2014-15 LEARNING SEMINAR: PERFECTOID SPACES

The aim of the seminar this year is to understand Scholze's remarkable paper on perfectoid spaces (including preliminary background from Huber's work on adic spaces, which provides the context for the basic constructions), and to see how it leads to applications in p-adic analytic étale cohomology under extremely natural hypotheses without any battling against integral models (in contrast with all prior work on such matters).

The miracle of perfectoid spaces is that they provide *functors* (called *tilting*) between geometric objects in characteristic 0 and in characteristic p; in the case of a single point this essentially recovers a construction of Fontaine and Wintenberger that underlies p-adic Hodge theory. Subsequent work by Scholze has demonstrated without a doubt that perfectoid spaces are a powerful new tool across many aspects of algebraic number theory.

The main prerequisite, in addition to fluency with scheme-theoretic algebraic geometry, is a solid command of classical rigid-analytic geometry (such as the properties of affinoid algebras and the work of Tate and Kiehl on coherent sheaf theory on rigid-analytic spaces, as developed in the book *Non-archimedean analysis* by Bosch, Güntzer, Remmert or the recent Springer LNM 2105 book *Lectures on formal and rigid geometry* by Bosch.) The Arizona Winter School article "Several approaches to non-archimedean geometry" provides a crash-course on the relevant background in rigid geometry, but isn't really a substitute for a deeper study of the classical framework (though strictly speaking, the theory of adic spaces is self-contained given some basic properties of affinoid algebras, much as EGA requires a lot of input from commutative algebra but nothing from Weil's "Foundations").

In the fall quarter we will discuss adic spaces, then in the winter and spring turn to Scholze's paper that introduced perfectoid spaces, and at the end give applications to étale cohomology on analytic spaces.

Lecture notes will be prepared by Masullo and posted at the course website. All lectures in the fall will be given by Conrad. In subsequent terms we will have talks given by some other participants too.

1. Fall quarter: adic spaces

We will initially focus on the foundational work [H1] and [H2] of R. Huber, along with one key point from Chapter 2 of Huber's book [H3]. The informal notes [W] by Wedhorn provide helpful explanations for some background in valuation theory and a variety of detailed which are treated a bit tersely in Huber's work.

References:

(H1) R. Huber, *Continuous Valuations*, Math. Z. **212** (1993), 455–477.
(H2) R. Huber, *A generalization of formal schemes and rigid-analytic varieties*, Math. Z. **217** (1994), 513–551.
(H3) R. Huber, *Étale cohomology of rigid-analytic varieties and adic spaces*, Friedr. Vieweg & Sohn, 1996.
(W) T. Wedhorn, *Adic spaces*, unpublished notes.

LECTURE 1 (September 26, Conrad): Overview of adic spaces, their relation with rigid-analytic spaces, and some examples.

LECTURE 2 (October 3, Conrad): Review of valuation rings (with some constructions), Riemann–Zariski spaces, and valuation spectra (as topological spaces).

LECTURE 3 (October 10, Conrad): Constructible subsets of topological spaces, the constructible topology (with examples from schemes), spectral spaces and their properties, the spectrality of Spv(A).

LECTURE 4 (October 17, Conrad): Vertical and horizontal specialization, examples thereof, and the ubiquity of these constructions to account for general specialization among points in Spv(A).

LECTURE 5 (October 24, Conrad): Non-archimedean rings, power-boundeness, Huber rings, Tate rings, rings of definition, operations on Huber rings (e.g., completion, topological localization).

LECTURE 6 (October 31, Conrad): More detail and examples of operations on Huber rings, especially localization.

LECTURE 7 (November 7, Conrad): Yet more discussion of topological localization, universal properties, various examples (especially related to Tate rings). Analytic points.

LECTURE 8 (November 14, Conrad): Exploring the topology on Cont(A) in terms of horizontal and vertical specialization, and the special features of the subspace of analytic points. Preparation for proof of spectrality of Cont(A).

LECTURE 9 (November 21, Conrad): Microbial valuations, geometric visualization of Spv(A,J) for general rings A and suitable ideals J, "algebraic" description of Cont(A) inside Spv(A) for Huber rings A. Proof that Spv(A,J) and retraction map from Spv(A) are spectral. Spectrality of Cont(A).

LECTURE 10 (December 5, Conrad): Cofinality with higher rank valuation rings, adic affinoid spectra as topological subspace of Cont(A), Nullstellensatz-like motivation for Spa(A,A^+). Spectrality of Spa(A,A^+), and summary of main properties to be proved for this construction.

LECTURE 11 (December 12, Conrad): Points of the adic closed unit disc, rational domains in Spa, behavior with respect to completion, non-emptiness characterization.

2. WINTER QUARTER: PERFECTOID RINGS, TILTS, AND ALMOST-MATHEMATICS

We discuss some basic definitions and theorems concerning perfectoid rings, especially the sheaf property and the tilting functor, as well as some technical tools: Faltings' almost-mathematics and the cotangent complex.

References:

(BV) Buzzard, Verberkmoes, *Stably uniform affinoids are sheaf*, preprint.
(F) Fontaine, *Perfectoides, presque pureté et monodromie-poids*, Seminaire Bourbaki 1057 (2011-12).
(GR) Gabber, Ramero, *Almost ring theory*, Springer LNM 1800 (2003).
(S1) Scholze, *Perfectoid spaces*, IHES **116** (2012), 245–313.
(S2) Scholze, *Perfectoid spaces: a survey*, Current Developments in Math., 2012.

Lecture 12 (January 5, Conrad): Huber rings associated to rational domains, good properties of power-bounded subring in the rigid-analytic case, structure presheaf and its A^+-analogue, properties of stalks.

Lecture 13 (January 12, Warner): Uniformity, statement of Huber's sheafyness results under noetherian hypotheses, proof of sheafyness of structure presheaf for complete uniform Tate rings after Buzzard-Verberkmoes, and various counterexamples to sheafyness. Example of a mixed-characteristic adic space associated to the formal affine line over $\mathbf{Z}_p$.

Lecture 14 (January 19, Conrad): Basic generalities on adic spaces

Lecture 15 (January 26, Conrad): Points and lft morphisms

Lecture 16 (February 2, Conrad): Rigid geometry and perfectoid rings

Lecture 17 (February 9, Conrad): The tilting functor

Lecture 18 (February 16, Masullo): Almost-mathematics I

Lecture 19 (February 23, Masullo): Almost-mathematics II

Lecture 20 (March 2, 9, Venkatesh): Cotangent complex I

Lecture 21 (March 20, Litt): Cotangent complex II

3. Spring quarter: perfectoid spaces and global tilting

We discuss the tilting equivalence, globalization via perfectoid spaces, the almost purity theorem, and an application of perfectoid spaces to the étale cohomology of smooth proper analytic spaces.

Reference (for final two meetings):

(S3) Scholze, *p-adic Hodge theory for rigid-analytic varieties*, Forum of Math. Π (2013), 1–77.

Lecture 22 (April 8, Masullo): Deformation theory of almost algebras

Lecture 23 (April 13, Masullo): Affinoid tilting equivalence I

Lecture 24 (April 20, Masullo): Affinoid tilting equivalence II

Lecture 25 (April 27, Masullo): Finite étale tilts (modulo almost-purity) and Fontaine-Wintenberger

Lecture 26 (May 4, Masullo: Global perfectoid spaces, étale topology, and global tilting

Lecture 27 (May 11, Masullo): Strongly étale morphisms, almost purity, and étale tilting equivalence

Lecture 28 (May 18, Venkatesh): Pro-étale methods

Lecture 29 (May 25, Bellovin): Finiteness for étale cohomology of analytic spaces