Adic Formalism, Part 2

Brian Lawrence

March 1, 2017

These are notes for Stanford's Number Theory Learning Seminar on étale cohomology. This is a continuation of Sheela's lecture; the goal is to extend our earlier results from torsion to ℓ-adic sheaves.

The one hard result in today's notes is Theorem 4.2, which shows that the definition of cohomology in the ℓ-adic setting is reasonable.

1 Review

We recall the following definitions.

Definition 1.1. *A sheaf of abelian groups (on the étale site of a noetherian scheme) is* locally constant constructible *(or* lcc *for short) if, locally on the étale site, it's a constant sheaf and the stalks are finite.*

A sheaf $\mathcal{F}$ of abelian groups on X_{et} is constructible *if there is a stratification of X such that the restriction of $\mathcal{F}$ to each stratum is lcc.*

For the rest of this discussion fix a complete DVR Λ with maximal ideal m, finite residue field and uniformizer ℓ, and let $\Lambda_n = \Lambda/\ell^n\Lambda$. (Feel free to assume $\Lambda = \mathbb{Z}_\ell$.)

Definition 1.2. *A* constructible strictly m-adic sheaf *is a projective system $(\mathcal{F}_n)_{n>0}$ of constructible sheaves on X_{et}, where $\mathcal{F}_n$ is a Λ_n-module and the natural map $\mathcal{F}_{n+1}/m^n\mathcal{F}_{n+1} \to \mathcal{F}_n$ is an isomorphism. A* constructible m-adic sheaf *is a projective system $(\mathcal{F}_n)_{n>0}$ of constructible sheaves on X_{et}, where $\mathcal{F}_n$ is a Λ_n-module, which is Artin-Rees isomorphic to a constructible m-adic sheaf. Morphisms of constructible strictly m-adic sheaves are defined in the Artin-Rees sense.*

In general (working in either the category of modules or the category of sheaves), we say a projective system A_n is strictly m-adic *if $\ell^n A_n = 0$ and the natural map $A_{n+1}/m^n A_{n+1} \to A_n$ is an isomorphism.*

The analogue of the lcc condition is that the requirement that an m-adic sheaf be *lisse*.

Definition 1.3. *A constructible m-adic sheaf $\mathcal{F}$ is* lisse *if it is Artin-Rees isomorphic to a strictly m-adic sheaf $(\mathcal{F}_n)$, where each $\mathcal{F}_n$ is lcc.*

In general, lisse sheaves are not locally constant for the étale topology. The reader who finds this upsetting is advised to go learn about the pro-étale topology of Bhatt-Scholze instead.

2 Technical Properties of Constructibility

We now prove that some familiar facts about constructible sheaves carry over to the m-adic setting.

Proposition 2.1. *Let $\mathcal{F}$ be a constructible m-adic sheaf on X. Then there is a stratification of X such that the restriction of $\mathcal{F}$ to each stratum is lisse.*

Proof. Take $\mathcal{F}_n$ a strictly m-adic system representing $\mathcal{F}$. We need to produce a stratification which makes $\mathcal{F}_n$ an lcc sheaf – the difficulty is that a single stratification has to work for all n.

Recall the specialization criterion for a constructible sheaf (in the finite setting) to be lcc: Given a constructible sheaf $\mathcal{G}$ on X_{et}, and any two geometric points p and q such that q is a generization of p, one defines a generization map $\mathcal{G}_p \to \mathcal{G}_q$ on stalks. Then $\mathcal{G}$ is lcc if and only if all such generization maps are isomorphisms.

One easily deduces the following: given a short exact sequence

$$0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to 0,$$

of constructible sheaves, if $\mathcal{G}_1$ and $\mathcal{G}_3$ are lcc, then $\mathcal{G}_2$ is as well.

Return to our m-adic sheaf $\mathcal{F}$. We want to apply the result just mentioned to the exact sequence

$$0 \to \ell^{n-1}\mathcal{F}_n \to \mathcal{F}_n \to \mathcal{F}_{n-1} \to 0.$$

If we know that $\mathcal{F}_1$ and each $\ell^{n-1}\mathcal{F}_n$ are lcc, then by induction we can show that each $\mathcal{F}_n$ is lcc, which is what we need.

But note that $\ell^{n-1}\mathcal{F}_n$ is a quotient of $\mathcal{F}_1$ by the natural map

$$\mathcal{F}_1 \cong \mathcal{F}_n/\ell \to \ell^{n-1}\mathcal{F}_n/\ell^n \cong \ell^{n-1}\mathcal{F}_n.$$

Let $\mathcal{K}_n$ be the kernel of the map

$$\mathcal{F}_1 \to \ell^{n-1}\mathcal{F}_n.$$

Then the $\mathcal{K}_n$'s form an increasing chain of subsheaves of $\mathcal{F}_1$, so they eventually stabilize; hence only finitely many constructible sheaves arise as the sheaves

$$\ell^{n-1}\mathcal{F}_n.$$

Hence we can find a stratification of X that makes them all lcc simultaneously; this completes the proof. □

Proposition 2.2. *Constructible m-adic sheaves are Noetherian: Let $\mathcal{F}$ be a constructible m-adic sheaf, and let $\mathcal{G}^k$ be an ascending chain of subsheaves; then the $\mathcal{G}^k$ eventually stabilize.*

Proof. We may take X connected, with η a generic geometric point. The generic fiber $\mathcal{F}_\eta$ is noetherian, so the generic fibers $\mathcal{G}^k_\eta$ eventually stabilize. Choose some k for which $\mathcal{G}^n_\eta = \mathcal{G}^k_\eta$ for all $n > k$; choose some U for which the natural map $\mathcal{G}^k(U) \to \mathcal{G}^k_\eta$ is an isomorphism, and use noetherian induction on X. □

Proposition 2.1 has some useful consequences.

Proposition 2.3. *Any constructible m-adic sheaf $\mathcal{F}$ fits in an exact sequence*

$$0 \to \mathcal{F}^{tor} \to \mathcal{F} \to \mathcal{F}^{tf} \to 0$$

where $\mathcal{F}^{tor}$ is a constructible torsion sheaf and $\mathcal{F}^{tf}$ is torsion-free.

Proof. If $\mathcal{F}$ were lisse we'd be done by the theory of the étale π_1. We know that $\mathcal{F}$ is lisse after restricting to a stratification by Proposition 2.1. But this isn't enough because we need to glue the pieces together...

Let $\mathcal{F}^{\mathrm{tor}}_n$ be the m^n-torsion subsheaf of $\mathcal{F}$. These form an ascending chain of subsheaves of $\mathcal{F}$; we know this must eventually terminate in some $\mathcal{F}^{\mathrm{tor}}$. Then one can check the required properties after restricting to a stratification. □

Proposition 2.4. *A constructible m-adic sheaf $\mathcal{F}$ is trivial if and only if it has vanishing stalks at all geometric points.*

Proof. Reduce to the lisse case, which is trivial. □

Proposition 2.5. *A complex of constructible m-adic sheaves is exact if and only if it is exact on stalks.*

Proof. Reduce to the lisse case, which is trivial. □

Proposition 2.6. *If $\mathcal{F} \to \mathcal{G}$ is a map of constructible m-adic sheaves, then the kernel and cokernel are constructible as well.*

Proof. Reduce to the lisse case, which was settled in the previous lecture. □

Now we give a technical result on the behavior of constructibility in exact sequences.

Proposition 2.7. *Suppose*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is a short exact sequence in the Artin-Rees category of Λ-sheaves, and $\mathcal{F}_1$ and $\mathcal{F}_3$ are constructible m-adic. Suppose further that $\mathcal{F}_2$ satisfies the following condition: there exist constants e, ν, N such that the map

$$m^{n+\nu}\mathcal{F}_n \to \mathcal{F}_{n-e}$$

is the zero map whenever $n > N$. Then $\mathcal{F}_2$ is itself constructible m-adic.

Proof. This follows from the corresponding fact (Theorem 3.11) in Sheela's notes from last week. □

3 $\mathbb{Q}_\ell$-sheaves

One defines the category of constructible $\mathbb{Q}_\ell$-sheaves in the same way as last week.

Definition 3.1. *The objects of the category of constructible $\mathbb{Q}_\ell$-sheaves are the same as the objects of the category of constructible $\mathbb{Z}_\ell$-sheaves. Given two such objects $\mathcal{F}$ and $\mathcal{G}$, we define the set of $\mathbb{Q}_\ell$-sheaf homomorphisms as*

$$\mathrm{Hom}_{\mathbb{Q}_\ell}(\mathcal{F},\mathcal{G}) = \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathcal{F},\mathcal{G}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

We may write $\mathcal{F} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ for the $\mathbb{Q}_\ell$-sheaf obtained from the $\mathbb{Z}_\ell$-sheaf $\mathcal{F}$.

We say a $\mathbb{Q}_\ell$-sheaf is lisse *if it is isomorphic to $\mathcal{F} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, with $\mathbb{F}$ a lisse $\mathbb{Z}_\ell$-sheaf.*

The stalk *of $\mathcal{F} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ at a geometric point $x \in X$ is $(\mathcal{F} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)_x = \mathcal{F}_x \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.*

One easily verifies a number of basic properties of $\mathbb{Q}_\ell$-sheaves.

Proposition 3.2. *The $\mathbb{Q}_\ell$-sheaves on X form an abelian category.*

Proof. Build kernels, cokernels and images in the category of $\mathbb{Z}_\ell$-sheaves. □

Proposition 3.3. *Any constructible $\mathbb{Q}_\ell$-sheaf can be expressed as $\mathcal{F} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, where $\mathcal{F}$ is a constructible $\mathbb{Z}_\ell$-sheaf whose stalks are flat over $\mathbb{Z}_\ell$.*

Proof. Use Proposition 2.3, and note that if $\mathcal{F}$ is a constructible torsion sheaf, then $\mathcal{F} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ vanishes. □

Proposition 3.4. *A constructible $\mathbb{Q}_\ell$-sheaf vanishes if and only if all its stalks vanish.*

Proof. In order that the object $\mathcal{F} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ vanish it is necessary and sufficient that the identity morphism and the zero morphism on this object coincide. This is equivalent to asking that multiplication by ℓ^n kill $\mathcal{F}$, for sufficiently large n; in terms of the Artin-Rees category, the condition is that there exist k and n such that the map $\mathcal{F} \to \mathcal{F}[k]$ induced by multiplication by ℓ^n vanish. This last condition can clearly be checked on stalks. □

Proposition 3.5. *For any map $f : \mathcal{F} \to \mathcal{G}$ of constructible $\mathbb{Q}_\ell$-sheaves, and any geometric point $x \in X$, let $f_x : \mathcal{F}_x \to \mathcal{G}_x$ be the induced map on stalks. Then the kernel of f_x is the stalk of the kernel of f; the image of f_x is the stalk of the image of f; and the cokernel of f_x is the stalk of the cokernel of f.*

A sequence $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ of constructible $\mathbb{Q}_\ell$-sheaves is exact if and only if the corresponding sequence of stalks is exact, for every geometric point $x \in X$.

Proof. One verifies the first claim by reducing to the corresponding fact about $\mathbb{Z}_\ell$-sheaves. The second claim follows from the first, since two sub-$\mathbb{Q}_\ell$-sheaves of $\mathcal{F}_2$ agree if and only if all their stalks agree. □

Proposition 3.6. *Let X be a connected noetherian scheme with geometric point x. The stalk at x defines a fully faithful and essentially surjective functor from lisse $\mathbb{Q}_\ell$-sheaves on X_{et} to continuous $\mathbb{Q}_\ell$-linear representations of $\pi_1(X, x)$ on finite-dimensional $\mathbb{Q}_\ell$-vector spaces.*

Proof. We have already proven the corresponding result for $\mathbb{Z}_\ell$-sheaves. To finish the proof we need only verify that if $\pi_1(X, x)$ acts continuously on a finite-dimensional $\mathbb{Q}_\ell$ vector space V, then V admits a $\pi_1(X, x)$-stable lattice Λ. But $\pi_1(X, x)$ is profinite, and one can construct the lattice by a standard averaging trick. □

4 Cohomological Functors

We need to define $R^i f_*$ and $R^i f_!$ on constructible m-adic sheaves. The only option is to define them term-by-term; we'll have to show that this "works." The hard part is to show that the result is again a constructible m-adic sheaf.

Definition 4.1. *Suppose given a morphism $f : X \to S$ of schemes and a constructible m-adic sheaf $\mathcal{F} = (\mathcal{F}_n)$ on S. Then we define the projective system $R^i f_* \mathcal{F}$ by $(R^i f_* \mathcal{F})_n = R^i f_* \mathcal{F}_n$. If $X \to S$ is compactifiable we define similarly $(R^i f_! \mathcal{F})_n = R^i f_! \mathcal{F}_n$.*

Theorem 4.2. *If f is proper then $R^i f_* \mathcal{F}$ is a constructible m-adic sheaf.*

Proof. This follows the argument in Freitag and Kiehl (I.12.14-15).

We assume that $\mathcal{F}$ is strict and torsion-free.

By Godement resolutions, we can construct (functiorially in $\mathcal{F}_n$) an acyclic resolution

$$\mathcal{C}(\mathcal{F}_n)$$

of finite length.

First, we recall the construction of Godement resolutions. Suppose $\mathcal{F}$ is a sheaf. The key idea is to construct an injective map $\mathcal{F} \to \mathcal{G}(\mathcal{F})$ into a flabby sheaf which depends functorially on $\mathcal{F}$. The construction is quite simple: for any open U, we take $\mathcal{G}(\mathcal{F})(U)$ to be the product of all the stalks $\mathcal{F}_x$, with $x \in U$.

The Godement resolution of a sheaf $\mathcal{F}$ is constructed recursively as follows: given some partial complex

$$0 \to \mathcal{F} \to \mathcal{C}_1(\mathcal{F}) \to \cdots \to \mathcal{C}_k(\mathcal{F}),$$

let $\mathcal{K}$ denote the cokernel of the right-hand map, and take

$$\mathcal{C}_{k+1}(\mathcal{F}) = \mathcal{G}(\mathcal{K}).$$

We apply this construction to each of the $\mathcal{F}_n$ coming from the strict m-adic sheaf $\mathcal{F}$. We get the projective system of complexes $\mathcal{H}_n^\bullet = f_* \mathcal{C}(\mathcal{F}_n)$, whose cohomology gives the thing we want to prove is constructible m-adic. It satisfies the following properties:

- $\ell^n \mathcal{H}_n^r = 0$
- The sheaves $\mathcal{H}_n^r$ are Λ_n-flat.
- All the cohomology sheaves $H^i(\mathcal{H}_n^\bullet)$ are constructible.
- The complexes are uniformly bounded: there exists r such that $\mathcal{H}_n^i = 0$ if $i < -r$ or $i > r$.
- For all n, the mapping

$$\mathcal{H}_{n+1}^\bullet / \ell^n \mathcal{H}_{n+1}^\bullet \to \mathcal{H}_n^\bullet$$

  is a quasi-isomorphism.

We will deduce the required result from this. We proceed in several steps. Recall that at the moment we have a projective system of complexes of ℓ-torsion sheaves.

First we prove a result for a projective system of complexes of *modules.* (The argument in Freitag-Kiehl is terse and difficult to follow, so we provide a detailed argument.)

Lemma 4.3. *(Freitag-Kiehl I.12.5)*

Let M_n^i be a projective system of complexes of Λ-modules. (The complexes are drawn as rows below.)

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & M_2^0 & \longrightarrow & M_2^1 & \longrightarrow & M_2^2 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & M_1^0 & \longrightarrow & M_1^1 & \longrightarrow & M_1^2 & \longrightarrow & \cdots \end{array}$$

Suppose M_n^i satisfies the following properties.

- $\ell^n M_n^i = 0$
- *Each M_n^i is Λ_n-flat.*
- *All the cohomology modules $H^i(M_n)$ are finite.*
- *The complexes are uniformly bounded: there exist R and r such that $\mathcal{H}_n^i = 0$ if $i < R$ or $i > r$.*
- *For all n, the mapping*

$$M_{n+1} / \ell^n M_{n+1} \to M_n$$

  is a quasi-isomorphism.

Then there is a bounded complex $K^\bullet$ of finitely generated free Λ-modules together with quasi-isomorphisms $K^\bullet/\ell^n K^\bullet \to K_n^\bullet$ compatible up to homotopy with the maps $K_{n+1}^\bullet \to K_n^\bullet$. In particular, the projective systems $H_n^i(K_i^\bullet)$ (for fixed i and varying n) satisfy the Artin-Rees condition.

Proof. The proof is by reverse induction on r.

Step 1. From the hypothesis $K_n^{r+1} = 0$ for every n we deduce that the natural map $H^r(K_{n+1}^\bullet)/\ell^n H^r(K_{n+1}^\bullet) \to H^r(K_n^\bullet)$ is an isomorphism, as follows.

$$\begin{array}{ccccc}
K_{n+1}^{r-1} & \xrightarrow[d_{n+1}^r]{} & K_{n+1}^r & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \phi_n^r} & & \\
K_n^{r-1} & \xrightarrow[d_n^r]{} & K_n^r & \longrightarrow & 0
\end{array}$$

We may assume that $K_n^\bullet = K_{n+1}^\bullet/\ell^n K_{n+1}^\bullet$. From the surjectivity of the vertical arrows we get that

$$\mathrm{Im}(d_n^r) = \phi_n^r(\mathrm{Im}(d_{n+1}^r)).$$

Hence, we have

$$\begin{aligned}
H^r(K_n^\bullet) &= K_n^r/\,\mathrm{Im}(d_n^r) \\
&= K_{n+1}^r/(\ell^n K_{n+1}^r, \mathrm{Im}(d_{n+1}^r)) \\
&= H^r(K_n^\bullet)/\ell^n H^r(K_{n+1}^\bullet).
\end{aligned}$$

Hence, the homology objects in degree r form a strictly m-adic system (see Sheela's notes).

Step 2. Let M denote the inverse limit $\varprojlim_n H^r(K_n^\bullet)$, the Λ-module represented by the strictly m-adic system $H^r(K_n^\bullet)$. This M is a finitely-generated Λ-module, hence admits a surjection from a finite free Λ-module F. Let I denote the kernel

$$0 \to I \to F \to M \to 0.$$

There are natural maps from M to the cohomology modules $H^r(K_n^\bullet) = K_n^r/\,\mathrm{Im}(d_n^r)$. We can lift these to maps from the free module F to K_n^r, compatibly with the surjections $K_{n+1}^r \to K_n^r$. From this one obtains the complex shown below.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & K_{n+1}^{r-2} & \xrightarrow[d_{n+1}^r]{} & K_{n+1}^{r-1} \oplus F/\ell^{n+1}F & \xrightarrow[d_{n+1}^r]{} & K_{n+1}^r & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \phi_n^r} & & \\
\cdots & \longrightarrow & K_n^{r-2} & \xrightarrow[d_n^r]{} & K_n^{r-1} \oplus F/\ell^n F & \xrightarrow[d_n^r]{} & K_n^r & \longrightarrow & 0
\end{array}$$

(In building this complex we use the zero map $K_n^{r-2} \to F/\ell^n F$.)

This projective system of complexes has been rigged so that the r-th cohomology of every row vanishes. To apply the inductive hypothesis, we will create another system having zeroes in the r-th column. To do this, let L_n^{r-1} be the kernel of the map

$$K_n^{r-1} \oplus F/\ell^n F \to K_n^r.$$

The new complex is as shown.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & K_{n+1}^{r-2} & \xrightarrow[d_{n+1}^r]{} & L_n^{r-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & K_n^{r-2} & \xrightarrow[d_n^r]{} & L_n^{r-1} & \longrightarrow & 0 \end{array}$$

Applying the inductive hypothesis to this, we obtain a complex

$$\cdots \to K^{r-2} \to K^{r-1} \to 0,$$

equipped with compatible maps to the complex in the n-th row of the diagram above, for every n.

Step 3. We now construct a complex

$$\cdots \to K^{r-2} \to K^{r-1} \to K^r = F \to 0$$

that will satisfy the conditions desired. First note that there are natural maps, for every n,

$$K^{r-1} \to L_n^{r-1} \subseteq K_n^{r-1} \oplus F/\ell^n F \to F/\ell^n F.$$

These maps combine to give a map $K^{r-1} \to F$, which we use to define our complex.

Note that the composite map $K^{r-2} \to F$ is indeed the zero map, because it is determined by the maps $K_n^{r-2} \to F/\ell^n F$ at finite level, and these maps were taken to be zero.

To specify the maps from this complex $K^\bullet$ to the complexes $K_n^\bullet$ given, we need only describe the maps in column $r-1$. These are given by

$$K^{r-1} \to L_n^{r-1} \subseteq K_n^{r-1} \oplus F/\ell^n F \to K_n^{r-1}.$$

Now all we have to prove is that the map $K^\bullet/\ell^n K^\bullet \to K_n^\bullet$ induces isomorphisms on cohomology. This is clear except for H^{r-1}, and H^r. For H^r, one verifies that the image of $K^{r-1} \to F$ is exactly I (the kernel of $F \to M$). Hence we have $H^r(K^\bullet) = F/I = M$ and $H^r(K^\bullet/\ell^n K^\bullet) = M/\ell^n M$. On the other hand we have seen that the modules $H^r(K_n^\bullet)$ form a strict m-adic system, so the natural map $M/\ell^n M \to H^r(K_n^\bullet)$ is an isomorphism.

Finaly, we consider the H^{r-1} terms. Consider the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{\ker(K^{r-1}/\ell^n \to F/\ell^n)}{\operatorname{im}(K^{r-2}/\ell^n \to K^{r-1}/\ell^n)} & \longrightarrow & \frac{K^{r-1}/\ell^n}{\operatorname{im}(K^{r-2}/\ell^n \to K^{r-1}/\ell^n)} & \longrightarrow & I/(I \cap \ell^n F) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \frac{\ker(K_n^{r-1} \to K_n^r)}{\operatorname{im}(K_n^{r-2} \to K_n^{r-1})} & \longrightarrow & \frac{\ker(K_n^{r-1} \oplus F/\ell^n \to K_n^r)}{\operatorname{im}(K_n^{r-2} \to K_n^{r-1})} & \longrightarrow & I/(I \cap \ell^n F) & \longrightarrow & 0 \end{array}$$

One verifies that the rows are exact. The right-hand vertical arrow is clearly an isomorphism. By the inductive hypothesis, the middle vertical arrow is an isomorphism as well. Hence, the vertical arrow on the left is an isomorphism.

This completes the proof of Lemma 4.3. □

Now we return to the setting of Theorem 4.2. Recall that we constructed (by Godement resolution) a projective system of complexes of sheaves $\mathcal{H}_n^r$ satisfying the sheafy analogues of the hypotheses to Lemma 4.3.

Lemma 4.4. *There is a nonempty open set U on which all of the cohomology sheaves $H^i(\mathcal{H}_n^\bullet)$ are locally constant.*

Proof. (This is proven in I.12.14 of Freitag-Kiehl.)

The difficulty is that we need to force infinitely many constructible sheaves to be locally constant on some U. Recall that our complexes are bounded in i: we know that H^i vanishes for i greater than some uniform bound, by dimensional vanishing. The issue is that there are infinitely many possible values of n.

We start with the exact sequence

$$0 \to \ell^n \mathcal{H}_{n+1}^i \to \mathcal{H}_{n+1}^i \to \mathcal{H}_n^i \to 0,$$

which gives rise to the long exact sequence

$$\cdots \to H^i(\ell^n \mathcal{H}_{n+1}^\bullet) \to H^i(\mathcal{H}_{n+1}^\bullet) \to H^i(\mathcal{H}_n^\bullet) \to \cdots .$$

Since $\mathcal{H}_{n+1}^\bullet$ is flat over $\Lambda/\ell^{n+1}\Lambda$, we have

$$\ell^n \mathcal{H}_{n+1}^\bullet \cong \mathcal{H}_{n+1}^\bullet / \ell \mathcal{H}_{n+1}^\bullet.$$

We know that this is quasi-isomorphic to $\mathcal{H}_1^\bullet$; in particular, it is independent of n. We can find some U on which the cohomology modules $H^i(\mathcal{H}_1^\bullet)$ are locally constant.

The images of the connecting maps

$$H^i(\mathcal{H}_{n+1}^\bullet) \to H^{i+1}(\ell^n \mathcal{H}_{n+1}^\bullet) \cong H^{i+1}(\mathcal{H}_1^\bullet)$$

form an increasing sequence in n, thanks to the multiplication-by-ℓ maps

$$\mathcal{H}_n^\bullet \to \mathcal{H}_{n+1}^\bullet.$$

Therefore, by Proposition 2.2, these images eventually stabilize, say to the value $\mathcal{M}^i$, for $n \geq n_0$.

Choose U on which all the cohomology modules for $n \leq n_0$ are locally constant, as is the image $\mathcal{M}^i$ of the connecting homomorphism described above, for each i. Then by the long exact sequence all cohomology modules $H^i(\mathcal{H}_n^\bullet)$ are locally constant on U.

This proves Lemma 4.4. □

For simplicity, fix i and let $\mathcal{H}_n$ denote the sheaf $H^i(\mathcal{H}_n^\bullet)$. By Lemma 4.4 and noetherian induction, we know that there is a stratification of S such that $\mathcal{H}_n$ becomes locally constant on the strata. Finally, by Lemma 4.3, for any geometric point s of S, the fibers $(\mathcal{H}_n)_s$ satisfy the Artin-Rees condition.

To finish we need a general fact about Artin-Rees categories, which we will not prove. (This is Remark I.12.8 in Freitag-Kiehl.)

Lemma 4.5. *Let A_n be a projective system of objects in either the category of Λ-modules or the category of sheaves of Λ-modules on S. This A_n is Artin-Rees isomorphic to a strictly m-adic object if and only if it satisfies the following two conditions.*

1. *There is some $r > 0$ such that $\mathrm{Im}(A_{n+t} \to A_n) = \mathrm{Im}(A_{n+r} \to A_n)$ for all n and all $t \geq r$. Call this image $B_n = \mathrm{Im}(A_{n+r} \to A_n)$.*
2. *There is some $s > 0$ such that $B_{n+t}/\ell^n B_{n+t} \cong B_{n+s}/\ell^n B_{n+s}$ for all n and all $t \geq s$.*

In this case, let $C_n = B_{n+s}/\ell^n B_{n+s}$. The objects C_n give a strict m-adic system which is Artin-Rees isomorphic to the original

We apply Lemma 4.5 to the sheaves $\mathcal{H}_n$. To check conditions 1 and 2 it is enough to restrict to the strata of our stratification; so we may assume that all the $\mathcal{H}_n$ are locally constant. But locally constant sheaves are equivalent to representations of π_1 via the fiber functor. So we can check conditions 1 and 2 on a fiber; and we know that the fibers satisfy the Artin-Rees condition.

Hence, the projective system of sheaves $\mathcal{H}_n$ is Artin-Rees isomorphic to a strictly m-adic system, and the theorem is proven. □