this map is injective. On the other hand, it is surjective because $\Phi_{A^0/\varpi A^0}$ is surjective - starting with anything in A^0, we can build a corresponding element on the left hand side.

Definition 17.2.9 Define the *tilt* of A to be:

$$A^\flat = A^{\flat 0}[1/\varpi^\flat] \supset A^{\flat 0}$$

with the $\varpi^\flat$-adic topology on $A^{\flat 0}$. This is a complete Tate ring with $\varpi^\flat$ as a pseudo-uniformizer and $A^{\flat 0}$ the ring of definition.

To conclude that $A^\flat$ is perfectoid, we must show that $A^{\flat 0}$ is actually the ring of power-bounded elements of $A^\flat$ (it is obviously contained in it, as it is visibly a ring of definition). For concreteness, we remark that this is the set of p-power compatible sequences in A (just as $A^{\flat 0}$ was the set of p-power compatible sequences in A^0). Since

$$(\varpi^\flat)^\# = \varpi_0 \in \varpi(A^0)^\times \subset A^\times,$$

then $A = A^0[1/\varpi] = A^0[1/(\varpi^\flat)^\#]$. Therefore, $\varpi^\flat \in A^{\flat 0}$ maps to $((\varpi^\flat)^\#, ((\varpi^\flat))^{1/p})^\#, \ldots,)$.

This being a multiplicative identification, it extends uniquely to a multiplicative map

$$A^\flat \to \varprojlim_{x \mapsto x^p} A$$

using $(**)$ and that is sufficies to check that the first term in the sequence is power-bounded, as the rest follow automatically from this.

We now want to show that $A^\flat$ is independent of the choice of ϖ and $\varpi^\flat$ and that its ring of power-bounded elements is precisely $A^{\flat 0}$.

Proposition 17.2.10 *For any two $\varpi, \varpi' \in A^0$ and $\varpi^\flat, (\varpi')^\flat$ are associated choices in $A^{\flat 0}$, then $A^{\flat 0}[1/\varpi^\flat] \simeq A^{\flat 0}[1/(\varpi')^\flat]$.*

Proof. We have already seen that $(\varpi^\flat)^\#$ differs from ϖ by a unit in A^0. Then we can replace ϖ by $(\varpi^\flat)^\#$, so we can assume that $\varpi = (\varpi^\flat)^\#$ (that is, we have compatible p-power roots of ϖ, and the sequence of these is $\varpi^\flat$).

The key is that we can now use topological nilpotence and the multiplicativity of all the identifications. By topological nilpotence in A^0, we can find $N \gg 0$ such that $(\varpi')^N$ lies in ϖA^0, say $(\varpi')^N = \varpi \cdot a$ for $a \in A^0$. (Note that a is a unit in A because ϖ, ϖ' are.) Using the specified p-power roots of ϖ, ϖ' giving $\varpi^\flat, (\varpi')^\flat$, we get such a sequence for a as well. Calling it $a^\flat$, and using the multiplicativity, we get:

$$((\varpi')^\flat)^N = \varpi^\flat a^\flat.$$

Likewise, one also has

$$(\varpi^\flat)^M \in (\varpi')^\flat \cdot A^{\flat 0}.$$

This concludes the proof. □

It remains to show that $A^{\flat 0}$ is actually the ring of power-bounded elements in $A^\flat$.

Proposition 17.2.11 *We have $A^{\flat 0} = (A^\flat)^0$.*

Proof. Certainly we have: $A^{\flat 0} \subset (A^\flat)^0$. Recall that we constructed the map:

$$A^{\flat 0} \to A^0$$

sending $x = (\overline{a_n})_n \mapsto \lim_{n\to\infty} a_n^{p^n} =: x^\#$. We likewise denote $A^\flat = \varprojlim_{x \mapsto x^p} A \to A$ the map taking the 0-th component as $x \mapsto x^\#$.

We have a map $A^\flat \to A$ sending $x \mapsto x^\#$. The key point is that the diagram

$$\begin{array}{ccc} A^\flat & \longrightarrow & A \\ \uparrow & & \uparrow \\ A^{\flat 0} & \longrightarrow & A^0 \end{array}$$

is *cartesian*, as if $a^{(0)}$ is power-bounded, then so are the rest of the terms in any p-power sequence.

Suppose $\xi \in A^\flat$ is power-bounded. We then want to conclude that $\xi^{(0)} \in A^0$ (which implies that $\xi \in A^{\flat 0}$ by the cartesian property). By definition, ξ is power-bounded if and only if

$$\{\xi^n\}_{n \geq 1} \subset (\varpi^\flat)^{-N} A^{\flat 0}$$

for some uniform N. As this is a purely multiplicative statement, we can apply $\#$: $\{(\xi^\#)^n\}_{n\geq 1} \subset \varpi^{-N} A^0$, and by the same argument this is equivalent to $\xi^\#$ being power-bounded in A. We conclude. □

We saw that

$$A^{\flat 0}/\varpi^\flat A^{\flat 0} \simeq A^0/\varpi A^0.$$

Corollary 17.2.12 *If A^0 is local, with maximal ideal $\mathfrak{m}$, then $A^{\flat 0}$ is also local, say with unique maximal ideal maximal ideal $\mathfrak{m}^\flat$ and canonically $A^{\flat 0}/\mathfrak{m}^\flat \simeq A^0/\mathfrak{m}$.*

Proof. It suffices to show that ϖ lies in every maximal ideal of A^0 and $\varpi^\flat$ lies in every maximal ideal of $A^{\flat 0}$ (the "Jacobson radical"). Recall that this is equivalent to $1 + \varpi x$ being a unit of $(A^0)^\times$ for all $x \in A^0$, and $1 + \varpi^\flat y \in (A^{\flat 0})^\times$ for all $y \in A^{\flat 0}$. But this is true (using the usual "geometric series" expansion) since A^0 is ϖ-adically separated and complete, and similarly for $A^{\flat 0}$. □

Example 17.2.13 Let K be a perfectoid ring that is a field, e.g. a perfectoid field (but recall that the latter is, a priori, more stringent). Then

$$(K^\flat)^\times = (\varprojlim_{x \mapsto x^p} K)^\times = \varprojlim_{x \mapsto x^p} K^\times = K^\flat - \{0\}.$$

This is obviously perfect. Note that if $\mathrm{char}(K) = p$, then $K^\flat = K$.

Now assume that K is a perfectoid field, with $|\cdot| : K \to \mathbf{R}_{\geq 0}$ defining the topology (i.e. we can choose an inclusion $K^\times/K^{0\times} \hookrightarrow \mathbf{R}^\times_{>0}$) as ordered abelian groups. (This isn't quite canonical; it would be better to say that the value groups are canonically isomorphic, but this is more concrete.) We claim that $K^{0\flat}$ is also a rank 1 valuation ring and we have a *canonical* absolute value $|\cdot|^\flat : K^\flat \to \mathbf{R}_{\geq 0}$ inducing $K^{\flat 0}$, such that $|K^{\flat\times}| = |K^\times|$.

Proof. We know that $\varpi^\flat \in K^{\flat 0}$ is a topologically nilpotent element defining the topology making $K^{\flat 0}$ complete. Define

$$|\cdot|_\flat : K^\flat \to |K|$$

by $x \mapsto |x^\#|$. This is obviously multiplicative and definite. Also, $|x|_\flat \leq 1 \iff x^\# \in K^0 \iff x \in K^{\flat 0}$. We need to check that

$$||x+y||_\flat \leq \max(|x|_\flat, |y|_\flat).$$

If x or y is 0 then this is trivial, so without loss of generality $x, y \neq 0$ and $|x^\#| \leq |y^\#|$ (so $x/y \in K^{\flat 0}$). Divide by $|y|_\flat = |y^\#|$, so

$$\left(\frac{x+y}{y}\right)^\# = \left(1 + \frac{x}{y}\right)^\# \in K^0$$

so $|x+y|_\flat \leq |y|_\flat = \max(|x|_\flat, |y|_\flat)$. □

Proposition 17.2.14 *If K is a perfectoid field, then* $\mathrm{Cont}(K) \cong \mathrm{Cont}(K^\flat)$ *via*

$$v \mapsto v^\flat : x \mapsto v(x^\#).$$

Proof. $\mathrm{Cont}(K)$ is exactly the set of open valuation subrings of K^0. But this must contain topologically nilpotent elements, hence the entire maximal ideal. These are just the valuation subrings of the residue fields, but then you can go to the other side.

To verify that $v^\flat$ is a valuation (the multiplicativity and 0 axiom are trivial, as K is a field)

$$v^\flat(x+y) \leq \max(v^\flat(x), v^\flat(y))$$

one uses the argument from before. □

We conclude with a crucial remark.

Remark 17.2.15 As we shall discuss, the whole theory of perfectoid spaces lies on a crucial passage between arbitrary characteristic and positive characteristic $p > 0$, which is exactly the following. Let A be a perfectoid ring, and $\varpi \in A$ a pseudo uniformizer as in Definition 17.1.1. Then, given the constructions and results presented above, we have a natural isomorphism:

$$A^0/\varpi \simeq A^{\flat 0}/\varpi^\flat.$$

We'll see that the marvelous correspondence between perfectoid spaces in arbitrary characteristic and characteristic p, relies on a process of "almost integral" extension which then passes through the above isomorphism. The observation that in a highly ramified at p algebra over $\mathbf{Q}_p$, as $\mathbf{Q}_p(p^{1/p^\infty})^\wedge$, p is extremely close to being zero, goes back to Deligne and roughly expresses exactly the isomorphism:

$$\mathbf{Z}_p[p^{1/p^\infty}]/p \simeq \mathbf{F}_p[\![t]\!](t^{1/p^\infty})/t,$$

where $\varpi = p$ and $\varpi^\flat = t$.

References

[Sch] P. Scholze, *Perfectoid spaces.*

[Fo] J. - M. Fontaine, *Perfectoides, presque pureté et monodromie-poids.* Séminaire Bourbaki.