Basic Adic Formalism

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1 Motivation

Consider the étale cohomology of a normal Noetherian scheme X (which we further assume connected, hence irreducible) with coefficients in a field F of characteristic 0. We will now show that the higher étale cohomology vanishes.

Let η be the generic point of X; we note that the constant sheaf $\underline{F}$ on X is equal to $\eta_* F$ (i.e., the natural map $\underline{F} \to \eta_* \eta^* \underline{F}$ is an isomorphism). This is because any E which is quasicompact and étale over X must also be normal and Noetherian, so the connected components of E are its irreducible components, with their generic points exactly the points of the fiber E_η.

Hence we reduce to showing that the higher étale cohomology of $\eta_* F$ vanishes.

Consider the Leray spectral sequence

$$E_2^{i,j} : H^i_{\acute{e}t}(X, R^j \eta_* F) \to H^{i+j}(\eta, F).$$

We note that $E_2^{i,0} = H^i(X, \underline{F})$.

Since $R^j \eta_* F$ is the sheafification of the presheaf $E \mapsto H^j(E_\eta, F)$, and E_η has finitely many points (all of which are closed since E_η is η-finite), we may compute $H^i_{\acute{e}t}(E_\eta, F)$ as Galois cohomology, which vanishes when $j > 0$.

Furthermore, for $i + j > 0$, we note that $H^{i+j}(\eta, F)$ vanishes for the same reasons (again we compute via Galois cohomology).

It is then clear that the higher étale cohomology vanishes as we desired to show.

As a corollary, note that $H^i(X, \mathbb{Z}_\ell)$ is torsion for $i > 0$. To prove this, we note that $H^i(X, \mathbb{Z}_\ell)[1/\ell] \simeq \varinjlim_\ell H^i(X, \mathbb{Z}_\ell)$, where the maps in the direct limit are multiplication by ℓ. Since we may pass the direct limit inside the cohomology, we find $\varinjlim_\ell H^i(X, \mathbb{Z}_\ell) \simeq H^i(X, \mathbb{Q}_\ell) = 0$ by the discussion above. Hence $H^i(X, \mathbb{Z}_\ell)$ must be torsion.

This example shows that making a useful cohomology theory with coefficients in a field F of characteristic 0 by taking coefficients in the constant sheaf $\underline{F}$ is not appropriate. We look to abelian varieties for a better approach.

Now we consider a smooth proper group scheme $A \to S$ with connected fibers of dimension g; in §6.1 (see Cor. 6.5) of [2] it is shown that such a group scheme is commutative.

Now take ℓ a prime which is invertible on S, and consider the map $[\ell^n] : A \to A$ given by multiplication by ℓ^n on A. By checking on geometric fibers of $A \to S$, we see that $[\ell^n]$ is finite étale with degree ℓ^{2ng}.

The scheme-theoretic 'kernel' of this map is the fiber product of the maps $[\ell^n]$ against the zero map from S to A. We call this kernel $A[\ell^n]$; it is a finite étale commutative S-group with geometric fibers $(\mathbb{Z}/\ell^n\mathbb{Z})^{2g}$. The diagram below shows the fiber product.

$$\begin{array}{ccc} A[\ell^n] & \longrightarrow & S \\ \downarrow & & \downarrow \\ A & \xrightarrow{\ell^n} & A \end{array}$$

In the case of $S = \operatorname{Spec} k$ for k a field, it is known that the **Tate module** $T_\ell(A) = \varprojlim A[\ell^n](k_s)$ (where k_s is the separable closure of k) is a useful object of study, such as for analyzing $\operatorname{End}_k(A)$ and for studying the action of $\operatorname{Gal}(k_s/k)$ on $T_\ell(A)$ when k finite.

When $k = \mathbb{C}$, we note that $A(\mathbb{C}) = V/\Lambda$ for a vector space V and lattice Λ. Then we have an isomorphism $A(\mathbb{C})[\ell^n] \simeq \Lambda/\ell^n\Lambda$ (compatible with change in n), so in fact $T_\ell(A) = \mathbb{Z}_\ell \otimes \Lambda = H_1(A(\mathbb{C}), \mathbb{Z}_\ell)$, which has $\mathbb{Z}_\ell$-dual $H^1(A(\mathbb{C}), \mathbb{Z}_\ell)$. Hence we have $H^1(A(\mathbb{C}), \mathbb{Z}_\ell) = \varprojlim H^1(A(\mathbb{C}), \mathbb{Z}/\ell^n\mathbb{Z})$.

One might expect the ℓ-adic cohomology groups in the more general case to similarly be usefully defined by the inverse limit of an appropriate projective system of cohomologies with finite coefficients.

We choose not to pass to the inverse limit at the sheaf level for the following reason. Consider for example the inverse limit sheaf $\varprojlim \mu_{\ell^{n+1}}$ on $(\operatorname{Spec} \mathbb{Q})_{ét}$. Since any number field has only finitely many roots of unity it is clear that $\left(\varprojlim \mu_{\ell^{n+1}}\right)(K) = \{1\}$ for any number field K. The corresponding projective system, however, is not trivial.

The same occurs if we take the inverse limit of the projective system $A[\ell^n](K)$ for A an abelian variety over $\mathbb{Q}$, since $A(K)_{tor}$ is finite for any number field K; therefore $\left(\varprojlim A[\ell^n]\right)(K)$ is also trivial.

This inspires us to seek a formalism to work with projective systems without passing to the inverse limit, since often taking the inverse limit for étale sheaves causes a loss of data.

We will also need this to define ℓ-adic higher direct image 'sheaves' $R^i f_* \mathbb{Z}_\ell$ as a projective system $(R^i f_*(\mathbb{Z}/\ell^n\mathbb{Z}))_{n\geq 0}$ for a proper map f (and likewise for $R^i f_!$ for f separated of finite type).

2 Adic Projective Systems

We now consider the category of finite modules over a Noetherian ring A which is I-adically complete and separated for an ideal $I \subseteq A$. Define $A_n := A/I^{n+1}$.

Given a finite A-module M, we may define a projective system $(M_n)_{n\geq 0}$ with $M_n := M/I^{n+1}M$. It is clear that each M_n is a finite A_n-module and that the natural map $A_n \otimes_{A_{n+1}} M_{n+1} \to M_n$ is an isomorphism.

This motivates the definition of a special property for projective systems:

Definition 2.1. *A projective system $(M_n)_{n\geq 0}$ of finite modules over the A_n is* **strictly I-adic** *if the natural map $A_n \otimes_{A_{n+1}} M_{n+1} \to M_n$ is an isomorphism for all $n \geq 0$.*

We now consider the category of strictly I-adic projective systems (M_n) over the A_n; clearly the operation

$M \rightsquigarrow \big(M/I^{n+1}M\big)$ is a functor from the category of finite A-modules to this category. This gives an equivalence of categories (see [1],0_I,7.2.9) via the inverse functor $(M_n) \rightsquigarrow \varprojlim M_n$.

However, this equivalence of categories does not serve our purpose: we cannot work directly with strict projective systems without passing to an inverse limit because certain module operations (when applied termwise) are incompatible with the strict I-adic property.

For example, given an A-linear map of finite A-modules $T : M \to N$ (with corresponding map of projective systems $(T_n)_{n\geq 0}$) we have an isomorphism $\operatorname{coker}(T)_n \simeq \operatorname{coker}(T_n)$, but in general we do not get a similar isomorphism between kernels. In fact, $(\ker(T_n))$ may not be a strictly I-adic projective system.

For example, let the map $T : \mathbb{Z}_\ell \to \mathbb{Z}^2/\ell^2\mathbb{Z}$ be reduction by ℓ^2 on $\mathbb{Z}_\ell$ as a $\mathbb{Z}_\ell$-module; we note that $\ker T_0 = \ker T_1 = 0$ and $\ker T_n \simeq \mathbb{Z}/\ell^{n-2}\mathbb{Z}$; it is simple to check that $(\ker T_n)$ is not strictly I-adic.

It therefore makes sense to consider more than just the strictly I-adic projective systems. The categorical equivalence given above with inverse limits motivates the idea of a **null system**: a projective system whose ν-fold composites of transition maps vanish for ν some fixed large integer depending on the projective system.

Note that a null system will have inverse limit 0. (The converse is not true; let $A = M_n = \mathbb{Z}_\ell$ for all n, with transition maps given by multiplication by ℓ.) Now we again consider the issue that the natural maps $f_n : \ker(T)_n \to \ker(T_n)$ are not necessarily isomorphisms. However, we claim that the projective systems $(\ker f_n)$ and $(\operatorname{coker} f_n)$ are null systems.

We now show that $(\ker f_n)$ is in fact a null system; the proof for $(\operatorname{coker} f_n)$ is similar.

Proposition 2.2. *Let T be an A-linear map of finite A-modules $T : M \to N$ (with corresponding map of projective systems $(T_n)_{n\geq 0}$), with the natural maps $f_n : \ker(T)_n \to \ker(T_n)$. The projective system $(\ker f_n)_{n\geq 0}$ is a null system.*

Proof. We have an exact sequence

$$0 \to \ker T \to M \to \operatorname{im} T \to 0,$$

which gives an exact sequence

$$\operatorname{Tor}^1_A(\operatorname{im} T, A_n) \to (\ker T)_n \to M_n \to (\operatorname{im} T)_n \to 0.$$

Considering the latter exact sequence, we see that it suffices to show that $\big(\operatorname{Tor}^1_A(\operatorname{im} T, A_n)\big)_{n\geq 0}$ is a null system, since the map $(\ker T)_n \to M_n$ is the same as the natural map f_n composed with the inclusion $\ker T_n \hookrightarrow M_n$.

We claim more generally that for any finite A-module M' that $\big(\operatorname{Tor}^1_A(M', A_n)\big)$ forms a null system. To show this we choose a presentation $M' \simeq F/K$ for F a finite free A-module; then we have $\operatorname{Tor}^1_A(M', A_n) \simeq (I^{n+1}F \cap K)/I^{n+1}K$ for all n (and this isomorphism is compatible with the transition maps of the projective system).

To exploit this, we recall the Artin-Rees Lemma:

Lemma 2.3 (Artin-Rees)**.** *For R a Noetherian ring with ideal I and M a finitely generated R-module with submodule N, there exists $k > 0$ such that for all $n > k$ we have $I^nM \cap N = I^{n-k}(I^kM \cap N)$.*

Now applying the Artin-Rees Lemma we can choose $e > 0$ such that for $m > e$,

$$I^mF \cap K = I^{m-e}(I^eF \cap K) \subseteq I^{m-e}K. \tag{2.4}$$

It is then clear that in sufficiently large degrees, e-fold composites of the transition maps on $\mathrm{Tor}^1_A(M', A_n) \simeq (I^{n+1}F \cap K)/I^{n+1}K$ must vanish. Hence $(\ker f_n)$ is a null system as we desired to show. $\square$

It is now clear that to develop an appropriate 'abelian category' formalism, we must create a category of projective systems with a mechanism that causes null systems to be zero objects and maps with null systems for kernel and cokernel to be isomorphisms.

3 Artin-Rees Categories

We now consider the category of projective systems $M_\bullet = (M_n)_{n\in\mathbb{Z}}$ of A-modules such that $M_n = 0$ for $n \ll 0$. We make no further assumptions on the M_n - in particular, the M_n may not be finite A-modules, and it is not necessarily the case that $I^{n+1}M_n = 0$ for any n.

We begin by defining a 'shift' in keeping with the ideas of last section. Let $M_\bullet[d] := (M_{n+d})_{n\in\mathbb{Z}}$ for any $d \in \mathbb{Z}$. We get a map of projective systems $i_{M_\bullet,d} : M_\bullet[d] \to M_\bullet$ given by the d-fold composite of transition maps.

Note that $i_{M_\bullet,d}$ induces an isomorphism of inverse limits, which motivates the definition of the 'morphisms' in the following category:

Definition 3.1. *The* **Artin-Rees category** *of A-modules has objects $M_\bullet = (M_n)_{n\in\mathbb{Z}}$ which are projective systems of A-modules with $M_n = 0$ for $n \ll 0$, and the morphisms in this category are the elements of the set*

$$\mathrm{Hom}_{\text{A-R}}(M_\bullet, N_\bullet) := \varinjlim \mathrm{Hom}(M_\bullet[d], N_\bullet) \tag{3.2}$$

where $\mathrm{Hom}(M_\bullet[d], N_\bullet)$ is the A-module of maps of projective systems, and the transition maps in the direct limit are composition with $i_{M_\bullet[d],1}$.

Composition of morphisms in the Artin-Rees category is defined via shifts in the obvious way. For example, the Artin-Rees morphism $j_{M_\bullet,d} : M_\bullet \to M_\bullet[d]$ represented by the identity map in $\mathrm{Hom}(M_\bullet[d], M_\bullet[d])$ provides an inverse to $i_{M_\bullet,d}$ in the Artin-Rees category.

The Artin-Rees category is an abelian category, since forming kernels, images, and cokernels of morphisms termwise (after applying a shift on the source of the morphism if necessary) satisfies the categorical requirements.

Now we describe the zero objects in the Artin-Rees category.

Definition 3.3. *An object $M_\bullet$ in the Artin-Rees category is a* **null system** *if for some $\nu \geq 0$ the map $M_{n+\nu} \to M_n$ vanishes for all n; equivalently, for some $d \geq 0$ the map $i_{M_\bullet,d}$ vanishes as an Artin-Rees morphism.*

To see the equivalence in the definition above, we need only check the definition of a morphism in the Artin-Rees category. Because $i_{M_\bullet,d}$ is an isomorphism for all $d \geq 0$, its vanishing is equivalent to $M_\bullet$ being a zero object - hence we have a category in which null systems are precisely the zero objects.

We now wish to check the compatibility of the Artin-Rees category (specifically, null systems) with inverse limits in the following lemma.

Lemma 3.4. *For $X_\bullet, Y_\bullet$ projective systems over a ring A with $X_n = Y_n = 0$ for $n \ll 0$ and a map of projective systems $f_\bullet : X_\bullet \to Y_\bullet$ such that $(\ker f_n)_{n\in\mathbb{Z}}, (\mathrm{coker}\, f_n)_{n\in\mathbb{Z}}$ are null systems, $f_\bullet$ gives an isomorphism*

$$\varprojlim X_n \simeq \varprojlim Y_n. \tag{3.5}$$

Proof. By hypothesis, we may choose e, e' so that e-fold and e'-fold composites of transition maps vanish on $(\ker f_n)_{n\in\mathbb{Z}}$, $(\operatorname{coker} f_n)_{n\in\mathbb{Z}}$ respectively.

We will construct maps $Y_{n+e+e'} \to X_n$ that compose on both sides with the maps f_n to give $i_{X_\bullet,e+e'}, i_{Y_\bullet,e+e'}$ on each term; this shows that $f_\bullet$ can be composed on both sides with another map to give an isomorphism on inverse limits, so in fact $f_\bullet$ gives an isomorphism on inverse limits.

We construct a commutative diagram with exact rows below.

$$
\begin{array}{ccccccc}
 & & X_{n+e+e'} & \xrightarrow{f_{n+e+e'}} & Y_{n+e+e'} & \longrightarrow & \operatorname{coker} f_{n+e+e'} \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 0} \\
\ker f_{n+e} & \longrightarrow & X_{n+e} & \xrightarrow{f_{n+e}} & Y_{n+e} & \longrightarrow & \operatorname{coker} f_{n+e} \\
\downarrow{\scriptstyle 0} & & \downarrow & & \downarrow & & \\
\ker f_n & \longrightarrow & X_n & \xrightarrow{f_n} & Y_n & &
\end{array}
$$

Note that the leftmost and rightmost vertical maps are 0 by our choice of e, e'.

We wish to construct a map $Y_{n+e+e'} \to X_n$. For $y_{n+e+e'} \in Y_{n+e+e'}$, we see that its image in $\operatorname{coker} f_{n+e}$ vanishes; hence we can lift $y_{n+e+e'}$ to $x_{n+e} \in X_{n+e}$ such that $f_{n+e}(x_{n+e})$ is the image of $y_{n+e+e'}$ in Y_{n+e}.

However, x_{n+e} is defined only up to adding an element of $\ker f_{n+e}$. Since $\ker f_{n+e}$ vanishes in $\ker f_n \subseteq X_n$, we pass to X_n to resolve the ambiguity.

Hence by diagram chasing we get a map $Y_{n+e+e'} \to X_n$; the proof that it satisfies the desired properties is omitted, since it is clear from further diagram chasing.

□

In summary, the Artin-Rees category is an abelian category in which null systems are the zero objects and isomorphisms induce isomorphisms on inverse limits. (Artin-Rees morphisms which induce isomorphisms on inverse limits are not necessarily isomorphisms in the Artin-Rees category. For example, if we take $A = M_n = \mathbb{Z}_\ell$ and $N_n = \mathbb{Z}/\ell^2\mathbb{Z}$ for all n, with the transition maps in $M_\bullet, N_\bullet$ multiplication by ℓ, the map $f_\bullet : M_\bullet \to N_\bullet$ given by reducing modulo ℓ^2 on each term automatically gives an isomorphism on inverse limits since $\varprojlim M_n = \varprojlim N_n = 0$. However, we may check easily that $(\ker f_n)$ is not a null system.)

3.1 Artin-Rees Adic Objects

We now wish to define strictly I-adic objects in the Artin-Rees category. Henceforth we assume A/I (hence A/I^n for $n \geq 1$) is Artinian.

Definition 3.6. *An object $M_\bullet$ in the Artin-Rees category is* **strictly I-adic** *if $M_n = 0$ for $n < 0$ and M_n is finite over $A_n = A/I^{n+1}$, with the natural map $A_n \otimes_{A_{n+1}} M_{n+1} \to M_n$ an isomorphism for $n \geq 0$.*

The Artinian property of A/I forces the M_n to be of finite length.

Definition 3.7. *An object $M_\bullet$ in the Artin-Rees category is* **Artin-Rees I-adic** *if it is isomorphic in the Artin-Rees category to a strictly I-adic object.*

Note that Artin-Rees I-adic objects need not be strictly I-adic even in large degrees, as in the following example.

Example 3.8. *Consider M a finite A-module with associated strict object $M_\bullet = \big(M/I^{n+1}M\big)$. We define $M'_\bullet$ by $M'_n = A_{n-d}^{\oplus(n-d)} \oplus M/I^{n-d}M$ for $n > d$ and $M'_n = 0$ for $n \leq d$, with transition maps that are 0 on the $A_{n-d}^{\oplus(n-d)}$ and projection on the $M/I^{n-d}M$.*

It is clear that $M'_\bullet$ is not strictly I-adic, but we may construct the obvious Artin-Rees isomorphism $M'_\bullet \to M_\bullet[-d]$ and compose with $i^{-1}_{M_\bullet[-d],d}$ to see that $M'_\bullet$ is Artin-Rees isomorphic to $M_\bullet$, so $M_\bullet$ is Artin-Rees I-adic.

For $M_\bullet$ a strictly I-adic object, any map from M_{n+d} to an A_n-module will factor through the natural isomorphism $A_n \otimes_{A_{n+d}} M_{n+d} \to M_n$; hence Artin-Rees morphisms of strict I-adic objects $M_\bullet, M'_\bullet$ factor through $i_{M_\bullet,d}$ and an ordinary map of projective systems $M_\bullet \to M'_\bullet$. That is, for $M_\bullet, M'_\bullet$ strict I-adic, we have the natural isomorphism

$$\mathrm{Hom}(M_\bullet, M'_\bullet) \simeq \mathrm{Hom}_{\text{A-R}}(M_\bullet, M'_\bullet).$$

This is very useful.

We now wish to verify that the Artin-Rees category gives us the mechanism to deal with projective systems termwise rather than by passing to and from inverse limits. We continue to assume A/I is Artinian.

Theorem 3.9. *The full subcategory of Artin-Rees I-adic objects is stable under the formation of kernels and cokernels, with $M_\bullet \rightsquigarrow \varprojlim M_n$ giving an equivalence of categories with the category of finite A-modules.*

Proof. Consider an Artin-Rees morphism $f : M_\bullet \to N_\bullet$ of Artin-Rees I-adic objects. Note that we can compose with Artin-Rees isomorphisms so that $M_\bullet, N_\bullet$ are strict I-adic objects and $f_\bullet$ is an ordinary map of projective systems (as shown above).

The case of kernels was handled with the Artin-Rees Lemma in the previous section, as in Proposition 2.2; applying Lemma 3.4 makes it clear that forming kernels in the Artin-Rees category is compatible with forming kernels in the category of finite A-modules. Images may also be handled with a combination of the Artin-Rees Lemma and Lemma 3.4.

The case of cokernels is similar and in fact easier, since for a map of projective systems $f_\bullet$ we have an isomorphism $\operatorname{coker} f_n \simeq (\operatorname{coker} f)_n$. ☐

We note that the property of being Artin-Rees I-adic is dependent on I. For example, even though I^2 and I give the same topology on A, note that for $I \neq 0$ the Artin-Rees object $\big(A/I^{2n}\big)_{n\in\mathbb{Z}}$ (define $A/I^{2n} = 0$ for $n < 0$) is not Artin-Rees I-adic, due to the following lemma.

Lemma 3.10. *For $M_\bullet$ Artin-Rees I-adic, there exist $e, \nu \in \mathbb{Z}$ with $e \geq 0$ such that for $n \gg 0$ the image of $I^{n+1+\nu}M_n$ vanishes in M_{n-e}.*

Proof. By hypothesis, we can find a strict I-adic object $N_\bullet$ and $f_\bullet \in \mathrm{Hom}_{\text{A-R}}(M_\bullet, N_\bullet)$ an Artin-Rees isomorphism with inverse $g_\bullet \in \mathrm{Hom}_{\text{A-R}}(N_\bullet, M_\bullet)$.

We pick d, d' so that we may represent $f_\bullet$ and $g_\bullet$ by termwise maps $f_n : M_n \to N_{n-d}, g_n : N_n \to M_{n-d'}$. Passing to larger d, d' if necessary, the composition of maps $g_{n-d} \circ f_n$ is the $(d + d')$-fold composite of transition maps on each term.

Because $N_\bullet$ is strictly I-adic, we observe that N_n is killed by I^{n+1} as an A-module. Therefore the image of f_n is killed by I^{n+1-d} since $\operatorname{im} f_n \subseteq N_{n-d}$. So the image of the $(d + d')$-fold composite of transition maps is also killed by I^{n+1-d}, from which it is clear that $I^{n+1-d}M_n$ vanishes in $M_{n-d-d'}$. We let $\nu = -d, e = d + d'$ to complete the proof. □

The preceding lemma motivates the hypothesis of the following theorem concerning exact sequences in the Artin-Rees category.

Theorem 3.11. *Given a short exact sequence in the Artin-Rees category*

$$0 \to M'_\bullet \to M_\bullet \to M''_\bullet \to 0 \tag{3.12}$$

where $M'_\bullet, M''_\bullet$ are Artin-Rees I-adic, if there exist $e \geq 0$ and ν such that the image of $I^{n+1+\nu}M_n$ vanishes in M_{n-e}, then $M_\bullet$ is Artin-Rees I-adic.

We note that the hypothesis on $M_\bullet$ is satisfied when there exists ν such that $I^{n+1+\nu}M_n = 0$ for $n \gg 0$. This stronger condition is invariant under shift (though ν may change) and invariant under the application of A-linear functors termwise. Furthermore, it will hold in the cases we need for applications of Theorem 3.11.

Proof. We specify the convention that $I^r = A$ for $r \leq 0$.

By hypothesis the natural map $M_\bullet \to (M_n/I^{n+1+\nu}M_n)$ has null kernel and cokernel systems, so it is an Artin-Rees isomorphism.

Therefore by composing maps in (3.12) with Artin-Rees isomorphisms, we may assume that $M'_\bullet, M''_\bullet$ are strict I-adic and that there exists $d \geq 0$ such that for any $n \in \mathbb{Z}$ we have $I^{n+1+d}M_n = 0$. Furthermore, we may assume that the maps of (3.12) correspond to maps of projective systems $f_\bullet : M'_\bullet[d] \to M_\bullet, h_\bullet : M_\bullet \to M''_\bullet$ by again composing with Artin-Rees isomorphisms.

Define M, M', M'' by taking the inverse limits of $M_\bullet, M'_\bullet, M''_\bullet$ respectively. By the strict I-adic property, M' and M'' are finite A-modules.

We now state the **Mittag-Leffler criterion**: an inverse limit of short exact sequences of modules

$$0 \to N'_n \to N_n \to N''_n \to 0$$

is short exact if the left term (N'_n) has the property that for each n, the decreasing system of images $N'_{n+\nu} \to N'_n$ stabilizes for $\nu \gg 0$ (which may depend on n).

Note that the Mittag-Leffler criterion is always satisfied when the terms N'_n have finite length.

The system of short exact sequences

$$0 \to \ker h_n \to M_n \to \operatorname{im} h_n \to 0 \tag{3.13}$$

gives a short exact sequence in the inverse limit. Indeed, the maps $M'_n \to \ker h_n$ have null kernel and cokernel systems and the M'_n have finite length (since A/I Artinian), ensuring that the M'_n satisfy the Mittag-Leffler criterion. Hence the $\ker h_n$ satisfy the Mittag-Leffler criterion as well, so we may pass to the inverse limit of (3.13) and get a short exact sequence as claimed.

By Lemma 3.4 we note that $\varprojlim \ker h_n = \varprojlim M'_n$.

Similarly, we may pass to the inverse limit of the short exact sequences

$$0 \to \operatorname{im} h_n \to M''_n \to \operatorname{coker} h_n \to 0 \tag{3.14}$$

since each M''_n has finite length. Hence we get an exact sequence

$$0 \to M' \xrightarrow{f} M \xrightarrow{h} M'' \to 0. \tag{3.15}$$

It then follows that M is a finite A-module.

Now consider our original short exact sequence (3.12) in the Artin-Rees category. Our assumptions at the beginning of the proof tell us that the maps $f_\bullet, h_\bullet$ in (3.12) are maps of projective systems with the nth term an A_{n+d}-module for all $n \geq 0$.

We may similarly take our short exact sequence of finite A-modules (3.15) and use Theorem 3.9 to get a short exact sequence of strictly I-adic objects in the Artin-Rees category, which forms the first row of this Artin-Rees commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \left(M'/I^{n+1+d}M'\right) & \longrightarrow & \left(M/I^{n+1+d}M\right) & \longrightarrow & \left(M''/I^{n+1+d}M''\right) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M'_\bullet[d] & \xrightarrow{f_\bullet} & M_\bullet & \xrightarrow{h_\bullet} & M''_\bullet & \longrightarrow & 0
\end{array}$$

The outer vertical maps are Artin-Rees isomorphisms; hence by the Snake Lemma in the Artin-Rees category, the center vertical map is as well. Hence $M_\bullet$ is Artin-Rees isomorphic to a strictly I-adic object, so $M_\bullet$ is Artin-Rees I-adic as desired.

□

4 Artin-Rees category of K-vector spaces

Assume A is a domain with fraction field K; we continue to assume A/I is Artinian, so A is a local ring.

We can consider K-vector spaces in terms of lattices as follows. Any finite-dimensional K-vector space V can be written as $V = K \otimes_A L$ where L is a finite free A-module; there are obviously many choices of lattice L that give the same K-vector space.

Furthermore, for any finite A-modules L, L' we have an isomorphism

$$K \otimes_A \mathrm{Hom}_A(L, L') \simeq \mathrm{Hom}_K(K \otimes_A L, K \otimes_A L').$$

This isomorphism will motivate the definition of the **Artin-Rees category of K-vector spaces** as a category whose objects are the Artin-Rees I-adics defined above. For clarity's sake, we write $K \otimes M_\bullet$ for the object in the Artin-Rees category of K-vector spaces corresponding to the Artin-Rees I-adic $M_\bullet$. The morphisms are elements of the set

$$\mathrm{Hom}_{\text{A-R},K}(K \otimes M_\bullet, K \otimes N_\bullet) := K \otimes_A \mathrm{Hom}_{\text{A-R}}(M_\bullet, N_\bullet). \tag{4.1}$$

Using Theorem 3.9 and the isomorphism $K \otimes_A \mathrm{Hom}_A(L, L') \simeq \mathrm{Hom}_K(K \otimes_A L, K \otimes_A L')$ we note that the Artin-Rees category of K-vector spaces admits a fully faithful and essentially surjective functor $K \otimes M_\bullet \rightsquigarrow K \otimes_A \varprojlim M_n$ to the category of finite-dimensional K-vector spaces.

Unlike in Theorem 3.9, there is not a natural functor from K-vector spaces back to the corresponding Artin-Rees category.

5 ℓ-adic Sheaves

We now consider sheaves on the étale site of a scheme. We let Λ be a complete local Noetherian ring with maximal ideal $\mathfrak{m}$, and X a Noetherian scheme. (We require X Noetherian so we may discuss constructible sheaves and so that the direct limit in the definition of the Artin-Rees morphism sets is appropriate.)

Define $\Lambda_n := \Lambda/\mathfrak{m}^{n+1}$ for $n \geq 0$.

We may define the **Artin-Rees category of Λ-sheaves on** $X_{ét}$ in the same way as Definition 3.1. This will again yield an abelian category with kernels and cokernels formed termwise (possibly after a shift).

We now define strict adic objects in this category.

Definition 5.1. *For X a Noetherian scheme, a* **strictly $\mathfrak{m}$-adic sheaf on** $X_{ét}$ *is $\mathscr{F}_\bullet = (\mathscr{F}_n)_{n\in\mathbb{Z}}$ in the Artin-Rees category of Λ-sheaves on $X_{ét}$ such that $\mathscr{F}_n = 0$ for $n < 0$, $\mathscr{F}_n$ is a Λ_n-module for $n \geq 0$, and the natural maps $\mathscr{F}_{n+1}/\mathfrak{m}^{n+1}\mathscr{F}_{n+1} \to \mathscr{F}_n$ are isomorphisms for all $n \geq 0$.*

If for all n we have $\mathscr{F}_n$ constructible, then $\mathscr{F}_\bullet$ is a **constructible strictly $\mathfrak{m}$-adic sheaf**.

As before, general adic objects in this category are defined by isomorphism with a strict object:

Definition 5.2. *$\mathscr{F}_\bullet$ in the Artin-Rees category of Λ-sheaves on $X_{ét}$ is an* **$\mathfrak{m}$-adic sheaf** *if it is Artin-Rees isomorphic to a strictly $\mathfrak{m}$-adic sheaf.*

Similarly, $\mathscr{F}_\bullet$ in the Artin-Rees category of Λ-sheaves on $X_{ét}$ is a **constructible $\mathfrak{m}$-adic sheaf** *if it is Artin-Rees isomorphic to a constructible strictly $\mathfrak{m}$-adic sheaf.*

Note that for $\mathscr{F}_\bullet$ constructible $\mathfrak{m}$-adic, it is not necessarily the case that each $\mathscr{F}_n$ is constructible.

Consider again the example of an abelian scheme $A \to S$ given in section 1, with the $A[\ell^n]$ constructed as in that section. The basic example of a strict I-adic object is $\mathscr{F}_n = A[\ell^n]$ with $\Lambda = \mathbb{Z}_\ell$. The stalk of $\mathscr{F}_n$ at a geometric point $\overline{s}$ is $(\mathscr{F}_n)_{\overline{s}} = (\mathbb{Z}/\ell^n\mathbb{Z})^{2g}$.

We may apply additive functors (ex: sheaf tensor product, higher direct images $R^i f_*, R^i f_!$) termwise in the Artin-Rees category as one might expect. In some cases we may have to impose certain restrictions, such as in the case of the bifunctor $\mathscr{H}om(\mathscr{F}_\bullet, \mathscr{G}_\bullet)$, which will be constructible $\mathfrak{m}$-adic when $\mathscr{F}_\bullet$ is constructible $\mathfrak{m}$-adic and there exists $d \geq 0$ such that $\mathscr{G}_n$ is a Λ_{n+d}-sheaf for all $n \geq 0$.

Henceforth we require that the residue field $\Lambda_0 := \Lambda/\mathfrak{m}$ is finite, so that we may have nonzero constructible $\mathfrak{m}$-adic sheaves.

An ordinary constructible Λ-module $\mathscr{F}$ on $X_{ét}$ is in fact a Λ_ν-module for a fixed $\nu \geq 0$. Hence we see that the category of constructible Λ-modules on $X_{ét}$ embeds fully faithfully into the category of constructible $\mathfrak{m}$-adic sheaves via the functor $\mathscr{F} \rightsquigarrow \left(\mathscr{F}/\mathfrak{m}^{n+1}\mathscr{F}\right)$.

We now define the property of stable images; this will provide a quasi-inverse to the inclusion of constructible Λ-modules into the Artin-Rees category.

Definition 5.3. *$\mathscr{G}_\bullet$ in the Artin-Rees category has* **stable images** *if there exists $d_0 \geq 0$ such that for each n, the image of $\mathscr{G}_{n+d}$ in $\mathscr{G}_n$ is the same for all $d \geq d_0$.*

If we let $\overline{\mathscr{G}_n}$ be the stable image in degree n and the natural maps $\overline{\mathscr{G}_{n+1}} \to \overline{\mathscr{G}_n}$ are isomorphisms for all $n \gg 0$, then we say that $\mathscr{G}_\bullet$ has **terminal stable images**.

Since an ordinary constructible Λ-module $\mathscr{F}$ on $X_{\acute{e}t}$ is in fact a Λ_ν-module for a fixed $\nu \geq 0$, we see that the corresponding Artin-Rees object $\mathscr{F}_\bullet$ has $\mathscr{F}_n = \mathscr{F}$ for $n \geq \nu$. It is therefore clear that $\mathscr{F}_\bullet$ has terminal stable images and that the terminal stable image is constructible.

The inclusion functor given above is therefore an equivalence onto the full subcategory of Artin-Rees objects with a constructible terminal stable image.

Note that the formation of the stable image in a fixed degree is not in general functorial in $\mathscr{G}_\bullet$ for reasons of shifting. However, on the full subcategory of Artin-Rees objects with terminal stable images, formation of the terminal stable image is in fact functorial and provides the quasi-inverse to the inclusion functor discussed above.

The property of being constructible $\mathfrak{m}$-adic can in fact be checked over a stratification or étale-locally, as stated in the following crucial theorem.

Theorem 5.4. *For $\{S_i\}$ a stratification of X and $\{U_j\}$ an étale cover, an object $\mathscr{F}_\bullet$ in the Artin-Rees category of Λ-modules on $X_{\acute{e}t}$ is constructible $\mathfrak{m}$-adic if and only if all the $\mathscr{F}_\bullet|_{S_i}$ or all the $\mathscr{F}_\bullet|_{U_j}$ are constructible $\mathfrak{m}$-adic.*

Proof. We may assume the U_j are quasicompact. One direction of the proof is immediate. Therefore, assume now that all the $\mathscr{F}_\bullet|_{S_i}$ or all the $\mathscr{F}_\bullet|_{U_j}$ are constructible $\mathfrak{m}$-adic.

Define for $n \geq 0$ an Artin-Rees object $\widetilde{\mathscr{F}_\bullet}^n := \mathscr{F}_\bullet / \mathfrak{m}^{n+1}\mathscr{F}_\bullet$. We wish to show that $\widetilde{\mathscr{F}_\bullet}^n$ has terminal stable image which is a constructible Λ_n-module.

Using the functoriality of the construction of the terminal stable image, it is sufficient to check this property for all the $\mathscr{F}_\bullet|_{S_i}$ or all the $\mathscr{F}_\bullet|_{U_j}$. In each of these cases we may pass to an Artin-Rees isomorphic object, so we reduce to the case $\mathscr{F}_\bullet$ is constructible strict $\mathfrak{m}$-adic. In that case it is clear that $\widetilde{\mathscr{F}_\bullet}^n$ has terminal stable image which is a constructible Λ_n-module.

Let $\mathscr{F}'_n$ be the terminal stable image of $\widetilde{\mathscr{F}_\bullet}^n$; note that since the construction of the terminal stable image is functorial, we get maps $\mathscr{F}'_{n+1} \to \mathscr{F}'_n$ over $\Lambda_{n+1} \to \Lambda_n$. Furthermore, by checking étale-locally or over a stratification, we can see that the natural maps $\mathscr{F}'_{n+1}/\mathfrak{m}^{n+1}\mathscr{F}'_{n+1} \to \mathscr{F}'_n$ are in fact isomorphisms.

Hence $\mathscr{F}'_\bullet = (\mathscr{F}'_n)_{n \in \mathbb{Z}}$ (let $\mathscr{F}'_n = 0$ for $n < 0$) is a constructible strict $\mathfrak{m}$-adic object.

Again working either étale-locally or over a stratification, we may show the existence of an integer e independent of n such that the terminal stable image of $\widetilde{\mathscr{F}_\bullet}^n$ is achieved in degree $n + e$; once again we use passage to an Artin-Rees isomorphic strict object to show étale-locally or over a stratification that such an e exists, though the value of e is not preserved by Artin-Rees isomorphism.

This gives an Artin-Rees morphism $\mathscr{F}_\bullet[e] \to \mathscr{F}'_\bullet$. Passing to a larger e if necessary, we may as before work étale-locally or over a stratification and use passage to an Artin-Rees isomorphic strict object to show that this morphism is an isomorphism. It is also necessary to note that the construction of $\mathscr{F}'_\bullet$ is functorial in $\mathscr{F}_\bullet$, so passage to an Artin-Rees isomorphic object does not affect the claim.

It then follows that $\mathscr{F}_\bullet$ is isomorphic to a constructible strict $\mathfrak{m}$-adic object, so it is constructible $\mathfrak{m}$-adic as we desired to show. □

We define for $\mathscr{F}_\bullet$ constructible $\mathfrak{m}$-adic **the stalk of a constructible $\mathfrak{m}$-adic sheaf at a geometric point**

$\overline{x}$, $(\mathscr{F}_\bullet)_{\overline{x}} := \varprojlim (\mathscr{F}_n)_{\overline{x}}$. We note that by Lemma 3.4 that $(\mathscr{F}_\bullet)_{\overline{x}}$ is functorial in $\mathscr{F}_\bullet$ in the Artin-Rees category, and furthermore $(\mathscr{F}_\bullet)_{\overline{x}}$ is a finite Λ-module.

Definition 5.5. *An Artin-Rees object $\mathscr{F}_\bullet$ is* **lisse $\mathfrak{m}$-adic** *if it is Artin-Rees isomorphic to a strictly $\mathfrak{m}$-adic object $\mathscr{G}_\bullet$ with each $\mathscr{G}_n$ a locally constant constructible (lcc) Λ_n-module for all $n \geq 0$.*

By Theorem 5.4. we note that the property of being lisse $\mathfrak{m}$-adic is étale-local.

Of future importance will be the case of $f : X \to S$ a smooth proper map of schemes (where S is Noetherian) and the system $(R^i f_*(\mathbb{Z}/\ell^n\mathbb{Z}))$ of lcc sheaves; we wish this system to be lisse ℓ-adic. If we replace f_* with $f_!$ where f is just separated and finite-type, we will obtain a constructible ℓ-adic result instead.

For $\mathscr{F}_\bullet$ a lisse $\mathfrak{m}$-adic sheaf, we have an analogue to the dictionary between local systems and representations of π_1 in the topological case.

First we review the topological case.

Given a 'reasonable' connected topological space X with covering space $f : Y \to X$ and deck transformation group G, one may show that locally constant sheaves on X are in correspondence with locally constant sheaves $\mathscr{F}$ on Y that are equipped with isomorphisms $\alpha_g : g^*\mathscr{F} \to \mathscr{F}$ for all $g \in G$ that respect the group composition law.

This correspondence is given by taking the pullback of sheaves on X in one direction; the inverse is given by $\mathscr{F} \mapsto (f_*\mathscr{F})^G$ (taking the G-invariants of the push-forward). The inverse nature of these functors under the evident maps may be checked locally on X, so one may assume Y is a split cover, for which the proof is trivial.

Now if we take Y to be the universal cover $\widetilde{X}$ of X, we note that any locally constant sheaf on $\widetilde{X}$ is in fact a constant sheaf; hence the correspondence described above takes locally constant sheaves on X to sets or abelian groups with a G-action given by the α_g. Note that the deck transformation group G is $\pi_1(X, x_0)$, the fundamental group.

Given a sheaf $\mathscr{F}$ on X and $q : \widetilde{X} \to X$ the universal cover as described above, for any point x with $\tilde{x}$ a lift to $\widetilde{X}$, we have an isomorphism of stalks $\mathscr{F}_x \simeq (q^*\mathscr{F})_{\tilde{x}}$. Furthermore, since any locally constant sheaf $\mathscr{G}$ on $\widetilde{X}$ is constant (since we have isomorphisms $\mathscr{G}(\widetilde{x}) \simeq \mathscr{G}_{\tilde{x}}$ for all points $\tilde{x} \in \widetilde{X}$), we may describe the $\pi_1(X, x_0)$ action as follows. Fix a lift $\tilde{x_0}$ of x. A loop g in $\pi_1(X, x_0)$ lifts to a path from $\tilde{x}_0$ to another lift $\tilde{x}_1$ of x_0; note that all such paths with fixed endpoint $\tilde{x}_1$ are homotopic in $\widetilde{X}$, so they respect the homotopy class of paths in $\pi_1(X, x_0)$. Furthermore, the definition of deck transformations makes it clear that $g\tilde{x}_0 = \tilde{x}_1$.

Since $q^*\mathscr{F}$ is constant, we may describe the action of $g \in \pi_1(X, x_0)$ via the natural isomorphisms

$$\mathscr{F}_{x_0} \simeq (q^*\mathscr{F})_{\tilde{x}_0} \simeq \Gamma(\widetilde{X}, q^*\mathscr{F}) \simeq (q^*\mathscr{F})_{\tilde{x}_1} \simeq \mathscr{F}_{x_0}.$$

The details of showing that this composition of isomorphisms ρ_g is compatible with composition of group elements and describe an action on $\mathscr{F}_{x_0}$ are omitted.

This gives the desired identification of locally constant sheaves and $\pi_1(X, x_0)$-representations in the topological case.

Now we return to X a connected Noetherian scheme. In the case of $\mathscr{F}_\bullet$ lisse $\mathfrak{m}$-adic on X, Grothendieck's equivalence between lcc sheaves and finite monodromy representations gives a continuous linear action of the étale fundamental group $\pi_1(X, \overline{x})$ on the stalk $(\mathscr{F}_\bullet)_{\overline{x}}$.

For any finite Λ-module M with a continuous linear action of $\pi_1(X, \overline{x})$, we may write $M \simeq \varprojlim M/\mathfrak{m}^{n+1}M$ and give a compatible continuous linear action of π_1 on $M/\mathfrak{m}^{n+1}M$.

Hence by Theorem 3.9 the stalk functor at $\overline{x}$ is an equivalence of categories between lisse $\mathfrak{m}$-adic sheaves and continuous linear representations of $\pi_1(X, \overline{x})$ on finite Λ-modules, analogously to the topological case.

Remark 5.6. For a constructible $\mathbb{Z}_\ell$-sheaf $\mathscr{F}_\bullet$ on X we will define $H^i_{ét}(X, \mathscr{F}_\bullet) = \varprojlim H^i_{ét}(X, \mathscr{F}_n)$. Later we will see that $H^i_{ét}(X, \mathscr{F}_\bullet)$ is a finitely generated $\mathbb{Z}_\ell$-module for X separated of finite type over a separably closed field. (The proof will require the use of relative considerations with higher direct images Rf_*.)

For X separated of finite type over $\mathbb{C}$, in order to have an ℓ-adic version of the Artin Comparison for even $\mathscr{F}_\bullet = \mathbb{Z}_\ell$, we will need to show that for all i,

$$H^i(X(\mathbb{C}), \mathbb{Z}_\ell) \simeq \varprojlim H^i(X(\mathbb{C}), \mathbb{Z}/\ell^n\mathbb{Z}),$$

and similarly for cohomology with compact supports. These equalities for such classical cohomology are *not* obvious when X is not proper and smooth, and their proof will rest on alterations or resolutions of singularities.

References

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