Cohomology with Proper Supports and Ehresmann's Theorem

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1 Topological Motivation

Let $f : X \to S$ be a map of locally compact Hausdorff topological spaces. If $\mathcal{F}$ is a sheaf on X, then

$$(f_!\mathcal{F})(U) = \left\{ s \in \Gamma(f^{-1}(U), \mathcal{F}) : \mathrm{supp}(s) \text{ is proper over } U \right\}$$

for any open $U \subseteq S$. One can verify by hand that if f is an open immersion (i.e., inclusion of an open subset), then this is the same as extension by zero. Note that when f is proper, $f_! = f_*$. Note also that in general $(g \circ f)_! = g_! \circ f_!$.

If f has a compactification, i.e., if f can be factorized as

$$\begin{array}{ccc} X & \overset{j}{\hookrightarrow} & \overline{X} \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \overline{f}} \\ & & S \end{array}$$

where j is an open immersion and $\overline{f}$ is proper, then one has $f_! = \overline{f}_* \circ j_!$. Since $j_!$ sends soft sheaves[1] to soft sheaves [3, Ch.III, 7.2] and is exact [3, Ch.II, 6.3],

$$\mathbf{R}f_! \simeq \mathbf{R}\overline{f}_* \circ \mathbf{R}j_! \simeq \mathbf{R}\overline{f}_* \circ j_!. \tag{1.1}$$

2 Cohomology with Proper Supports

In our étale cohomology setting, however, $j_!$ no longer sends injectives to injectives, and the isomorphism (1.1) may not hold. Consider the following example:

Example 2.1. Consider $f : \mathbf{A}^1_{\mathbf{C}} \to \mathbf{C}$. A closed subset of $\mathbf{A}^1_{\mathbf{C}}$ is either the entire line $\mathbf{A}^1_{\mathbf{C}}$ or a finite set of closed points, so for any sheaf $\mathcal{F}$ on $\mathbf{A}^1_{\mathbf{C}}$, if a section $s \in \Gamma(\mathbf{A}^1_{\mathbf{C}}, \mathcal{F})$ has proper support, then $\mathrm{supp}(s)$ must be a finite set. Therefore,

$$\Gamma_c(\mathbf{A}^1_{\mathbf{C}}, \bullet) = \bigoplus_{a \in \mathbf{A}^1(\mathbf{C})} \Gamma_{\{a\}}(\mathbf{A}^1_{\mathbf{C}}, \bullet)$$

($\Gamma_{\{a\}}$ will be defined in Lecture 11). If $R^\nu\Gamma_c$ is the derived functor of Γ_c, then

$$R^2\Gamma_c(\mathbf{A}^1_{\mathbf{C}}, \mathbf{Z}/n\mathbf{Z}) = \bigoplus_{a \in \mathbf{A}^1(\mathbf{C})} H^2_{\{a\}}(\mathbf{A}^1_{\mathbf{C}}, \mathbf{Z}/n\mathbf{Z}) = \bigoplus_{a \in \mathbf{A}^1(\mathbf{C})} (\mathbf{Z}/n\mathbf{Z})$$

which is highly infinite (the statement $H^2_{\{a\}}(\mathbf{A}^1_{\mathbf{C}}, \mathbf{Z}/n\mathbf{Z}) = \mathbf{Z}/n\mathbf{Z}$ follows from the relative purity theorem to be proved in Lecture 11). On the other hand, for the compactification $j : \mathbf{A}^1_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$, $H^2_{\text{ét}}(\mathbf{P}^1_{\mathbf{C}}, j_!(\mathbf{Z}/n\mathbf{Z}))$ is finite (Theorem 3.1 in Lecture 4).

[1] A sheaf $\mathcal{F}$ on a locally compact space X is *soft* if for any compact subset K and any section s defined in some neighborhood U of K, there is a section t over X which agrees with s in a neighborhood V of K (possibly smaller than U). An injective sheaf is flabby [3, Ch.II, 7.3], a flabby sheaf is soft, and a soft sheaf is acyclic for $f_!$ [3, Ch.VII, 1.6].

Instead of using derived functors as the definition of $\mathbf{R}f_!$, we will use the RHS of (1.1). For any scheme X, denote by $\mathbf{D}^+(X)$ the derived category of bounded below complexes of abelian sheaves, and by $\mathbf{D}^+(X, \mathrm{tor})$ the full subcategory of $\mathbf{D}^+(X)$ containing complexes $\mathcal{F}^\bullet$ whose cohomologies $\mathcal{H}^\nu(\mathcal{F}^\bullet)$ are torsion.

Definition 2.2. Suppose $f : X \to S$ is separated and finite type, and S is qcqs. For any compactification (i.e., a factorization of f into

$$\begin{array}{ccc} X & \xhookrightarrow{j} & \overline{X} \\ & \searrow^{f} & \downarrow{\scriptstyle \overline{f}} \\ & & S \end{array} \tag{2.1}$$

where j is an open immersion and $\overline{f}$ is proper), we define $\mathbf{R}f_! : \mathbf{D}^+(X, \mathrm{tor}) \to \mathbf{D}^+(S, \mathrm{tor})$ by

$$\mathbf{R}f_! = \mathbf{R}\overline{f}_* \circ j_!.$$

If $\mathcal{F}$ is a torsion sheaf on X, then we define $R^\nu f_! \mathcal{F} = \mathcal{H}^\nu(\mathbf{R}f_!\mathcal{F})$ (or equivalently, $R^\nu f_! = R^\nu \overline{f}_* \circ j_!$). If S is the spectrum of a separably closed field, then we define

$$\mathbf{R}\Gamma_c(X, \mathcal{F}) = \Gamma(S, \mathbf{R}f_!\mathcal{F}) \quad \text{and} \quad H^\nu_{c,\text{ét}}(X, \mathcal{F}) = H^\nu\left(\mathbf{R}\Gamma_c(X, \mathcal{F})\right) = \Gamma(S, R^\nu f_!\mathcal{F}).$$

Moreover, we write $f_! = R^0 f_!$ and $\Gamma_c = H^0_{c,\text{ét}}$.

Proposition 2.3. *In the above definition, $\mathbf{R}f_!\mathcal{F}^\bullet$ is well-defined up to a unique isomorphism (independent of any auxiliary choice of compactifications).*

Proof Sketch. The details regarding independence of compactifications are discussed in [2, Ch.I, §8] (from the paragraph after Lemma 8.4 to Definition 8.6). The compactification (2.1) exists because of Nagata compactification theorem.

The independence of compactifications depends on the following lemma.

Lemma 2.4 ([2, Ch.I, 8.5])**.** *Suppose we have a commutative diagram*

$$\begin{array}{ccc} X & \xhookrightarrow{i} & \overline{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ Y & \xhookrightarrow{j} & \overline{Y} \end{array} \tag{2.2}$$

where i and j are open embeddings and f is proper. There is a natural transformation $j_! f_* \to \overline{f}_* i_!$ inducing a natural transformation

$$j_! \circ \mathbf{R}f_* \to \mathbf{R}\overline{f}_* \circ i_! \tag{2.3}$$

between functors $\mathbf{D}^+(X, \mathrm{tor}) \to \mathbf{D}^+(\overline{Y}, \mathrm{tor})$. If $\overline{f}$ is proper, then (2.3) *is in fact an isomorphism.*

Here is a rough sketch of the proof of Lemma 2.4: in the case where the square (2.2) is cartesian, the natural transformation is

$$j_! f_* \longrightarrow \overline{f}_* \overline{f}^* j_! f_* \longrightarrow \overline{f}_* i_! f^* f_* \longrightarrow \overline{f}_* i_!$$

where the first and the third maps are adjunction while the second map is the compatibility of base change and extension-by-zero. In this special case the isomorphism of (2.3) can be proved by proper base change. In the general case one can factorize i and f by

$$\begin{array}{ccccc} X & \xhookrightarrow{i_1} & \overline{X} \times_{\overline{Y}} Y & \xhookrightarrow{i_2} & \overline{X} \\ & \searrow^{f} & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle \overline{f}} \\ & & Y & \xhookrightarrow{j} & \overline{Y} \end{array} \tag{2.4}$$

and apply the result in the previous case to the Cartesian square in (2.4).

Now we go back to the proof of independence of compactifications. Suppose we have two compactifications:

$$\begin{array}{ccc} X & \xhookrightarrow{j_1} & \overline{X}_1 \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \overline{f}_1} \\ & & S \end{array} \qquad \begin{array}{ccc} X & \xhookrightarrow{j_2} & \overline{X}_2 \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \overline{f}_2} \\ & & S \end{array}$$

and we are trying to show that they yield isomorphic $\mathbf{R}f_!$. One can get a third one which dominates both of them:

$$\begin{array}{ccc} & & \overline{X}_3 \\ & {\scriptstyle j_3}\nearrow & \downarrow{\scriptstyle g} \\ X & \xhookrightarrow{j_i} & \overline{X}_i \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \overline{f}_i} \\ & & S \end{array} \qquad \overline{f}_3 : \overline{X}_3 \to S$$

and use Lemma 2.4 to conclude that $(j_i)_! \simeq \mathbf{R}g_* \circ (j_3)_!$ and hence

$$\mathbf{R}(\overline{f}_1)_* \circ (j_1)_! \simeq \mathbf{R}(\overline{f}_1)_* \circ \mathbf{R}g_* \circ (j_3)_! \simeq \mathbf{R}(\overline{f}_3)_* \circ (j_3)_! \simeq \mathbf{R}(\overline{f}_2)_* \circ \mathbf{R}g_* \circ (j_3)_! \simeq \mathbf{R}(\overline{f}_2)_* \circ (j_2)_!.$$

The construction of this isomorphism depends on the choice of this "dominating compactification", but one can prove that the isomorphism obtained is independent of the choice of this "dominating compactification" by picking another compactification which dominates everything to reduce to special cases. □

Remark. If f is étale, by Zariski's main theorem one can pick a compactification (2.1) with $\overline{f}$ finite. Since $\overline{f}_*$ is exact in this case (Lemma 2.4.2 in Lectures 7 − 8), one has $\mathbf{R}f_! = \overline{f}_* \circ j_!$ and one can easily see that this coincides with the definition of $f_!$ before (§3.3 in Lecture 2).

Using the uniqueness in the definition, we can extend the definition to the case where S is not qcqs: cover S by affine opens U_i and define $\mathbf{R}f_!$ on each U_i, and we can glue the constructions together due to the uniqueness.

Below is a list of properties of the higher proper pushforwards:

Theorem 2.5 (see [2, Ch.I, 8.7] for compactifiable case)**.** *(1) (Base Change) If we have a Cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

where f is finite type and separated, then there is a natural isomorphism $g^* \circ \mathbf{R}f_! \simeq \mathbf{R}(f')_! \circ h^*$.

(2) (Composition) If $f_1 : X \to Y$ and $f_2 : Y \to Z$ are finite type and separated, then there is a natural isomorphism $\mathbf{R}(f_2 \circ f_1)_! \simeq \mathbf{R}(f_2)_! \circ \mathbf{R}(f_1)_!$. This gives a Leray spectral sequence

$$E_2^{pq} = R^p(f_2)_! \circ R^q(f_1)_! \Rightarrow R^{p+q}(f_2 \circ f_1)_!.$$

(3) (Excision) Suppose $f : X \to S$ is finite type and separated, and $\mathcal{F}^\bullet \in \mathbf{D}^+(X, \mathrm{tor})$. Suppose Z is a closed subscheme of X and $U = Z \backslash X$. Let $f_Z : Z \to S$ and $f_U : U \to S$ be the corresponding restrictions of f. Then, there is a long exact excision sequence

$$\cdots \longrightarrow \mathbf{R}^\nu(f_U)_!(\mathcal{F}^\bullet|_U) \longrightarrow \mathbf{R}^\nu f_! \mathcal{F}^\bullet \longrightarrow \mathbf{R}^\nu(f_Z)_!(\mathcal{F}^\bullet|_Z) \longrightarrow \mathbf{R}^{\nu+1}(f_U)_!(\mathcal{F}^\bullet|_U) \longrightarrow \cdots.$$

In addition, the sequence is compatible with base change.

Proof Sketch. Roughly speaking, one constructs the relevant maps and isomorphisms in the qcqs case using compactifications, verifies that the constructions are independent of these auxiliary stuff and then glues these constructions to extend to the general case. We will only give a rough sketch for the qcqs case.

For (1), suppose f has a compactification (2.1). Then $\mathbf{R}\overline{f}_*$ is compatible with base change by proper base change, and one can check that $j_!$ is compatible with base change by hand.

For (2), suppose

$$\begin{array}{ccccc} X & \xhookrightarrow{j_1} & \overline{X} & \xhookrightarrow{i} & W \\ & {\scriptstyle f_1}\searrow & \downarrow{\scriptstyle \overline{f}_1} & & \downarrow{\scriptstyle g} \\ & & Y & \xhookrightarrow{j_2} & \overline{Y} \\ & & & {\scriptstyle f_2}\searrow & \downarrow{\scriptstyle \overline{f}_2} \\ & & & & Z \end{array}$$

where $\overline{f}_1 \circ j_1$ is a compactification of f_1, $\overline{f}_2 \circ j_2$ is a compactification of f_2 and $g \circ i$ is a compactification of $j_2 \circ \overline{f}_1$. Then Lemma 2.4 says $(j_2)_! \circ \mathbf{R}(\overline{f}_1)_* \simeq \mathbf{R}g_* \circ i_!$, which implies the result.

For (3), let $i : Z \to X$ and $j : U \to X$ be the inclusions. If $\mathcal{F}$ is a sheaf on X, then we have an exact sequence

$$0 \longrightarrow j_!j^*\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_*i^*\mathcal{F} \longrightarrow 0 \tag{2.5}$$

and the result follows from applying $R^\bullet f_!$. □

3 A Vanishing Result

In this section we will prove

Theorem 3.1. *Suppose $f : X \to S$ is separated and finite type, and let $d = \sup_{s\in S} \dim X_s$. If $\mathcal{F}$ is a torsion sheaf on X, then*

$$R^\nu f_!\mathcal{F} = 0 \quad \textit{for } \nu > 2d. \tag{3.1}$$

In general, if $\mathcal{F}^\bullet \in \mathbf{D}^+(X, \mathrm{tor})$ and $\mathcal{H}^\nu(\mathcal{F}^\bullet) = 0$ for $\nu \geq r$, then

$$\mathbf{R}^\nu f_!\mathcal{F}^\bullet = 0 \quad \textit{for } \nu \geq r + 2d. \tag{3.2}$$

Proof. We wish to show that, for suitable ν, $\mathbf{R}^\nu f_!\mathcal{F}^\bullet$ has zero stalk everywhere, so by the compatibility of base change with higher proper pushforwards we may assume $S = \mathrm{Spec}(\Omega)$, where Ω is a separably closed field. In particular, S is noetherian and so is X (since f is finite type), so by noetherian induction X has a stratification by affine subschemes, and by excision it suffices to prove the result for each of these subschemes. Therefore, we may assume X is affine as well.

By Noether normalization lemma we can factorize f as

$$X \xrightarrow{g} \mathbf{A}^d_S \longrightarrow \mathbf{A}^{d-1}_S \longrightarrow \cdots \longrightarrow S,$$

where g is finite, so by compatibility of composition and higher proper pushforwards, it suffices to deal with the finite case and the case of an affine line. Clearly (3.1) implies (3.2), so we will just prove (3.1). For the case f is finite, we wish to show that f_* is exact, but this is just Lemma 2.4.2 in Lectures 7 – 8. For the case of affine line, we again base change to separably closed fields Ω, and we wish to show $H^\nu_{c,\text{ét}}(\mathbf{A}^1_\Omega, \mathcal{F}) = 0$. Using the obvious compactification $\mathbf{P}^1_\Omega$, the result follows from Theorem 3.1 in Lecture 4, which implies that $H^\nu_{\text{ét}}(\mathbf{P}^1_\Omega, \mathcal{G}) = 0$ for $\nu \geq 3$ and any torsion abelian sheaf $\mathcal{G}$ on $\mathbf{P}^1_\Omega$. □

4 Ehresmann's Fibration Theorem in Topology

The Ehresmann's fibration theorem asserts that

Theorem 4.1. *Every proper surjective submersion $f : X \to S$ between manifolds is a $\mathcal{C}^\infty$ fiber bundle.*

Locally around a point $s \in S$, say for some neighborhood U of s, $f^{-1}(U)$ looks like $f^{-1}(s) \times U$. In particular, if $B_1 \supseteq B_2$ are open balls around s contained in U, then for any constant sheaf $\mathcal{F}$ the restriction maps

$$H^\nu(f^{-1}(B_1), \mathcal{F}) \longrightarrow H^\nu(f^{-1}(B_2), \mathcal{F}) \longrightarrow H^\nu(f^{-1}(s), \mathcal{F})$$

are isomorphisms, so we have isomorphisms $(R^\nu f_*\mathcal{F})_s \simeq H^\nu(f^{-1}(s), \mathcal{F})$ induced by the restriction maps. If B_1 contains another point s', then the two restriction maps

$$H^\nu(f^{-1}(s), \mathcal{F}) \xleftarrow{\sim} H^\nu(f^{-1}(B_1), \mathcal{F}) \xrightarrow{\sim} H^\nu(f^{-1}(s'), \mathcal{F})$$

identify the stalks $(R^\nu f_*\mathcal{F})_s$ and $(R^\nu f_*\mathcal{F})_{s'}$, and the identification does not change even if we replace B_1 by another open ball containing both points. Thus we have

Proposition 4.2. *For f as in Theorem 4.1, $R^\nu f_*\mathcal{F}$ is locally constant for any constant sheaf $\mathcal{F}$ on X.*

5 Finiteness Theorem and Analogue of Ehresmann's Theorem

In this section, we will prove the following analogue of Proposition 4.2:

Theorem 5.1. *Let $f : X \to S$ be smooth and proper, and let $\mathcal{F}$ be an lcc abelian sheaf on X whose torsion order is invertible on S. Then, $R^\nu f_*\mathcal{F}$ is lcc.*

In addition, we will prove the following finiteness result:

Theorem 5.2. *Let $f : X \to S$ be finite type and separated, and let $\mathcal{F}$ be a constructible abelian sheaf on X whose torsion order is invertible on S. Suppose S is noetherian. Then, $R^\nu f_!\mathcal{F}$ is constructible.*

Before beginning the proof, we mention the following criterion for lcc sheaves.

Lemma 5.3. *For a sheaf $\mathcal{F}$ over a noetherian scheme S, the following are equivalent:*

(1) $\mathcal{F}$ is lcc.

(2) $\mathcal{F}$ is constructible and all specialization maps are isomorphisms.

(3) All stalks of $\mathcal{F}$ are finite and all specialization maps are isomorphisms.

The equivalence of (1) and (2) is Theorem 4.1 in Lecture 3, and the proof of the equivalence of (1) and (3) is essentially the same and will be omitted.

In the course of the proof, we will need the following lemma:

Lemma 5.4. *Let $f : X \to S$ be smooth and proper, and let $\mathcal{F}$ be an lcc abelian sheaf on X whose torsion order is invertible on S. Suppose S is noetherian. Then, all specialization maps for $R^\nu f_*\mathcal{F}$ are isomorphisms.*

Proof of Lemma 5.4. Let $\overline{s}$ and $\overline{\eta}$ be geometric points of S where $\overline{s}$ is a specialization of $\overline{\eta}$, and let s and η be the respective underlying physical points. This means that $\overline{\eta}$ factors through $\mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{S,\overline{s}})$ (the second paragraph of §4 in Lecture 3), so by base-changing S to $\mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{S,\overline{s}})$ we may assume S is the spectrum of a strictly henselian local ring, and $s = \overline{s}$ is the closed point. Let $i : X_s \to X$ and $j : X_{\overline{\eta}} \to X$ be the respective base changes of the geometric points s and η to X. Since $\Gamma(X_{\overline{\eta}}, \mathcal{G}_{\overline{\eta}}) \simeq \Gamma(X, j_*j^*\mathcal{G})$ naturally in $\mathcal{G}$, we have $\mathbf{R}\Gamma(X_{\overline{\eta}}, \mathcal{F}_{\overline{\eta}}) \simeq \mathbf{R}\Gamma(X, \mathbf{R}j_*j^*\mathcal{F})$ and hence a Leray spectral sequence:

$$E_2^{pq} = H^p_{\text{ét}}(X, R^q j_*j^*\mathcal{F}) \Rightarrow H^{p+q}_{\text{ét}}(X_{\overline{\eta}}, \mathcal{F}_{\overline{\eta}}).$$

By proper base change, $H^p_{\text{ét}}(X, R^q j_* j^* \mathcal{F}) \simeq H^p_{\text{ét}}(X_s, i^* R^q j_* j^* \mathcal{F})$, so the spectral sequence can be rewritten as

$$E_2^{pq} = H^p_{\text{ét}}(X_s, i^* R^q j_* j^* \mathcal{F}) \Rightarrow H^{p+q}_{\text{ét}}(X_{\overline{\eta}}, \mathcal{F}_{\overline{\eta}}). \tag{5.1}$$

Now consider

$$H^n_{\text{ét}}(X_s, i^* \mathcal{F}) \longrightarrow H^n_{\text{ét}}(X_s, i^* j_* j^* \mathcal{F}) \longrightarrow H^n_{\text{ét}}(X_{\overline{\eta}}, \mathcal{F}_{\overline{\eta}}), \tag{5.2}$$

where the first map is induced by the adjunction $\mathcal{F} \to j_* j^* \mathcal{F}$, and the second is the edge homomorphism for the spectral sequence (5.1). One can verify that the composition is equal to the specialization map, so it suffices to show that both maps in (5.2) are isomorphisms. Therefore, we need to prove

- the adjunction map $i^* \mathcal{F} \to i^* j_* j^* \mathcal{F}$ is an isomorphism; and
- for $q > 0$, $i^* R^q j_* j^* \mathcal{F} = 0$ (so that the spectral sequence (5.1) only has one non-zero row).

Equivalently, we want to prove that, for any geometric point $\overline{x}$ of X_s,

- the adjunction map $(i^* \mathcal{F})_{\overline{x}} \to (i^* j_* j^* \mathcal{F})_{\overline{x}}$ of stalks is an isomorphism; and
- for $q > 0$, the stalk $(i^* R^q j_* j^* \mathcal{F})_{\overline{x}} = 0$.

Via the map $i : X_s \to X$, we may think of $\overline{x}$ as geometric points of X above s, and now we want to show that $\mathcal{F}_{\overline{x}} \to (j_* j^* \mathcal{F})_{\overline{x}}$ is an isomorphism and, for $q > 0$, $(R^q j_* j^* \mathcal{F})_{\overline{x}} = 0$. Suppose that, étale locally around $\overline{x}$ in X, $\mathcal{F} = \underline{M}$ for some finite abelian group M. Let $U = \operatorname{Spec}(\mathcal{O}^{\text{sh}}_{X,\overline{x}})$, and consider the diagram

$$\begin{array}{ccccc} & & U_{\overline{\eta}} & \xrightarrow{\varphi_{\overline{\eta}}} & X_{\overline{\eta}} \\ & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j} \\ \overline{x} & \longrightarrow & U & \xrightarrow{\varphi} & X \end{array}$$

in which the square is Cartesian. By smooth base change, for $q \geq 0$,

$$\varphi^* R^q j_* j^* \mathcal{F} \simeq R^q (j')_* (\varphi_{\overline{\eta}})^* j^* \mathcal{F} \simeq R^q (j')_* \underline{M}_{U_{\overline{\eta}}} \quad \Rightarrow \quad (R^q j_* j^* \mathcal{F})_{\overline{x}} \simeq H^q_{\text{ét}}(U_{\overline{\eta}}, \underline{M}_{U_{\overline{\eta}}}).$$

Now we just have to prove that, for each direct summand $\mathbf{Z}/n\mathbf{Z}$ of M,

- the canonical map $\mathbf{Z}/n\mathbf{Z} \to H^0_{\text{ét}}(U_{\overline{\eta}}, \mathbf{Z}/n\mathbf{Z})$ is an isomorphism; and
- for $q > 0$, $H^q_{\text{ét}}(U_{\overline{\eta}}, \mathbf{Z}/n\mathbf{Z}) = 0$.

In other words, it is enough to show that $X \to S$ is locally acyclic (Definition 2.2.4 in Lectures 7 – 8), but this just follows from Theorem 3.3.1 in Lectures 7 – 8. □

Proof of Theorem 5.2. We will proceed in three steps.

Case 1: f is finite. By Noetherian induction and base change it suffices to prove the statement after base-changing to some étale $S' \to S$. In particular, by base-changing to some non-empty open subsets we may assume S is irreducible. We may also assume f is surjective because otherwise we can just base change to $S \backslash f(X)$. Let $\overline{\eta}$ be a geometric point above the generic point η of S, and consider the base change $f_{\overline{\eta}} : X_{\overline{\eta}} \to \overline{\eta}$ of f to $\overline{\eta}$ (note that $X_{\overline{\eta}}$ is non-empty by the surjectivity assumption). This map $f_{\overline{\eta}}$ is a finite map to the spectrum of a separably closed field, and such maps factor as

$$X_{\overline{\eta}} \xrightarrow{g_{\overline{\eta}}} \overline{\eta} \sqcup \overline{\eta} \sqcup \cdots \sqcup \overline{\eta} \xrightarrow{h_{\overline{\eta}}} \overline{\eta}$$

where $g_{\overline{\eta}}$ is a universal homeomorphism and $h_{\overline{\eta}}$ is the obvious map mapping each connected component $\overline{\eta}$ to $\overline{\eta}$ by identity (informally speaking, $g_{\overline{\eta}}$ is the "purely inseparable" part while $h_{\overline{\eta}}$ is the "separable" part of $f_{\overline{\eta}}$). One can descend this factorization to some étale neighborhood $S' \to S$ around $\overline{\eta}$ and get

$$X \times_S S' \xrightarrow{g} S' \sqcup S' \sqcup \cdots \sqcup S' \xrightarrow{h} S'$$

where g is a universal homeomorphism and h is the obvious map. By the previous discussion we may replace S with S' and hence it suffices to prove the statement for g and h separately. For g, this follows from Remark 3.4.5 in Lecture 2. For h this is even more trivial: if S'_i denotes the i^{th} copy of S' in $S' \sqcup S' \sqcup \cdots \sqcup S'$, then

$$h_*\mathcal{F} \simeq \bigoplus_i \mathcal{F}|_{S'_i}.$$

Case 2: X/S is a smooth curve (i.e., f is smooth with fibers of pure dimension 1). By base change, we may assume S is affine. We first show that there exists a resolution

$$\cdots \longrightarrow \mathcal{G}^{-2} \longrightarrow \mathcal{G}^{-1} \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{F} \longrightarrow 0,$$

where each $\mathcal{G}^q$ is a finite direct sum of sheaves of the form $g_!\underline{M}_{\widetilde{X}}$ such that $g : \widetilde{X} \to X$ is étale and $M = \mathbf{Z}/m\mathbf{Z}$ for some m invertible on S. We have to find a finite direct sum of this form mapping surjectively to any constructible $\mathcal{F}$. It is clear this can be done when $\mathcal{F}$ is locally constant constructible, and now we can just apply noetherian induction and the constructibility of $\mathcal{F}$ to construct such a finite direct sum.

We claim that with such a resolution, we have a spectral sequence

$$E_1^{pq} = R^p f_! \mathcal{G}^q \Rightarrow R^{p+q} f_! \mathcal{F}. \tag{5.3}$$

To see this, let $\mathcal{I}^{\bullet\bullet}$ be a Cartan-Eilenberg resolution of $\mathcal{G}^\bullet$, so in particular for each q,

$$0 \longrightarrow \mathcal{G}^q \longrightarrow \mathcal{I}^{0q} \longrightarrow \mathcal{I}^{1q} \longrightarrow \mathcal{I}^{2q} \longrightarrow \cdots$$

is an injective resolution. Consider the double complex $\mathcal{C}^{\bullet\bullet}$ defined by

$$\mathcal{C}^{pq} = \begin{cases} \mathcal{I}^{pq} & \text{if } p < 2 \\ \ker(\mathcal{I}^{2q} \to \mathcal{I}^{3q}) & \text{if } p = 2 \\ 0 & \text{if } p > 2 \end{cases}$$

Now we use the row and column filtrations to compute $\mathbf{R}^\bullet f_!(\mathrm{Tot}(\mathcal{C}))$. If we use the column filtration, we get

$$E_2^{pq} = \begin{cases} R^p f_! \mathcal{F} & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases} \Rightarrow \mathbf{R}^{p+q} f_!(\mathrm{Tot}(\mathcal{C})).$$

Since the second page only has one non-zero row, $R^p f_! \mathcal{F} \simeq \mathbf{R}^p f_!(\mathrm{Tot}(\mathcal{C}))$. If we now use the row filtration, since $R^p f_! \mathcal{G}^q = 0$ for $p > 2$ (Theorem 3.1), we get

$$E_1^{pq} = R^p f_! \mathcal{G}^q \Rightarrow \mathbf{R}^{p+q} f_!(\mathrm{Tot}(\mathcal{C})) \simeq R^{p+q} f_! \mathcal{F},$$

which is precisely the spectral sequence we want.

In the spectral sequence (5.3), the first page only has three non-zero columns, so it suffices to prove that all terms in the first page are constructible. Since $\mathcal{G}^q$ is a finite direct sum of sheaves of the form $g_!\underline{M}_{\widetilde{X}}$, it suffices to prove that

$$R^p f_!(g_! \underline{M}_{\widetilde{X}} \simeq R^p (f \circ g)_! \underline{M}_{\widetilde{X}}$$

is constructible, so we may replace X by $\widetilde{X}$ and $\mathcal{F}$ by $\underline{M}_{\widetilde{X}}$. In other words, we assume $\mathcal{F}$ is constant.

Using noetherian induction on S and base change, we can assume $S = \mathrm{Spec}(A)$ where A is an integral domain. Moreover, we may further

- base-change S to some dense open; or
- base-change S by some universal homeomorphism $T \to S$ with T noetherian.

We claim that, by doing so, we can get a compactification

$$\begin{array}{ccc} X & \xhookrightarrow{j} & \overline{X} \\ & \underset{f}{\searrow} & \downarrow{\scriptstyle \overline{f}} \\ & & S \end{array}$$

where j is an open immersion and $\overline{f}$ is a smooth projective curve, such that $\overline{X} \backslash X \to S$ is finite. If L is the perfect closure of the generic residue field $\kappa(\eta)$ of S, then clearly $X_L \to L$ has a compactification $X_L \to \overline{X}_L \to L$ given by the regular completion $\overline{X}_L$ of X_L. We can then descend this to $X_K \to \overline{X}_K \to K$ for some finite purely inseparable extension $K/\kappa(\eta)$, spread this construction out to some dense open U of S and absorb the field extension $K/\kappa(\eta)$ into some universal homeomorphism $T \to S$.

Since $\overline{X} \backslash X \to S$ is finite, the corresponding statement for $\overline{X} \backslash X$ is covered by case 1. Since the sheaf $\underline{M}_{\overline{X}}$ on $\overline{X}$ restricts to $\underline{M}_X = \mathcal{F}$ on X, by excision, it suffices to prove that $R^\nu \overline{f}_! \underline{M}_{\overline{X}} = R^\nu \overline{f}_* \underline{M}_{\overline{X}}$ is constructible. In fact, we will prove that $R^\nu \overline{f}_* \underline{M}_{\overline{X}}$ is lcc.

Since $\overline{f}$ is smooth and projective, we can apply Lemma 5.4 and conclude that the specialization maps for $R^\nu \overline{f}_* \underline{M}_{\overline{X}}$ are isomorphisms. By Lemma 5.3, it remains to show that it has finite stalks: if $\alpha : \mathrm{Spec}(\Omega) \to S$ is a geometric point, then

$$(R^\nu \overline{f}_* \underline{M}_{\overline{X}})_\alpha = H^\nu_{\text{ét}}(\overline{X}_\Omega, \underline{M}_{\overline{X}_\Omega}),$$

and it is finite by Theorem 3.1 in Lecture 4.

Case 3: general case. By base change and excision we may assume X and S are affine, and then one can factorize $f : X \to S$ as

$$X \xrightarrow{g} \mathbf{A}^n_S \longrightarrow \mathbf{A}^{n-1}_S \longrightarrow \mathbf{A}^{n-2}_S \longrightarrow \cdots \longrightarrow S,$$

where g is a closed immersion, so it suffices to prove the statement for each of these maps in the factorization. However, g is finite (case 1) and all others are smooth curves (case 2). $\square$

Proof of Theorem 5.1. One can write $\mathcal{F} = \underline{X'}$ for some finite étale $X' \to X$, and by noetherian descent we may assume that $X' \to X \to S$ is the base change of $X'_0 \to X_0 \to S_0$, where S_0 is noetherian, $X_0 \to S_0$ is smooth and proper, and $X'_0 \to X_0$ is finite étale. By proper base change, we may then replace S by S_0 (i.e., assume S is noetherian). By Theorem 5.2, $R^\nu f_* \mathcal{F} = R^\nu f_! \mathcal{F}$ is constructible, and by Lemma 5.4, the specialization maps for $R^\nu f_* \mathcal{F}$ are isomorphisms, so by Lemma 5.3, $R^\nu f_* \mathcal{F}$ is lcc. $\square$

Let me end by pointing out that Delinge proved the following version of Theorem 2.5(1), 3.1 and 5.2 for ordinary higher pushforwards (under extra mild conditions):

Theorem 5.5 ([1, Th. finitude, 1.1, 1.9])**.** *Let $f : X \to S$ be a finite type and separated map, where S is finite type over a regular base of dimension ≤ 1. Let $\mathcal{F}$ be a constructible abelian sheaf on X whose torsion order is invertible on S. Then the sheaves $R^\nu f_* \mathcal{F}$ are constructible, and $R^\nu f_* \mathcal{F} = 0$ for $\nu > \dim S + 2 \dim X$. Moreover, there exists a dense open $U \subseteq S$ (depending on $\mathcal{F}$) such that the formation of $R^\nu f_* \mathcal{F}|_U$ is compatible with base change on U.*

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