Overview

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October 5, 2016

1 Introduction

Let X be a finite type scheme over $\mathbf{Z}$. The *zeta function* of X is

$$\zeta_X(s) = \prod_{x \in |X|} (1 - q_x^{-s})^{-1}$$

where $|X|$ is th set of closed points and $q_x = \#\kappa(x)$ is the size of the residue field at x. By very crude estimates, this converges for

$$\mathrm{Re}(s) > 1 + \frac{\dim X}{2}$$

(uniformly on closed half-planes therein).

Now suppose X is separated of finite type over a finite field κ of size q. There is an "absolute q-Frobenius" $F_{X,q}\colon X \to X$ which on affines corresponds to the map of rings

$$\begin{aligned} A &\leftarrow A \\ a^q &\leftarrow\!\shortmid a \end{aligned}$$

that commutes with all κ-algebra homomorphisms (and so globalizes); on the underlying topological space $F_{X,q}$ is the identity map since for a prime ideal $\mathfrak{p} \subset A$ we have $a \in \mathfrak{p}$ if and only if $a^q \in \mathfrak{p}$. As Weil observed:

$$X(\kappa) = \{x \in X(\overline{\kappa})\colon F_{X,q}(x) = x\} = (\Gamma_{F_{X,q}} \cap \Delta_{X/\kappa})(\overline{\kappa}).$$

Recall that for *compact smooth manifolds* M, for any $f : M \to M$ whose graph is transverse to the diagonal $\Delta \subset M \times M$ (i.e., the fixed points are "isolated"), the number of fixed points is given by $\sum(-1)^i \mathrm{Tr}(\mathrm{H}^i(f))$.

In the above setting, informally $\Gamma_{F_{X,q}}$ is transverse to $\Delta_{X/\kappa}$ because the tangent line to $y = x^q$ at any geometric point in characteristic $p > 0$ consists of horizontal

*Notes taken by Tony Feng

lines whereas the tangent line to $y = x$ at any geometric point is complementary to that. Motivated by this, Weil's dream was to construct a cohomology theory $X \rightsquigarrow \mathrm{H}^\bullet(X, K)$ from the category of smooth projective varieties X over an algebraically closed field to the category of finite-dimensional vector spaces over a field K of characteristic 0 such that (among other things) if $f\colon X \to X$ has Γ_f transverse to $\Delta_{X/k}$ at their geometric intersection points then

$$\#\,\mathrm{Fix}(f) = \sum_{i=0}^{2\dim X} (-1)^i\,\mathrm{Tr}(\mathrm{H}^i(f)) \in K.$$

With some work, this would imply that $\zeta(X, s)$ is a rational function in q^{-s} for smooth projective X over a finite field κ of size q.

Dwork proved this rationality statement, using p-adic analysis, for *arbitrary* X of finite type over κ (not necessarily projective, nor smooth). Later, Grothendieck went far beyond this by constructing a good theory of $\mathrm{H}^i_c(Y, \mathbf{Q}_\ell)$ for *any* scheme Y which is separated and finite type over a separably closed field k of characteristic $\neq \ell$ such that for X separated and finte type over κ we have

$$\#X(\kappa) = \sum_{i=0}^{2d}(-1)^i\,\mathrm{Tr}(\phi|\mathrm{H}^i_c(X_{\overline{\kappa}}, \mathbf{Q}_\ell))$$

where ϕ is the effect of $(F_{X,q})_{\overline{\kappa}}$.

Remark 1.1. There is also an action of $1_X \otimes_\kappa F_{\overline{\kappa},q}$ on $X_{\overline{\kappa}}$. Note that this is "Galois-theoretic" rather than "geometric" (it is not a morphism over $\overline{\kappa}$). The composition

$$(1_X \otimes_\kappa F_{\overline{\kappa},q}) \circ (F_{X,q})_{\overline{\kappa}}$$

on $X_{\overline{\kappa}}$ is the q-Frobenius of this scheme, and it is a general fact for *any* $\overline{\kappa}$-scheme Z of finite type that the effect of its p-Frobenius on $\mathrm{H}^i_c(Z, \mathbf{Q}_\ell)$ is the identity (we'll later see something much more general concerning the invisibility of absolute Frobenius maps on cohomology), so the action of the "geometric" ϕ on cohomology coincides with the *inverse* of the induced action of $F_{\overline{\kappa},q}$. That's why this inverse as an element of $\mathrm{Gal}(\overline{\kappa}/\kappa)$ (i.e. the qth-root automorphism) is called the "geometric q-Frobenius" element.

The Grothendieck-Lefschetz trace formula is a vast generalization of the above formula for $\#X(\kappa)$ (involving cohomology of rather more general ℓ-adic sheaves), the generality of which is essential to its proof, and it yields the cohomological formula

$$\zeta(X, s) = \prod_{i=0}^{2d} \det(1 - \phi t \mid \mathrm{H}^i_c(X_{\overline{\kappa}}, \mathbf{Q}_\ell))^{(-1)^{i+1}}$$

where $t = q^{-s}$. Strictly speaking, this means that the right side as an element of $\mathbf{Q}_\ell(t)$ actually lies in $\mathbf{Q}(t)$ and then when we then substitute q^{-s} for t we recover $\zeta(X, s)$. The "Riemann hypothesis" implies that there is no cancellation among these factors, and more specifically:

Theorem 1.2 (Deligne)**.** *For X smooth and proper over κ,*

$$\det(1 - \phi t \mid \mathrm{H}^i(X_{\overline{\kappa}}, \mathbf{Q}_\ell)) \in \mathbf{Q}[t]$$

with roots in $\mathbf{C}$ on $\{|z| = q^{-i/2}\}$.

In Weil II, Deligne proved an "RH inequality" for eigenvalues of a suitable q-Frobenius operation on $\mathrm{H}^i_c(X_{\overline{\kappa}}, \mathcal{F}_{\overline{\kappa}})$ with $\mathcal{F}$ any "constructible $\overline{\mathbf{Q}}_\ell$-sheaf" on X which satisfies an "RH inequality" for its stalks at closed points (a condition we'll define much later, and which holds for $\mathcal{F} = \overline{\mathbf{Q}}_\ell$), with X any separated scheme of finite type over κ. This is a vast generalization of the usual RH equality in the sense that for X smooth and proper Poincaré duality is available in a very concrete form that identifies the RH-inequality in each degree with the same inequality in the *other direction* in the complementary cohomological degree, from which one deduces all inequalities or equalities, recoveing the usual RH as above in the smooth proper case.

Remark 1.3. In nearly all applications (such as to exponential sums), it is the Weil II result applicable for X neither proper nor smooth that is the truly useful result. The inequalities that arise also illuminate the true role of the smoothness and properness conditions in the usual RH as conjectured by Weil.

2 Basic aims

We want to define cohomology groups $\mathrm{H}^\bullet(X, A)$ for any scheme X and finite abelian group A, with the following properties.

1. If X is connected then

$$\begin{aligned} \mathrm{H}^1(X, A) &= \mathrm{Hom}_{\mathrm{cont}}(\pi_1(X, \overline{x}), A) \\ &= \{A\text{-torsors } E \to X \text{ for étale topology}\}/\text{isom.} \end{aligned}$$

   Strictly speaking, one first shows

$$\mathrm{H}^1(X, A) = \check{\mathrm{H}}^1(X, A)$$

   by general facts relating derived functor and Čech cohomology, and the latter is directly related to the torsor description via descent theory, which in turn coincides with the π_1-description by the definition of the fundamental group and the finiteness of A. (One needs A to be finite or else difficulties with effective descent arise; we'll come back to this later.)

2. For X separated and finite type over $k = k_s$ with $\mathrm{char}(k) \nmid n$, both $\mathrm{H}^i(X, \mathbf{Z}/n\mathbf{Z})$ and $\mathrm{H}^i_c(X, \mathbf{Z}/n\mathbf{Z})$ are finite groups. (We'll prove finiteness for coefficients in any n-torsion constructible abelian sheaf. The need to go beyond constant coefficients when proving general finiteness results is analogous to the fact that

proving finiteness results for cohomology of vector bundles on proper schemes over a noetherian ring really requires the much wider framework of cohomology of coherent sheaves, without which the powerful formalism of exact sequence arguments would be lost. The constructible abelian sheaves will turn out to be precisely the noetherian objects in the category of abelian torsion sheaves for the étale topology on a noetherian scheme.)

We will also show that such cohomology groups (and variants with more general coefficient sheaves) are unaffected by scalar extension to any separably closed extension field over k (e.g., $\overline{\mathbf{Q}} \to \mathbf{C}$).

3. If X is separated and finite type over $\mathbf{C}$, and A is a finite abelian group, then $\mathrm{H}^\bullet(X, A) \simeq \mathrm{H}^\bullet_{\mathrm{top}}(X(\mathbf{C}), A)$ (the usual derived-functor cohomology) and $\mathrm{H}^\bullet_c(X, A) \simeq \mathrm{H}^\bullet_{\mathrm{top},c}(X(\mathbf{C}), A)$, which is compatible with the usual structures (δ-functoriality when formulated for more general coefficient sheaves, excision sequences, cup product, Poincaré duality in the smooth case, etc.)

Remark 2.1. (i) The étale fundamental group $\pi_1(X, \overline{x})$, which we will discuss next time, will "unify" the traditional notion of π_1 with Galois theory, explaining the formal similarities between the two theories (via Grothendieck's notion of "Galois category", which we will not discuss since it is not really necessary for our purposes). As an illustration, once the definitions are made we will see quite easily that $G_{K,S} = \pi_1(\mathrm{Spec}\, \mathcal{O}_{K,S}, \mathrm{Spec}\,(\overline{K}))$ for any global field K and non-empty finite set S of places of K (containing the archimedean places).

(ii) For $X = \mathrm{Spec}\, k$, with k a field, it will turn out that the category of sheaves of sets for the étale topology on X is naturally equivalent to the category of discrete $\mathrm{Gal}(k_s/k)$-sets, and in particular that the category of abelian sheaves for the étale topology on X is equivalent to the category of discrete $\mathrm{Gal}(k_s/k)$-modules. This equivalence will identify the functors of global sections and $\mathrm{Gal}(k_s/k)$-invariants, so passing to derived functors will thereby identity the δ-functor $\mathrm{H}^\bullet_{\text{ét}}(k, \cdot)$ with Galois cohomology for k (using (iii) below).

This will provide an essential tool to compute the étale cohomology of curves using Brauer groups of function fields and the theory of abelian varieties. Many deep theorems in the foundations of étale cohomology involve ultimately reducing to computations in the cohomology of curves.

Curiously, the choice of k_s on the Galois side is not seen on the étale cohomology side! This explains in a very satisfying manner the classical observation that Galois cohomology is functorial in k without reference to k_s because conjugation on a profinite group induces a trivial effect on the associated group cohomology.

(iii) Let Γ be a profinite group. Consider $\mathrm{Mod}_{\mathrm{disc}}(\Gamma)$, the category of discrete Γ-modules. Often one makes the definition

$$\mathrm{H}^{\bullet}(\Gamma, M) = \varinjlim_{\text{open } \Gamma' \triangleleft \Gamma} \mathrm{H}^{\bullet}(\Gamma/\Gamma', M^{\Gamma'})$$

but then one has to argue by hand that it is a δ-functor in M (a property that is generally not true at any finite layer of the direct limit).

It is an instructive exercise to prove that this profinite group cohomology is an *erasable* δ-functor and moreover that $\mathrm{Mod}_{\mathrm{disc}}(\Gamma)$ has enough injectives (!), so by Grothendieck's theorem on erasable δ-functors it follows that the bare-hands δ-functor structure on profinite group cohomology must be the derived functor of $(\cdot)^{\Gamma}$. This makes Galois cohomology an instance of the theory of derived functors, underlying its link to étale cohomology over fields as in (ii).

3 Étale maps

We now start with the development of the mathematics. We first consider a construction that may initially look silly but is a gentle warm-up to the idea of the étale topology and will actually show up much later when we discuss the Artin comparison isomorphism relating étale and topological cohomology over $\mathbf{C}$.

Example 3.1. Let X be a topological space, and consider the category $\mathrm{Shv}(X)$ of sheaves of sets on X. The subcategory $\mathrm{Ab}(X)$ of abelian group objects is just the category of sheaves of abelian groups.

We define $X_{\text{ét}}$ to be the category of maps $(f \colon U \to X)$ with f a local homeomorphism, and morphisms in this category are commutative triangles

$$\begin{array}{ccc} U & \xrightarrow{\quad h \quad} & U' \\ & {\scriptstyle f}\searrow \qquad \swarrow{\scriptstyle f'} & \\ & X & \end{array}$$

(Note that h is then necessarily also a local homeomorphism.)

A *cover* of $(U \to X)$ is a collection of maps

$$\begin{array}{ccc} U_i & \xrightarrow{\quad h_i \quad} & U \\ & \searrow \qquad \swarrow & \\ & X & \end{array}$$

such that $\bigcup_i h_i(U_i) = U$. (Note that each $h_i(U_i)$ is open in U.)

Remark 3.2. For

$$\begin{array}{ccc} (U' \to X) & & (U'' \to X) \\ & \searrow \quad \swarrow & \\ & (U \to X) & \end{array}$$

in $X_{\text{ét}}$, the usual topological fiber product $U' \times_U U''(\subset U' \times U'')$ with its evident structure map to X (that is obviously a local homeomorphism) *is also* a fiber product in $X_{\text{ét}}$. That is, $X_{\text{ét}}$ admits fiber products in a very hands-on manner.

We define a *sheaf* to be a (contravariant) functor $\mathcal{F}\colon X_{\text{ét}} \to \text{Sets}$ such that for any cover $(U_i \xrightarrow{h_i} U)$ in $X_{\text{ét}}$ (suppressing the structure maps to X from the notation to keep things uncluttered),

$$\mathcal{F}(U) \to \prod_k \mathcal{F}(U_k) \rightrightarrows \prod_{(i,j)} \mathcal{F}(U_i \times_U U_j)$$

(whose pair of maps on the right sends $(s_k)_{k\in I}$ to $(\mathcal{F}(p_1)(s_i))_{(i,j)}$ and $(\mathcal{F}(p_2)(s_j))_{(i,j)}$ respectively) is an *equalizer diagram* of sets. That is, the first map is an injection whose image consists of (s_k) with the same image under both maps on the right.

We stress that in the product on the right we consider *all* ordered pairs (i, j). In ordinary sheaf theory, or more specifically when U is an open subset of X with U_i and U_j open subsets of U then the fiber product collapses to $U_i \cap U_j$ and we would usually avoid redundancy by not distinguishing (i, j) from (j, i) (and would generally drop the case $i = j$). However, in the fiber-product setup it is absolutely essential that we retain contant with all ordered pairs (i, j) and even $U_i \times_U U_i$ can be very different from U_i. The significance of this will become apparent later.

Let $\text{Shv}(X_{\text{ét}})$ be the category of such $\mathcal{F}$ (with natural transformations as morphisms). The following result is an easy exercise that the reader should check.

Proposition 3.3. *For $\mathcal{F} \in \text{Shv}(X)$, $\mathcal{F}_{\text{ét}} : X_{\text{ét}} \to \text{Set}$ defined by $\mathcal{F}_{\text{ét}}(U \xrightarrow{h} X) := \Gamma(U, h^*(\mathcal{F}))$ is a sheaf and $\mathcal{F} \rightsquigarrow \mathcal{F}_{\text{ét}}$ is an equivalence $\text{Shv}(X) \to \text{Shv}(X_{\text{ét}})$ with quasi-inverse $\mathcal{G} \rightsquigarrow \mathcal{G}|_X$.*

Proposition 3.3 may make the introduction of $X_{\text{ét}}$ look completely absurd, but it is not a total waste of time. Later when we relate étale cohomology for finite type $\mathbf{C}$-schemes X and topological cohomology for the space of $\mathbf{C}$-points, sheaves for the étale topology on the scheme side will "analytify" to objects in $\text{Shv}(X(\mathbf{C})_{\text{ét}})$ (and so Proposition 3.3 will provide the link back to cohomology on $X(\mathbf{C})$).

We're now going to define an algebraic analogue of "local homeomorphism", which is the notion of "étale map". (The best introductory reference on this notion is Chapter 2 of the book *Néron Models*, which provides precise arguments and references to

EGA for various results stated below without proof.) There are many equivalent definitions of étale morphism, and perhaps the most concrete for getting started is in the spirit of the inverse function theorem:

Definition 3.4. A map of schemes $f\colon X \to S$ is *étale* if Zariski-locally on X it has the form

$$\begin{array}{ccc} \operatorname{Spec} B & \hookrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \hookrightarrow & S \end{array}$$

where $B = A[T_1, \ldots, T_n]/(f_1, \ldots, f_n)[1/J_f]_h$ for $J_f := \det(\partial_{x_j} f_i)$ and some element $h \in A[T_1, \ldots, T_n]$.

This definition has some deficiencies because it is too explicit; one would also like to know some further properties and characterizations of this notion applicable to abstract situations. In the special case that $S = \operatorname{Spec}(k)$ for a a field k it is clear from the classical description of smooth schemes over algebraically closed fields via a Jacobian criterion that an étale k-scheme has geometric fibers that are smooth of dimension 0, which is to say are a disjoint union of copies of $\operatorname{Spec}(\overline{k})$.

Hence, an étale k-scheme is necessarily $\coprod \operatorname{Spec}(k_i)$ for finite separable extension fields k_i/k, and by the primitive element theorem all such disjoint unions are k-étale. In particular, since étaleness as defined above is visibly preserved by any base change we see that the actual fibers X_s (not just geometric fibers!) of an étale morphism $X \to S$ are given as a disjoint union of spectra of finite separable extensions of $k(s)$. To go further, we need a deep result:

Theorem 3.5 (Structure theorem for étale morphisms)**.** *If $f\colon X \to S$ is étale, then for each $x \in X$ there are open neighborhoods around x and $f(x)$ on which f restricts to a map of the form*

$$\operatorname{Spec}(A[T]/(g)[1/g']_h) \to \operatorname{Spec} A$$

for some monic $g \in A[T]$.

Note that this immediately implies by inspection that an étale f is flat (a fact that is not at all obvious from the initial definition), since $A[T]/(g)$ is free over A for g monic. The proof of the structure theorem lies quite deep, as in rests on Zariski's Main Theorem (and really that quasi-finite separated maps factor locally on the base as an open immersion into a scheme finite over the base; finiteness allows one to access Nakayama's Lemma to boostrap from the primitive element theorem on fibers). With flatness and the fibral description in hand, we get immediately via Nakayama's Lemma on fibers the result:

Corollary 3.6. *If $f\colon X \to S$ is étale then f is flat and locally of finite presentation with discrete geometric fibers and $\Omega^1_{X/S} = 0$. Moreover, f is open.*

The openness of étale maps is a special case of the general fact that flat maps locally of finite presentation are open. The following result summarizes the crucial equivalences:

Theorem 3.7. *The following conditions on a map $f\colon X \to S$ are equivalent.*

(i) *f is étale.*

(ii) *f is flat, locally finitely presented, and $\Omega^1_{X/S} = 0$.*

(iii) *f is locally finitely presented and formally étale (i.e., $X(R) \to X(R/J)$ is bijective for any* $\mathrm{Spec}\,(R) \to S$ *and ideal $J \subset R$ satisfying $J^2 = 0$).*

For (ii), in the presence of being flat and locally of finite presentation, the vanishing of $\Omega^1_{X/S}$ is equivalent to the vanishing of each $\Omega^1_{X_s/k(s)}$ due to Nakayama's Lemma over each local ring on X. The vanishing of this fibral Ω^1 is equivalent to its vanishing on the geometric fiber at s, where it expresses that $X_{\overline{s}}$ is "discrete" (i.e., a disjoint union of copies of $\overline{s}$).

Given what we have already discussed, the hard part is to show that (iii) implies flatness. First one shows (by considerations on geometric fibers, where formal étaleness is easily seen to imply discreteness) that (iii) implies quasi-finiteness, and then leverages Zariski's Main Theorem and infinitesimal considerations to deduce flatness.

Remark 3.8. The property of a map of schemes being locally finitely presented has a *functorial* characterization [EGA IV$_3$, 8.14.2]: $\varinjlim X(R_i) \xrightarrow{\sim} X(\varinjlim R_i)$ for any inverse system of affine schemes $\{\mathrm{Spec}\ R_i \to S\}$ over S. (The proof of this characterization involves a very clever trick, but it is not deep as the above results resting on Zariski's Main Theorem.) Thus, condition (iii) in Theorem **??** is characterized entirely in terms of the functor of points, and as such provides the standard method for checking étaleness of maps between abstract schemes (such as in the consideration of moduli schemes).