

9 Lecture 9: Spectrality of $\text{Cont}(A)$

9.1 Introduction

This lecture is devoted to prove spectrality of $\text{Cont}(A)$. Next time this will be used to prove spectrality, and in particular quasi-compactness, of affinoid adic spectra (once these are defined). The way to achieve spectrality of $\text{Cont}(A)$ is by giving an “algebraic” description of the notion of continuous valuations on a Huber ring A , achieved in terms of cofinality of values of valuations. The idea is to give an algebraic description of continuity in terms of A and the ideal of A generated by the set A^{00} of topologically nilpotent elements of A . Continuity is stable under horizontal specializations, and the idea is to exploit the algebraic input lying behind horizontal specializations.

More properties of analytic points

Recall Proposition 8.3.2 and its Corollary 8.3.3, summarized in the following:

Proposition 9.1.1 *Let A be a Huber ring, and choose a couple of definition (A_0, I) . The point $v \in \text{Spv}(A)$ is analytic if and only if $v(I) \neq 0$. In particular, if A is Tate then*

$$\text{Cont}(A) = \text{Cont}(A)_{\text{an}}.$$

The reader should keep in mind the following:

Remark 9.1.2 The statement $v(I) \neq 0$ is equivalent to saying that v has non-open support. Without mentioning the couple (A_0, I) , one may simply rephrase this by saying that there exists a topologically nilpotent element $a \in A^{00}$ which is not killed by v . In other words,

$$\{\mathfrak{p} \in \text{Spec}(A) \mid \mathfrak{p} \text{ open}\} = V(A^{00} \cdot A)$$

and v is in $\text{Cont}(A)$ if and only if $v \notin V(A^{00} \cdot A)$. We stress once more the fact that A^{00} is not in general an ideal of A , and that the ideal it generates in A may well be A itself: Tate rings contain topologically nilpotent units by definition. If A is Tate then

$$A^{00} \cdot A = A,$$

so $V(A^{00} \cdot A) = \emptyset$; this rephrases the last statement in Proposition 9.1.1.

Rank-1 valuations v and w on a field K coincide as points in $\text{Spv}(K)$ if (and only if) they induce the same valuation topology. Indeed, in the rank-1 case the valuation ring is characterized in terms of the topology on the fraction field as the set of power-bounded elements. But for higher-rank valuations on a field it can (and often does!) happen that its topology coincides with that induced by a rank-1 valuation. We saw an instance of this in Example 6.2.1, and the following characterization of such all such cases underlies the importance of rank-1 valuations (and hence of rigid-analytic geometry!) in the general theory of adic spaces:

Proposition 9.1.3 *Let K be a field, and $v \in \text{Spv}(K)$ nontrivial. Let $R \subset K$ be the valuation ring of v . The following conditions are equivalent:*

- (1) *There exists a rank-1 valuation w on K defining the same topology on K as v does.*
- (2) *There exists a nonzero topologically nilpotent element in K .*
- (3) *R has a prime ideal of height equal to 1.*

Proof. Assuming (1) holds, any $\varpi \neq 0$ in K such that $w(\varpi) < 1$ is topologically nilpotent because w is of rank one, so (2) holds. Next assume (2) holds, so the valuation topology is non-trivial and makes K a Huber ring by Proposition 6.2.3. More specifically, if $\varpi \in K^\times$ is topologically nilpotent then we may replace it by a suitable power so that ϖ lies in the open subring R and then R is a ring of definition with $I = \varpi R$ as an ideal of definition. Note that I is a proper ideal (equivalently, $\varpi \notin R^\times$) since all elements of I are topologically nilpotent. Let \mathfrak{p} be minimal among prime ideals of R over I (i.e., it corresponds to a generic point of $\text{Spec}(R/I)$). This is a nonzero prime (since $I \neq 0$, as $\varpi \neq 0$), and we claim that \mathfrak{p} has height 1, which is to say that it is minimal as a nonzero prime ideal of the domain R . In other words, if \mathfrak{q} is a prime ideal of R strictly contained in \mathfrak{p} then we claim that $\mathfrak{q} = 0$. The set of *all* ideals of a valuation ring is totally ordered by inclusion. (Indeed, if J, J' are ideals in a valuation ring A and there exists $a \in J$ and $a' \in J'$ with $a \notin J'$ and $a' \notin J$ then we reach a contradiction because either $a \in a'A \subset J'$ or $a' \in aA \subset J$.) Since $I \not\subseteq \mathfrak{q}$ (as otherwise \mathfrak{p} would not be minimal over I), it follows that $\mathfrak{q} \subsetneq I = \varpi R$. In particular, $\varpi \notin \mathfrak{q}$. Every $x \in \mathfrak{q}$ can be written as $x = \varpi y$ for $y \in R$. Rather generally, if such an x has the form $\varpi^n y_n$ for some $n \geq 1$ and $y_n \in R$ then the membership $\varpi^n y_n = x \in \mathfrak{q}$ forces $y_n \in \mathfrak{q}$ since \mathfrak{q} is prime and $\varpi \notin \mathfrak{q}$. But then $y_n \in I = \varpi R$, so $y_n = \varpi y_{n+1}$ for some $y_{n+1} \in R$, so $x = \varpi^{n+1} y_{n+1}$. This calculation shows that $\mathfrak{q} \subseteq \bigcap_{n \geq 1} I^n$, and this intersection vanishes since the I -adic topology of R is the v -adic topology, which is separated. This proves $\mathfrak{q} = 0$ as claimed, so we have shown that (2) implies (3).

Finally, assume (3) and let \mathfrak{p} be a height-1 prime ideal of R , so $R_{\mathfrak{p}}$ is a valuation ring of K containing R . The value group $K^\times/R_{\mathfrak{p}}^\times$ is a quotient of the value group K^\times/R^\times of v . Let $w : K \rightarrow (K^\times/R_{\mathfrak{p}}^\times) \cup \{0\}$ be the associated valuation. We claim that w is rank-1, which is to say that its nontrivial value group is order-isomorphic to a subgroup of $\mathbf{R}_{>0}^\times$ (or equivalently \mathbf{R} in additive notation). We need to show that Γ_w is order-isomorphic to a subgroup of $\mathbf{R}_{>0}^\times$. By [Mat, Thm. 10.6] it is equivalent to show that Γ_w satisfies the Archimedean axiom. This means that for *any* $\gamma \in \Gamma_w$ with $\gamma > 1$, every $\gamma' \in \Gamma_w$ satisfies $\gamma' < \gamma^n$ for some $n > 0$ (equivalently, $1/\gamma$ is cofinal in Γ_w). There is a very useful necessary and sufficient condition for a totally ordered abelian group G to satisfy the Archimedean axiom: it should have no nontrivial proper convex subgroups. The necessity of this condition is clear, and for sufficiency suppose there are no such convex subgroups of G and consider $g \in G$ with $g > 1$. Define Δ to be the set of elements $h \in G$ that satisfy $g^{-n} < h < g^n$ for some $n > 0$ (which may depend on h). Clearly Δ contains 1 and is stable under inversion and multiplication, so it is a subgroup of G , and by design Δ is convex. Also, $\Delta \neq 1$ since $g \in \Delta$ (as $g > 1$). The hypothesis on G then forces $\Delta = G$, as desired. Thus, to prove that w is a rank-1 valuation we just need to check that its nontrivial value group has no nontrivial proper convex subgroups. But in general there is a natural inclusion-preserving bijection between the set of prime ideals of a valuation ring and the set of convex subgroups of its value group, explained by combining both parts of [Wed, Prop. 2.14]. Thus, since $R_{\mathfrak{p}}$ has no nonzero non-maximal prime ideals (as \mathfrak{p} has height 1), it follows that Γ_w has no nonzero proper convex subgroups.

To conclude the proof that (3) implies (1), it suffices to show that the w -topology on K coincides with the initial v -topology associated to R . Rather generally, any pair of non-trivial valuations on a field whose valuation rings do *not* general the field as a ring (e.g., this applies when one of the valuation rings contains the other) define the same topology on the field [Bou, Ch. VI, §7.2, Prop. 3]. (This reference also proves the more difficult converse that we do not need.) \square

Definition 9.1.4 We call a valuation v on a field K *microbial* if it satisfies the equivalent conditions in Proposition 9.1.3.

We can now prove the following:

Proposition 9.1.5 *Let A be a Huber ring and choose $v \in \text{Cont}(A)_{\text{an}}$. Then v is microbial and has a unique rank-1 generization w inside $\text{Cont}(A)_{\text{an}}$. Moreover, v has no proper generization inside*

$\text{Cont}(A)_{\text{an}}$ if and only if it is a rank-1 point.

Proof. If v were trivial then its support \mathfrak{p}_v would be $\{v < 1\}$ and hence would be open by continuity of v , yet \mathfrak{p}_v is non-open by analyticity. Thus, v is a non-trivial valuation. By Proposition 9.1.1 the non-open prime \mathfrak{p}_v is not contained in $V(A^{00} \cdot A)$. Hence, there exists some nonzero topologically nilpotent element $a \in A^{00}$ such that $v(a) \neq 0$. In particular the image of a in A/\mathfrak{p}_v is nonzero and topologically nilpotent for the quotient topology and thus for the v -topology since v is *continuous* on A . It follows that the residue field $\kappa(v)$ equipped with its non-trivial valuation arising from v satisfies (2) in Proposition 9.1.3, so v is microbial. In particular, v admits a rank-1 vertical generization w . Since w is non-trivial it inherits continuity from v , and its support is equal to \mathfrak{p}_v so it is also analytic. So we have produced a rank-1 generization of v inside $\text{Cont}(A)_{\text{an}}$.

By Proposition 8.3.10, all generizations inside $\text{Cont}(A)_{\text{an}}$ are vertical, so to prove uniqueness of w we just need to check that v has no other rank-1 *vertical* generization inside $\text{Spv}(A)$. If v' is a rank-1 vertical generization of v then it gives rise to a rank-1 valuation ring inside $\kappa(v)$ that contains the valuation ring of v , and the two resulting valuation topologies on $\kappa(v)$ must coincide (as we saw at the end of the proof of Proposition 9.1.3). But w also gives the same topology, and we have seen earlier the elementary fact that rank-1 points in $\text{Spv}(\kappa(v))$ coincide if and only if they induce the same topology on $\kappa(v)$. Thus, $v' = w$ as desired.

If v does not have rank 1 then we have produced a proper generization of it inside $\text{Cont}(A)_{\text{an}}$, and if v is rank 1 then any point $v' \in \text{Cont}(A)_{\text{an}}$ generizing v must be vertical and so corresponds to a rank-1 valuation ring of $\kappa(v)$ that contains the rank-1 valuation ring associated to v . But it is elementary that there are no proper subrings of a field that strictly contain a rank-1 valuation ring, so the valuation rings for v and v' inside $\kappa(v)$ are equal, forcing $v' = v$ as points of $\text{Spv}(A)$. \square

Example 9.1.6 Let K be a field, and v be a microbial valuation on K . Endow K with the valuation topology induced by v , thus making it into a non-archimedean ring (in fact a topological field). We claim $\text{Cont}(K)$ is the subspace of $\text{Spv}(K)$ consisting of those valuations w on K whose valuation topology on K coincides with that induced by v .

Since every valuation on a field has support (0) , and this is not v -open since v is nontrivial, it follows that

$$\text{Cont}(K) = \text{Cont}(K)_{\text{an}}.$$

The microbial hypothesis on v implies that K contains a topologically nilpotent unit, so by continuity every $w \in \text{Cont}(K)$ admits a topologically nilpotent unit in K for the w -topology and thus is itself microbial. By Proposition 9.1.3, w therefore admits a rank-1 generization that defines the same topology as w . Thus, to prove that all such w induces the same topology as v we may assume v has rank 1 and just need to check that (i) $\text{Cont}(K)$ has no rank-1 points aside from v and (ii) any rank-1 point of $\text{Spv}(K)$ whose topology coincides with that of v is equal to v .

Assertion (ii) is immediate from the fact that a rank-1 point is recovered from its associated topology on K (in such cases the valuation ring is precisely the set of power-bounded elements). As for (i), if a rank-1 valuation v' on K is continuous for the v -topology then any topologically nilpotent $a \in K$ for the v -topology must satisfy $v'(a) < 1$ since $v'(a^n) \rightarrow 0$ for the order topology on $\Gamma_{v'} \cup \{0\}$. But all elements of \mathfrak{m}_v are topologically nilpotent for the v -topology since v has rank 1, so $\mathfrak{m}_v \subseteq \mathfrak{m}_{v'}$. Thus,

$$K - R_v = (\mathfrak{m}_v - \{0\})^{-1} \subseteq (\mathfrak{m}_{v'} - \{0\})^{-1} = K - R_{v'},$$

giving that $R_{v'} \subseteq R_v$. But $R_v \neq K$ and $R_{v'}$ is a *rank-1* valuation ring, forcing $R_{v'} = R_v$ since there are no rings strictly between a rank-1 valuation ring and its fraction field (always false in higher rank!).

For a Huber ring A and (A_0, I) a couple of definition for A , Remark 8.4.3 gave

$$\text{rad}(I \cdot A) = \text{rad}(A^{00} \cdot A).$$

Since $I \cdot A$ is a finitely generated ideal of A , this equality of radicals has geometric meaning: recall from Proposition 3.2.1 that for any commutative ring A an open subset $U \subset \text{Spec}(A)$ is quasi-compact if and only if $\text{Spec}(A) - U$ has the form $\text{Spec}(A/J)$ for some finitely generated ideal J of A , which is to say that the unique radical ideal defining the closed complement of U is also the radical of a finitely generated ideal. (In particular, $\text{Spec}(A) - V(A^{00} \cdot A)$ is quasi-compact.)

The setup for “algebraic” description of continuity of valuations on Huber rings

Definition 9.1.7 If Γ is a totally ordered abelian group and $H \subset \Gamma$ is a subgroup, we say that $\gamma \in \Gamma \cup \{0\}$ is *cofinal for H* if for all $h \in H$ there exists a sufficiently large integer $n \geq 0$ such that $\gamma^n < h$.

Note that 0 is always cofinal for *any* subgroup $H \subseteq \Gamma$.

Now let us consider a commutative ring A , and an ideal $J \subset A$ such that its radical is equal to the radical of a *finitely generated* ideal. We have seen that an example of such an ideal is $A^{00} \cdot A$ in a Huber ring A . For Huber rings A , the points of $\text{Spv}(A)$ over $\text{Spec}(A) - V(A^{00} \cdot A)$ are precisely the analytic points, so for general commutative rings A and J as above we will imitate this and focus on the preimage under the natural continuous map $\text{Spv}(A) \rightarrow \text{Spec}(A)$ of the quasi-compact open set $\text{Spec}(A) - V(J)$; these are the valuations v on A such that $v(J) \neq 0$.

The picture is a Cartesian diagram of topological spaces:

$$\begin{array}{ccc} \text{Spv}(A) & \longleftarrow & \{v \in \text{Spv}(A) \mid v(J) \neq 0\} \\ \downarrow & & \downarrow \\ \text{Spec}(A) & \longleftarrow & \text{Spec}(A) - V(J) \end{array}$$

For $v \in \text{Spv}(A)$ we shall study the property of $v(a)$ of being cofinal in Γ_v for $a \in J$, and in order to do so we shall separately consider two cases: $v(J) \cap c\Gamma_v = \emptyset$, and $v(J) \cap c\Gamma_v \neq \emptyset$. To that end, we first recall some notions related to horizontal specialization and give geometric interpretations of when $v(J) \cap c\Gamma_v$ is empty or non-empty.

For $v \in \text{Spv}(A)$ the horizontal specializations of v in $\text{Spv}(A)$ are given by

$$w|_H : a \mapsto \begin{cases} v(a) & \text{if } a \in H \\ 0 & \text{if } a \notin H \end{cases}$$

for H a convex subgroup of Γ_v containing the characteristic subgroup $c\Gamma_v$, and we had the following:

Theorem 9.1.8 *There is a bijection*

$$\{\text{horizontal specializations of } v\} \leftrightarrow \{v\text{-convex primes}\},$$

via the formation of supports. Moreover, for two such specializations w and w' of v , w specializes to w' if and only if $\mathfrak{p}_w \subset \mathfrak{p}_{w'}$ (i.e., \mathfrak{p}_w specializes to $\mathfrak{p}_{w'}$ in $\text{Spec}(A)$). An inverse is given by assigning to any v -convex prime \mathfrak{q} of A the valuation

$$w_{\mathfrak{q}} : a \mapsto \begin{cases} v(a) & \text{if } a \notin \mathfrak{q}, \\ 0 & \text{if } a \in \mathfrak{q}. \end{cases}$$

In geometric terms, $v|_{c\Gamma_v}$ is to be viewed as the “most special” among the horizontal specializations of v . Also, keep in mind that the above bijection is *not* between the set of v -convex primes of A and the set of convex subgroups of Γ_v containing $c\Gamma_v$ (see Example 4.4.1!).

Remark 9.1.9 Let $v \in \text{Spv}(A)$. For any convex subgroup H of Γ_v , the containment

$$v(J) \cap H \subseteq v(\text{rad}(J)) \cap H$$

is an equality, because Γ_v/H is torsion-free (in fact even totally ordered). In particular, the condition of whether $v(J) \cap H$ is empty or not only depends on J through its radical.

Remark 9.1.10 Let $v \in \text{Spv}(A)$, and let H be a convex subgroup of Γ_v containing $c\Gamma_v$. Then we have that $v(J) \cap H$ is nonempty if and only if $v|_H$ does *not* kill J , which is in turn equivalent to saying that $v|_H$ lies over $\text{Spec}(A) - V(J)$. Note that in the case $v(J) \cap H \neq \emptyset$, necessarily $v(J) \neq 0$!

When $v(J) \neq 0$, if $v(J) \cap H$ is empty then $v|_H$ kills J and hence lies over the vanishing locus of J .

[making picture!]

The upshot of Remark 9.1.10 is that the condition

$$v(J) \cap c\Gamma_v \neq \emptyset$$

means exactly that *all* horizontal specializations of v lie over $\text{Spec}(A) - V(J)$.

Understanding the condition $v(J) \cap c\Gamma_v \neq \emptyset$

The next lemma says that checking whether $v(J) \cap c\Gamma_v$ is empty or not is equivalent to checking whether $v(J) \cap v(A)_{\geq 1}$ is empty or not.

Lemma 9.1.11 *Let A be a ring, J an ideal of A and $v \in \text{Spv}(A)$. Then $v(J) \cap c\Gamma_v \neq \emptyset$ if and only if $v(J) \cap v(A)_{\geq 1} \neq \emptyset$.*

Proof. Certainly if $v(J) \cap v(A)_{\geq 1}$ is nonempty, so is $v(J) \cap c\Gamma_v$. Conversely, suppose there exists $a \in J$ such that $v(a) \in c\Gamma_v$. If $v(a) \geq 1$ we are done. Let's assume $v(a) < 1$. By definition of $c\Gamma_v$ there exist $b, b' \in A$ such that $v(b), v(b') \geq 1$ and

$$\frac{v(b')}{v(b)} \leq v(a) < 1.$$

It follows

$$1 \leq v(b') \leq v(b)v(a) = v(ab),$$

and since $ab \in J$ we conclude. □

Remark 9.1.12 We can draw two consequences from the assumption $v(J) \cap c\Gamma_v \neq \emptyset$:

- (1) For *any* convex subgroup $H \subset \Gamma_v$ containing $c\Gamma_v$, some element of $v(J)$ (e.g., anything in $v(J) \cap v(A)_{\geq 1} \neq \emptyset$) is not cofinal for H .
- (2) Every horizontal specialization of v lies over $\text{Spec}(A) - V(J)$ (since if $v(a) \in c\Gamma_v$ then for any convex subgroup H of Γ_v containing $c\Gamma_v$ clearly $(v|_H)(a) = v(a) \neq 0$). Conversely, if this property holds for all horizontal specializations then by considering the most special case $H = c\Gamma_v$ we see that $v(J) \cap c\Gamma_v$ is non-empty. So this gives a precise geometric visualization of the condition $v(J) \cap c\Gamma_v \neq \emptyset$.

Understanding the condition $v(J) \cap c\Gamma_v = \emptyset$

The following result will be given a nice geometric interpretation via horizontal specialization in §9.2, and it is a striking contrast with both parts of Remark 9.1.12.

Proposition 9.1.13 *Let A be a ring, and $v \in \text{Spv}(A)$ with value group Γ_v . Let J be an ideal of A such that its radical is equal to the radical of a finitely generated ideal, and assume $v(J) \cap c\Gamma_v = \emptyset$ (i.e., the most special horizontal specialization $v|_{c\Gamma_v}$ lies over $V(J)$).*

- (1) *There exist convex subgroups $H \subseteq \Gamma_v$ containing $c\Gamma_v$, with the property that all elements of $v(J)$ are cofinal for H , and among all such H there is one H_J that contains all others.*
- (2) *If moreover $v(J) \neq 0$ (so $c\Gamma_v \neq \Gamma_v$) then necessarily $H_J \neq c\Gamma_v$, $v(J) \cap H_J \neq \emptyset$, and H_J is contained in every convex subgroup H of Γ_v satisfying $v(J) \cap H \neq \emptyset$. Geometrically, if v lies over $\text{Spec}(A) - V(J)$ then so does the horizontal specialization $v|_{H_J}$ and it is the “unique most special” such specialization in the sense that every horizontal specializations of v over $\text{Spec}(A) - V(J)$ is a horizontal generization of $v|_{H_J}$.*

Proof. If $v(J) = 0$ then certainly $v(J) \cap c\Gamma_v = \emptyset$. In this case $H_J = \Gamma_v$ does the job.

We now assume $v(J) \neq 0$. Let H be a convex subgroup of Γ_v containing $c\Gamma_v$ and disjoint from $v(J)$ (e.g., $c\Gamma_v$ is one such H). Note that $v(J) \cap H = \emptyset$ if and only if we have

$$v(\text{rad}(J)) \cap H = \emptyset.$$

Indeed, if $a \in \text{rad}(J)$ and $v(a) \in H$ then $v(a^n) \in H$ for any $n > 0$, so take n large enough to ensure $a^n \in J$. It follows that our problem only depends on J through its radical, so we may assume J itself is finitely generated: say

$$J = (a_1, \dots, a_n).$$

Since $v(J) \neq 0$, some $v(a_i)$ is non-zero. The valuation of the generators a_1, \dots, a_n of J attains a maximum, which we call h , so $h \neq 0$. Note that $h \notin H$ since $h \in v(J)$. In particular, $h < 1$ since $v(A)_{\geq 1} \subseteq c\Gamma_v \subseteq H$ and likewise $h \notin c\Gamma_v$. We call

$$H_J := \{\gamma \in \Gamma_v \mid h^n \leq \gamma \leq h^{-n} \text{ for some } n \geq 0\};$$

this is the convex subgroup “generated” by h inside Γ_v , and $h \in H_J$ since $h < 1$; in particular, $v(J) \cap H_J \neq \emptyset$.

We now show that H_J satisfies the required properties.

Step 1: H_J strictly contains $c\Gamma_v$ We saw above that necessarily $h < 1$. Since h does not lie in the convex subgroup $c\Gamma_v$, it follows that $h < \gamma$ for all $\gamma \in c\Gamma_v$. (Indeed, if for some such γ we have $\gamma \leq h$ then the h is sandwiched between γ and 1, forcing $h \in c\Gamma_v$ by convexity.) Applying inversion to the relation $h < c\Gamma_v$, we obtain

$$h < c\Gamma_v < h^{-1},$$

and being H_J a convex subgroup of Γ_v by construction, necessarily $c\Gamma_v \subseteq H_J$. The inclusion is strict, since $h \notin c\Gamma_v$.

Step 2: $v(J)$ is cofinal for H_J By design, for any $\gamma \in H_J$ there exists $n > 0$ such that $h^n < \gamma$, so $v(a_i)^n \leq h^n < \gamma$ for all i . Hence, the set

$$\{a \in A \mid v(a) \text{ cofinal for } H_J\}$$

contains a_1, \dots, a_n . We want this set to contain the ideal J generated by the a_i 's, so it suffices for this set to be an ideal itself. We cannot merely argue by “ A -linearity” since $v(A)$ might not be ≤ 1 inside Γ_v . Nonetheless, since $c\Gamma_v$ is strictly contained in H_J the hypotheses of the following surprising lemma are satisfied for H_J and thereby gives the required ideal property:

Lemma 9.1.14 *Let A be a ring, and $v \in \text{Spv}(A)$. Let H be a subgroup of Γ_v which strictly contains $c\Gamma_v$. Then*

$$\mathfrak{a} := \{a \in A \mid v(a) \text{ is cofinal for } H\}$$

is a radical ideal of A .

The surprise is that cofinality relative to H defines an ideal even when $v(a) > 1$ for some $a \in A$.

Proof. Let $a, b \in \mathfrak{a}$. We want to show $v(a + b)$ is again cofinal for H . By the ultrametric inequality,

$$v(a + b) \leq \max\{v(a), v(b)\}$$

and hence, since both $v(a)$ and $v(b)$ are cofinal for H , so is $v(a + b)$. Now let $a \in \mathfrak{a}$, and $b \in A$. If $v(b) \leq 1$, then $v(ab) \leq v(a)$, and we are done. Suppose $v(b) > 1$. Then $v(b)$ is contained in $c\Gamma_v \subseteq H$. We now use the assumption that $c\Gamma_v$ is strictly contained in H , and that $v(b) > 1$. There exists $h_0 \in H$ not in $c\Gamma_v$, and by convexity of $c\Gamma_v$ we know that h_0 is either strictly larger or strictly smaller than all elements of $c\Gamma_v$. Passing to its inverse if necessary, we can arrange that

$$h_0 < c\Gamma_v.$$

It follows $h_0^{-1} > c\Gamma_v$, and we have $v(a)^n h_0^{-1} < 1$ for sufficiently large $n \geq 0$. Finally,

$$v(ab)^{n+N} = v(a)^{n+N} v(b)^{n+N} < v(a)^{n+N} h_0^{-1} < v(a)^N$$

for $N \geq 1$, and we conclude that $v(ab)$ is cofinal for H since $v(a)$ is cofinal for H . Thus, \mathfrak{a} is indeed an ideal of A .

If $a^n \in \mathfrak{a}$ then for any $h \in H$ we have $v(a^n)^m < h^n$ for all large m , so $v(a)^m < h$ for all large m , so $a \in \mathfrak{a}$. That is, \mathfrak{a} is radical. \square

Step 3: H_J is maximal Indeed, if H is any convex subgroup of Γ_v containing $c\Gamma_v$ and such that $v(J)$ is cofinal for H , then in particular h is cofinal for H . It follows $H_{\leq 1} \subseteq H_J$ by convexity of H_J , and then $H \subseteq H_J$.

It remains to address the minimality property for H_J . Clearly $v(J) \cap H_J \neq \emptyset$, as h is in the intersection. We want to prove that H_J is contained in every convex subgroup $H \subseteq \Gamma_v$ satisfying

$$v(J) \cap H \neq \emptyset.$$

We instead briefly sketch the idea, and refer the reader to [Wed, Lemma 7.2]. The assumption $v(J) \cap c\Gamma_v = \emptyset$ is used again to deduce that any such H strictly contains $c\Gamma_v$, and convexity of H and the choice of h do the job to show H_J is contained in H . \square

9.2 The key construction

Using Remark 9.1.12 and Proposition 9.1.13, we are led to make the following definition.

Definition 9.2.1 Let A be a ring, and $v \in \text{Spv}(A)$, with value group Γ_v . Let J be an ideal of A whose radical coincides with the radical of a finitely generated radical. We set:

$$c\Gamma_v(J) := \begin{cases} H_J & \text{if } v(J) \cap c\Gamma_v = \emptyset \\ c\Gamma_v & \text{if } v(J) \cap c\Gamma_v \neq \emptyset. \end{cases}$$

The “reasonableness” of the definition when $v(J) \cap c\Gamma_v$ is non-empty (which can only happen when $v(J) \neq 0$) is explained by the minimality aspect of Proposition 9.1.13(3), even though that proposition only concerns cases with $v(J)$ disjoint from $c\Gamma_v$. Note also that always $c\Gamma_v(A) = c\Gamma_v$ (since $1 \in v(A)$).

The geometric idea underlying the definition of $c\Gamma_v(J)$ is that the associated horizontal specialization $v|_{c\Gamma_v(J)}$ of v is the one that is “just barely” still over $\text{Spec}(A) - V(J)$ (with the understanding that it is equal to v when v itself lies over $V(J)$). We summarize the basic properties of $c\Gamma_v(J)$, unifying the two cases of its dichotomous definition, in the following proposition (see especially part (3)).

Proposition 9.2.2 For A and J as above and $v \in \text{Spv}(A)$, the subgroup $c\Gamma_v(J)$ of Γ_v satisfies the following properties:

- (1) $c\Gamma_v(J)$ is convex, and $c\Gamma_v \subseteq c\Gamma_v(J)$.
- (2) $c\Gamma_v(J) = \Gamma_v$ if and only if every proper horizontal specialization of v lies over $V(J) \subset \text{Spec}(A)$.
- (3) $c\Gamma_v(J)$ is minimal with respect to the property of being a convex subgroup of Γ_v that contains $c\Gamma_v$ and meets $v(J)$.
- (4) If $v(J) \cap c\Gamma_v = \emptyset$ then $c\Gamma_v(J)$ is maximal among the convex subgroups of Γ_v that contain $c\Gamma_v$ and relative to which all elements of $v(J)$ are cofinal.

There is no ambiguity about the use of “minimal” and “maximal” above since the collection of convex subgroups of a totally ordered abelian group is itself totally ordered under inclusion (see [Wed, Rem. 1.10], an elementary proof by contradiction).

Proof. (1) is a consequence of Proposition 9.1.13 and the definition of $c\Gamma_v$, as are (3) and (4). But (2) requires an argument (using (3) crucially), as follows.

If $v(J) = 0$ then (2) is trivially true since v lies already over $V(J)$. Suppose instead $v(J) \neq 0$. First assume $c\Gamma_v(J) = \Gamma_v$. Consider a proper horizontal specialization $v|_H$, so H is a proper convex subgroup of Γ_v containing $c\Gamma_v$. By (3), $v(J) \cap H$ must be empty, which implies exactly $v|_H(J) = 0$; i.e., every proper horizontal specialization of v lies over $V(J)$. Conversely, suppose every proper horizontal specialization of v lies over $V(J)$. Then for every proper convex subgroup $H \subsetneq \Gamma_v$ containing $c\Gamma_v$ the specialization $v|_H$ has support in $V(J)$, so $v(J) \cap H$ must be empty. This says that among all proper convex subgroups of Γ_v containing $c\Gamma_v$ none meet $v(J)$. Hence, Γ_v is the only convex subgroup of Γ_v containing $c\Gamma_v$ and meeting $v(J) \neq 0$, so by (3) it must equal $c\Gamma_v(J)$. \square

Let us recast the geometric meaning of Proposition 9.2.2 in terms that will be useful for what follows. By part (1), $c\Gamma_v(J)$ is a convex subgroup containing $c\Gamma_v$, so $v|_{c\Gamma_v(J)}$ makes sense as a horizontal specialization of v . In particular, the assignment

$$r : v \longmapsto v|_{c\Gamma_v(J)}$$

makes sense as a map of sets $\mathrm{Spv}(A) \rightarrow \mathrm{Spv}(A)$. By part (2), $r(v) = v$ if and only if all *proper* horizontal specializations of v lie over $V(J)$; this is a vacuous condition if there are no proper horizontal specializations, and it is a trivial condition if v itself lies over $V(J)$. Part (3) says that when v lies over $\mathrm{Spec}(A) - V(J)$ then $r(v)$ is the “most special” among specializations of v which lie over $\mathrm{Spec}(A) - V(J)$. So the map r should be viewed over $\mathrm{Spec}(A) - V(J)$ as carrying each v to the “last” among its horizontal specializations that do not lie over $V(J)$. If $v(J) \cap c\Gamma_v$ is nonempty then even the most special among all horizontal specializations, namely $v|_{c\Gamma_v}$, does not lie over $V(J)$. But if $v(J) \cap c\Gamma_v$ is empty then all proper horizontal specializations of $r(v)$ lie over $V(J)$ but $r(v)$ does not. So r acts as the identity on the part of $\mathrm{Spv}(A)$ over $V(J)$ and in general the geometric interpretation (and the *transitivity* of the relation “horizontal specialization”) makes it clear that $r(r(v)) = r(v)$ for any v whatsoever.

To summarize, r is a retraction of the entire space $\mathrm{Spv}(A)$ onto the union of the locus over $V(J)$ and the subset of the locus over $\mathrm{Spec}(A) - V(J)$ that is its “interior edge” relative to horizontal specialization.

The preceding considerations motivate interest in the following definition:

Definition 9.2.3 For a ring A and ideal J such that $\mathrm{rad}(J)$ is the radical of a finitely generated ideal, define

$$\mathrm{Spv}(A, J) := \{v \in \mathrm{Spv}(A) \mid r(v) = v\} = \{v \in \mathrm{Spv}(A) \mid c\Gamma_v(J) = \Gamma_v\}.$$

Thus, r is a retraction of $\mathrm{Spv}(A)$ onto $\mathrm{Spv}(A, J)$. As we have seen above,

$$\mathrm{Spv}(A, J) \supseteq \{v \in \mathrm{Spv}(A) \mid v(J) = 0\}.$$

Since the case $J = A$ is relevant to the case of Tate rings later on, we note that the space $\mathrm{Spv}(A, A) := \{v \mid c\Gamma_v = \Gamma_v\}$ can be described geometrically as the locus of points $v \in \mathrm{Spv}(A)$ with no proper horizontal specializations (since if H is any *proper* convex subgroup of the group Γ_v and $H \supseteq c\Gamma_v$ then necessarily $v|_H \neq v$: some $a \in A - \mathfrak{p}_v$ must satisfy $v(a) \notin H$ since Γ_v is generated by $v(A - \mathfrak{p}_v)$, so $\mathrm{supp}(v|_H)$ contains a whereas $\mathrm{supp}(v)$ does not contain a).

Here is a concrete description of $\mathrm{Spv}(A, J)$ in the spirit of continuity conditions for valuations on Huber rings (though presently A is just a commutative ring):

Lemma 9.2.4 Let a_1, \dots, a_n satisfy $\mathrm{rad}(a_1, \dots, a_n) = \mathrm{rad}(J)$. For $v \in \mathrm{Spv}(A)$, the following are equivalent:

- (1) $c\Gamma_v(J) = \Gamma_v$ (i.e., $v \in \mathrm{Spv}(A, J)$).
- (2) $\Gamma_v = c\Gamma_v$ or $v(a)$ is cofinal for Γ_v for all $a \in J$.
- (3) $\Gamma_v = c\Gamma_v$ or $v(a_1), \dots, v(a_n)$ are all cofinal for Γ_v .

Proof. (1) and (2) are equivalent by Definition 9.2.3 and Proposition 9.1.13. (2) trivially implies (3). On the other hand, by Lemma 9.1.14 and Proposition 9.2.2, (3) implies (2) as well: indeed either $\Gamma_v = c\Gamma_v$, or if not then the assumptions of Lemma 9.1.14 are satisfied. \square

The technical importance of $\mathrm{Spv}(A, J)$ is due to:

Proposition 9.2.5 Let A and J be as above, and give $\mathrm{Spv}(A, J)$ the subspace topology from $\mathrm{Spv}(A)$. Then:

- (1) $X := \mathrm{Spv}(A, J)$ is a spectral space.

(2) A base of quasi-compact open sets for the topology of X is given by

$$X(T/s) := \{v \in X \mid v(g_1) \leq v(s) \neq 0, \dots, v(g_n) \leq v(s) \neq 0\}$$

for non-empty finite sets $T = \{g_1, \dots, g_n\}$ such that $J \subseteq \text{rad}(T \cdot A)$.

(3) The retraction $r : \text{Spv}(A) \rightarrow \text{Spv}(A, J)$ is a spectral map (in particular, continuous).

(4) If $v \in \text{Spv}(A)$ lies over $\text{Spec}(A) - V(J)$ then so does $r(v)$.

Proof. The proof of (4) has been given in our discussion of the geometric meaning of r . We divide the proof of the rest into steps. As we have seen a few times already, we may and do assume J is finitely generated.

Proof of (2) apart from quasi-compactness: Choose any such T and s , so

$$X(T/s) = X \cap \text{Spv}(A)(T/s)$$

is open in X . To check that these open sets are stable under finite intersection, first note that

$$X(T/s) = X \left(\frac{T \cup \{s\}}{s} \right),$$

so we are reduced to the case in which T contains s . Pick two finite nonempty subsets T and T' in A , and $s \in T$ and $s' \in T'$ with the property

$$J \subseteq \text{rad}(T \cdot A) \cap \text{rad}(T' \cdot A).$$

If we define $T'' := \{t \cdot t', t \in T \text{ and } t' \in T'\}$ then $(ss' \in T'' \text{ and})$ the ideal $\text{rad}(T'' \cdot A)$ contains J since it contains J^2 . Thus, $X(T''/ss')$ makes sense and clearly

$$X(T/s) \cap X(T'/s') \subseteq X(T''/ss').$$

The reverse inclusion holds because if $v(tt') \leq v(ss') \neq 0$ for some $t \in T$ and $t' \in T'$ then it is not possible that both $v(t) > v(s)$ and $v(t') > v(s')$.

For the topological base assertion in (2) it remains to show that given $v \in \text{Spv}(A, J)$ and any open neighbourhood U of v in $\text{Spv}(A)$, there exists an open neighbourhood of v in $U \cap \text{Spv}(A, J)$ given by such an $X(T/s)$. We first assume $c\Gamma_v = \Gamma_v$. By definition of the topology on $\text{Spv}(A)$, we can choose $t_1, \dots, t_n, s \in A$ such that

$$v \in \text{Spv}(A) \left(\frac{t_1, \dots, t_n}{s} \right) \subseteq U.$$

We claim we can arrange the choice of such t_i 's and s such that one of the t_i 's is 1 (so trivially $J \subseteq \text{rad}(t_1, \dots, t_n)$).

Assume first $v(s) \geq 1$. Then we simply append an extra $t' := 1$ to the t_i 's.

Let now $v(s) < 1$. It follows $v(s^{-1}) = v(s)^{-1} \geq 1$, and $v(s)^{-1} \in \Gamma_v = c\Gamma_v$. We claim that $v(s)^{-1} \leq v(a)$ for some $a \in A$. In this case we will have

$$1 \leq v(as)$$

and therefore

$$v \in X \left(\frac{t_1 a, \dots, t_n a, 1}{sa} \right) \subseteq U,$$

as required. Since $v(s) \in \Gamma_v = c\Gamma_v$, there exist $a', a \in A$ such that $v(a'), v(a) \geq 1$ and $v(a')/v(a) \leq v(s)$. If $v(as) \geq 1$ then we are done, so we may assume $v(as) < 1$, or equivalently $v(s) < 1/v(a)$. Thus,

$$v(s) \geq v(a')/v(a) > v(a')v(s),$$

so $v(a') < 1$, a contradiction.

Now assume $\Gamma_v \neq c\Gamma_v$. Let $\{a_1, \dots, a_m\}$ be generators for J . Since $v(s) \neq 0$, by Lemma 9.2.4 (3) we know that for sufficiently large $k \geq 0$,

$$v(a_i)^k \leq v(s) \quad \text{for all } i = 1, \dots, m.$$

This implies

$$v \in X \left(\frac{t_1, \dots, t_n, a_1^k, \dots, a_m^k}{s} \right) \subseteq U.$$

This new collection of “numerators” clearly does the job.

Proof of (3): We let $s \in A$, $T \subset A$ be a finite nonempty subset of A , and $I \subset \text{rad}(T \cdot A)$. Let

$$U := X(T/s) \quad \text{and} \quad U' := \text{Spv}(A)(T/s).$$

We know that U' is quasi-compact from our work on the spectrality of $\text{Spv}(A)$. We claim $U' = r^{-1}(U)$, which will achieve continuity of r due to the settled part of (2), and also the spectrality once U is known to be quasi-compact (the part of (2) not yet proved). We know $U \subseteq U'$, and every point of $r^{-1}(U)$ is a horizontal generization of a point of U . Since U' is open, it is stable under generization and hence contains $r^{-1}(U)$.

Now choose $w \in U'$ and we have to show $r(w) \in U$. If w lies over $V(J)$ then $r(w) = w \in U' \cap X = X(T/s) = U$. Assume now $w(J) \neq 0$. Thus, w lies over $\text{Spec}(A) - V(J)$, so the same holds for $r(w)$. That is, $r(w)$ does not kill J . But $J \subseteq \text{rad}(T \cdot A)$ by hypothesis, so $r(w)$ cannot kill T . That is, $r(w)(t_0) \neq 0$ for some $t_0 \in T$. We have by assumption

$$w(t) \leq w(s) \neq 0$$

for all $t \in T$.

It is a general fact that if $v' = v|_H$ is a horizontal specialization of $v \in \text{Spv}(A)$ and $a, b \in A$ satisfy $v(a) \leq v(b)$ then $v'(a) \leq v'(b)$. Indeed, this is obvious if $v'(b) \neq 0$ (i.e., $v(b) \in H$) since in such cases $v'(b) = v(b)$ whereas always $v'(a)$ is equal either to $v(a)$ or to 0 (depending on whether or not $v(a) \in H$). So we just need to rule out the possibility $v(b) \notin H$ yet $v(a) \in H$. But $v(A)_{\geq 1} \subseteq c\Gamma_v \subseteq H$, so if $v(b) \notin H$ then $v(b) < 1$, so convexity of H and the inequalities $v(a) \leq v(b) < 1$ would force $v(a) \notin H$. Applying this general conclusion to the horizontal specialization $v' = r(w)$, we have $r(w)(t) \leq r(w)(s)$ for all $t \in T$. Taking $t = t_0$ thereby prevents $r(w)(s)$ from vanishing, so $r(w) \in U' \cap X = U$.

End of the proof Consider the Boolean algebra generated by the subsets $X(T/s)$ of X , and endow X with the topology that this algebra generates (so each $X(T/s)$ is open and closed for this new topology). The key lies in (3), where we proved that for any member U of this algebra, $r^{-1}(U)$ is *constructible* in the spectral space $\text{Spv}(A)$. Using Theorem 3.3.9 (Hochster’s criterion) one deduces that X is spectral *and* that every $X(T/s)$ is quasi-compact, so also r is spectral. See the last paragraph of the proof of [H1, Prop. 2.6] for the details. (This argument uses the quasi-compactness of the *constructible topology* of spectral spaces, applied to the spectral space $\text{Spv}(A)$.) \square

9.3 The Main Theorem

Recall we started seeking a suitable algebraic description of continuity for valuations on A . Now A shall be a Huber ring, and our constructions from the previous section do finally produce the desired description! It turns out that cofinality captures continuity when we choose the appropriate J :

Theorem 9.3.1 *Let A be a Huber ring, and (A_0, I) a couple of definition. Then we have:*

$$\text{Cont}(A) = \{v \in \text{Spv}(A, A^{00} \cdot A) \mid v(a) < 1 \text{ for all } a \in A^{00}\}.$$

Note first of all that $\text{rad}(A^{00} \cdot A) = \text{rad}(I \cdot A)$, so since $I \cdot A$ is a finitely generated ideal of A it follows that the construction explained throughout the previous section is applicable.

The Theorem implies that $v \in \text{Spv}(A)$ is continuous if and only if for every $a \in I$ the value $v(a)$ is cofinal in Γ_v (not just $v(a) < 1$; in Example 10.2.3 we will illustrate that this is weaker than cofinality for higher-rank v). In fact, we can push this a bit further in terms of a finite generating set of I as an ideal of A_0 , subject to (necessary) boundedness for v on the entirety of A_0 ; see Corollary 9.3.3.

Granting Theorem 9.3.1 for a moment, we finally prove spectrality of $\text{Cont}(A)$:

Corollary 9.3.2 *Let A be a Huber ring, and (A_0, I) a couple of definition. Then $\text{Cont}(A)$ is closed in $\text{Spv}(A, I \cdot A)$, which is spectral. In particular, $\text{Cont}(A)$ is spectral and thus quasi-compact.*

Proof. Indeed,

$$\text{Spv}(A, I \cdot A) - \text{Cont}(A) = \bigcup_{a \in I} \text{Spv}(A, I \cdot A) \left(\frac{1}{a} \right),$$

which is open. By §3.3 (Lecture 3), since $\text{Cont}(A)$ is a closed subspace of a spectral space, it is spectral. \square

We now prove Theorem 9.3.1. The reader may refer also to [Wed, Thm. 7.10], and to [H1, Thm. 3.1].

Proof of Theorem 9.3.1.

Step 1: $\text{Cont}(A) \subseteq$ right side. Let $v \in \text{Cont}(A)$. For all $a \in A^{00}$ the continuity of v implies both $v(a) < 1$ and the cofinality of $v(a)$ for Γ_v . Now Lemma 9.2.4 yields $c\Gamma_v(A^{00} \cdot A) = \Gamma_v$, so $v \in \text{Spv}(A, A^{00} \cdot A)$.

Step 2: cofinality. To show that the right side is contained in the left side, choose $v \in \text{Spv}(A, I \cdot A)$ such that $v(a) < 1$ for all $a \in I$. We first check that $v(a)$ is cofinal for Γ_v for all $a \in I$. This is immediate if $c\Gamma_v \neq \Gamma_v$ by Lemma 9.2.4(2). Assume now $c\Gamma_v = \Gamma_v$. We have to show that for any γ in Γ_v and a in I , if n is large enough (depending on a and γ) then $v(a)^n < \gamma$. The case $\gamma \geq 1$ is trivial (as $v(a) < 1$ by hypothesis), so assume $\gamma < 1$. But $\gamma \in c\Gamma_v$, so it is bounded below by a fraction $v(b')/v(b)$ with b, b' in A such that $v(b), v(b') \geq 1$. Hence,

$$1/v(b) \leq v(b')/v(b) \leq \gamma.$$

Thus, we are reduced to show that for large enough n (such n depending on a) we have $v(a)^n < 1/v(b)$, or equivalently $v(a^n b) < 1$. But $a^n b \rightarrow 0$ in A as $n \rightarrow \infty$ since a is in I , so for n sufficiently large we have $a^n b \in I$ and hence (by our initial hypotheses on v) $v(a^n b) < 1$, as required.

Step 3: continuity. We finally deduce continuity for v as in Step 2. Let $T := \{a_1, \dots, a_n\}$ be a set of generators for I as an ideal of A_0 . Since Γ_v is totally ordered, upon reindexing we can assume

$$v(a_1) \geq v(a_2) \geq \dots \geq v(a_n).$$

For any m -fold product a of the a_i 's we have $v(a) \leq v(a_1)^m$, so by the established cofinality of $v(a_1)$ in Γ_v it follows that for all $\gamma \in \Gamma_v$ and sufficiently large $m \geq 0$ we have

$$v(a) < \gamma.$$

Since $v(b) < 1$ for all $b \in I$ by hypothesis (!), for such $m \geq 0$ we have

$$v(T^m \cdot I) < \gamma.$$

But $T^m \cdot I = I^{m+1}$, so this displayed inequality expresses exactly that v is continuous. \square

Corollary 9.3.3 *Let A be a Huber ring with couple of definition (A_0, I) , and let $\{a_1, \dots, a_n\}$ be a finite generating set of I as an A_0 -module. For $v \in \text{Spv}(A)$, the following are equivalent:*

- (1) v is continuous;
- (2) $v(a)$ is cofinal in Γ_v for all $a \in I$,
- (3) $\gamma_i := v(a_i)$ is cofinal in Γ_v for all i and when $\gamma := \max_i \gamma_i \neq 0$ then $v(a) < 1/\gamma$ for all $a \in A_0$.

This is an ‘‘algebraic’’ description of continuity (up to the fact that the specification of (A_0, I) is of topological nature). beware that it is *not* necessary for $v \in \text{Cont}(A)$ that $v(a) \leq 1$ for all $a \in A_0$; we will give pervasive counterexamples with higher-rank v in Example 10.2.4.

Proof. Trivially (1) implies (2) due to topological nilpotence of elements of I . To see that (2) implies (3) we first note that cofinal elements of Γ_v must be < 1 , so since $aa_i \in I$ for all $a \in A_0$ we see that $v(aa_i) < 1$ for all i and all $a \in A_0$. This implies $v(a) < 1/\gamma_i$ whenever $\gamma_i \neq 0$. Thus, if $\gamma := \max_i \gamma_i$ is nonzero then necessarily $v(a) < 1/\gamma$ for all $a \in A_0$.

Finally, assume γ_i is cofinal in Γ_v for all i and moreover that if some γ_i is nonzero (so $\gamma \neq 0$) then in fact $v(a) < 1/\gamma$ for all $a \in A_0$. We want to show that v is continuous. If all γ_i vanish then certainly $v(I) = \{0\}$, so v is trivially continuous (as it factors through the discrete quotient $A/I \cdot A$ of A). Thus, now we assume some γ_i is nonzero, so $\gamma \neq 0$ and by hypothesis $v(a) < 1/\gamma$ for all $a \in A_0$. Hence, for any i we have

$$v(aa_i) = \gamma_i v(a) \leq \gamma v(a) < 1,$$

so every A_0 -linear combination b of the a_i 's satisfies $v(b) < 1$. Such linear combinations exhaust I , so $v(b) < 1$ for all $b \in I$, and hence $v(a) < 1$ for all $a \in A^{00}$ (as any such a satisfies $a^m \in I$ for some $m > 0$, since I is open around 0). By Theorem 9.3.1, continuity of such v is reduced to checking that $v \in \text{Spv}(A, A^{00} \cdot A)$. Since $\text{rad}(\sum Aa_i) = \text{rad}(A^{00} \cdot A)$, by Lemma 9.2.4 we are done because $v(a_i)$ is cofinal in Γ_v for all i by hypothesis. \square

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