Poincaré duality for étale cohomology

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1 Statement of Poincaré duality

1.1 Poincaré duality for complex manifolds

In these notes we will discuss the Poincaré duality theorem for smooth varieties over algebraically closed fields. This story is parallel to Poincaré duality for topological manifolds, so let's recall how that goes. Although this is probably familiar in some form, we want to emphasize the formulation that will generalize well to étale cohomology.

For a real topological manifold M of (pure) dimension d, the cup product gives a bilinear map

$$H^i(M;\mathbf{R}) \times H_c^{d-i}(M;\mathbf{R}) \to H_c^d(M;\mathbf{R}) \tag{1.1}$$

If M is orientable, then a *choice* of orientation defines a map

$$\int : H_c^d(M;\mathbf{R}) \xrightarrow{\sim} \mathbf{R} \tag{1.2}$$

The Poincaré duality theorem says that for connected M, in which case (1.2) is an isomorphism, (1.2) identifies $H^i(M,\mathbf{R})$ with $H_c^{d-i}(M;\mathbf{R})^*$ via (1.1).

The map (1.2) is called a *trace* map. I like to think of this as an integration map because in terms of de Rham cohomology, we can interpret it as the map given on the cohomology class of a compactly supported form $[\eta]$ by

$$[\eta] \mapsto \int_M \eta.$$

Remark 1.1. We emphasize the following features of this trace map:

- This map depends implicitly on the choice of orientation, since that enters in defining integration.
- The map is well-defined for compactly-supported forms; for general forms the integral doesn't need to converge .

This discussion involved a *choice* of orientation. Let's try to abstract out that choice. The issue arose in defining the trace map. We define the *orientation sheaf* $\mathbb{O}_{\mathbf{R}}$ for M to be the local system whose fiber at $x \in M$ is $H^d(M, M \setminus x; \mathbf{R}) \simeq H^{d-1}(S^{d-1}; \mathbf{R})$, which is *non-canonically* isomorphic to $\mathbf{R}$. Then we have a *canonical* map

$$\int : H^d(M, \mathbb{O}_{\mathbf{R}}) \xrightarrow{\sim} \mathbf{R}.$$

defining a *canonical* perfect pairing

$$H^i(M; \mathbf{R}) \times H_c^{d-i}(M; \mathbb{O}_{\mathbf{R}}) \to \mathbf{R} \tag{1.3}$$

identifying $H^i(M; \mathbf{R})$ with the $\mathbf{R}$-dual of $H_c^{d-i}(M, \mathbb{O}_{\mathbf{R}})$. In these terms, an orientation is simply a compatible choice of orientations, i.e. a non-vanishing *global section* of the local system $\mathbb{O}_{\mathbf{R}}$. Thus a choice of orientation is a choice of trivialization $\mathbb{O}_{\mathbf{R}} \simeq \underline{\mathbf{R}}$, which converts (1.3) into the composition of (1.1) and (1.2).

Now suppose M is a *complex* manifold, of pure dimension d over $\mathbf{C}$. Then M is *automatically* orientable. Indeed, we can choose a compatible family of frames as follows. For every $x \in M$, choose a basis $v_1, \ldots, v_d$ for $T_x M$ over $\mathbf{C}$. Now, take $(v_1, iv_1, v_2, iv_2, \ldots, v_d, iv_d)$ as an ordered basis for $T_x M$ over $\mathbf{R}$. It is easily checked that holomorphic transition functions automatically preserve this orientation.

We seemingly made no choices in the above description; for this reason, it is often said that complex manifolds are *canonically* oriented. But from an algebraic perspective, we actually *did* make a choice, of $i = \sqrt{-1}$. Choosing the opposite square root changes the orientation by $(-1)^d$. In other words, we have just seen that the complex structure defines an isomorphism

$$\underline{\mathbf{R}} \xrightarrow{\sim} \mathbb{O}_{\mathbf{R}}$$

but the "Galois action" on $\mathbb{O}_{\mathbf{R}}$ isn't trivial. This doesn't really matter for complex geometers or topologists, but when studying étale cohomology we need to be careful about such matters, since we want the topological story to fit into a purely algebraic theory.

Example 1.2. Just consider the case $M = \mathbf{C}$ (so $d = 1$). We have just seen that a choice of orientation on $\mathbf{C}$ is equivalent to a choice of $\sqrt{-1}$, which is negated by the complex conjugation. This means that we can give the following concrete model for the orientation sheaf. Let $\mathbf{Z}(1) := \ker(\exp\colon \mathbf{C} \to \mathbf{C}^\times)$. Then $\mathbf{R}(1) := \mathbf{R} \otimes_{\mathbf{Z}} \mathbf{Z}(1)$ is the orientation sheaf for $\mathbf{C}$, so $\mathbf{R}(d) := \mathbf{R}(1)^{\otimes d}$ is the orientation sheaf for a complex n-manifold.

1.2 Poincaré duality for smooth varieties

Let X be a connected smooth variety of dimension n over an *algebraically closed* field k. (This last assumption is essential. Over general fields k, even the "point" $X = \operatorname{Spec} k$ won't have a Poincaré duality - its étale cohomlogy is just Galois cohomology.) Then X should be thought of as analogous to a complex manifold of dimension n. In particular, setting $\Lambda(d) := \Lambda \otimes \mu_n^{\otimes d}$, there will be an isomorphism of groups

$$H^{2d}_{\text{ét}}(X; \Lambda(d)) \simeq \Lambda$$

inducing a perfect pairing of groups

$$H^i_{\text{ét}}(X; \Lambda(d)) \times H^{2n-i}_{\text{ét}}(X; \Lambda) \to H^{2n}_{\text{ét}}(X; \Lambda(d)) \simeq \Lambda.$$

In these notes we will explain the proofs of these statements, following Verdier [**?**].

Remark 1.3. The cup product in étale cohomology is reviewed in §B

Remark 1.4. An important case will be when X is base changed from some variety X_0 defined over a base field k_0 which is not algebraically closed, and whose algebraic closure is k. Then $X = X_0 \times_{k_0} k$ will have an action of $\operatorname{Gal}(k/k_0)$ through the second factor (contravariant due to Spec), inducing an action of $\operatorname{Gal}(k/k_0)$ on $H^i_{\text{ét}}(X; \Lambda)$ (now covariant). In this case, each cohomology group is equipped with the additional structure of a Galois module, and everything is Galois equivariant due to naturality.

To motivate the intervention of $\Lambda(d)$, we need to identify the orientation sheaf for complex manifolds. Previously we found that an orientation essentially amounted to an orientation on the *roots of unity*, so the orientation sheaf with Λ-coefficients for a d-dimensional connected complex manifold is again $\Lambda(d)$.

Remark 1.5. When we eventually study the zeta functions of varieties over finite fields, we'll see that Poincaré duality explains the functional equation.

2 The Trace map

We would like to construct a "trace isomorphism"

$$\int : H^{2d}_c(X, \Lambda(d)) \xrightarrow{\sim} \Lambda$$

for smooth, connected varieties over separably closed fields k, with $n \in k^\times$. Unfortunately, we don't have a concrete understanding of étale cohomology, except in some low degrees. So let's content ourselves for now with a case that we *do* understand: the cohomology of curves.

2.1 The case of curves

First suppose X is a smooth projective curve (over separably closed k). We want to find a natural map

$$H_c^2(X, \mu_n) = H_c^2(X, \mu_n) \xrightarrow{\sim} \Lambda.$$

We know that one should study this cohomology group using the Kummer sequence:

$$1 \to \mu_n \to \mathbf{G}_m \xrightarrow{[n]} \mathbf{G}_m \to 1.$$

This induces a long exact sequence in cohomology:

$$\ldots \to H^1_{\text{ét}}(X, \mathbf{G}_m) \xrightarrow{n} H^1_{\text{ét}}(X, \mathbf{G}_m) \to H^2_{\text{ét}}(X, \mu_n) \to H^2_{\text{ét}}(X, \mathbf{G}_m) \to \ldots$$

Now, we know that $H^2_{\text{ét}}(X, \mathbf{G}_m) = 0$ (vanishing of the Brauer group for curves), and

$$H^1_{\text{ét}}(X, \mathbf{G}_m) \simeq \mathrm{Pic}(X) \simeq \mathbf{Z} \times \mathrm{Pic}^0(X).$$

Since $\mathrm{Pic}^0(X)$ is divisible, being the group of geometric points of an abelian variety (namely the Jacobian of X), this gives us a natural isomorphism

$$H^1_{\text{ét}}(X, \mu_n) \simeq H^1_{\text{ét}}(X, \mathbf{G}_m)/nH^1_{\text{ét}}(X, \mathbf{G}_m) \xrightarrow{\sim} \Lambda.$$

Now suppose that X is not necessarily projective. By definition, we compute compactly supported cohomology using a compactification $j \colon X \hookrightarrow \overline{X}$:

$$H_c^i(X, \mu_n) := H^i(\overline{X}, j_! \mu_n).$$

How can we compute this? We'll employ a very useful construction, which is also applicable to more general situations. Let $i \colon Y \hookrightarrow Z$ be a closed embedding and $j \colon W := Z \setminus Y \hookrightarrow Z$ the inclusion of the complement, with the natural open subscheme structure. For any étale sheaf $\mathscr{F}$ on Z, we have the following short exact sequence:

$$0 \to j_! j^* \mathscr{F} \to \mathscr{F} \to i_* i^* \mathscr{F} \to 0. \tag{2.1}$$

Applying this to $\mathscr{F} = \mu_{n,\overline{X}}$, so $j^* \mathscr{F} = \mu_{n,X}$, we obtain the short exact sequence

$$0 \to j_! \mu_{n,X} \to \mu_{n,\overline{X}} \to i_* i^* \mu_{n,Z} \to 0.$$

In our setting Z is a finite collection of points, so $i_* i^* \mu_{n,Z}$ is the sum of skyscraper sheaves on points. In particular, since we are over an algebraically closed field the higher cohomology vanishes, giving an isomorphism

$$H^2(\overline{X}, j_! j^* \mu_n) \simeq H^2(\overline{X}, \mu_n) \simeq \Lambda$$

by our analysis of the projective case.

Remark 2.1. The most important instance of this construction is when $X = \mathbf{A}^1_k$, since this is the basic "Euclidean" case.

Remark 2.2. This discussion applies equally well to compact connected Riemann surfaces. A priori, this gives two different trace maps

$$H^2_c(X, \mu_n) \simeq \Lambda$$

one via the degree of line bundles and one via reduction mod n of Poincaré duality. For a smooth variety over $\mathbf{C}$, one has an identification between the étale cohomology and singular cohomology, and we want the trace maps to be compatible. So one should check that the maps really are the *same*. We leave this as an exercise for the reader.

2.2 The general formulation

To establish the general case, we'll want to factor an arbitrary map locally into curve fibrations, thus reducing to the curve case. This strategy requires us to allow bases other than Spec k. So we see that it will be essential to have a more general framework, to give us more flexibility.

Let $f : X \to S$ be any smooth quasi-projective map with fibers of pure dimension d. Furthermore, we consider a general étale sheaf of Λ-modules $\mathscr{F}$ on S. For any such sheaf, we define $\mathcal{F}(d) := \mathcal{F} \otimes_\Lambda \mu_n^{\otimes d}$.

Remark 2.3. For the purpose of the defining the trace map we can stick to $\mathscr{F} = \underline{\Lambda}$, but for proving the perfect duality statement it *will* be necessary to consider more general $\mathscr{F}$, so we should work generally from the start.

We want to define a trace map

$$\mathrm{tr}_{X/S}(\mathscr{F}) : R^{2d} f_!(f^*\mathscr{F}(d)) \xrightarrow{\sim} \mathscr{F}.$$

We already know what this should be in certain cases. We'll begin building axioms which we want $\mathrm{tr}_{X/S}$ to satisfy.

(A1) When $S = \mathrm{Spec}\ k$ for separably closed k and $X = \mathbf{A}^1_k$, $\mathscr{F} = \underline{\Lambda}$, the map $\mathrm{tr}_{X/S}$ is the one we constructed in the preceding section §2.1.

It is also easy to see what this map should be when f is étale. In that case, we are seeking a map

$$f_! f^* \mathscr{F} \to \mathscr{F}.$$

This is just the counit associated to the adjunction $(f_!, f^*)$.

Remark 2.4. One should really think of $f_!$ as left adjoint to "$f^!$", which we have not defined yet. If f is étale then $f^! = f^*$, which "explains" the above adjunction.

(A2) When f is étale, $\mathrm{tr}_{X/S}(\mathscr{F})$ is the natural counit $f_! f^* \mathscr{F} \to \mathscr{F}$.

Finally, we'll want the trace map to have good functoriality properties, so we demand the following properties.

(A3) $\mathrm{tr}_{X/S}$ is functorial in $\mathscr{F}$.

(A4) $\mathrm{tr}_{X/S}$ is compatible with base change.

Let's clarify what this means. Suppose we have a cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[g]{} & S \end{array}$$

For a sheaf $\mathscr{F}$ on S, we have the trace map

$$R^{2d} f_!(f^* \mathscr{F}(d)) \to \mathscr{F}.$$

Pulling back by g^* gives a map

$$g^* R^{2d} f_!(f^* \mathscr{F}(d)) \to g^* \mathscr{F} \tag{2.2}$$

We can rewrite this using proper base change:

$$g^* R^{2d} f_!(f^* \mathscr{F}(d)) = R^{2d}(f')_!(g'^* f^* \mathscr{F}(d)) = R^{2d}(f')_!(f')^*(g^* \mathscr{F}). \tag{2.3}$$

The axiom says that (2.2), with the identification of (2.3), should coincide with $\mathrm{tr}_{X'/S'}(g^* \mathscr{F})$.

(A5) $\mathrm{tr}_{X/S}$ is compatible with compositions.

This also requires some explanation. Consider a diagram

$$\begin{array}{c} X' \\ \downarrow{\scriptstyle f'} \\ X \\ \downarrow{\scriptstyle f} \\ S \end{array}$$

where f has relative dimension d and f' has relative dimension d'. For a sheaf $\mathscr{F}$ on S, we have a map

$$R^{2d} f_!(f^* \mathscr{F}(d)) \to \mathscr{F}.$$

Also, we have the trace map

$$R^{2d'}(f')_!(f')^*(f^* \mathscr{F}(d))(d') \to f^* \mathscr{F}(d).$$

Stringing these together gives a map

$$R^{2d} f_! R^{2d'} (f')_! (f')^* (f^* \mathscr{F}(d))(d') \to R^{2d} f_! f^* \mathscr{F}(d) \to \mathscr{F}. \tag{2.4}$$

Now, it happens that we have an isomorphism

$$R^{2d'}(f')_! R^{2d} f_! \xrightarrow{\sim} R^{2(d+d')}(f' \circ f)_!$$

which comes from the degeneration at the "upper right corner" of the spectral sequence of composition for $(f \circ f)_!$, *because* $R^i f_! = 0$ for $i > 2d$ and $R^i f'_! = 0$ for $i > 2d'$ are the top homological dimensions for f and f'. Using this above in (2.4) gives a map

$$R^{2d+2d'}(f')_!(f')^*(f^* \mathscr{F}(d))(d') \to R^{2d} f_! f^* \mathscr{F}(d) \to \mathscr{F}$$

which we ask to coincide with trace map $\mathrm{tr}_{X'/S}(\mathscr{F})$.

Theorem 2.5. *There exists a unique trace map satisfying properties (A1)-(A5). Moreover, it enjoys the following additional properties:*

- *When the geometric fibers of f are connected and non-empty,* $\mathrm{tr}_{X/S}$ *is an isomorphism.*
- *When $S = \mathrm{Spec}\ k$ (k algebraically closed), X is a smooth curve, and $\mathscr{F} = \Lambda$, the map*

  $$\mathrm{tr}_{X/S} \colon H^2_c(X, \mu_n) \to \Lambda$$

  agrees with the one constructed in §2.1.
- *The trace maps are compatible for different n.*

It suffices to establish this theorem for $\mathscr{F} = \underline{\Lambda}$, because there is a canonical isomorphism

$$R^{2d} f_!(\Lambda(d) \otimes f^* \mathscr{F}) \xrightarrow{\sim} R^{2d} f_!(\Lambda(d)) \otimes \mathscr{F}. \tag{2.5}$$

This follows from Proposition A.1 in the Appendix.

2.3 Construction of the map

The first step is to show that the trace map can be defined locally on the source. (We should believe this by analogy to the topological trace map, which we thought of as like an "integration" map.) Once this is done, the structure theory of smooth maps tells us that we can factor any smooth map $f \colon X \to S$ Zariski-locally as a composition of an étale map with the natural projection $\mathbf{A}^d_S \to S$:

$$f \xrightarrow{\text{étale}} \mathbf{A}^d_S \to S.$$

By compatibility with compositions, it suffices to define the trace map in the étale case and for $\mathbf{A}^1_S \to S$. We already know how the trace map must be defined in for étale morphisms. Then we just have to define it for the map $\mathbf{A}^1_S \to S$, which is similar to what we considered above.

Step 1: reduction to local case. The first step is to show that it suffices to define the trace map locally, i.e. if the trace map is defined for each member of a Zariski cover $\{U_\alpha \to X\}$, then we can define it for X. Since we're dealing in noetherian situations, we only need to worry about finite covers. Therefore, it suffices to consider the case of a cover with two members: $\{U_1 \xrightarrow{j_1} X,\ U_2 \xrightarrow{j_2} X\}$. What we want is a kind of "Mayer-Vietoris" sequence:

$$\ldots \to H^{2d}_c(U_{12}, \mathscr{G}) \to H^{2d}_c(U_1, \mathscr{G}) \oplus H^{2d}_c(U_2, \mathscr{G}) \to H^{2d}_c(X, \mathscr{G}) \to 0. \tag{2.6}$$

Given such, we can define a trace maps for $\mathscr{G} = f^*\mathscr{F}$ in terms of the local trace maps:

$$\begin{array}{ccccccc} H^{2d}_c(U_{12}, f^*\mathscr{F}) & \longrightarrow & H^{2d}_c(U_1, f^*\mathscr{F}) \oplus H^{2d}_c(U_2, f^*\mathscr{F}) & \longrightarrow & H^{2d}_c(X, f^*\mathscr{F}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \vdots & & \\ \mathscr{F} & \longrightarrow & \mathscr{F} \oplus \mathscr{F} & \longrightarrow & \mathscr{F} & & \end{array}$$

To justify this, we use the short exact sequence of sheaves

$$0 \to j_{12!}(\mathscr{G}|_{U_{12}}) \to j_{1!}(\mathscr{G}|_{U_1}) \oplus j_{2!}(\mathscr{G}|_{U_2}) \to \mathscr{G} \to 0.$$

(Exactness is easy to check by looking at stalks, since we know the effect on stalks of $j_!$. Over any geometric point, it boils down to simply the "inclusion-exclusion principle".) The associated long exact sequence of cohomology is

$$\ldots \to Rf_! \circ j_{12!}(\mathscr{G}|_{U_{12}}) \to Rf_! \circ j_{1!}(\mathscr{G}|_{U_1}) \oplus Rf_! \circ j_{2!}(\mathscr{G}|_{U_2}) \to Rf_!\mathscr{G} \to 0$$

Since extension-by-zero is exact (so its higher direct images vanish), and an easy argument using the spectral sequence for composition, this agrees with

$$\ldots \to R(f \circ j_{12!})(\mathscr{G}|_{U_{12}}) \to R(f \circ j_1)_!(\mathscr{G}|_{U_1}) \oplus R(f \circ j_2)!(\mathscr{G}|_{U_2}) \to Rf_!\mathscr{G} \to 0$$

Now suppose that we have defined trace maps for U_1 and U_2, so we have maps

$$R(f \circ j_1)_!(f^*\mathscr{F}|_{U_1}) \to \mathscr{F}$$
$$R(f \circ j_2)!(f^*\mathscr{F}|_{U_2}) \to \mathscr{F}$$

We then define a trace map

$$Rf_!(f^*\mathscr{F}) \to \mathscr{F}$$

as the sum of the ones for U_1 and U_2.

$$\begin{array}{ccccccc} j_{12!}(f^*\mathscr{F}|_{U_{12}}) & \longrightarrow & j_{1!}(f^*\mathscr{F}|_{U_1}) \oplus j_{2!}(f^*\mathscr{F}|_{U_2}) & \longrightarrow & f^*\mathscr{F} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \vdots & & \\ \mathscr{F} & \longrightarrow & \mathscr{F} \oplus \mathscr{F} & \longrightarrow & \mathscr{F} & & \end{array}$$

We now reach a subtle technical point.

This is well-defined because the contribution of $j_{12!}(f^*\mathscr{F}|_{U_{12}})$ vanishes, which boils down to the fact that the two trace maps induced by the two compositions

$$\begin{array}{ccccc} & & U_{12} & & \\ & \swarrow & & \searrow & \\ U_1 & & & & U_2 \\ & \searrow & & \swarrow & \\ & & S & & \end{array}$$

are equal. This is not immediate from the construction (which we have yet to give), but will be shown in due course.

Step 2: the étale case. In this case we are forced to define

$$R^0 f_!(f^*\mathscr{F}) = f_! f^*\mathscr{F} \to \mathscr{F}$$

to be the natural adjunction map.

Step 3: affine space fibrations. Thanks to Steps 1 and 2, we are reducing to defining a trace map for a morphism $f \colon \mathbf{A}^n_S \to S$. By compatibility with compositions, we can reduce to the case $f \colon \mathbf{A}^1_S \to S$.

To compute $R^1 f_!(\mu_n)$, we choose a compactification $j \colon \mathbf{A}^1_S \hookrightarrow \mathbf{P}^1_S \xrightarrow{\overline{f}} S$. By definition,

$$R^1 f_!(\mu_n) = R^1\overline{f}_*(j_!\mu_n).$$

By the exact sequence

$$0 \to j_! j^! \mu_n \to \mu_n \to i_* i^* \mu_n$$

where $i \colon S \to \mathbf{P}^1_S$ is the inclusion of ∞, we have by the excision long exact sequence

$$R^2 f_!(\mu_n) = R^2\overline{f}_*(\mu_n) = \underline{\Lambda}_S$$

which follows from the same computation as before, because $R^1\overline{f}_*(\mathbf{G}_m) \simeq \underline{\mathbf{Z}}_S$.

Well-definedness. Our "construction" involved choices at several steps, and it is not clear that it is well-defined. We sketch how to complete the argument. A detailed exposition can be found in [**?**] §2.1.

- The first choice was in choosing the open subscheme $U \hookrightarrow X$. When comparing the constructions for different open subschemes, we may intersect them and thus assume that one is contained in the other. Then we can restrict the "affine chart" $U \to \mathbf{A}^n_S$, reducing to the next step.

- The next choice was the étale map $U \to \mathbf{A}^n_S$. This amounts to a choice of d (local) functions on X with independent differentials. We need to check that different such factorizations yield the same trace map.

  First we argue that permutations of the coordinates of $\mathbf{A}^d_S$ don't affect the trace map. This is because the action of S_d extends through the embedding $S_d \hookrightarrow \mathrm{GL}_d$, and the latter group is connected, so doesn't admit any non-trivial action on a discrete space.

  Then, by permuting the coordinates and induction, we reduce to checking that two different factorizations

$$\begin{array}{ccc} X & \longrightarrow & \mathbf{A}^1_S \\ \downarrow & & \downarrow \\ \mathbf{A}^1_S & \longrightarrow & S \end{array}$$

  yield the same trace map. To see this, we reduce to the case where $S = \mathrm{Spec}\ k$ is an algebraically closed field by taking stalks. This case boils down to showing that our construction is well-defined for any curve, as it can be identified with taking a line bundle to its relative degree (modulo n). It would be a useful exercise to the reader to chase through the construction and see that this is the case.

- Finally, we should check that $R^{2d} f_! f^* \Lambda(d) \to \Lambda(d)$ is an isomorphism if f has geometrically connected fibers. This is clear step-by-step, since we already know the case of curves.

3 Derived categories

To proceed in our discussion of Poincaré duality, we need the formalism of derived categories. Therefore we will give a utilitarian introduction to derived categories.

3.1 Motivation

Let's first try to indicate why we need to talk about derived categories. So far we have been content to deal with "cohomological" functors $H^*, R^* f_*, R^* f_!$, etc. The common feature of all of these functors is that they are cooked up by taking cohomology of some sort of complex. The point of the derived category is to work with the complex directly, rather than its cohomology. In other words, we will instead study *complexes* of sheaves $Rf_*, Rf_!$, etc. which are complexes of sheaves, whose homology sheaves are the known $R^* f_*, R^* f_!$.

It is clear that the complex contains strictly more information than its cohomology, so we are certainly not losing information here. What is less clear is that this would be useful. Indeed, we will hardly ever be able to compute the complexes. But it is a fact of life that all sorts of proofs can be better expressed using the complexes directly.

Let's give a couple of examples to illustrate what we mean. We know that derived functors behave in rather complicated ways with respect to composition. For example, for higher direct images of a composition of pushforwards we have a spectral sequence

$$R^p f_*(R^q g_*) \implies R^{p+q}(f \circ g)_*.$$

This is a rather complicated thing! But at the level of the derived category, the pushforward behaves as nicely as could be hoped with respect to composition:

$$R(f \circ g)_* = Rf_* \circ Rg_*.$$

Similarly, we will have

$$R(f \circ g)_! = Rf_! \circ Rg_!.$$

These can be unpackaged to recover the spectral sequence.

As another example, we used in the previous section the identity

$$R^{2d} f_!(\Lambda(d) \otimes f^* \mathscr{F}) \xrightarrow{\sim} R^{2d} f_!(\Lambda(d)) \otimes \mathscr{F}.$$

This is a rather delicate statement: it holds only for the top higher direct image R^{2d}, and only because we are using the sheaf $\Lambda(d)$. The conceptual explanation is that it is a shadow of a general equality in the derived category:

$$Rf_!(\mathscr{G} \otimes^L f^* \mathscr{F}) \xrightarrow{\sim} Rf_!(\mathscr{G}) \otimes^L \mathscr{F}.$$

Here $\otimes^L$ is the "derived tensor product", which as we'll see is necessary even to formulate the Künneth Theorem.

Finally, why did we get away before without the derived category? It was basically because we were dealing *only with top-degree cohomology*. For morphisms f, f' of cohomological dimension d, d' we are lucky enough that

$$R^{2d}(f \circ f')_! = R^{2d} f_! \circ R^{2d'} f'_!.$$

So for studying the trace map, we only need to consider the top-degree cohomology, which does behave nicely because the spectral sequence is particularly degenerate there. However, if we want to consider all degrees (and we certainly do, for Poincaré duality) then it's better to work in the derived category.

To summarize, one could say that derived constructions, which work at the level of complexes, "capture" information that is lost in usual constructions without assumptions of projectivity, injectivity, flatness, etc. Instead of the perspective that tensor product is not left-exact, and the failure of left exactness is measured by derived functors, we adopt the uniform point of view of the "derived tensor product" which is always "well-behaved".

3.2 Formalism of derived categories

Having hopefully motivated the need for and uses of the derived category, it's time to tell you about the basic formalism of the derived category.

First of all, what *is* it? Roughly speaking, a derived category is a category constructed by starting out with some other category, and inverting a specified collection of morphisms. Let's make this concrete in our case of interest.

For an abelian category $\mathcal{A}$, we'll define $C(\mathcal{A})$ to be the category of chain complexes of objects in $\mathcal{A}$,

$$\ldots \to A_i \xrightarrow{d_i} A_{i+1} \xrightarrow{d_{i+1}} \ldots \quad A_i \in \mathcal{A}$$

and the morphisms are maps of chain complexes (i.e. level-wise maps between objects, which are compatible with the differentials).

Remark 3.1. The main examples to keep in mind are when $\mathcal{A}$ is the category of abelian groups, or the category of sheaves of Λ-modules on X. These are the only two examples which will come up for us.

A *quasi-isomorphism* in $C(\mathcal{A})$ is a morphism that induces isomorphisms on all cohomology groups. The derived category $D(\mathcal{A})$ is then obtained by "inverting" the quasi-isomorphisms in $C(\mathcal{A})$. This is sort of analogous to "localization" in *non*-commutative ring theory. There are set-theoretic concerns in doing this, but they can be ignored for the purposes of our discussion; see [**?**] §10.

So what sort of structure does $D(\mathcal{A})$ have? It is *not* an abelian category. A possibly useful analogy is to think of $D(\mathcal{A})$ as the homotopy category of *CW complexes*. Note that I don't say "homotopy category of topological spaces"; that would be analogous to a category usually called $K(\mathcal{A})$, which is obtained from $C(\mathcal{A})$ by inverting homotopy equivalences. The passage from $K(\mathcal{A})$ to $D(\mathcal{A})$ should be thought of as an elimination of pathologies, by inverting maps which "should be" isomorphisms but are not for pathological reasons.

Shifts. For a complex $\mathcal{K}_\bullet$ and $n \in \mathbf{Z}$, we define $\mathcal{F}[n]_\bullet$ by

$$\mathcal{F}[n]_i := \mathcal{F}_{n+i}$$

with $d[n]_i = (1)^n d_{n+i}$.

Derived functors. From now on, we'll specialize to the case where A is the category of étale sheaves of Λ-modules on a scheme X. We'll now discuss the form of "derived functors" on $D(\mathcal{A})$, in the examples that we need. For technical reasons, one usually needs to restrict to certain subcategories, which we now introduce.

Definition 3.2. We define

- $D^+(\mathcal{A})$ to be the full subcategory of objects equivalent to a chain complex C_* such that $H^i(C_*) = 0$ for $i \ll 0$,
- $D^-(\mathcal{A})$ to be the full subcategory of objects equivalent to C_* such that $H^i(C_*) = 0$ for $i \gg 0$,
- $D^b(\mathcal{A})$ to be the full subcategory of objects with vanishing $H^i(C_*)$ for $i \gg 0$ or $i \ll 0$.

We're now going to define the functor Rf_* that was alluded to above. The definition is simple: for a complex $C_\bullet \in D_+(A)$, we pick an injective resolution

$$C_\bullet \xrightarrow{\sim} I_\bullet$$

meaning a quasi-isomorphism with $I_\bullet \in D_+(\mathcal{A})$ consisting of levelwise injective objects, and set

$$Rf_*(C_\bullet) := f_*(I_\bullet).$$

It's evident that the cohomology sheaves of $Rf_*(C_\bullet)$ are exactly the hypercohomology sheaves $R^\bullet f_*(C_\bullet)$.

Remark 3.3. The resolution $C_\bullet \xrightarrow{\sim} I_\bullet$ can be thought of as a section of $K(A) \to D(\mathcal{A})$. A priori we need to restrict the definition to $D_+(A)$ in order to guarantee that any two choices of injective resolutions are actually homotopic: if you go back to the proof of this homological algebra fact, you'll see that it's important for the chain complex to have an "end".

In particular, the terminology "derived functor" is a bit misleading in that there is no actual functor being derived. For instance, f_* is not a functor on $D(\mathcal{A})$ - if it were, it would send quasi-isomorphisms to isomorphisms.

We can similarly define $Rf_!$ for a compactifiable morphism $f\colon X \to S$. To do so, we find a compactification

$$\begin{array}{ccc} X & \xrightarrow{j} & \overline{X} \\ & {}_{f}\searrow & \downarrow{\scriptstyle \overline{f}} \\ & & S \end{array}$$

and define

$$Rf_! := R\overline{f}_* \circ j_!.$$

As promised, these satisfy a very nice compatibility with composition, which is already *used* in the construction of the Grothendieck spectral sequence. To illustrate, let's show that naturally

$$R(f \circ g)_* = Rf_* \circ Rg_*.$$

Let's test both sides against some chain complex $C_\bullet$. To compute the left side, we take some injective resolution $C_\bullet \xrightarrow{\sim} I_\bullet$ and then form

$$(f \circ g)_* I_\bullet = f_*(g_* I_\bullet).$$

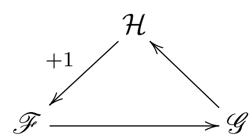
To compute the right side, we can start with the same injective resolution. We then need to resolve $g_* I_\bullet$ by an injective resolution again, and then apply f_*. But g_* preserves injectives, since it is a right adjoint with an exact left adjoint g^*. So the right side is manifestly also computed functorially in $D(\mathcal{A})$ as $f_*(g_* I_\bullet)$.

The proof of the composition relation for $f_!$ is a little more technical, and is relegated to the appendix (Proposition A.2).

Distinguished triangles. If $D(\mathcal{A})$ is not an abelian category, how can you express a statement like "a short exact sequence of sheaves induces a long exact sequence of cohomology"? The derived category carries the structure of a *triangulated category*, wherein the role of "short exact sequences" is played by "distinguished triangles". This means that there are certain triples of complexes and maps

$$\mathscr{F} \xrightarrow{u} \mathscr{G} \xrightarrow{v} \mathcal{H} \xrightarrow{w} \mathscr{F}[1]$$

satisfying a certain list of axioms. This is sometimes denoted by a "triangle" (hence the name)



We'll just highlight the axioms that are relevant to our discussion.

- A rotation of a distinguished triangle is distinguished.
- For any map $u \colon \mathscr{F} \to \mathscr{G}$, there is an $\mathcal{H}$ fitting into a distinguished triangle

  $$\mathscr{F} \to \mathscr{G} \to \mathcal{H} \xrightarrow{+1} \mathscr{F}[1]$$

  This $\mathcal{H}$ is called the "mapping cone".
- If there are maps between the first two members of two distinguished triangles

  $$\begin{array}{ccccccc} \mathscr{F}_\bullet & \xrightarrow{u} & \mathscr{G}_\bullet & \xrightarrow{v} & \mathcal{H}_\bullet & \xrightarrow{w} & \mathscr{F}_\bullet[1] \\ \downarrow & & \downarrow & & \vdots & & \downarrow \\ \mathscr{F}'_\bullet & \xrightarrow{u'} & \mathscr{G}_\bullet & \xrightarrow{v'} & \mathcal{H}_\bullet & \xrightarrow{w'} & \mathscr{F}_\bullet[1] \end{array}$$

  then there is a (not necessarily unique) map between the third.

It may be helpful to discuss an analogy. We advocated thinking of $D(\mathcal{A})$ as analogous to a homotopy category of CW complexes. A map of spaces $f\colon M \to N$ doesn't have a kernel or cokernel. However, one can make sense of N/M as the mapping cone of f. And if one takes the "mapping cone of the map to the mapping cone", then one visibly obtains the suspension. (Draw a picture!)

A *cohomological functor* from $D(\mathcal{A})$ to an abelian category transforms any distinguished triangle into an exact sequence. In particular, $R^i f_*$ is a cohomological functor, which means that for a distinguished triangle

$$\mathscr{F} \to \mathscr{G} \to \mathcal{H} \xrightarrow{+1} \mathscr{F}[1]$$

we get a long exact sequence

$$\ldots \to R^i f_* \mathscr{F} \to R^i f_* \mathscr{G} \to R^i f_* \mathcal{H} \to R^{i+1} f_* \mathscr{F} \to \ldots$$

There are a few very useful distinguished triangles worth mentioning.

- Let $D(X)$ be the derived category of sheaves of Λ-modules on X. Let $i\colon Y \hookrightarrow Z$ be the inclusion of a closed subscheme, and $j\colon U \hookrightarrow Z$ be the inclusion of the open complement. Then for any $\mathscr{F} \in D(X)$ there is a distinguished triangle

$$j_! j^* \mathscr{F} \to \mathscr{F} \to i_* i^* \mathscr{F}$$

  There is also a distinguished triangle

$$i_! i^! \mathscr{F} \to \mathscr{F} \to Rj_* j^* \mathscr{F}$$

  but we have not yet discussed the meaning of $i^!$.

- For any complex $\mathscr{F}_\bullet$, we can define the "truncation functor"

$$\tau_{\geq n}(\mathscr{F}_\bullet)_i = \begin{cases} 0 & i < n-1 \\ \operatorname{coker} d_{n-1} & i = n-1 \\ \mathscr{F}_i & i \geq n \end{cases}$$

  The point of this construction is that there is a map $\mathscr{F} \to \tau_{\geq n}$, which is an isomorphism on H^i for $i \geq n$ and 0 for $i < n$. There is an analogous truncation $\tau_{\leq n}(\mathscr{F}_\bullet)$, whose definition I leave as an exercise. Then there is a distinguished triangle

$$\tau_{<n} \mathscr{F}_\bullet \to \mathscr{F}_\bullet \to \tau_{\geq n} \mathscr{F}_\bullet.$$

  In practice the purpose of this is to cut up complexes into sheaves, so we can prove statement about complexes from the case of sheaves using the 5-Lemma. For instance, we know that for a *sheaf* $\mathscr{F}$ the base change map is an isomorphism:

$$g^* R f_! \mathscr{F} \xrightarrow{\sim} R(f')_! (g')^* \mathscr{F}.$$

  The same holds for any bounded above and below *complex* $\mathscr{F}_\bullet$, by using this distinguished triangle. This is a trick we'll use again in the future.

Hom in the derived category. The derived category has an "internal hom", meaning that for two objects C, C' we can produce another object of the derived category denoted $\mathrm{R}\mathscr{H}om(C, C')$. To define it, pick an injective resolution $C'_\bullet \xrightarrow{\sim} I_\bullet$. Define

$$\mathrm{R}\mathscr{H}om(C_\bullet, C'_\bullet)_k = \bigoplus_i \mathscr{H}om(C_i, I_{i+k})$$

with the differential being

$$d_k(\varphi) = d_I \circ \varphi + (-1)^k \varphi \circ d_C.$$

This is a complex of sheaves on X.

We can also define $\mathrm{RHom}(C_\bullet, C'_\bullet) \in D(\mathrm{Ab})$ by

$$\mathrm{RHom}(C_\bullet, C'_\bullet)_k = \bigoplus_i \mathrm{Hom}(C_i, I_{i+k})$$

with analogous differential. Thus, $\mathrm{RHom} = R\Gamma \circ \mathrm{R}\mathscr{H}om$.

Next we introduce the Ext sheaves and groups. We define

$$\mathscr{E}xt^i(C_\bullet, C'_\bullet) = \mathcal{H}^i(\mathrm{R}\mathscr{H}om(C_\bullet, C'_\bullet))$$

to be the homology sheaves of $\mathrm{R}\mathscr{H}om$. We define

$$\mathrm{Ext}^i = H^i(\mathrm{RHom}(C_\bullet, C'_\bullet))$$

to be the cohomology groups of RHom.

Remark 3.4. Note that while $\mathrm{Hom} = \Gamma \circ \mathscr{H}om$, it is *not* true that $\mathrm{Ext}^i = \Gamma \circ \mathscr{E}xt^i$, because Γ is not exact. Instead, these are related by a spectral sequence.

4 The Duality Theorem

Recall that we want to prove the perfectness of the pairing

$$H^i(X, \Lambda) \times H^i_c(X, \Lambda(d)) \to H^{2d}_c(X, \Lambda(d)) \xrightarrow{\mathrm{Tr}} \Lambda \tag{4.1}$$

for smooth X of pure dimension d over a separably closed field. To get a handle on this issue, we'll need to "upgrade" this statement in two ways.

1. The first is to formulate a "local" version of this "global" statement, or in other words to formulate the duality at the level of sheaves. As we saw in the discussion of the trace map, a sheaf-theoretic formulation is more robust with respect to geometric constructions.

2. The second is to formulate a statement at the level of the derived category, i.e. at the level of complexes rather than cohomology.

Therefore, the first step is to recast our work on the trace map in terms of the derived category.

4.1 Derived trace map

In general, $f_!$ does not have a sheaf-theoretic right adjoint. However, $Rf_!$ has a right adjoint $f^!$ in the derived category. For a smooth morphism f of relative dimension d, we give an ad-hoc definition:

$$f^!\mathscr{F} = f^*\mathscr{F}[2d](d).$$

We'll need a version of the trace map in the derived category, which will take the form

$$Rf_!f^!\mathscr{F} \to \mathscr{F}.$$

This is just the counit for the adjunction $(Rf_!, f^!)$. To define this map, it will suffice to show that there is a morphism

$$Rf_!f^!\mathscr{F} \to R^{2d}f_!f^!\mathscr{F}[-2d]$$

with the latter sheaf regarded as a complex in the derived category concentated in degree 0, since we can then compose this with the trace map that we already constructed.

Now, $R^{2d}f_!f^!\mathscr{F}$ is just a homology sheaf of the complex $Rf_!f^!\mathscr{F}$. Usually a complex doesn't map to its own cohomology, but one exception is when the complex is bounded above - then it maps to its top cohomology. (This is a special case of the truncation $\mathscr{G} \to \tau_{\geq 2d}\mathscr{G}$.) So it suffices to show that if $\mathscr{F}$ is a sheaf on X, then $Rf_!f^!\mathscr{F}$ is represented by a complex with support in degrees $[-2d, 0]$. This a consequence of the fact that $f_!$ has cohomological dimension $2d$. It ultimately boils down to a nifty trick in homological algebra, as follows.

Take a compactification $j\colon X \hookrightarrow \overline{X}$ and an injective resolution of $j_!\mathscr{F}$ by a complex $I_\bullet$, which we can assume has the form

$$I^{-2d} \to I_{-2d+1} \to \ldots \to I_0 \xrightarrow{d_0} I^1 \to \ldots$$

Now, consider the truncated complex

$$I^{-2d} \to I_{-2d+1} \to \ldots \to I_{-1} \to Z_0 := \ker d_0.$$

This is still a resolution of $\mathscr{F}$, but perhaps not by injectives. However, we don't *need* a resolution by injectives - we can just as well compute $R\overline{f}_*(j_!\mathscr{F})$ with a complex of f_*-acyclic sheaves. Therefore, it suffices to show that Z_0 is f_*-acyclic. To see this, we break the complex up into a bunch of short exact sequences. Let $Z_i = \ker d_i$. Since the complex is exact above degree $-2d$, we have short exact sequences

$$\begin{aligned} 0 \to Z_{-1} \to &I_{-1} \to Z_0 \to 0 \\ 0 \to Z_{-2} \to &I_{-2} \to Z_{-1} \to 0 \\ &\vdots \\ 0 \to Z_{-2d} \to &I_{-2d} \to Z_{-2d+1} \to 0 \end{aligned}$$

Therefore, since $R^i f_*$ is a cohomological functor we have

$$\begin{aligned} R^1\overline{f}_* Z_0 &\simeq R^2\overline{f}_* Z_{-1} \\ R^2\overline{f}_* Z_{-1} &\simeq R^3\overline{f}_* Z_{-2} \\ &\vdots \\ R^{2d}\overline{f}_* Z_{-2d+1} &\simeq R^{2d+1}\overline{f}_* Z_{-2d} \end{aligned}$$

So $R^1\overline{f}_* Z_0 \simeq R^{2d+1}\overline{f}_* Z_{-2d} = 0$. The higher direct images vanish similarly.

4.2 Formulation of the duality theorem

Let $H_\bullet \in D_-(X)$ and $K_\bullet \in D_+(X)$. Then we have defined $\mathrm{R}\mathscr{H}om(H, K) \in D(X)$.

Now consider a smooth, quasiprojective morphism $f: X \to S$. We claim that there is a map

$$Rf_*\mathrm{R}\mathscr{H}om(H, K) \to \mathrm{R}\mathscr{H}om(Rf_!H, Rf_!K). \tag{4.2}$$

This map is essentially functoriality of the construction $Rf_!$. It can be described as follows. Take a compactification $j: X \hookrightarrow \overline{X}$ and injective resolutions

$$\begin{aligned} j_!H &\xrightarrow{\sim} I \\ j_!K &\xrightarrow{\sim} J \end{aligned}$$

To compute the left side of (4.2), we need to replace $j_!K$ by an injective resolution on X. Fortunately, j^*J is a complex of injectives, since $j^* = j^!$ for open embeddings has the exact left adjoint $j_!$. Therefore,

$$j_!\mathrm{R}\mathscr{H}om(H, K) = j_!\mathscr{H}om(H, j^*J) = \mathscr{H}om(j_!H, J) = \mathscr{H}om(I, J).$$

Now apply $R\overline{f}_*$, to obtain a map

$$R\overline{f}_*(j_!\mathrm{R}\mathscr{H}om(H, K)) = R\overline{f}_*\mathscr{H}om(H, J) \to \mathscr{H}om(R\overline{f}_*I, R\overline{f}_*J) = \mathrm{R}\mathscr{H}om(Rf_!H, Rf_!K).$$

This is the map we promised:

$$Rf_*\mathrm{R}\mathscr{H}om(H, K) \to \mathrm{R}\mathscr{H}om(Rf_!H, Rf_!K) \in D(X).$$

This will an incarnation of the cup product. Now let's extract the "global" and then "cohomological" versions of this map. Taking global sections (i.e. applying $\mathrm{R}\Gamma$), we get a map

$$\mathrm{RHom}(H, K) \to \mathrm{RHom}(Rf_!H, Rf_!K) \in D(\mathrm{Ab}).$$

Taking cohomology, we get a map

$$\mathrm{Ext}^p(H, K) \to \mathrm{Ext}^p(Rf_!H, Rf_!K) \in \mathrm{Ab}.$$

Finally, let's bring in the trace map. If $H = \mathscr{F}$ is a sheaf on X and $K = f^!\mathscr{G}$ for a sheaf $\mathscr{G}$ on S, then we get maps

$$\Delta^1_{X/S}\colon Rf_*\mathrm{R}\mathscr{H}om(\mathscr{F}, f^!\mathscr{G}) \to \mathrm{R}\mathscr{H}om(Rf_!\mathscr{F}, \mathscr{G}). \tag{4.3}$$

$$\Delta^2_{X/S}\colon \mathrm{RHom}(\mathscr{F}, f^!\mathscr{G}) \to \mathrm{RHom}(Rf_!\mathscr{F}, \mathscr{G}). \tag{4.4}$$

$$\Delta^3_{X/S}\colon \mathrm{Ext}^p(\mathscr{F}, f^!\mathscr{G}) \to \mathrm{Ext}^p(Rf_!\mathscr{F}, \mathscr{G}). \tag{4.5}$$

Theorem 4.1 (Grothendieck)**.** *The maps $\Delta^i_{X/S}$ are all isomorphisms.*

In fact, the statement that any one of the $\Delta^i_{X/S}$ is an isomorphism implies that all of them are. It is obvious that $\Delta^1_{X/S}$ isomorphism $\implies$ $\Delta^2_{X/S}$ isomorphism $\iff$ $\Delta^3_{X/S}$ isomorphism. To go the other way, one uses that the sheaves $\mathscr{E}xt^i$ can be recovered as the sheafification of Ext^i on opens.

We shall show shortly that the Poincaré duality statement is captured in the assertion that $\Delta^3_{X/S}$. However, it's more natural to *prove* the fact that $\Delta^1_{X/S}$ is an isomorphism, for reasons we've discussed: it's a *local* statement (at the level of sheaves versus global sections), and a statement about complexes (as opposted to cohomology).

Poincaré duality. Let's see what the theorem says for $S = \mathrm{Spec}\, k$ with k an algebraically field, and f a smooth map of pure relative dimension d. Take $\mathscr{F} = \underline{\Lambda}_X$ and $\mathscr{G} = \underline{\Lambda}_S$. The theorem tells us that we have an isomorphism

$$\mathrm{Ext}^{-p}(\mathscr{F}, \Lambda(d)[2d]) \xrightarrow{\sim} \mathrm{Ext}^{-p}(Rf_!\mathscr{F}, \Lambda).$$

The right hand side can be unravelled as

$$\begin{aligned} \mathrm{Ext}^{-p}(Rf_!\mathscr{F}, \Lambda) &= H^{-p}(\mathrm{RHom}(Rf_!\mathscr{F}, \Lambda)) \\ &= \mathrm{Hom}(H^p(Rf_!\mathscr{F}), \Lambda) \\ &= \mathrm{Hom}(H^p_c(X, \mathscr{F}), \Lambda). \end{aligned}$$

In the second line we used the *injectivity* of $\Lambda = \mathbf{Z}/n\mathbf{Z}$ as a module over itself to see that $\mathrm{Hom}(-, \Lambda)$ commutes with taking cohomology.

On the other hand, the left side can be unravelled as

$$\mathrm{Ext}^{-p}(Rf_!\mathscr{F}, \Lambda(d)[2d]) = \mathrm{Ext}^{2d-p}(\Lambda, \Lambda(d)) = H^{2d-p}(X, \Lambda(d))$$

where we have identified Ext with cohomology because in the category of Λ-modules, $\mathrm{Hom}(\Lambda, -) = \Gamma(-)$. To summarize, the duality theorem tells us that the canonical map

$$H^{2d-p}(X, \Lambda(d)) \to \mathrm{Hom}(H^p_c(X, \mathscr{F}), \Lambda)$$

is an isomorphism. It is not obvious that this is equivalent to the formulation (4.1). In fact we haven't defined the cup product, but the best way to see this is by invoking an axiomatic characterization of the cup product, rather than chasing definitions. Note that this crucially requires having a functorial definition for a very general class of sheaves $\mathcal{F}$, so even if we are mainly interested in the case of constant coefficients we are naturally led to consider general sheaves!

4.3 Reduction to the case of curves

In this section, we'll indicate a reduction of the proof that $\Delta^1_{X/S}$ is an isomorphism to the case when $S = \operatorname{Spec} k = \overline{k}$, and X is a smooth projective curve. This final can is then handled by lifting to characteristic 0, invoking the Lefschetz principle to reduce to the case $k = \mathbf{C}$, and then applying the Artin comparison theorem with the topological Poincaré duality on Riemann surfaces. These things have been or will be discussed elsewhere in the seminar.

The reduction will follow a familiar pattern. First, we need to verify some "formal properties" of the statement: that it can be checked étale locally, and that it is compatible with composition. (As discussed, the latter is where the machinery of the derived category really kicks in.) Second, we need to check it in some elemental cases: for étale maps, and for the projection map $\mathbf{A}^1_S \to S$. The statement for étale maps will be easy, and the statement for general constructible sheaves on $\mathbf{A}^1$ will be bartered for the statement for *constant* étale sheaves on general curves.

Compatibility with compositions. Suppose have morphisms $f\colon X \to S$ and $g\colon Y \to X$. Let $\mathscr{F}$ be a sheaf on S and $\mathscr{G}$ a sheaf on Y.

Lemma 4.2. *If the two maps*

$$\Delta^1(f, Rg_!\mathscr{G}, \mathscr{F})\colon Rf_*\mathrm{R}\mathscr{H}om(Rg_!\mathscr{G}, \mathscr{F}) \to \mathrm{R}\mathscr{H}om(Rf_!Rg_!\mathscr{G}, \mathscr{F})$$
$$\Delta^1(g, \mathscr{G}, f^!\mathscr{F})\colon Rg_*\mathrm{R}\mathscr{H}om(\mathscr{G}, f^!\mathscr{F}) \to \mathrm{R}\mathscr{H}om(Rg_!\mathscr{G}, f^!\mathscr{F})$$

are both isomorphisms, then so is

$$\Delta^1(f\circ g, \mathscr{G}, \mathscr{F})\colon R(f\circ g)_*\mathrm{R}\mathscr{H}om(\mathscr{G}, (g\circ f)^!\mathscr{F}) \to \mathrm{R}\mathscr{H}om(R(f\circ g)_!\mathscr{G}, \mathscr{F})$$

Proof. Using that $Rf_! \circ Rg_! = R(f\circ g)_!$ and $Rf_* \circ Rg_* = R(f\circ g)_*$, we can compare $\Delta^1(g, \mathscr{G}, f^!\mathscr{F}) \circ Rg_*\Delta^1(f, Rg_!\mathscr{G}, \mathscr{F})$ and $\Delta^1(f\circ g, \mathscr{G}, \mathscr{F})$ as maps

$$R(f\circ g)_*\mathrm{R}\mathscr{H}om(\mathscr{G}, (g\circ f)^!\mathscr{F}) \to \mathrm{R}\mathscr{H}om(R(f\circ g)_!\mathscr{G}, \mathscr{F})$$

and we find that they agree, using the compatibility of the trace map with compositions. □

The étale case. The étale case is *used* to deduce the compatibility with localization, so let's do that first.

Lemma 4.3. $\Delta^1_{X/S}$ *is an isomorphism when* $f\colon X \to S$ *is étale.*

Proof. In this case $f_!$ is exact, so $Rf_! = f_!$. Also, $f^! = f^*$. In this case $\Delta^2_{X/S}$ is the adjunction for $(f_!, f^*)$ for open embeddings f. $\square$

Localization on the source. Now we argue that if $\Delta^1_{X/S}$ is an isomorphism for smooth morphisms of the form

$$W \xrightarrow{\text{étale}} \mathbf{A}^d_S \to S \tag{4.6}$$

then it is an isomorphism for X. Using that $p_! p^! \mathscr{F} \to \mathscr{F}$ is surjective for any étale cover p, we can find a resolution of $\mathscr{F}$ of the form

$$\ldots \to (p_1)_! \mathscr{P}_1 \to (p_0)_! \mathscr{P}_0 \to \mathscr{F} \to 0$$

where each $p_i\colon W_i \to X$ is an étale cover of X of the special form (4.6), and each $\mathscr{P}_i$ is a sheaf on W_i. This lets us replace $\mathscr{F}$ by a complex of the form

$$\ldots \to (p_1)_! \mathscr{P}_1 \to (p_0)_! \mathscr{P}_0 \to 0.$$

Then we reduce to the case $\mathscr{F} = p_! \mathscr{P}$ using truncation functors. Now, we want to know if the map

$$\Delta^1_{X/S}\colon Rf_* \mathrm{R}\mathscr{H}om(p_!\mathscr{P}, \mathscr{G}) \to \mathrm{R}\mathscr{H}om(R(f_!)Rp_!\mathscr{P}, \mathscr{G})$$

is an isomorphism. We claim that it suffices to show that

$$\Delta^1_{W/S}\colon R(f \circ p)_* \mathrm{R}\mathscr{H}om(\mathscr{P}, \mathscr{G}) \to \mathrm{R}\mathscr{H}om(Rf_! Rp_! \mathscr{P}, \mathscr{G})$$

is an isomorphism. Indeed, we have a factorization

$$\begin{array}{ccc} R(f \circ p)_* \mathrm{R}\mathscr{H}om(\mathscr{P}, \mathscr{G}) & & \\ \downarrow{\scriptstyle \Delta^1_{W/X}} & \searrow^{\Delta^1_{W/S}} & \\ Rf_* \mathrm{R}\mathscr{H}om(p_!\mathscr{P}, \mathscr{G}) & \xrightarrow[\Delta^1_{X/S}]{} & \mathrm{R}\mathscr{H}om(Rf_! Rp_! \mathscr{P}, \mathscr{G}) \end{array}$$

and we just saw that $\Delta^1_{W/X}$ is an isomorphism.

By arguments already discussed, this reduces us to the case where f is the natural projection map $\mathbf{A}^1_S \to S$ (since we have already handled the étale case), and $\mathscr{F}$ and $\mathscr{G}$ are sheaves (by arguments using truncation). By limit arguments, we may assume that $\mathscr{F}$ and $\mathscr{G}$ are constructible. We next want to trade this case for a scenario in which the map f is a more general curve fibration, but in which $\mathscr{F}$ can be assumed to be the constant sheaf $\underline{\Lambda}$.

To get this, we use that $\mathscr{F}$ can be replaced by a complex of the form

$$\ldots \to \underline{\Lambda}_{W_0!} \to \underline{\Lambda}_{W_1!} \to \mathscr{F}.$$

where W_i is an étale cover of X. here $\underline{\Lambda}_{W_{i!}}$ means proper pushforward from $\mathcal{F}|_{W_i}$, via $\mathcal{F}|_{W_i} \to X \xrightarrow{f} S$. The justification for this is dual to earlier arguments embedding $\mathscr{F}$ into pushforwards of constant sheaves from étale covers. The earlier arguments used that $\mathscr{F} \to p_*p^*\mathscr{F}$ is an injection for any étale cover $\mathscr{F}$; here we use that $p_!p^!\mathscr{F} \to \mathscr{F}$ is a surjection.

Summary. The upshot of the discussion so far has reduced us to the following situation:

- X, S are affine and noetherian.
- $f\colon X \to S$ is a smooth, quasiprojective map of pure relative dimension 1.
- $\mathscr{F} = \underline{\Lambda}_X$,
- $\mathscr{G}$ is a constructible sheaf.

We want to know if the map

$$\Delta^1_{X/S}\colon Rf_*\mathrm{R}\mathscr{H}om(\mathscr{F}, f^!\mathscr{G}) \to \mathrm{R}\mathscr{H}om(Rf_!\mathscr{F}, \mathscr{G})$$

is an isomorphism. Since $Rf_!(\Lambda)$ and $\mathscr{G}$ are constructible, they are constant on an étale neighborhood of a generic point η of S. Therefore, the homology sheaves $R^i\mathscr{H}om(Rf_!\Lambda, \mathscr{G})$ are constant on an étale neighborhood of η. We would like to know the same for the left hand side. Unfortunately, we don't have a base change result for the morphism f, since it is not proper. However, we can get around this in nice situations using the Gysin sequence, i.e. the purity theorem that Bogdan explained last time.

To explain this, we consider the following general situation. Let $i\colon Y \hookrightarrow Z$ be regular embedding of codimension d and $U = Z \setminus Y$ the open complement. For any sheaf $\mathscr{F}$ on Z, we define a sheaf $\underline{\Gamma_Y}(\mathscr{F})$ to be $\ker(\mathscr{F} \to j_*j^*\mathscr{F})$ ("sections supported on Y"). Its global sections are denoted

$$\Gamma_Y(\mathscr{F}) = \ker(\mathscr{F}(Z) \to \mathscr{F}(U)).$$

The purity theorem tells us that

$$R^q\underline{\Gamma_Y}(\mathscr{F}) = \begin{cases} 0 & \nu \neq 2d, \\ i^*\mathscr{F}(-d) & q = 2d. \end{cases}$$

Remark 4.4. There is an exact triangle in the derived category

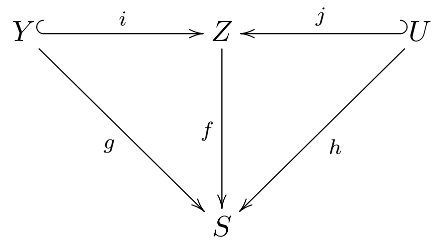
$$i_! i^! \mathscr{F} \to \mathscr{F} \to Rj_* j^* \mathscr{F}$$

So one can think of $\underline{\Gamma_Y} = R^0(i_! i^!)$. The purity theorem tells us that

$$i_! i^! \mathscr{F} = i_!(i^* \mathscr{F}(-d)[-2d]).$$

Note that this is an "extension" of the formula $f^! = f^*(d)[2d]$ for a smooth map of relative dimension d.

Now consider the diagram



This implies that there is a long exact sequence

$$\ldots \to R^{q-2d} g_*(i^* \mathscr{F}(-d)) \to R^q f_* \mathscr{F} \to R^q h_*(\mathscr{F}|_U) \to R^{q-2d+1} g_*(i^* \mathscr{F}(-d)) \to \ldots$$

In particular, the base change and constructibility results for the *proper* pushforwards $R^q f_* \mathscr{F}$ and $R^{q-2d} g_*(i^* \mathscr{F}(-d))$ carry over to $R^q h_*(\mathscr{F}|_U)$.

Now we want to apply the preceding discussion to our situation of a relative curve $X \to S$. We may not be able to find a globally smooth relative compactification with smooth complement (as required for the Purity Theorem), but we can do so at least over the generic point: $X_\eta \hookrightarrow \overline{X}_\eta$. After passing to a finite extension, we can assume that the complement $\overline{X}_\eta \setminus X_\eta$ is just a collection of points. Then by spreading out, we obtain a compactification $X_V \to \overline{X}_V$ over some *(Zariski) open neighborhood* of η with the property that $\overline{X}_V \setminus X_V$ is smooth.

We can also extend the constant sheaf Λ on X_V to the constant sheaf Λ on $\overline{X}_V$. Furthermore, the sheaf $\mathscr{G}$ was assumed constructible, so $R^i \mathscr{H}om(\mathscr{F}, f^! \mathscr{G})$ is constructible. By smooth base change for Zariski open subsets (which is really much simpler than smooth base change), we deduce:

Proposition 4.5. *$Rf_*(f^! \mathscr{G})$ is constant on an étale neighborhood of η, and the stalk $Rf_*(f_! \mathscr{G})_{\overline{\eta}}$ is canonically isomorphic to $Rf_{\overline{\eta},*}(f^!_{\overline{\eta}} \mathscr{G}_{\overline{\eta}})$.*

At this point we bring in our assumption that $\Delta^1_{X/S}$ is an isomorphism whenever $S = \operatorname{Spec} \overline{k}$ and X is a smooth curve. As discussed above, the case where X is *projective* can be established by lifting to characteristic 0, and using the Artin comparison theorem. We can then deduce the result for general smooth curves by choosing a compactification $j\colon X \hookrightarrow \overline{X}$, and using the Gysin exact sequence

$$H^0_c(\overline{X} \setminus X, \mu_n) \to H^1_c(X, \mu_n) \to H^1(\overline{X}, \mu_n) \to 0$$

together with the Gysin sequence

$$0 \to H^1(\overline{X}, \mu_n) \to H^1(X, \mu_n) \to H^0(\overline{X} \setminus X, \mu_n)$$

By Proposition 4.5, this tells us that the map

$$\Delta^1_{X/S}\colon Rf_* \mathrm{R}\mathscr{H}om(\Lambda, f^! \mathscr{G}) \to \mathrm{R}\mathscr{H}om(Rf_! \Lambda, \mathscr{G})$$

is an isomorphism on stalks at $\overline{\eta}$.

The desired result can then be deduced from noetherian induction on the support of $\mathscr{G}$.[1]

A Some technical results concerning proper pushforward

Here we gather some technical results which were used in the main body of the notes.

A.1 A projection formula

Proposition A.1. *Let $f\colon X \to S$ be a quasiprojective morphism. For $\mathscr{F} \in D_-(S)$ and $\mathscr{G} \in D(X)$ we have a natural isomorphism*

$$\mathscr{F} \otimes^L Rf_! \mathscr{G} \simeq Rf_!(f^* \mathscr{F} \otimes^L \mathscr{G}).$$

Proof. By passing to a compactification of f if necessary, we may and do assume that f is proper. First let's define the map. By adjunction it suffices to define a map

$$f^*(\mathscr{F} \otimes Rf_* \mathscr{G}) \to f^* \mathscr{F} \otimes \mathscr{G},$$

which is the one obtained from the counit $f^* Rf_* \mathscr{G} \to \mathscr{G}$.

First we claim that if $\mathscr{F}$ is flat and $\mathscr{G}$ is f_*-acyclic, then

1. $f_*(f^* \mathscr{F} \otimes \mathscr{G}) = \mathscr{F} \otimes f_* \mathscr{G}$
2. $R^\nu f_*(f^* \mathscr{F} \otimes \mathscr{G}) = 0$ for $\nu > 0$.

It suffices to checks these statements on stalks. By proper base change, we may assume that S is the spectrum of a separably closed field. Then $\mathscr{F}$ is a constant sheaf, with value group a flat Λ-modules. Therefore it is projective, hence a direct

[1] There was an error in my original attempt to flesh out this step. I hope to put in a complete and correct explanation eventually.

summand of a free module. So it suffices to see the result when $\mathscr{F}$ is a free Λ-module, in which case both assertions amount to the compatibility of H^ν with direct sums.

Now, resolve $\mathscr{F}$ by a complex of flat sheaves and $\mathscr{G}$ by a complex of f_*-acyclic sheaves. By the claim, we know that $f^*\mathscr{F} \otimes \mathscr{G}$ is f_*-acyclic, so that

$$Rf_*(f^*\mathscr{F} \otimes \mathscr{G}) = f_*(f^*\mathscr{F} \otimes \mathscr{G}).$$

Then using the second property, we are done.

$\square$

A.2 Compatibility with composition

Proposition A.2. *We have*

$$R(f \circ g)_! = Rf_! \circ Rg_!$$

Proof. We choose a diagram of relative compactifications:

$$\begin{array}{ccccc} Y & \xhookrightarrow{\ i\ } & \overline{Y} & \xhookrightarrow{\ j\ } & \overline{\overline{Y}} \\ {\scriptstyle g}\downarrow & \swarrow {\scriptstyle \overline{g}} & & & \downarrow{\scriptstyle \overline{\overline{g}}} \\ X & & \xhookrightarrow{\quad k \quad} & & \overline{X} \\ {\scriptstyle f}\downarrow & & \swarrow {\scriptstyle \overline{f}} & & \\ S & & & & \end{array}$$

Let $\mathscr{F}$ be a sheaf on Y. To start computing, choose an injective resolution $j_!\mathscr{F} \xrightarrow{\sim} J_\bullet$. Then $j^*U_\bullet$ is an injective resolution of $i_!\mathscr{F}$. By definition,

$$R(f \circ g)_!\mathscr{F} = (\overline{f} \circ \overline{\overline{g}})_* J_\bullet.$$

On the other hand,

$$Rf_! \circ Rg_!\mathscr{F} = Rf_!(\overline{g}_* I_\bullet) = R\overline{f}_*(k_!\overline{g}_* I_\bullet).$$

Since $j_! I_\bullet = J_\bullet$, the result will follow if we know that

$$\overline{\overline{g}}_* j_! I_\bullet = k_! \overline{g}_* I_\bullet.$$

Therefore it suffices to establish the following general lemma. $\square$

Lemma A.3. *Consider a commutative diagram*

$$\begin{array}{ccc} Y & \xhookrightarrow{\ j\ } & \overline{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \overline{g}} \\ X & \xhookrightarrow[\ k\ ]{} & \overline{X} \end{array}$$

where j and k are open embeddings and g is proper. There is a natural transformation

$$k_! g_* \to \overline{g}_* j_!$$

and the induced natural transformation in the derived category

$$k_! R g_* \to R\overline{g}_* j_!$$

is an isomorphism if $\overline{g}$ is proper.

Proof. First we define the natural transformation

$$k_! g_* \to \overline{g}_* j_!$$

By adjunction, it suffices to define a natural transformation

$$\overline{g}^* k_! g_* \to j_!$$

By compatibility of extension-by-zero with pullbacks, we have $\overline{g}^* k_! = j_! g^*$. Applying the counit gives

$$\overline{g}^* k_! g_* = j_! g^* g_* \to j_!$$

as desired. Moreover, by compatibility of extension-by-zero with base change we have that $k_! g_* \to \overline{g}_* j_!$ is an isomorphism over X.

The induced natural transformation on the derived category can be described as follows. For $\mathscr{F} \in D_+(Y)$, pick an injective resolution $j_! \mathscr{F} \xrightarrow{\sim} J_\bullet$. This induces an injective resolution $I_\bullet := j^! J_\bullet$ of $\mathscr{F}$. Applying the preceding construction to $I_\bullet$, we obtain a map

$$k_! R f_* \mathscr{F} = k_! j_* I \to \overline{f}_* j_! I \simeq \overline{f}_* J = R\overline{f}_* j_! \mathscr{F}$$

which is an isomorphism over X.

Now assume that $\overline{f}$ is proper. Then by the proper base change theorem, the sheaf $R\overline{f}_* i_! \mathscr{F}$ is only supported over X, so the map is an isomorphism. □

B The cup product

Here we review the construction of the cup product for derived functors. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of Λ-modules on X, the goal is to define a cup product

$$H^p(X, \mathcal{F}) \otimes H^q(X, \mathcal{G}) \to H^{p+q}(X, \mathcal{F} \otimes_\Lambda \mathcal{G}).$$

The basic idea is to take tensor products of injective resolutions of $\mathcal{F}$ and $\mathcal{G}$ and try to make them into an injective resolution for $\mathcal{F} \otimes \mathcal{G}$. However, some care must be taken because the naïve tensor product is not functorial in the derived category - it must be replaced by the derived tensor product $\overset{L}{\otimes}$. The definition, then, is that we have a map of complexes in the derived category

$$\mathcal{F} \overset{L}{\otimes} \mathcal{G} \to \mathcal{F} \otimes \mathcal{G}$$

inducing

$$H^*(X, \mathcal{F}) \otimes H^*(X, \mathcal{G}) \to H^*(X, \mathcal{F} \overset{L}{\otimes} \mathcal{G}) \to H^*(X, \mathcal{F} \otimes \mathcal{G}).$$

Let us spell out what is actually going on here. First let $P^\bullet(\mathcal{F})$ denote a projective resolution of $\mathcal{F}$. Then we can form an injective resolution of $P^\bullet(\mathcal{F})$ by projectives, for instance by the Godement resolution. So we have an isomorphism in the derived category

$$\mathcal{F} \sim I^\bullet(P^\bullet(\mathcal{F})).$$

Also, let $I^\bullet(\mathcal{G})$ be an injective resolution of $\mathcal{G}$. We let $J^\bullet$ be an injective resolution of $I^\bullet(P^\bullet(\mathcal{F})) \otimes I^\bullet(\mathcal{G})$, so $J^\bullet$ is an injective model of $\mathcal{F} \overset{L}{\otimes} \mathcal{G}$. The map

$$I^\bullet(P^\bullet(\mathcal{F})) \otimes I^\bullet(\mathcal{G}) \to J^\bullet$$

induces

$$\begin{aligned} H^p(X, \mathcal{F}) \otimes H^q(X, \mathcal{G}) &= H^p(X, I^\bullet(P^\bullet(\mathcal{F}))) \otimes H^q(X, I^\bullet(\mathcal{G})) \\ &\to H^{p+q}(X, J^\bullet) = H^{p+q}(X, P^\bullet(\mathcal{F}) \otimes \mathcal{G}) \to H^{p+q}(X, \mathcal{F} \otimes \mathcal{G}). \end{aligned}$$

This is the cup product.