ADIC ARTIN COMPARISON

ARPON RAKSIT

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The goal is to extend the Artin comparison theorem from the setting of torsion coefficients to the setting of adic coefficients. We will essentially be following the presentation of [Conrad, §§1.4.7–1.4.8], and we should repeat the note from there that the main argument of §1 is due to Deligne.

0.1 NOTATION — Throughout, Λ is a complete discrete valuation ring with uniformizer ℓ, characteristic zero fraction field K, and finite residue field $\Lambda_0 := \Lambda/\ell\Lambda$.

§1 Adic analytification

1.1 NOTATION — Throughout this section we let X be a finite-type $\mathbf{C}$-scheme.

1.2 DEFINITION — Suppose given an object $\mathscr{F}_\bullet$ in the Artin-Rees category of Λ-sheaves on X. We may analytify $\mathscr{F}_n \rightsquigarrow \mathscr{F}_n^{\mathrm{an}}$, and obtain an object $\mathscr{G}_\bullet$ in the Artin-Rees category of Λ-sheaves on X^{an}. We then define

$$\mathscr{F}_\bullet^{\mathrm{an}} := \lim \mathscr{G}_\bullet = \lim \mathscr{F}_n^{\mathrm{an}},$$

a sheaf of Λ-modules on X^{an}. As taking limits is a functor on the Artin-Rees category of Λ-sheaves on X^{an}, this construction defines a functor from the Artin-Rees category of Λ-sheaves on X to the category of sheaves of Λ-modules on X^{an}.

Our aim in this section is to demonstrate that this adic analytification construction has good properties. The key result is the following.

1.3 LEMMA — Let $\mathscr{F}_\bullet$ be a constructible Λ-sheaf on X. Then, for $x \in X(\mathbf{C})$, the canonical map

$$\iota_x\colon \left(\mathscr{F}_\bullet^{\mathrm{an}}\right)_x = \left(\lim \mathscr{F}_n^{\mathrm{an}}\right)_x \longrightarrow \lim \left(\mathscr{F}_n^{\mathrm{an}}\right)_x \simeq \lim\,(\mathscr{F}_n)_x = (\mathscr{F}_\bullet)_x$$

is an isomorphism.

Before proving this lemma, let us give some consequences.

1.4 COROLLARY — The analytification functor $\mathscr{F}_\bullet \rightsquigarrow \mathscr{F}_\bullet^{\mathrm{an}}$ defined in (1.2) is exact.

PROOF — On both sides of the functor exactness may be checked on stalks, so this is immediate from (1.3). □

1.5 COROLLARY — Let $j\colon U \hookrightarrow X$ be an open immersion.

(*a*) Let $\mathscr{F}_\bullet$ be a constructible Λ-sheaf on U. Then the canonical map

$$j_!^{\mathrm{an}}(\mathscr{F}_\bullet^{\mathrm{an}}) \longrightarrow \lim j_!^{\mathrm{an}}(\mathscr{F}_n^{\mathrm{an}}) = \left(j_!(\mathscr{F}_\bullet)\right)^{\mathrm{an}}$$

is an isomorphism.

(*b*) Let $\mathscr{F}_\bullet$ be a constructible Λ-sheaf on X. Then the canonical map

$$(j^{\mathrm{an}})^*(\mathscr{F}_\bullet^{\mathrm{an}}) \longrightarrow \lim (j^{\mathrm{an}})^*(\mathscr{F}_n^{\mathrm{an}}) = \big(j^*(\mathscr{F}_\bullet)\big)^{\mathrm{an}}$$

is an isomorphism.

Proof — (*a*) It follows from (1.3) that the map induces isomorphisms on stalks.

(*b*) This is immediate from $(j^{\mathrm{an}})^*$ preserving limits. □

1.6 Corollary — Let $i\colon Z \hookrightarrow X$ be an closed immersion.

(*a*) Let $\mathscr{F}_\bullet$ be a constructible Λ-sheaf on Z. Then the canonical map

$$i_*^{\mathrm{an}}(\mathscr{F}_\bullet^{\mathrm{an}}) \longrightarrow \lim i_*^{\mathrm{an}}(\mathscr{F}_n^{\mathrm{an}}) = (i_*(\mathscr{F}_\bullet))^{\mathrm{an}}$$

is an isomorphism.

(*b*) Let $\mathscr{F}_\bullet$ be a constructible Λ-sheaf on X. Then the canonical map

$$(i^{\mathrm{an}})^*(\mathscr{F}_\bullet^{\mathrm{an}}) \longrightarrow \lim (i^{\mathrm{an}})^*(\mathscr{F}_n^{\mathrm{an}}) = \big(i^*(\mathscr{F}_\bullet)\big)^{\mathrm{an}}$$

is an isomorphism.

Proof — (*a*) This is immediate from pushforward i_*^{an} preserving limits.

(*b*) Let $j\colon U \hookrightarrow X$ be the open complement of $i\colon Z \hookrightarrow X$. We have an exact sequence

$$0 \longrightarrow j_! j^* \mathscr{F}_\bullet \longrightarrow \mathscr{F}_\bullet \longrightarrow i_* i^* \mathscr{F}_\bullet \longrightarrow 0$$

which by (1.4) analytifies to an exact sequence

$$0 \longrightarrow \big(j_! j^* \mathscr{F}_\bullet\big)^{\mathrm{an}} \longrightarrow \mathscr{F}_\bullet^{\mathrm{an}} \longrightarrow \big(i_* i^* \mathscr{F}_\bullet\big)^{\mathrm{an}} \longrightarrow 0.$$

We also have an exact sequence

$$0 \longrightarrow j_!^{\mathrm{an}}(j^{\mathrm{an}})^* \mathscr{F}_\bullet^{\mathrm{an}} \longrightarrow \mathscr{F}_\bullet^{\mathrm{an}} \longrightarrow i_*^{\mathrm{an}}(i^{\mathrm{an}})^* \mathscr{F}_\bullet^{\mathrm{an}} \longrightarrow 0.$$

By (1.5) and (*a*) we get

$$j_!^{\mathrm{an}}(j^{\mathrm{an}})^* \mathscr{F}_\bullet^{\mathrm{an}} \xrightarrow{\sim} \big(j_! j^* \mathscr{F}_\bullet\big)^{\mathrm{an}}, \quad i_*^{\mathrm{an}}\big(i^* \mathscr{F}_\bullet\big)^{\mathrm{an}} \xrightarrow{\sim} \big(i_* i^* \mathscr{F}_\bullet\big)^{\mathrm{an}}.$$

From all this we deduce that the canonical map

$$i_*^{\mathrm{an}}(i^{\mathrm{an}})^* \mathscr{F}_\bullet^{\mathrm{an}} \longrightarrow i_*^{\mathrm{an}}\big(i^* \mathscr{F}_\bullet\big)^{\mathrm{an}}$$

is an isomorphism, which implies the claim. □

1.7 Corollary — Suppose $\mathscr{F}_\bullet$ is a lisse Λ-sheaf on X. Then $\mathscr{F}_\bullet^{\mathrm{an}}$ is a local system of finite Λ-modules on X^{an}.

Proof — We may assume $\mathscr{F}_\bullet$ is lisse strictly ℓ-adic. Restricting to a (Zariski-)connected component of X, we may assume X is connected. Then all stalks of $\mathscr{F}_\bullet$ are (abstractly) isomorphic (since $(\mathscr{F}_\bullet)_x \simeq \lim (\mathscr{F}_n)_x$ this follows from the property holding for the locally constant constructible sheaves $\mathscr{F}_n$, which is a consequence of their specialization properties).

Now fix $x \in X(\mathbf{C})$. By (I.3) and strictness of $\mathscr{F}_\bullet$ we have

$$\left(\mathscr{F}_\bullet^{\mathrm{an}}\right)_x / \ell^{n+1} \left(\mathscr{F}_\bullet^{\mathrm{an}}\right)_x \simeq (\mathscr{F}_n)_x .$$

In particular we may find an open $U \subseteq X^{\mathrm{an}}$ containing x and local sections $s_1, \dots, s_r \in \mathscr{F}_\bullet^{\mathrm{an}}(U)$ such that $\{(s_\nu)_x\}$ projects to a Λ_0-basis of $(\mathscr{F}_0)_x$. By shrinking U we may assume $\mathscr{F}_0$ is constant on U so that in fact $\{(s_\nu)_y\}$ projects to a Λ_0-basis of $(\mathscr{F}_0)_y$ for all $y \in U$.

Now, for each $y \in U$, $\left(\mathscr{F}_\bullet^{\mathrm{an}}\right)_y \simeq (\mathscr{F}_\bullet)_y$ is a finite Λ-module, so by Nakayama's lemma $\{(s_\nu)_y\}$ generates $\left(\mathscr{F}_\bullet^{\mathrm{an}}\right)_y$. Again shrinking U if necessary, we may assume that the finitely many relations on $\{(s_\nu)_x\}$ in $\left(\mathscr{F}_\bullet^{\mathrm{an}}\right)_x$ are satisfied by $\{(s_\nu)_y\}$ in $\left(\mathscr{F}_\bullet^{\mathrm{an}}\right)_y$ for all $y \in U$.

We conclude that there is a finite Λ-module M and a surjection $\phi\colon \underline{M} \longrightarrow \mathscr{F}_\bullet^{\mathrm{an}}|_U$, for $\underline{M}$ the constant sheaf on U with value M, such that the induced map $\phi_x\colon M \longrightarrow \left(\mathscr{F}_\bullet^{\mathrm{an}}\right)_x$ is an isomorphism. Using (I.3) and our restriction to connected X, we know all the stalks

$$\left(\mathscr{F}_\bullet^{\mathrm{an}}\right)_y \simeq (\mathscr{F}_\bullet)_y, \quad y \in U$$

are isomorphic. It follows that ϕ must be an isomorphism at every $y \in U$, and hence an isomorphism. This proves $\mathscr{F}_\bullet^{\mathrm{an}}$ is locally constant, as desired. □

I.8 We now work towards proving (I.3), though we will only give the proof for the case that $\mathscr{F}_\bullet$ is lisse; the details for general case of $\mathscr{F}_\bullet$ constructible may be found in [Conrad, §I.4.7]. The proof will require some preliminaries.

I.8.I Remark — Before beginning the argument, it's perhaps worth pointing out where intuitively the difficulty lies. Taking $\mathscr{F}_\bullet$ to be lisse strictly ℓ-adic, we have a collection of local systems $\mathscr{G}_n := \mathscr{F}_n^{\mathrm{an}}$ on X^{an} and want to show that

$$(\lim \mathscr{G}_n)_x \simeq \lim (\mathscr{G}_n)_x$$

at each point $x \in X^{\mathrm{an}}$. This might seem easy as the $\mathscr{G}_n$ are locally constant. However, it is not easy, as we don't know that we may find a *single* open neighborhood of x on which *all* of the $\mathscr{G}_n$ are constant. More precisely, while it is clear we may accomplish this in the case that X is smooth, as then $x \in X^{\mathrm{an}}$ has a contractible neighborhood, it is not clear in the non-smooth case. Our strategy below is to use alterations to bootstrap from the smooth to the general case.

I.8.2 Lemma — Let M be a complex manifold and $D \subseteq M$ a normal crossings divisor. Then there exists a base of opens W in M around D such that, for all local systems $\mathscr{G}$ on M, the restriction map $\mathscr{G}(W) \longrightarrow \mathscr{G}(D)$ is an isomorphism.

Proof — As we may replace M with an arbitrary open in M around D, it suffices to find one such W. In the local picture, M is a polydisk and D the zero locus of a product of coordinate functions; both of these are contractible sets, on which any local system is constant, so the claim is clear.

We can glue to bootstrap to the global case. Choose open neighborhoods W'_d in M of each $d \in D$ which look like the local picture. Put a Riemannian metric on M

and for each $d \in D$ choose $r_d \geq 0$ small enough so that the open ball $W_d := \mathrm{B}_{r_d}(d)$ is geodesically convex and $\mathrm{B}_{3r_d}(d) \subseteq W'_d$. Set $W := \bigcup_{d\in D} W_d$.

Let $\mathscr{G}$ be a local system on M. For each $d \in D$ we know $\mathscr{G}(W'_d) \longrightarrow \mathscr{G}(W'_d \cap D)$ is an isomorphism. In fact W'_d and $W'_d \cap D$ are contractible, implying $\mathscr{G}|_{W'_d}$ is constant and $\mathscr{G}(W'_d \cap D) \longrightarrow \mathscr{G}(W_d \cap D)$ is injective. Also W_d is contractible (by convexity) so $\mathscr{G}(W'_d) \longrightarrow \mathscr{G}(W_d)$ is an isomorphism. We deduce that $\mathscr{G}(W_d) \longrightarrow \mathscr{G}(W_d \cap D)$ is injective for each $d \in D$, and it follows that $\mathscr{G}(W) \longrightarrow \mathscr{G}(D)$ is injective.

We now argue for surjectivity. Fix $s \in \mathscr{G}(D)$. There exist (unique) $\widetilde{s}(d) \in \mathscr{G}(W'_d)$ restricting to $s(d) := s|_{W'_d \cap D} \in \mathscr{G}(W'_d \cap D)$. We just need to glue these into a section $\widetilde{s} \in \mathscr{G}(W)$, so it suffices to show that $\widetilde{s}(d)|_{W_d \cap W_{d'}} = \widetilde{s}(d')|_{W_d \cap W_{d'}}$ for all $d, d' \in D$. If $W_d \cap W_{d'} = \emptyset$ this is trivial. Otherwise, by symmetry we may assume $r_{d'} \leq r_d$, and then $W_d \cap W_{d'}$ being nonempty implies that

$$W_{d'} = \mathrm{B}_{r_{d'}}(d') \subseteq \mathrm{B}_{3r_d}(d) \subseteq W'_d.$$

In particular $d' \in W'_d$, implying

$$\widetilde{s}(d)_d = \widetilde{s}(d)_{d'} = s(d)_{d'} = s_{d'} = s(d')_{d'} = \widetilde{s}(d')_{d'}.$$

Finally any $w \in W_d \cap W_{d'}$ admits paths to d and to d', so we get

$$\widetilde{s}(d)_w = \widetilde{s}(d)_d = \widetilde{s}(d')_{d'} = \widetilde{s}(d')_w,$$

proving the desired gluability. □

1.8.3 Lemma — Let $T' \longrightarrow T$ be a quotient map of topological spaces. Let $T'' := T' \times_T T'$. Let $\mathscr{G}$ be a sheaf of sets on T, and let $\mathscr{G}'$ and $\mathscr{G}''$ be the pullbacks of $\mathscr{G}$ to T' and T''. Then the sequence

$$\mathscr{G}(T) \longrightarrow \mathscr{G}'(T') \rightrightarrows \mathscr{G}''(T'')$$

is an equalizer sequence.

Proof — Let $\pi\colon E \longrightarrow T$ be the espace étalé associated to $\mathscr{G}$, so that elements of $\mathscr{G}(T)$ are given by (continuous) sections of π. Similarly take $\pi'\colon E' \longrightarrow T'$ and $\pi''\colon E'' \longrightarrow T''$ associated to $\mathscr{G}'$ and $\mathscr{G}''$; these are obtained by pulling back π to T' and T'', so $E' \simeq E \times_T T'$ and $E'' \simeq E \times_T T''$. The claim now follows from the universal property of a quotient map. □

1.8.4 Lemma — Suppose X is separated. Let Y be a (Zariski-)closed subset of X.

(*a*) Fix an open $U \subseteq X^{\mathrm{an}}$ around Y^{an}. There is an open $V \subseteq U$ around Y^{an} such that, for all local systems $\mathscr{G}$ on X^{an}, the restriction map

$$\mathrm{im}\,(\mathscr{G}(U) \longrightarrow \mathscr{G}(V)) \longrightarrow \mathscr{G}(Y^{\mathrm{an}})$$

is injective.

(*b*) There exists an open $U \subseteq X^{\mathrm{an}}$ around Y^{an} such that, for all local systems $\mathscr{G}$ on X^{an}, the restriction map $\mathscr{G}(U) \longrightarrow \mathscr{G}(Y^{\mathrm{an}})$ is surjective.

Proof — Let $g\colon \widetilde{X} \longrightarrow X$ be the normalization of X_{red}, and $\widetilde{Y} := g^{-1}(Y)$. Note that $\widetilde{X}$ is separated since X is. Thus we may apply de Jong's alterations theorem to each connected/irreducible component $\widetilde{X}_i$ of $\widetilde{X}$, together with the proper closed subset

$$\begin{cases} \widetilde{X}_i \cap \widetilde{Y} & \text{if } \widetilde{X}_i \cap \widetilde{Y} \neq \widetilde{X}_i \\ \emptyset & \text{otherwise.} \end{cases}$$

We obtain a smooth quasi-projective $\mathbf{C}$-scheme X_0 and a generically finite surjective proper map $f\colon X_0 \longrightarrow X$ such that $Y_0 := f^{-1}(Y) = Y_1 \amalg Y_2$ with Y_1 a union of some connected components of X_0 and Y_2 a strict normal crossings divisor in the remaining components.

Applying (I.8.2) with $M = X_0^{\mathrm{an}}$ and $D = Y_2^{\mathrm{an}}$, we find a base $\mathfrak{B}$ of opens $W \subseteq X_0^{\mathrm{an}}$ around Y_0^{an} for which restriction $\mathscr{G}(W) \longrightarrow \mathscr{G}(Y_0^{\mathrm{an}})$ is an isomorphism for all local systems $\mathscr{G}$ on X_0^{an}. With all this preparation in hand, we now address the two claims:

(*a*) Choose an open $W_0 \in \mathfrak{B}$ contained in $U_0 := (f^{\mathrm{an}})^{-1}(U)$. As f is proper, $f^{\mathrm{an}}\colon X_0^{\mathrm{an}} \longrightarrow X^{\mathrm{an}}$ is closed, so we may find an open $V \subseteq U$ containing Y^{an} such that $V_0 := f^{-1}(V) \subseteq W_0$.

Let $\mathscr{G}$ be any local system on X^{an}, set $\mathscr{G}_0 := (f^{\mathrm{an}})^*(\mathscr{G})$, and consider the commutative diagram

$$\begin{array}{ccccccc} \mathscr{G}(U) & & \longrightarrow & & \mathscr{G}(V) & \longrightarrow & \mathscr{G}(Y^{\mathrm{an}}) \\ \downarrow & & & & \downarrow & & \downarrow \\ \mathscr{G}_0(U_0) & \longrightarrow & \mathscr{G}_0(W_0) & \longrightarrow & \mathscr{G}_0(V_0) & \longrightarrow & \mathscr{G}_0(Y_0^{\mathrm{an}}). \end{array}$$

We want to show that an element of $\mathscr{G}(U)$ that dies in $\mathscr{G}(Y^{\mathrm{an}})$ already dies in $\mathscr{G}(V)$. To deduce this from the diagram, we observe that the analogous property holds for $\mathscr{G}_0(U_0), \mathscr{G}_0(W_0), \mathscr{G}(Y_0^{\mathrm{an}})$ since $W_0 \in \mathfrak{B}$, and that the map $\mathscr{G}(V) \longrightarrow \mathscr{G}(V_0)$ is injective, by stalk considerations, since f is surjective.

(*b*) Let $X_{00} := X_0 \times_X X_0$. Let Y_{00} be the closed subset $Y_0 \times_X Y_0 \subseteq X_{00}$ (putting, say, the reduced scheme structure on $Y_0 \subseteq X_0$ to form this fiber product).

Choose an open $W_0 \in \mathfrak{B}$. Let $W_{00} := W_0 \times_X W_0$, an open in X_{00} around Y_{00}. By (*a*) we may choose an open $V_{00} \subseteq W_{00}$ around Y_{00} such that, for all local systems $\mathscr{G}_{00}$ on X_{00}, the restriction map

I.8.4.I $$\operatorname{im}(\mathscr{G}(W_{00}) \longrightarrow \mathscr{G}(V_{00})) \longrightarrow \mathscr{G}(Y_{00}^{\mathrm{an}})$$

is injective. Again using that f is proper and hence f^{an} closed, we may find an open $U \subseteq X^{\mathrm{an}}$ whose preimage U_{00} in X_{00} is contained in V_{00}; this implies that also $U_0 := (f^{\mathrm{an}})^{-1}(U)$ is contained in W_0.

Now suppose given a local system $\mathscr{G}$ on X^{an} and $s \in \mathscr{G}(Y^{\mathrm{an}})$. Setting $\mathscr{G}_0 := (f^{\mathrm{an}})^*(\mathscr{G})$, we get a pullback $s_0 \in \mathscr{G}_0(Y_0^{\mathrm{an}})$. Since $W_0 \in \mathfrak{B}$ this extends (uniquely) to a section $\widetilde{s}_0 \in \mathscr{G}_0(W_0)$. To finish, we'd like to descend $\widetilde{s}_0|_{U_0}$ to U. Let $\mathscr{G}_{00}$ be the pull back of $\mathscr{G}$ to X_{00}^{an}. By the injectivity of (I.8.4.I) we deduce that the two pullbacks of $\widetilde{s}_0$ to $\mathscr{G}_{00}(V_{00})$ agree, and hence the two pullbacks of $\widetilde{s}_0|_{U_0}$ to $\mathscr{G}_{00}(U_{00})$ agree. We are then done by (I.8.3), as $f^{\mathrm{an}}\colon U_0 \longrightarrow U$ is a closed surjection, hence a quotient map, and $U_{00} \simeq U_0 \times_U U_0$. □

Proof of (I.3) for $\mathscr{F}_\bullet$ lisse — The question is local on X so we may assume X is separated. We are assuming $\mathscr{F}_\bullet$ is lisse, and we may moreover assume $\mathscr{F}_\bullet$ is lisse strictly ℓ-adic so that each $\mathscr{F}_n$ is locally constant constructible; then each $\mathscr{F}_n^{\mathrm{an}}$ is a local system on X^{an}.

Fix $x \in X(\mathbf{C})$. Applying (1.8.4) to $Y = \{x\} \subseteq X$, we may find a sequence of pairs of opens $V_m \subseteq U_m \subseteq X^{\mathrm{an}}$ around x such that:

(*a*) $U_{m+1} \subseteq V_m$ for $m \in \mathbf{N}$;

(*b*) $\{U_m\}_{m\in\mathbf{N}}$ is a base at x (the ability to choose a countable base follows from the fact that X^{an} has the topology of a subspace in some affine space $\mathbf{C}^d$);

(*c*) the restriction map

$$I_{n,m} := \operatorname{im}\left((\mathscr{F}_n^{\mathrm{an}}(U_m) \longrightarrow \mathscr{F}_n(V_m^{\mathrm{an}})\right) \longrightarrow \left(\mathscr{F}_n^{\mathrm{an}}\right)_x \simeq (\mathscr{F}_n)_x$$

is an isomorphism for all $m, n \in \mathbf{N}$.

We now show that the map $\iota_x\colon \left(\mathscr{F}_\bullet^{\mathrm{an}}\right)_x \longrightarrow (\mathscr{F}_\bullet)_x$ is bijective. For surjectivity, observe that by (*c*) every element of $(\mathscr{F}_\bullet)_x = \lim(\mathscr{F}_n)_x$ arises from an element of $\lim I_{n,m} \subseteq \mathscr{F}_\bullet^{\mathrm{an}}(V_m)$ for any fixed m. For injectivity, observe that by (*b*) any element of $s_x \in \left(\mathscr{F}_\bullet^{\mathrm{an}}\right)_x$ has a representative $s \in \mathscr{F}_\bullet^{\mathrm{an}}(U_m)$ for some $m \in \mathbf{N}$, and if it vanishes under ι_x then by (*c*) we must have $s|_{V_m} = 0$, implying $s_x = 0$. □

§2 Adic comparison

We now arrive at the main result, the adic Artin comparison isomorphism:

2.1 THEOREM — Let $f\colon X \longrightarrow S$ be a separated morphism between finite-type $\mathbf{C}$-schemes. Let $\mathscr{F}_\bullet$ be a constructible Λ-sheaf on X. Then, for $p \geq 0$, the canonical maps

$$\left(\mathrm{R}^p f_!(\mathscr{F}_\bullet)\right)^{\mathrm{an}} \xrightarrow{\alpha_1} \lim \mathrm{R}^p f_!^{\mathrm{an}}(\mathscr{F}_n^{\mathrm{an}}) \xleftarrow{\alpha_2} \mathrm{R}^p f_!^{\mathrm{an}}(\mathscr{F}_\bullet^{\mathrm{an}})$$

$$\left(\mathrm{R}^p f_*(\mathscr{F}_\bullet)\right)^{\mathrm{an}} \xrightarrow{\beta_1} \lim \mathrm{R}^p f_*^{\mathrm{an}}(\mathscr{F}_n^{\mathrm{an}}) \xleftarrow{\beta_2} \mathrm{R}^p f_*^{\mathrm{an}}(\mathscr{F}_\bullet^{\mathrm{an}})$$

are isomorphisms.

2.1.1 Alas, here we will only prove that $\alpha_1, \alpha_2, \beta_1$ are isomorphisms. We first give the (easy) argument for α_1, β_1.

PROOF OF (2.1) FOR α_1, β_1 — By definition we have

$$\left(\mathrm{R}^p f_!(\mathscr{F}_\bullet)\right)^{\mathrm{an}} \simeq \lim\left(\mathrm{R}^p f_!(\mathscr{F}_n)\right)^{\mathrm{an}}.$$

We may assume $\mathscr{F}_\bullet$ is a constructible strictly ℓ-adic sheaf, so that each $\mathscr{F}_n$ is constructible torsion. Then by the Artin comparison theorem for torsion coefficients we have canonical isomorphisms

2.1.1.1 $$\left(\mathrm{R}^p f_!(\mathscr{F}_n)\right)^{\mathrm{an}} \xrightarrow{\sim} \mathrm{R}^p f_!^{\mathrm{an}}(\mathscr{F}_n^{\mathrm{an}}).$$

As α_1 is precisely the map induced by these isomorphisms, it too is an isomorphism. The same argument, with $f_!$ and $f_!^{\mathrm{an}}$ replaced by f_* and f_*^{an}, demonstrates that β_1 is an isomorphism. □

2.2 We now work towards proving the claim for α_2, first establishing several tools that will be needed in the proof.

2.2.1 Terminology — Given a separated morphism of finite-type $\mathbf{C}$-schemes $f\colon X \longrightarrow S$, we will say "$\alpha_2(f)$ is an isomorphism" if for this fixed morphism f the map α_2 of (2.1) is an isomorphism for all constructible Λ-sheaves $\mathscr{F}_\bullet$ on X.

2.2.2 Lemma — Let $f\colon X \longrightarrow S$ be a separated morphism of finite type $\mathbf{C}$-schemes. For $s \in S(\mathbf{C})$ let $f_s\colon X_s \longrightarrow \mathrm{Spec}(\mathbf{C})$ be the fiber of f over s. Suppose $\alpha_2(f_s)$ is an isomorphism for all $s \in S(\mathbf{C})$. Then $\alpha_2(f)$ is an isomorphism.

Proof — It suffices to show for each $s \in S^{\mathrm{an}}$ that the map on stalks

$$(\alpha_2)_s\colon \left(\mathrm{R}^p f_!^{\mathrm{an}}(\mathscr{F}_\bullet^{\mathrm{an}})\right)_s \longrightarrow \left(\lim \mathrm{R}^p f_!^{\mathrm{an}}(\mathscr{F}_n^{\mathrm{an}})\right)_s$$

is an isomorphism. By (2.1.1.1) and (1.3) the canonical map

$$\left(\lim \mathrm{R}^p f_!^{\mathrm{an}}(\mathscr{F}_n^{\mathrm{an}})\right)_s \longrightarrow \lim \left(\mathrm{R}^p f_!^{\mathrm{an}}(\mathscr{F}_n^{\mathrm{an}})\right)_s$$

is an isomorphism. The claim thus follows from proper base change. □

2.2.3 Lemma — Let $S := \mathrm{Spec}(\mathbf{C})$. Suppose given a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\;g\;} & X \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle f} & \\ & S & \end{array}$$

of separated morphisms between finite type $\mathbf{C}$-schemes. Suppose that $\alpha_2(f), \alpha_2(g)$ are isomorphisms. Then $\alpha_2(h)$ is an isomorphism.

Proof — Let $\mathscr{F}_\bullet$ be a constructible Λ-sheaf on Y. For each $n \in \mathbf{N}$ we have a Leray spectral sequence

$$\mathrm{H}_\mathrm{c}^p(X^{\mathrm{an}}; \mathrm{R}^q g_!^{\mathrm{an}}(\mathscr{F}_n^{\mathrm{an}})) \quad\Longrightarrow\quad \mathrm{H}_\mathrm{c}^{p+q}(Y^{\mathrm{an}}; \mathscr{F}_n^{\mathrm{an}});$$

let us denote this spectral sequence by E_n. We may assume that $\mathscr{F}_\bullet$ is constructible strictly ℓ-adic, so each $\mathscr{F}_n$ (and hence $\mathscr{F}_n^{\mathrm{an}}$) is constructible. Then by the torsion Artin comparison isomorphism we have

$$\mathrm{R}^q g_!^{\mathrm{an}}(\mathscr{F}_n^{\mathrm{an}}) \simeq \left(\mathrm{R}^q g_!(\mathscr{F}_n)\right)^{\mathrm{an}},$$

which we know is constructible. We then similarly deduce that the cohomology groups

$$\mathrm{H}_\mathrm{c}^p(X^{\mathrm{an}}; \mathrm{R}^q g_!^{\mathrm{an}}(\mathscr{F}_n^{\mathrm{an}})), \quad \mathrm{H}_\mathrm{c}^{p+q}(Y^{\mathrm{an}}; \mathscr{F}_n^{\mathrm{an}})$$

are all finite. Therefore, for each $p, q \geq 0$ the inverse systems

$$\{\mathrm{H}_\mathrm{c}^p(X^{\mathrm{an}}; \mathrm{R}^q g_!^{\mathrm{an}}(\mathscr{F}_n^{\mathrm{an}}))\}_{n\in\mathbf{N}}, \quad \left\{\mathrm{H}_\mathrm{c}^{p+q}(Y^{\mathrm{an}}; \mathscr{F}_n^{\mathrm{an}})\right\}_{n\in\mathbf{N}}$$

satisfy the Mittag-Leffler condition. Taking inverse limits of these systems is therefore exact, so that $\lim E_n$ gives a spectral sequence

$$\lim \mathrm{H}_\mathrm{c}^p(X^{\mathrm{an}}; \mathrm{R}^q g_!^{\mathrm{an}}(\mathscr{F}_n^{\mathrm{an}})) \quad\Longrightarrow\quad \lim \mathrm{H}_\mathrm{c}^{p+q}(Y^{\mathrm{an}}; \mathscr{F}_n^{\mathrm{an}}).$$

We also have a Leray spectral sequence for $\mathscr{F}_\bullet$,

$$\mathrm{H}^p_{\mathrm{c}}(X^{\mathrm{an}};\mathrm{R}^q g^{\mathrm{an}}_!(\mathscr{F}^{\mathrm{an}}_\bullet)) \quad \Longrightarrow \quad \mathrm{H}^{p+q}_{\mathrm{c}}(Y^{\mathrm{an}};\mathscr{F}^{\mathrm{an}}_\bullet);$$

let us denote this one by $E_\bullet$. There is a map of spectral sequences $\alpha_2\colon E_\bullet \longrightarrow \lim E_n$ which on the initial page is given by the composition

$$\mathrm{H}^p_{\mathrm{c}}(X^{\mathrm{an}};\mathrm{R}^q g^{\mathrm{an}}_!(\mathscr{F}^{\mathrm{an}}_\bullet)) \xrightarrow{\alpha_2(g)} \mathrm{H}^p_{\mathrm{c}}(X^{\mathrm{an}};\lim \mathrm{R}^q g^{\mathrm{an}}_!(\mathscr{F}^{\mathrm{an}}_n)) \xrightarrow{\alpha_2(f)} \lim \mathrm{H}^p_{\mathrm{c}}(X^{\mathrm{an}};\mathrm{R}^q g^{\mathrm{an}}_!(\mathscr{F}^{\mathrm{an}}_n))$$

and on the abutment is given by $\alpha_2(h)$. Our hypothesis implies that the map of spectral sequences is an isomorphism on the initial page, and hence it must be on the abutments as well. □

2.2.4 Lemma — Let $S := \mathrm{Spec}(\mathbf{C})$. Let $f\colon X \longrightarrow S$ be a separated morphism of finite type $\mathbf{C}$-schemes. Let $j\colon U \hookrightarrow X$ be an open subscheme and let $i\colon Z \hookrightarrow X$ denote its closed complement (with the reduced scheme structure). Define

$$g := f\circ j\colon U \longrightarrow S, \quad h := f\circ i\colon Z \longrightarrow S,$$

and suppose $\alpha_2(g), \alpha_2(h)$ are isomorphisms. Then $\alpha_2(f)$ is an isomorphism.

Proof — Let $\mathscr{F}_\bullet$ be a constructible Λ-sheaf on X. For each $n \in \mathbf{N}$ we have an excision sequence

$$\cdots \longrightarrow \mathrm{H}^p_{\mathrm{c}}(U^{\mathrm{an}},\mathscr{F}^{\mathrm{an}}_n) \longrightarrow \mathrm{H}^p_{\mathrm{c}}(X^{\mathrm{an}},\mathscr{F}^{\mathrm{an}}_n) \longrightarrow \mathrm{H}^p_{\mathrm{c}}(Z^{\mathrm{an}},\mathscr{F}^{\mathrm{an}}_n) \longrightarrow \mathrm{H}^{p+1}_{\mathrm{c}}(U^{\mathrm{an}},\mathscr{F}^{\mathrm{an}}_n) \longrightarrow \cdots.$$

We may assume that $\mathscr{F}_\bullet$ is constructible strictly ℓ-adic, so each $\mathscr{F}_n$ is constructible. As in the proof of (2.2.3), torsion Artin comparison implies that all these cohomology groups are finite, and hence for each $p \geq 0$ the inverse systems

$$\{\mathrm{H}_{\mathrm{c}}(U^{\mathrm{an}},\mathscr{F}^{\mathrm{an}}_n)\}_{n\in\mathbf{N}}, \quad \{\mathrm{H}_{\mathrm{c}}(X^{\mathrm{an}},\mathscr{F}^{\mathrm{an}}_n)\}_{n\in\mathbf{N}}, \quad \{\mathrm{H}_{\mathrm{c}}(Z^{\mathrm{an}},\mathscr{F}^{\mathrm{an}}_n)\}_{n\in\mathbf{N}}$$

all satisfy the Mittag-Leffler condition. Taking inverse limits of these systems is therefore exact, so the sequence

$$\cdots \longrightarrow \lim \mathrm{H}^p_{\mathrm{c}}(U^{\mathrm{an}},\mathscr{F}^{\mathrm{an}}_n) \longrightarrow \lim \mathrm{H}^p_{\mathrm{c}}(X^{\mathrm{an}},\mathscr{F}^{\mathrm{an}}_n) \longrightarrow \lim \mathrm{H}^p_{\mathrm{c}}(Z^{\mathrm{an}},\mathscr{F}^{\mathrm{an}}_n) \longrightarrow \cdots$$

remains exact.

We also have an excision sequence for $\mathscr{F}_\bullet$, and, using (1.5) and (1.6), we get a commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathrm{H}^p_{\mathrm{c}}(U^{\mathrm{an}},\mathscr{F}^{\mathrm{an}}_\bullet) & \longrightarrow & \mathrm{H}^p_{\mathrm{c}}(X^{\mathrm{an}},\mathscr{F}^{\mathrm{an}}_\bullet) & \longrightarrow & \mathrm{H}^p_{\mathrm{c}}(Z^{\mathrm{an}},\mathscr{F}^{\mathrm{an}}_\bullet) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \alpha_2(g)} & & \downarrow{\scriptstyle \alpha_2(f)} & & \downarrow{\scriptstyle \alpha_2(h)} & & \\
\cdots & \longrightarrow & \lim \mathrm{H}^p_{\mathrm{c}}(U^{\mathrm{an}},\mathscr{F}^{\mathrm{an}}_n) & \longrightarrow & \lim \mathrm{H}^p_{\mathrm{c}}(X^{\mathrm{an}},\mathscr{F}^{\mathrm{an}}_n) & \longrightarrow & \lim \mathrm{H}^p_{\mathrm{c}}(Z^{\mathrm{an}},\mathscr{F}^{\mathrm{an}}_n) & \longrightarrow & \cdots
\end{array}$$

with exact rows. The claim now follows from the five-lemma. □

2.2.5 Lemma — Suppose $f\colon X \longrightarrow S$ is a finite morphism of finite-type $\mathbf{C}$-schemes. Then $\alpha_2(f)$ is an isomorphism.

Proof — In this situation $f_! = f_*$ and f_* is exact. So $\mathrm{R}^p f_!^{\mathrm{an}}$ vanishes for $p > 0$ and we need only check, for $\mathscr{F}_\bullet$ a constructible Λ-sheaf on X, that

$$\alpha_2\colon f_*^{\mathrm{an}}(\mathscr{F}_\bullet^{\mathrm{an}}) \longrightarrow \lim f_*^{\mathrm{an}}(\mathscr{F}_n^{\mathrm{an}})$$

is an isomorphism. This follows from the fact that f_* preserves limits. □

2.2.6 Lemma — Let T be a topological space. Let $\{\mathscr{F}_n\}_{n\in\mathbf{N}}$ be an inverse system of abelian sheaves on T. Assume that:

- (a) there is a basis $\mathfrak{U}$ of T such that:
  - (a.1) the inverse system $\{\mathscr{F}_n(U)\}_{n\in\mathbf{N}}$ satisfies the Mittag-Leffler criterion for all $U \in \mathfrak{U}$;
  - (a.2) $\mathrm{H}^p(U;\mathscr{F}_n) \simeq 0$ for all $U \in \mathfrak{U}$, $p > 0$, $n \in \mathbf{N}$;
- (b) for each $p \geq 0$ the inverse system of abelian groups $\{\mathrm{H}^p(T;\mathscr{F}_n)\}_{n\in\mathbf{N}}$ satisfies the Mittag-Leffler criterion.

Then, for $p \geq 0$, the canonical map

$$\mathrm{H}^p(T, \lim \mathscr{F}_n) \longrightarrow \lim \mathrm{H}^p(T, \mathscr{F}_n)$$

is an isomorphism.

Proof — Let Ab denote the category of abelian groups and Ab(T) the category of abelian sheaves on T; let $\mathrm{Ab}^{\mathbf{N}}$ and $\mathrm{Ab}(T)^{\mathbf{N}}$ denote the categories of inverse systems in Ab and Ab(T). As $\Gamma\colon \mathrm{Ab}(T) \longrightarrow \mathrm{Ab}$ preserves limits, we get a commutative diagram of left-exact functors

$$\begin{array}{ccc} \mathrm{Ab}(T)^{\mathbf{N}} & \xrightarrow{\lim} & \mathrm{Ab}(T) \\ {\scriptstyle\Gamma}\downarrow & & \downarrow{\scriptstyle\Gamma} \\ \mathrm{Ab}^{\mathbf{N}} & \xrightarrow{\lim} & \mathrm{Ab}. \end{array}$$

Taking (total) right-derived functors, we see that $\mathrm{R}\Gamma \circ \mathrm{R}\lim \simeq \mathrm{R}\lim \circ \mathrm{R}\Gamma$, so we have

$$\mathrm{R}\Gamma(T, \mathrm{R}\lim \mathscr{F}_n) \simeq \mathrm{R}\lim \mathrm{R}\Gamma(T, \mathscr{F}_n).$$

On the left-hand side, assumption (a) implies that $\mathrm{R}\lim \mathscr{F}_n \simeq \lim \mathscr{F}_n$ [Stacks, Tag 0BKS]. To address the right-hand side, consider the Grothendieck spectral sequence

$$\mathrm{R}^q \lim \mathrm{H}^p(T, \mathscr{F}_n) \quad \Longrightarrow \quad \mathrm{H}^{p+q}(\mathrm{R}\lim \mathrm{R}\Gamma(T, \mathscr{F}_n)).$$

In the same way, assumption (b) implies that $\mathrm{R}^q \lim \mathrm{H}^p(T, \mathscr{F}_n) \simeq 0$ for all $q > 0$, so the spectral sequence immediately degenerates. Combining this with the prior observations, we obtain isomorphisms

$$\mathrm{H}^p(T, \lim \mathscr{F}_n) \simeq \lim \mathrm{H}^p(T, \mathscr{F}_n),$$

which one should check arise from the canonical map, as desired.[1] □

[1] I learned this argument from a comment on http://mathoverflow.net/q/65249.

2.2.7 LEMMA — Let $S := \mathrm{Spec}(\mathbf{C})$. Suppose $f\colon X \longrightarrow S$ is a smooth morphism of finite-type $\mathbf{C}$-schemes. Then

$$\beta_2\colon \mathrm{H}^p(X^{\mathrm{an}};\mathscr{F}_\bullet^{\mathrm{an}}) \longrightarrow \lim \mathrm{H}^p(X^{\mathrm{an}};\mathscr{F}_n^{\mathrm{an}})$$

is an isomorphism for all lisse Λ-sheaves $\mathscr{F}_\bullet$ on X.

PROOF — It suffices to consider a lisse strictly ℓ-adic sheaf $\mathscr{F}_\bullet$; thus the sheaves $\mathscr{F}_n$ are locally constant constructible, and hence the sheaves $\mathscr{F}_n^{\mathrm{an}}$ are local systems. We now have the following:

(*a*) Since f is smooth, X^{an} is a complex manifold, and hence its topology has a basis $\mathfrak{U}$ consisting of contractible opens.

(*a*.1) The strictness of $\mathscr{F}_\bullet$ implies that the transition maps $\mathscr{F}_n \longrightarrow \mathscr{F}_{n-1}$ are surjective, by strictness of $\mathscr{F}_\bullet$. This implies that the same holds for the analytified transition maps $\mathscr{F}_n^{\mathrm{an}} \longrightarrow \mathscr{F}_{n-1}^{\mathrm{an}}$ (by exactness of analytification). As the sheaves $\mathscr{F}_n^{\mathrm{an}}$ are local systems, $\mathscr{F}_n^{\mathrm{an}}|_U$ is constant for each $U \in \mathfrak{U}$, so we also have that $\mathscr{F}_n^{\mathrm{an}}(U) \longrightarrow \mathscr{F}_{n-1}(U)$ is surjective.

(*a*.2) That $\mathscr{F}_n^{\mathrm{an}}$ are local systems also implies that $\mathrm{H}^p(U;\mathscr{F}_n^{\mathrm{an}}) \simeq 0$ for all $U \in \mathfrak{U}$, $p > 0$, $n \in \mathbf{N}$.

(*b*) The sheaves $\mathscr{F}_n$ being (locally constant) constructible also implies that the cohomology groups

$$\mathrm{H}^p(X^{\mathrm{an}};\mathscr{F}_n^{\mathrm{an}}) \simeq \mathrm{H}^p(X;\mathscr{F}_n)$$

are finite. Thus, for each $p \geq 0$, the system $\{\mathrm{H}^p(X^{\mathrm{an}};\mathscr{F}_n^{\mathrm{an}})\}$ satisfies the Mittag-Leffler criterion.

That β_2 is an isomorphism thus follows from (2.2.6). □

PROOF OF (2.1) FOR α_2 — By (2.2.2) we may reduce to the case that $S = \mathrm{Spec}(\mathbf{C})$. Then by noetherian induction and (2.2.4) we may reduce to the case that X is affine. By Noether normalization we have a factorization of $f\colon X \longrightarrow S$ as a composite

$$X \xrightarrow{g} \mathbf{A}^d_{\mathbf{C}} \longrightarrow \mathbf{A}^{d-1}_{\mathbf{C}} \longrightarrow \cdots \longrightarrow \mathbf{A}^1_{\mathbf{C}} \longrightarrow S$$

where g is finite. Now applying (2.2.3) and (2.2.5) and then again (2.2.2), we are reduced to the case that $f\colon X \longrightarrow S$ is a smooth curve over $\mathrm{Spec}(\mathbf{C})$. And again using noetherian induction and (2.2.4), we may reduce to the case that furthermore $\mathscr{F}_\bullet$ is lisse.

If f is proper then $f_! \simeq f_*$ and $\alpha_2 \simeq \beta_2$ so by (2.2.7) we are done. Otherwise X^{an} is a punctured Riemann surface. Let us remind ourselves that our goal is to show

$$\alpha_2\colon \mathrm{H}^p_{\mathrm{c}}(X^{\mathrm{an}},\mathscr{F}_\bullet^{\mathrm{an}}) \longrightarrow \lim \mathrm{H}^p_{\mathrm{c}}(X^{\mathrm{an}},\mathscr{F}_n^{\mathrm{an}})$$

is an isomorphism. For any sheaf $\mathscr{G}$ on X^{an} we have

$$\mathrm{H}^p_{\mathrm{c}}(X^{\mathrm{an}},\mathscr{G}) \simeq \operatorname*{colim}_{\Delta^*} \mathrm{H}^p_{X^{\mathrm{an}}\setminus\Delta^*}(X^{\mathrm{an}};\mathscr{G}),$$

where the colimit is over shrinking punctured-disk neighborhoods Δ^* of the punctures of X^{an}.

By considering the excision sequence

$$\cdots \longrightarrow \mathrm{H}^p_{X^{\mathrm{an}} \setminus \Delta^*}(X^{\mathrm{an}}; \mathscr{G}) \longrightarrow \mathrm{H}^p(X^{\mathrm{an}}; \mathscr{G}) \longrightarrow \mathrm{H}^p(\Delta^*; \mathscr{G}) \longrightarrow \cdots$$

and noting that the homotopy type of Δ^* does not change as it shrinks, we observe that the transition maps in the direct system $\{\mathrm{H}^p_{X^{\mathrm{an}} \setminus \Delta^*}(X^{\mathrm{an}}; \mathscr{G})\}_{\Delta^*}$ are isomorphisms when $\mathscr{G}$ is a local system. We may assume $\mathscr{F}_\bullet$ to be lisse strictly ℓ-adic, so the sheaves $\mathscr{F}_n^{\mathrm{an}}$ are local systems, and by (1.7) $\mathscr{F}_\bullet^{\mathrm{an}}$ is a local system. We conclude from this and the above excision sequence that it suffices to show that the canonical maps

$$\mathrm{H}^p(X^{\mathrm{an}}; \mathscr{F}_\bullet^{\mathrm{an}}) \longrightarrow \lim \mathrm{H}^p(X^{\mathrm{an}}; \mathscr{F}_n^{\mathrm{an}}), \quad \mathrm{H}^p(\Delta^*; \mathscr{F}_\bullet^{\mathrm{an}}) \longrightarrow \lim \mathrm{H}^p(\Delta^*; \mathscr{F}_n^{\mathrm{an}})$$

are isomorphisms. We're now done by (2.2.7). □

2.3 Remark — The adic Artin comparison isomorphism also holds for K- and $\overline{K}$-sheaves (recall K is the fraction field of Λ). To deduce this from the result for Λ-sheaves (2.1) one only needs to check that

$$\mathrm{R}^p f_!^{\mathrm{an}}(K \otimes_\Lambda \mathscr{F}_\bullet^{\mathrm{an}}) \simeq K \otimes_\Lambda \mathrm{R}^p f_!^{\mathrm{an}}(\mathscr{F}_\bullet^{\mathrm{an}}), \quad \mathrm{R}^p f_*^{\mathrm{an}}(K \otimes_\Lambda \mathscr{F}_\bullet^{\mathrm{an}}) \simeq K \otimes_\Lambda \mathrm{R}^p f_*^{\mathrm{an}}(\mathscr{F}_\bullet^{\mathrm{an}}),$$

and similarly with K replaced by $\overline{K}$. For pushforward with proper supports this follows from the fact that $\mathrm{R}^p f_!^{\mathrm{an}}$ preserves colimits, as $K, \overline{K}$ can be written as a colimit of finite free Λ-modules. As with (2.1), the situation for ordinary pushforward is more subtle and will not be discussed here.

References

[Conrad] Brian Conrad, *Étale cohomology*, unpublished notes.

[Stacks] The Stacks Project Authors, *Stacks Project*, http://stacks.math.columbia.edu, 2017.